Rigorous results for the Stigler-Luckock model for the evolution of an order book

Jan M. Swart

Institute of Information Theory and Automation of the ASCR (UTIA)
Pod vodárenskou věží 4
18208 Praha 8
Czech Republic
e-mail: swart@utia.cas.cz

May 5, 2016

Abstract

In 1964, G.J. Stigler introduced a stochastic model for the evolution of an order book on a stock market. This model was independently rediscovered and generalized by H. Luckock in 2003. In his formulation, traders place buy and sell limit orders of unit size according to independent Poisson processes with possibly different intensities. Newly arriving buy (sell) orders are either immediately matched to the best available matching sell (buy) order or stay in the order book until a matching order arrives. Assuming stationarity, Luckock showed that the distribution functions of the best buy and sell order in the order book solve a differential equation, from which he was able to calculate the position of two prices $J_- < J_+$ such that buy orders below J_- and sell orders above J_+ stay in the order book forever while all other orders are eventually matched. We extend Luckock's model by adding market orders, i.e., with a certain rate traders arrive at the market that take the best available buy or sell offer in the order book, if there is one, and do nothing otherwise. We give necessary and sufficient conditions for such an extended model to be positive recurrent and show how these conditions are related to the prices J_- and J_+ of Luckock.

MSC 2010. Primary: 82C27; Secondary: 60K35, 82C26, 60J05
Keywords. Continuous double auction, order book, rank-based Markov chain, self-organized criticality, Stigler-Luckock model.
Acknowledgments. Work sponsored by GAČR grants 16-15238S and P201/12/2613.

Contents

1 Introduction and results

1.1 Definition of the model

We will be interested in a stochastic model for traders interacting through an order book as is commonly used on a stock market or commodity market. In the more theoretical economic literature, the sort of trading system we are interested in is also known as the continuous double auction. In our specific model of interest, traders arrive according to independent Poisson processes and place either a buy or sell limit order for exactly one item of a certain stock or commodity. If the order book already contains a suitable offer, then the new limit order is immediately matched with the best available offer, i.e., a new buy limit order at a price x is cancelled against an existing sell limit order at the lowest possible price $x' \leq x$, if such a sell limit order exists, and vice versa for new sell limit orders. Orders that are not immediately matched stay in the order book until they are matched with a new incoming order, or, if such an order never comes, forever. This model, in discrete time and for a specific choice of the parameters, was invented by Stigler [Sti64] and, in its full generality, independently by Luckock [Luc03]; a special case of the model was again independently reinvented by Plačková in her master thesis [Pla11]. We will generalize the model by also allowing market orders, i.e., with a certain rate a trader arrives that takes the best available limit buy (sell) order in the order book, if such an order exists, and does nothing otherwise.

To formulate this model in more mathematical detail, let $I = (I_-, I_+) \subset \mathbb{R}$ a nonempty open interval, modelling the possible prices of limit orders, and let $\overline{I} := [I_-, I_+] \subset [-\infty, \infty]$ denote its closure. Let $\lambda_\pm : \overline{I} \to [0, \infty)$ be functions such that:

(A1) λ_- is nonincreasing and left-continuous, while λ_+ is nondecreasing and right-continuous.

(A2) $\lim_{x \downarrow I_-} \lambda_-(x) = \lambda_-(I_-)$ and $\lim_{x \uparrow I_+} \lambda_+(x) = \lambda_+(I_+)$.

We interpret $\lambda_-(x)$ and $\lambda_+(x)$ as the *demand* and *supply* functions, which describe how many items per time unit traders are willing to buy or sell at the price level x. More precisely, let $\mu_\pm$ be finite measures on $\overline{I}$ such that

$$\mu_-\big([x, I_+]\big) = \lambda_-(x) \quad \text{and} \quad \mu_+\big([I_-, x]\big) = \lambda_+(x) \qquad (x \in \overline{I}). \tag{1.1}$$

Then the restriction of μ_- (resp. μ_+) to I will be the Poisson intensity at which traders place buy (resp. sell) limit orders at a given price, while $\mu_-(\{I_+\})$ (resp. $\mu_+(\{I_-\})$) will be the Poisson intensity at which traders place buy (resp. sell) market orders. Note that $\mu_-(\{I_-\}) = 0 = \mu_+(\{I_+\})$ by assumption (A2).

We let τ_k ($k \geq 1$) denote the time when the k-th trader arrives at the market, we let $\sigma_k \in \{-,+\}$ be a random variable that indicates whether this trader wants to buy ($-$) or sell ($+$), and we let $U_k \in \overline{I}$ denote the price associated with this trader, where $U_k \in I$ for limit orders and $U_k = I_\pm$ for market orders. Then

$$\Pi = \{(U_k, \sigma_k, \tau_k) : k = 1, 2, \ldots\} \quad \text{with} \quad 0 < \tau_1 < \tau_2 < \cdots \tag{1.2}$$

is a Poisson point process on $\overline{I} \times \{-,+\} \times [0,\infty)$ with intensity $\mu \otimes \ell$, where ℓ is the Lebesgue measure on $[0,\infty)$ and μ is the finite measure on $\overline{I} \times \{-,+\}$ given by $\mu\big(\{\sigma\} \times A\big) = \mu_\sigma(A)$ for all $\sigma \in \{-,+\}$ and measurable $A \subset \overline{I}$. We let

$$|\mu_\pm| := \mu_\pm(\overline{I}) \quad \text{and} \quad |\mu| := \mu\big(\overline{I} \times \{-,+\}\big) = |\mu_-| + |\mu_+| \tag{1.3}$$

denote the total masses of the measures $\mu_\pm$ and μ. To avoid trivialities, we assume that $|\mu| \neq 0$. Our assumption that the point process Π in (1.2) is Poisson with intensity $\mu \otimes \ell$ implies that $(\tau_k - \tau_{k-1})_{k \geq 1}$ are i.i.d. exponentially distributed with mean $1/|\mu|$. Moreover, the random variables $(U_k, \sigma_k)_{k \geq 1}$ are i.i.d. with law $\overline{\mu} := |\mu|^{-1}\mu$ and independent of $(\tau_k)_{k \geq 1}$.

We represent the state of the order book at a time $t \geq 0$ by a signed counting measure on I, i.e., a measure of the form

$$\mathcal{X} = \sum_{x \in \mathrm{supp}(\mathcal{X})} n_x \delta_x, \tag{1.4}$$

where $\mathrm{supp}(\mathcal{X}) \subset I$ is a countable set, δ_x denotes the delta measure at x, and $n_x \in \mathbb{Z} \backslash \{0\}$ is an integer that indicates how many buy ($-$) or sell ($+$) limit orders there are in the order book at the price x. We let $\mathcal{S}_{\mathrm{ord}}$ denote the set of all signed measures of the form (1.4) such that moreover

$$\begin{array}{rl} \text{(i)} & \text{there are no } x, y \in I \text{ such that } x < y,\ \mathcal{X}(\{x\}) > 0,\ \mathcal{X}(\{y\}) < 0. \\ \text{(ii)} & \text{the set } \{x \in I : \mathcal{X}(\{x\}) < 0\} \text{ is a locally finite subset of } (I_-, I_+], \\ & \text{i.e., its only possible cluster point is } I_-. \\ \text{(iii)} & \text{the set } \{x \in I : \mathcal{X}(\{x\}) > 0\} \text{ is a locally finite subset of } [I_-, I_+), \\ & \text{i.e., its only possible cluster point is } I_+. \end{array} \tag{1.5}$$

For any $\mathcal{X} \in \mathcal{S}_{\mathrm{ord}}$, we let

$$\begin{aligned} M_-(\mathcal{X}) &:= \max\big(\{I_-\} \cup \{x \in I : \mathcal{X}(\{x\}) < 0\}\big), \\ M_+(\mathcal{X}) &:= \min\big(\{I_+\} \cup \{x \in I : \mathcal{X}(\{x\}) > 0\}\big), \end{aligned} \tag{1.6}$$

which can be interpreted as the highest bid and lowest ask price in the order book.

The state of our Markov process changes only at the times $\tau_1, \tau_2, \ldots$ and we denote the corresponding embedded Markov chain by

$$X_k := \mathcal{X}_{\tau_k} \qquad (k \geq 0) \qquad \text{with} \quad \tau_0 := 0. \tag{1.7}$$

Our previous informal description of the model then translates into the following definition. Given the initial state $X_0 \in \mathcal{S}_{\mathrm{ord}}$, we inductively define $(X_k)_{k \geq 1}$ as

$$X_k := L_{U_k, \sigma_k}(X_{k-1}) \qquad (k \geq 1), \tag{1.8}$$

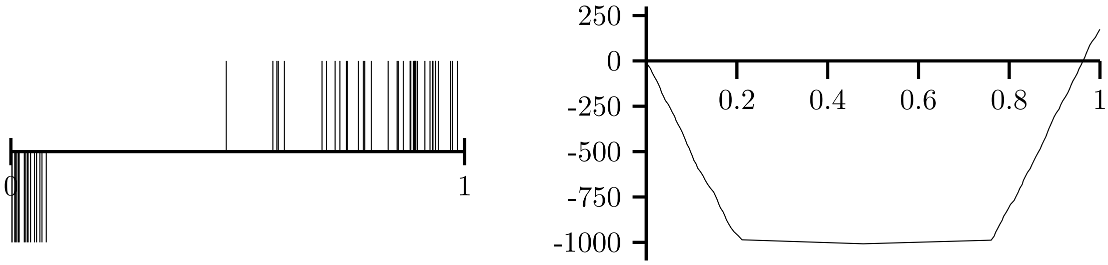

Figure 1: Simulation of the "uniform" Stigler-Luckock model with $I = (0,1)$, $\lambda_-(x) = 1 - x$, and $\lambda_+(x) = x$. On the left: the state X_{250} of the order book after the arrival of 250 traders, starting from the empty initial state. On the right: the distribution function $x \mapsto X_k([0,x])$ of the random signed measure X_k after the arrival of $k = 10,000$ traders.

where for each $(u,\sigma) \in \overline{I} \times \{-,+\}$, we define a "Luckock map" $L_{u,\sigma} : \mathcal{S}_{\mathrm{ord}} \to \mathcal{S}_{\mathrm{ord}}$ by

$$L_{u,\sigma}(\mathcal{X}) := \begin{cases} \mathcal{X} - \delta_{u \wedge M_+(\mathcal{X})} & \text{if } \sigma = -,\ u \wedge M_+(\mathcal{X}) \in I, \\ \mathcal{X} + \delta_{u \vee M_-(\mathcal{X})} & \text{if } \sigma = +,\ u \vee M_-(\mathcal{X}) \in I, \\ \mathcal{X} & \text{otherwise.} \end{cases} \tag{1.9}$$

For example, if $\sigma = +$, then this says that a new sell limit order is added at the price u, unless the current best buy offer $M_-(\mathcal{X})$ is higher than u, in which case this offer is taken, which amounts to adding a delta measure at $M_-(\mathcal{X})$. The rules for market orders are the same, except that these are not added to the order book if no suitable buy offer exists.

It is easy to see that $(X_k)_{k\geq 0}$ is a Markov chain; in fact, using terminology from [LPW09], we have just given a *random mapping representation* for it. We call the Markov chain in (1.8) or, more or less equivalently, the corresponding continuous-time Markov process $(\mathcal{X}_t)_{t\geq 0}$ the *Stigler-Luckock model* with parameters $\lambda_\pm$. In the special case that there are no market orders, this is the model introduced in [Luc03]. The authors [Sti64, Pla11] considered only the case that $\mu_- = \mu_+$ is the uniform distribution on a set of 10, resp. 100 prices. As we will see, the introduction of market orders is natural also from a mathematical point of view and helps us understand the model without market orders.

We will sometimes need the following stronger conditions on our demand and supply functions.

(A3) λ_- is nonincreasing, λ_+ is nondecreasing, and both are continuous on $\overline{I}$.

(A4) The function $\lambda_+ - \lambda_-$ is strictly increasing on $\overline{I}$.

(A5) The functions λ_- and λ_+ are > 0 on I.

(A6) The rates $\lambda_+(I_-)$ and $\lambda_-(I_+)$ of market orders are both > 0.

(A7) The rates $\lambda_+(I_-)$ and $\lambda_-(I_+)$ of market orders are both $= 0$.

In particular, (A3) implies (A1) and (A2). As shown in Appendix A.1, (A3) and (A4) are not really a restriction, since every Stigler-Luckock model satisfying (A1) and (A2) can be obtained as a function of a Stigler-Luckock model satisfying (A3) and, under mild extra assumptions, also (A4). Condition (A5) also comes basically without loss of generality, since sell orders on the right of the first point x where $\lambda_-(x) = 0$ are trivially never matched, and similarly for buy orders at the other end of the interval. The conditions (A6) and (A7) are restrictive, of course. Condition (A7) corresponds to the original model as introduced by Luckock. As we will see in Section 1.4, to understand the behavior of such a model, it is often useful to consider "restricted" models that are obtained by restricting the functions $\lambda_\pm$ to a subinterval J with $\overline{J} \subset I$. Under the condition (A5), such restricted models naturally satisfy (A6).

1.2 Luckcock's differential equation

The following theorem is essentially proved in [Luc03], but for completeness we will provide a proof in the present setting. Below, if $f : \overline{I} \to \mathbb{R}$ is a continuous function of bounded variation, then we let $\mathrm{d}f$ denote the signed measure on $\overline{I}$ such that $\mathrm{d}f\big((x, y]\big) := f(x)-f(y)$. If $g : \overline{I} \to \mathbb{R}$ is a bounded measurable function, then $g\,\mathrm{d}f$ denotes the measure $\mathrm{d}f$ weighted with g, i.e., $g\,\mathrm{d}f\big((x, y]\big) := \int_x^y g\,\mathrm{d}f$. We call any pair (f_-, f_+) of continuous functions of bounded variation such that (1.11) below holds a *solution to Luckock's equation.*

Theorem 1 (Luckock's differential equation) *Consider a Stigler-Luckock model with supply and demand functions $\lambda_\pm : \overline{I} \to [0, \infty)$ satisfying (A3). Assume that the model has an invariant law on $\mathcal{S}_{\rm ord}$ and let $(\mathcal{X}_k)_{k\geq 0}$ denote the corresponding stationary process. Then the functions $f_\pm : \overline{I} \to \mathbb{R}$ defined by*

$$f_-(x) := \mathbb{P}\big[M_-(X_k) \leq x\big] \quad \textit{and} \quad f_+(x) := \mathbb{P}\big[M_+(X_k) \geq x\big] \qquad (x \in \overline{I}) \tag{1.10}$$

are continuous and solve the equations

$$\begin{aligned} &\text{(i)} \quad f_-\mathrm{d}\lambda_+ + \lambda_-\mathrm{d}f_+ = 0, \\ &\text{(ii)} \quad f_+\mathrm{d}\lambda_- + \lambda_+\mathrm{d}f_- = 0, \\ &\text{(iii)} \quad f_-(I_+) = 1 = f_+(I_-). \end{aligned} \tag{1.11}$$

We remark that although Theorem 1 shows that the equilibrium distributions of the best buy and sell order in the order book can more or less be solved explicitly (depending on our ability to solve (1.11)), this does not automatically mean that Stigler-Luckock models as a whole are "solvable". For example, we do not know how to explicitly calculate the joint distribution of $M_-(\mathcal{X})$ and $M_+(\mathcal{X})$ (as opposed to its marginals). Also, it seems to be quite hard to get information about the equilibrium distribution of seemingly simple functions of the process like the number of sell (or buy) limit orders in a certain interval.

Theorem 1 motivates the study of solutions to Luckock's equation (1.11).

Proposition 2 (Solutions to Luckock's equation) *Assume (A3) and (A6). Then Luckock's equation has a unique solution (f_-, f_+). One has*

$$\begin{aligned} &\text{(i)} \quad f_-(I_-) \geq 0 \quad \Leftrightarrow \quad \Lambda_- := \frac{1}{\lambda_-(I_-)\lambda_-(I_+)} - \int_{I_-}^{I_+} \frac{1}{\lambda_+}\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) \geq 0, \\ &\text{(ii)} \quad f_+(I_+) \geq 0 \quad \Leftrightarrow \quad \Lambda_+ := \frac{1}{\lambda_+(I_-)\lambda_+(I_+)} + \int_{I_-}^{I_+} \frac{1}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) \geq 0. \end{aligned} \tag{1.12}$$

Both formulas also hold with the inequality signs reversed. The functions (f_-, f_+) also satisfy

$$\lambda_+(I_+) - f_-(I_-)\lambda_+(I_-) = \lambda_-(I_-) - f_+(I_+)\lambda_-(I_+). \tag{1.13}$$

If the solution (f_-, f_+) to Luckock's equation satisfies $f_-(I_-) \wedge f_+(I_+) \geq 0$, then we call such a solution *valid.* See Figure 3 for a plot of (f_-, f_+) for one particular model -in this particular example, (f_-, f_+) is not valid. By Theorem 1, a necessary condition for a Stigler-Luckock model to have an invariant law is that the solution to Luckock's equation is valid. We conjecture that this condition is also sufficient, but stop short of proving this. (Note however Theorem 3 below, which goes some way in this direction.)

If a Stigler-Luckock model has an invariant law, then the quantity in (1.13) can be interpreted as the *volume of trade*, i.e., the expected number of orders (of either type) that are matched per unit of time. Indeed, since the process has an invariant law, buy limit orders, which arrive at rate $\lambda_+(I_+) - \lambda_+(I_-)$, are all eventually matched, while $1 - f_-(I_-) = \mathbb{P}[M_-(\mathcal{X}_t) > I_-]$ is the fraction of buy market orders that are matched, so $\big(\lambda_+(I_+) - \lambda_+(I_-)\big) + \big(1 - f_-(I_-)\big)\lambda_+(I_-)$ is the total rate at which buy orders are matched, which equals the left-hand side of (1.13). The right-hand side of (1.13) has a similar interpretation in terms of sell orders.

1.3 Positive recurrence

Let $(X_k)_{k\geq 0}$ be a Stigler-Luckock model with discrete time (i.e., the embedded Markov chain from (1.7)), started in the empty initial state $X_0 = 0$, and let τ denote its first return time to 0, i.e., $\tau := \inf\{k > 0 : X_k = 0\}$. We say that a Stigler-Luckock model is *positive recurrent* if $\mathbb{E}[\tau] < \infty$, *transient* if $\mathbb{P}[\tau = \infty] > 0$, and *null recurrent* in the remaining case. The main result of the present paper is the following result, that gives a more or less complete characterization of positive recurrent Stigler-Luckock models. Below and in what follows, we let $\mathcal{S}^{\rm fin}_{\rm ord}$ denote the set of all finite configurations $\mathcal{X} \in \mathcal{S}_{\rm ord}$, i.e., those for which $\mathcal{X}^-$ and $\mathcal{X}^+$ are finite measures.

Theorem 3 (Positive recurrence) *Assume (A3) and (A6). Then a Stigler-Luckock model is positive recurrent if and only if the unique solution (f_-, f_+) to Luckock's equation satisfies $f_-(I_-) \wedge f_+(I_+) > 0$. If a Stigler-Luckock model is positive recurrent, then it has an invariant law ν that is concentated on $\mathcal{S}^{\rm fin}_{\rm ord}$. Moreover, the process started in any initial law that is concentated on $\mathcal{S}^{\rm fin}_{\rm ord}$ satisfies*

$$\big\|\mathbb{P}[X_k \in \cdot\,] - \nu\big\| \underset{k\to\infty}{\longrightarrow} 0, \tag{1.14}$$

where $\|\cdot\|$ denotes the total variation norm.

1.4 Restricted models

Assume that the demand and supply functions λ_- and λ_+ satisfy (A3) and (A5). Then, for each interval $J = (J_-, J_+)$ such that $\overline{J} = [J_-, J_+] \subset I$, the restrictions of λ_- and λ_+ to $\overline{J}$ satisfy (A6). We call the corresponding Stigler-Luckock model the *restricted model on J*. By Proposition 2, Luckock's equation has a unique solution for this restricted model, and by Theorem 3 we can read off from this solution whether the restricted model is positive recurrent. In the present section, for fixed I and $\lambda_\pm$, we investigate the set of all subintervals $\overline{J} \subset I$ for which the restricted model is positive recurrent.

We note that if $(X_k)_{k\geq 0}$ is a Stigler-Luckock model on I and $X_k\big|_J$ denotes the restriction of the random signed measure X_k to a subinterval $J \subset I$, then it is in general not true that $(X_k\big|_J)_{k\geq 0}$ is a Markov chain. In particular, this is not the same as the restricted model on J. Nevertheless, we will see that under suitable conditions, there exists a special subinterval $J \subset I$ (below, this is called the *critical window*) such that in the long run, we expect the model on I to behave basically like the model restricted to J, with all buy limit orders on the left of J and all sell limit orders on the right of J never being matched and as a result staying in the order book forever.

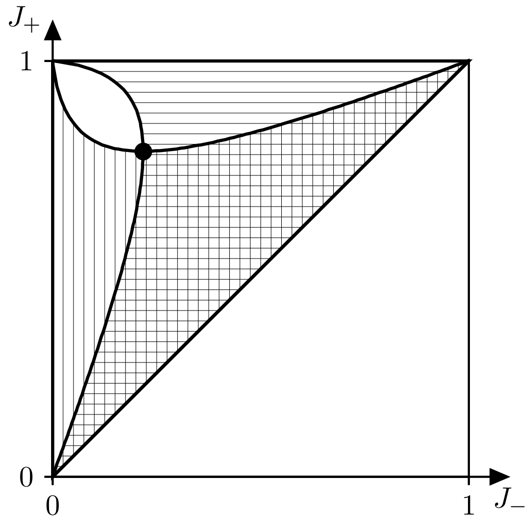

Figure 2: Restrictions of the uniform Stigler-Luckock model with $I = (0, 1)$, $\lambda_-(x) = 1 - x$, and $\lambda_+(x) = x$ to a subinterval (J_-, J_+). The solution (f_-, f_+) to Luckock's equation for the restricted model satisfies $f_+(J_+) > 0$ in the vertically striped area and $f_-(J_-) > 0$ in the horizontally striped area. The intersection of these areas corresponds to the set R of subintervals on which the restricted model is positive recurrent. The intersection of the curves $J_- = \phi_-(J_+)$ and $J_+ = \phi_+(J_-)$, indicated with a dot, corresponds to the critical window.

Assume (A3) and (A5) and for $I_- < J_- < J_+ < I_+$, let $\Lambda_-(J_-, J_+)$ and $\Lambda_+(J_-, J_+)$ denote the expressions in (1.12), calculated for the process restricted to the subinterval $\overline{J}$. For fixed $J_- \in I$ resp. $J_+ \in I$, we define

$$\begin{aligned} \phi_-(J_+) &:= \sup\big\{J_- \in (I_-, J_+) : \Lambda_-(J_-, J_+) \leq 0\big\}, \\ \phi_+(J_-) &:= \inf\big\{J_+ \in (J_-, I_+) : \Lambda_+(J_-, J_+) \leq 0\big\}, \end{aligned} \tag{1.15}$$

with the conventions $\sup \emptyset := I_-$ and $\inf \emptyset := I_+$. Let

$$R := \big\{(J_-, J_+) \in I \times I : J_- < J_+ \text{ and the restricted model on } J \text{ is positive recurrent}\big\}. \tag{1.16}$$

The following lemma says that the set R is bounded by the graphs of the functions $\phi_\pm$, as well as (trivially) the line $J_- = J_+$.

Lemma 4 (Positive recurrence of restricted models) *Assume (A3) and (A5). Then $\phi_-(J_+) < J_+$ and $J_- < \phi_+(J_-)$ for all $J_-, J_+ \in I$. Moreover, a point $(J_-, J_+) \in I \times I$ belongs to the set R from (1.16) if and only $\phi_-(J_+) < J_-$, $J_+ < \phi_+(J_-)$, and $J_- < J_+$.*

In Figure 2 we have pictured the set R and the graphs of the functions $\phi_\pm$ for the "uniform" model with $I = [0, 1]$, $\lambda_-(x) = 1 - x$, and $\lambda_+(x) = 1$. For this model, one can check that the solution of Luckock's equation for the restricted model on J satisfies $f_-(J_-) = 0$ if and only if $J_- = \phi_-(J_+)$, and likewise one has $f_+(J_+) = 0$ if and only if (J_-, J_+) lies on the graph $\{J_+ = \phi_+(J_-)\}$. The graphs of the functions $\phi_\pm$ intersect in a unique point, which in the light of (1.13) must satisfy $\lambda_-(J_-) = \lambda_+(J_+)$.

These observations motivate the following definition. Assume (A3), (A4), (A5) and (A7) and let $J = (J_-, J_+)$ be a subinterval such that $\overline{J} \subset I$. Then we say that J is a *critical window* if it satisfies the following conditions.

(i) The solution to Luckock's equation restricted to J satisfies $f_-(J_-) = 0 = f_+(J_+)$.

(ii) $\lambda_- < \lambda_-(J_-)$ on $(J_-, J_+]$ and $\lambda_+ < \lambda_+(J_+)$ on $[J_-, J_+)$.

We will see that such critical windows exist for a large class of models, and if they exist, they are unique. It follows from (A3)–(A5) and (A7) that there exists a unique point $x_{\rm W} \in I$ such that

$$\lambda_-(x_{\rm W}) = \lambda_+(x_{\rm W}). \tag{1.17}$$

Classical economic theory going back to Walras [Wal74] says that in an infinitely liquid market, the equilibrium price is $x_{\rm W}$, which is why we call $x_{\rm W}$ the *Walrasian price*. We call

$$V_{\rm W} := \lambda_-(x_{\rm W}) = \lambda_+(x_{\rm W}) = \sup_{x \in \overline{I}} \lambda_-(x) \wedge \lambda_+(x) \tag{1.18}$$

the *Walrasian volume of trade*. We also define

$$V_{\rm max} := \lambda_-(I_-) \wedge \lambda_+(I_+), \tag{1.19}$$

which is a natural upper bound on the volume of trade that is possible in any trading system. For any $V \in [V_{\rm W}, V_{\rm max}]$, we define

$$j_-(V) = \sup\big\{x \in \overline{I} : \lambda_-(x) \geq V\big\} \quad \text{and} \quad j_+(V) = \inf\big\{x \in \overline{I} : \lambda_+(x) \geq V\big\}. \tag{1.20}$$

We define a function $\Psi : [V_{\rm W}, V_{\rm max}] \to [-\infty, \infty)$ by

$$\Psi(V) := \frac{1}{V_{\rm W}^2} + \int_{V_{\rm W}}^{V} \Big\{\frac{1}{\lambda_+\big(j_-(W)\big)} + \frac{1}{\lambda_-\big(j_+(W)\big)}\Big\} \mathrm{d}\Big(\frac{1}{W}\Big). \tag{1.21}$$

Note that Ψ is nonincreasing since $1/W$ is a nonincreasing function. We will prove in Subsection 2.5 below that

$$\Psi(V) = \Lambda_-\big(j_-(V), j_+(V)\big) = \Lambda_+\big(j_-(V), j_+(V)\big) \qquad (I_- < j_-(V) \leq j_+(V) < I_+). \tag{1.22}$$

We call the quantity

$$V_{\rm L} := \sup\big\{V \in [V_{\rm W}, V_{\rm max}] : \Psi(V) \geq 0\big\} \tag{1.23}$$

Luckock's volume of trade.

Proposition 5 (Critical window) *Assume (A3)–(A5) and (A7) and set* $J = (J_-, J_+) := \big(j_-(V_{\rm L}), j_-(V_{\rm L})\big)$*. If the Stigler-Luckock model has a critical window, then it is* J*. Conversely, if* $\Psi(V_{\rm L}) = 0$ *and* $\overline{J} \subset I$*, then* J *is a critical window.*

Since $V_{\rm L}$ us usually much larger than $V_{\rm W}$, the Stigler-Luckock model is highly non-liquid. As such, it is not a realistic model of a real market, though it may be a useful first step towards building more realistic models. The special case where buy and sell limit orders are uniformly distributed on the unit interval is of some special interest. Numerically, the constant $V_{\rm L}$ from Lemma 6 is given by $V_{\rm L} \approx 0.78218829428020$.

Lemma 6 (Uniform model) *The Stigler-Luckock model with* $\overline{I} = [0, 1]$, $\lambda_-(x) = 1 - x$, *and* $\lambda_+(x) = x$ *has a critical window* (J_-, J_+) *which is given by* $1 - J_- = J_+ = V_{\rm L}$, *where* $V_{\rm L} = 1/z$ *with* z *the unique solution of the equation* $e^{-z} - z + 1 = 0$.

1.5 Discussion and open problems

In simulations (see Figure 1), the uniform Stigler-Luckock model of Lemma 6 shows interesting behavior. Starting from any finite initial state, it seems that

$$\liminf_{k\to\infty} M_-(X_k) = J_- \quad \text{and} \quad \limsup_{k\to\infty} M_+(X_k) = J_+ \quad \text{a.s.,} \tag{1.24}$$

where $J = (J_-, J_+)$ is the critical window. Moreover, it seems that if $X_k\big|_J$ denotes the restriction of X_k to J, then the law of $X_k\big|_J$ converges as $k \to \infty$ to a limit law that is concentrated on $\mathcal{S}_{\rm ord}$. It seems likely that this limit law is an invariant law for the restricted model on J. Indeed, (1.24) says that in the long run, the price of the best buy offer never drops below J_- and the price of the best sell offer never climbs above J_+, which allows us to treat limit sell orders at prices below J_- and limit buy orders above J_+ as market orders.

Proving the conjectures mentioned above, such as (1.24), remains an open problem. Theorems 1 and 3 allow us to conclude, however, that for each $\varepsilon > 0$, the restricted model on $(J_- - \varepsilon, J_+ + \varepsilon)$ does not have an invariant law while the restricted model on $(J_- + \varepsilon, J_+ - \varepsilon)$ is positive recurrent. Further motivation for the conjectures comes from the study of similar models. In [Swa15], a "one-sided canyon model" is studied that is in many ways similar to the Stigler-Luckock model except that there is only one type of points as opposed to the two types (buy and sell orders) of a Stigler-Luckock model. This "one-sided" model also has a critical window that can be calculated explicitly and in fact the analogues of the conjectures above have all been proved for this model, mainly due to the hugely simplifying fact that for this model, restricting the process to a smaller interval does again yield a Markov chain.

In this context, we also mention a model for email communication due to Gabrielli and Caldarelli [CG09]. This model is even simpler than the previous one since not only is the restriction of the process to a subinterval Markovian, but even just counting the number of points in a subinterval already yields a Markov chain. For this model, it has been possible to solve subtle questions about the behavior of the stationary process near the boundary of the critical window [FS15].

The models mentioned so far belong to a wider class of models that also includes the Bak Sneppen model [BS93] and its modified version from [MS12], as well as the branching Brownian motions with strong selection treated in [Mai13]. All these models implement some version of the rule "kill the lowest particle" and seem to exhibit self-organized criticality, although this has been rigorously proved only for some of the models.

As mentioned before, the Stigler-Luckock model describes an extremely non-liquid market, and (mainly) for that reason is not a realistic model for a real market, although it may perhaps be used as a first step towards more realistic models. In recent years, there has been considerable activity in the search for simple, yet realistic models for an order book. We refer the reader to Chapter 4 of the book [Sla13] and also to [Mas00, Kru12, Smi12, SRR16], and references therein, for a more complete view on this topic.

1.6 Methods

The results in Sections 1.2 and 1.4 are mainly a reworking of similar results already proved by Luckock in [Luc03], although Proposition 5 is a significant improvement over [Luc03, Prop. 4]. Nevertheless, Luckock already derived the differential equation (1.11) and showed how it could be used to calculate the critical window for a given model. Throughout his paper, however, he takes stationarity as a model assumption, where in fact, "stationarity" for him means the existence of two prices $J_- < J_+$ such that buy orders on the left of J_- and sell orders on the right of J_+ are never matched while the process inside (J_-, J_-) is stationary in law.

From a mathematical point of view, the existence of such a stationary process requires proof. Moreover, one would like to prove that the process started in an arbitrary finite initial state converges, in a suitable sense, to such a stationary state. For Luckock's original model, these problems remain open, but for positive recurrent processes with market orders, these questions are resolved by Theorem 3, which is the most important contribution of the present paper.

The proof of Theorem 3 is based on a Lyapunov function. Equip the space $\mathcal{S}_{\rm ord}$ with the topology of vague convergence and the associated Borel σ-algebra. For any bounded measurable function $F : \mathcal{S}_{\rm ord} \to \mathbb{R}$, write

$$GF(\mathcal{X}) := \int \big\{F\big(L_{u,\sigma}(\mathcal{X})\big) - F\big(\mathcal{X}\big)\big\}\,\mu\big(\mathrm{d}(u,\sigma)\big), \tag{1.25}$$

where $L_{u,\sigma}$ is the Luckock map defined in (1.9) and μ is the measure defined below (1.2). Then G is the generator of the continuous-time Markov process $(\mathcal{X}_t)_{t\geq 0}$.

It turns out that there is a useful and explicit formula for GF when F is a "linear" function of the form

$$F(\mathcal{X}) := \int_I w_-(x)\mathcal{X}^-(\mathrm{d}x) + \int_I w_+(x)\mathcal{X}^+(\mathrm{d}x) \qquad \big(\mathcal{X} \in \mathcal{S}^{\rm fin}_{\rm ord}\big), \tag{1.26}$$

where $w_\pm : \overline{I} \to \mathbb{R}$ are bounded "weight" functions such that w_- is left-continuous and w_+ is right-continuous. The values of w_- and w_+ in the boundary points I_- and I_+ are irrelevant for (1.26), but for notational convenience, we define $w_-(I_+)$ and $w_+(I_-)$ by left, resp. right continuity, and use the convention that

$$w_-(I_-) := 0 \quad \text{and} \quad w_+(I_+) := 0. \tag{1.27}$$

With this convention, the following lemma describes the action of the generator on linear functions of the form (1.26).

Lemma 7 (Generator on linear functionals) *Assume (A3). Then, for functions of the form (1.26), one has*

$$GF(\mathcal{X}) = q_-\big(M_-(\mathcal{X})\big) + q_+\big(M_+(\mathcal{X})\big) \qquad \big(\mathcal{X} \in \mathcal{S}^{\rm fin}_{\rm ord}\big), \tag{1.28}$$

where $q_\pm : \overline{I} \to \mathbb{R}$ are given by

$$\begin{aligned} q_-(x) &:= \int_x^{I_+} w_+ \mathrm{d}\lambda_+ - w_-(x)\lambda_+(x), \\ q_+(x) &:= -\int_{I_-}^{x} w_- \mathrm{d}\lambda_- - w_+(x)\lambda_-(x). \end{aligned} \tag{1.29}$$

If $w_\pm$ are supported on a compact set $K \subset I$, then (1.28) holds more generally for $\mathcal{X} \in \mathcal{S}_{\rm ord}$.

Proof We observe that $\int_{M_-(\mathcal{X})}^{I_+} w_+ \mathrm{d}\lambda_+$ is the rate at which $F(\mathcal{X})$ increases due to sell limit orders being added to the order book while $w_-\big(M_-(\mathcal{X})\big)\lambda_+\big(M_-(\mathcal{X})\big)$ is the rate at which $F(\mathcal{X})$ decreases due to buy limit orders being removed from the order book. In view of our convention (1.27), the latter term is zero when the order book contains no buy limit orders. The two terms in $q_+(M_+(\mathcal{X}))$in have similar interpretations. ∎

Formula (1.29) tells us how to calculate the functions $q_\pm$ from (1.28) from the weight functions $w_\pm$. It turns out that under the assumptions (A3) and (A6), one can uniquely solve the following inverse problem: if $q_\pm$ are given up to an additive constant, then find $w_\pm$ such that (1.29) holds. This is shown in Theorem 10 below and more specifically for indicator functions of the form $q_- = 1_{[I_-,z]}$ and $q_+ = 1_{[z,I_+]}$ in the following theorem, that moreover specifies the additive constant.

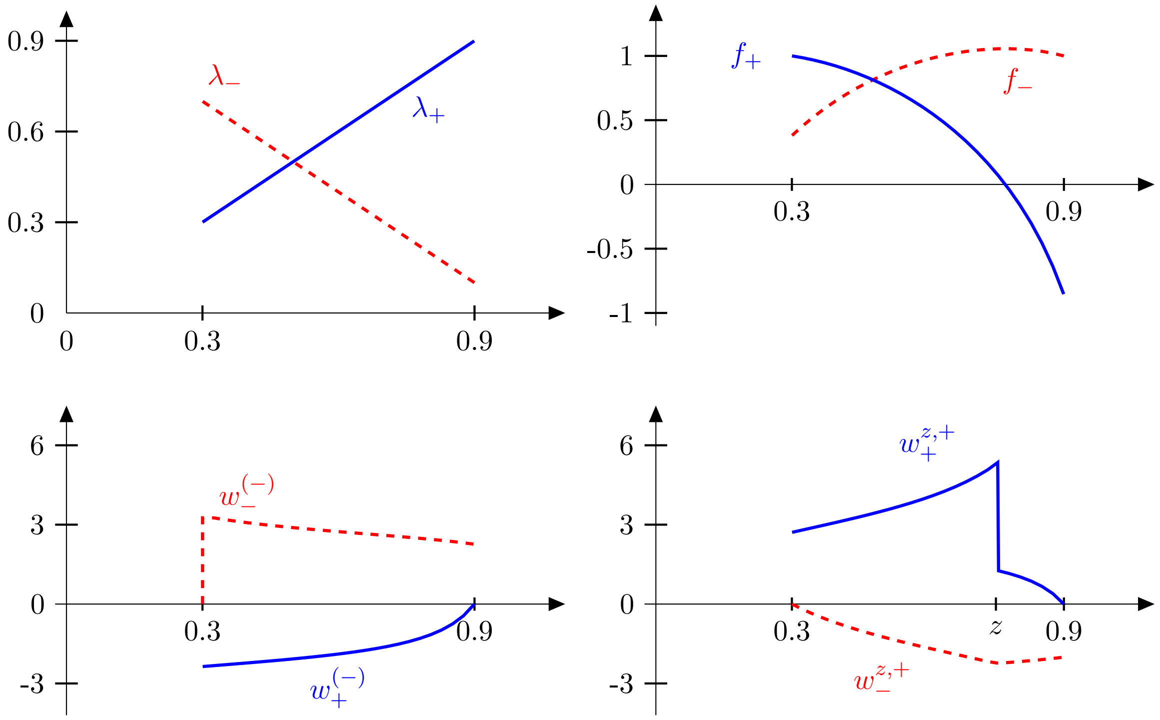

Figure 3: The solution (f_-, f_+) to Luckock's equation, as well as two examples of weight functions (w_-, w_+) as in Theorem 8. In this example, $I = (0.3, 0.9)$, $\lambda_-(x) = 1 - x$, and $\lambda_+(x) = x$. The lower left picture shows the weight functions $w_\pm^{(-)} = w_\pm^{I_-,-}$ while the lower right picture shows the weight functions $w_\pm^{z,+}$ for $z = 0.75$.

Theorem 8 (Special weight functions) *Assume (A3) and (A6). Then, for each* $z \in \overline{I}$, *there exist a unique pair of bounded weight functions* $(w_-^{z,-}, w_+^{z,-}) = (w_-, w_+)$ *such that* w_- *is left-continuous and* w_+ *is right-continuous, and the linear functional* $F^{z,-} = F$ *from (1.26) satisfies*

$$GF(\mathcal{X}) = 1_{\{M_-(\mathcal{X}) \leq z\}} - f_-(z) \qquad \big(\mathcal{X} \in \mathcal{S}_{\rm ord}^{\rm fin}\big), \tag{1.30}$$

where (f_-, f_+) *is the unique solution to Luckock's equation (1.11). Likewise, there exist a unique pair of weight functions* $(w_-^{z,+}, w_+^{z,+}) = (w_-, w_+)$ *such that the linear functional* $F^{z,+} = F$ *from (1.26) satisfies*

$$GF(\mathcal{X}) = 1_{\{M_+(\mathcal{X}) \geq z\}} - f_+(z) \qquad \big(\mathcal{X} \in \mathcal{S}_{\rm ord}^{\rm fin}\big), \tag{1.31}$$

Figure 3 shows plots of weight functions as in Theorem 8 together with the solution of Luckock's equation, for one explicit example of a Stigler-Luckock model. Theorem 8 is closely related to Luckock's result Theorem 1. Indeed, if a Stigler-Luckock model has an invariant law that is concentrated on $\mathcal{S}_{\rm ord}^{\rm fin}$, then the fact that the functions in (1.10) are given by the solution to Luckock's equation follows from Theorem 8 and the equilibrium equation $\mathbb{E}[GF(\mathcal{X}_t)] = 0$.

Theorem 8 is more powerful that Theorem 1, however, since it gives an interpretation to the solution to Luckock's equation even if such a solution is not valid. Also, we have fairly explicit expressions for the weight functions $(w_-^{z,\pm}, w_+^{z,\pm})$ (see Lemma 15 below), and their associated linear functions $F^{z,\pm}$ are useful also in a non-stationary setting. In particular, we will prove Theorem 3 by constructing a Lyapunov function from the functions $F^{I_-,-}$ and $F^{I_+,+}$ (see formula (3.5) below).

We hope that the linear functions $F^{z,\pm}$ from Theorem 8 will also prove useful in future work aimed at resolving the open problems mentioned in Section 1.5. In Appendix A.4, we have recorded some concrete ideas on how the functions $F^{z,\pm}$ could possible be used to attack the conjecture (1.24).

1.7 Outline

In Section 2, we investigate two differential equations: Luckock's equation (1.11) and a differential equation that allows one to solve the weight functions $w_\pm$ in terms of the functions $q_\pm$ from (1.29). In particular, we prove Theorem 8 in Subsection 2.3, Proposition 2 in Subsection 2.4, and Proposition 5 and Lemmas 4 and 6 in Subsection 2.5.

After the preparatory work on the differential equations in Section 2, the analysis of the Markov chain, which is contained in Section 3, is actually quite short. In particular, we prove Theorem 1 in Subsection 3.1 and Theorem 3 in Subsection 3.3.

The paper concludes with four appendices. In Appendix A.1, we show that the assumptions (A3) and (A4) can basically be made without loss of generality. Appendix A.2 collects some facts from the general theory of Markov chains needed to translate the properties of our Lyapunov function into properties of the Markov chain. In Appendix A.3 we have collected (without proof) some formulas for Stigler-Luckock models that take only finitely many values, and that are analogues to our integral formulas for continuous models but cannot easily be deduced from them. Appendix A.4 collects some concrete open problems with some ideas on how to approach them.

2 Analysis of the differential equations

2.1 Lebesgue-Stieltjes integrals

For any interval $J \subset [-\infty, \infty]$ that can be either closed, open, or half open, with left and right boundaries $J_- < J_+$, we let $B(J)$ denote the space of bounded measurable functions $f : J \to \mathbb{R}$ and we let $B_{\rm bv}(J)$ denote the space of functions $f \in B(J)$ that are of bounded variation. For each $f \in B_{\rm bv}(J)$ and $x \in J$, we define

$$f(x-) := \lim_{y \uparrow x} f(y) \quad (x \neq J_-) \quad \text{and} \quad f(x+) := \lim_{y \downarrow x} f(y) \quad (x \neq J_+), \tag{2.1}$$

where the limits exist by the assumption that f is of bounded variation. If $J_- \in J$, then we set $f(J_- -) := f(J_-)$, and we define $f(J_+ +)$ similarly. We let

$$B^{\pm}_{\rm bv}(J) := \big\{ f \in B_{\rm bv}(J) : f(x\pm) = f(x) \ \forall x \in J \big\} \tag{2.2}$$

denote the spaces of left $(-)$ and right $(+)$ continuous functions $f : J \to \mathbb{R}$ of bounded variation. Each $f \in B_{\rm bv}(J)$ defines a finite signed measure $\mathrm{d}f$ on J through the formula

$$\mathrm{d}f([x, y]) := f(y+) - f(x-) \qquad (x, y \in J, \ x \leq y). \tag{2.3}$$

The set of atoms of $\mathrm{d}f$ is the set $\mathcal{D}_f := \{x \in J : f(x-) \neq f(x+)\}$ of points of discontinuity of f. For each finite signed measure ρ on J we can find functions $f \in B^-_{\rm bv}(J)$ and $g \in B^+_{\rm bv}(J)$ such that $\mathrm{d}f = \rho = \mathrm{d}g$, and these functions are unique up to an additive constant. We equip $B^{\pm}_{\rm bv}(J)$ with a topology such that $f_n \to f$ if and only if $\mathrm{d}f_n$ converges weakly to $\mathrm{d}f$ and $f_n(x) \to f(x)$ for at least one (and hence every) point $x \in J \backslash \mathcal{D}_f$. It is known [Hoe77, page 182] that if J is a closed interval, then $f_n \to f$ in this topology if and only if:

$$
\begin{aligned}
&\text{(i)} \quad \sup_n \|\mathrm{d}f_n\| < \infty, \ \text{where } \|\cdot\| \text{ denotes the total variation norm,}\\
&\text{(ii)} \quad \mathrm{d}f_n(J) \to \mathrm{d}f(J),\\
&\text{(iii)} \quad \int_J \big|f_n(x) - f(x)\big|\,\mathrm{d}x \to 0, \ \text{i.e., } f_n \to f \text{ in } L^1 \text{ norm w.r.t. to the Lebesgue measure.}
\end{aligned}
\tag{2.4}
$$

In line with earlier notation, we write $g\,\mathrm{d}f$ to denote the measure $\mathrm{d}f$ weighted with a bounded measurable function g. We will make use of the *product rule* which says that

$$
\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f \qquad \big(f, g \in B_{\mathrm{bv}}(J),\ \mathcal{D}_f \cap \mathcal{D}_g = \emptyset\big), \tag{2.5}
$$

and also of the *chain rule* which tells us that if $f \in B_{\mathrm{bv}}(J)$ takes values in a compact interval K and $F : K \to \mathbb{R}$ is continuously differentiable, then

$$
\mathrm{d}(F \circ f) = (F' \circ f)\,\mathrm{d}f \qquad \big(f \in B_{\mathrm{bv}}(J),\ \mathcal{D}_f = \emptyset\big), \tag{2.6}
$$

where $(F \circ f)(x) := F(f(x))$ denotes the composition of F and f. All our integrals will be of Lebesgue type, which coincides with the Riemann-Stieltjes integral if both functions involved are of bounded variation and do not share points of discontinuity.

If $g : [a, b] \to [-\infty, \infty]$ is a function of bounded variation and $\psi : [a, b] \to [-\infty, \infty]$ is nondecreasing, then we write $\mathrm{d}g \ll \mathrm{d}\psi$ if $\mathrm{d}g$ is absolutely continuous with respect to ψ, i.e., if for each $s \leq t$, $\psi(s-) = \psi(t+)$ implies $g(s-) = g(t+)$. We will sometimes use the *substitution of variables rule*, which says that

$$
\int_a^b f\,\mathrm{d}g = \int_{\psi(a)}^{\psi(b)} (f \circ \psi^{-1})\,\mathrm{d}(g \circ \psi^{-1}) \qquad \big(f \in B[a, b],\ g \in B_{\mathrm{bv}}[a, b]\big), \tag{2.7}
$$

and which holds provided $\psi : [a, b] \to [-\infty, \infty]$ is a nondecreasing function such that $\mathrm{d}g \ll \mathrm{d}\psi$, and $\psi^{-1} : [\psi(a), \psi(b)] \to [a, b]$ is a right inverse of ψ.

As a general reference to these rules, we refer to [CB00, Section 6.2]. In the substitution of variables rule, the condition $\mathrm{d}g \ll \mathrm{d}\psi$ guarantees that $f \circ \psi^{-1} \circ \psi$ differs from f only on a set of measure zero under $\mathrm{d}g$.

We will need one more result that we formulate as a lemma. The result holds in any dimension but since we only need the two-dimensional case, for ease of notation, we restrict to two dimensions.

Lemma 9 (Integrals along curves) *Let $D \subset \mathbb{R}^2$ be a closed, convex set that is the closure of its interior. Let F, g_1, g_2, f_1, f_2 be continuous real functions on D such that f_1 and f_2 are moreover Lipschitz. Assume that for all x_1, x_1', x_2, x_2' such that $x_1 \leq x_1'$, $x_2 \leq x_2'$, and $(x_1, x_2), (x_1', x_2), (x_1, x_2') \in D$,*

$$
\begin{aligned}
F(x_1', x_2) - F(x_1, x_2) &= \int_{x_1}^{x_1'} g_1(\,\cdot\,, x_2)\,\mathrm{d}f_1(\,\cdot\,, x_2)\\
F(x_1, x_2') - F(x_1, x_2) &= \int_{x_2}^{x_2'} g_2(x_1, \,\cdot\,)\,\mathrm{d}f_2(x_1, \,\cdot\,).
\end{aligned}
\tag{2.8}
$$

Let $[t_-, t_+]$ be a closed interval and let $\gamma : [t_-, t_+] \to D$ be a continuous function of bounded variation. Then

$$
F\big(\gamma(t_+)\big) - F\big(\gamma(t_-)\big) = \int_{t_-}^{t_+} \big\{(g_1 \circ \gamma)\,\mathrm{d}(f_1 \circ \gamma) + (g_2 \circ \gamma)\,\mathrm{d}(f_2 \circ \gamma)\big\}. \tag{2.9}
$$

Proof (sketch) Formula (2.8) shows that (2.9) holds for any continuous function $[t_-, t_+] \mapsto \big(\gamma_1(t), \gamma_2(t)\big) \in D$ of bounded variation such that moreover either γ_1 or γ_2 is constant. It follows that (2.9) also holds for any finite concatination of such curves; call such curves simple. Then it is not hard to see that any $\gamma : [t_-, t_+] \to D$ that is continuous and of bounded variation can be approximated by simple curves $\gamma^{(n)}$ in such a way that $\gamma^{(n)}(t_-) \to \gamma(t_-)$ and $\mathrm{d}\gamma_i^{(n)}$ converges weakly to $\mathrm{d}\gamma_i$ for $i = 1, 2$. In particular, this implies that $\gamma^{(n)}$ converges uniformly to γ so by the continuity of g_i $(i = 1, 2)$, also $g_i \circ \gamma^{(n)}$ converges uniformly to $g_i \circ \gamma$. In view of (2.4), the Lipschitz continuity of f_i $(i = 1, 2)$ moreover implies that $\mathrm{d}(f_i \circ \gamma^{(n)})$ converges weakly to $\mathrm{d}(f_i \circ \gamma)$. Using this and the continuity of F, taking the limit in (2.8), which holds for $\gamma^{(n)}$, we obtain that the formula also holds for γ. ∎

2.2 The inverse problem

The main result of the present subsection is the following theorem.

Theorem 10 (Inverse problem) *Assume (A3) and (A6). Then, for each pair of functions* (g_-, g_+) *with* $g_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$, *there exists a unique pair of functions* $(w_-^{(g_-,g_+)}, w_+^{(g_-,g_+)}) = (w_-, w_+)$ *with* $w_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$ *and* $w_\pm(I_\pm) = 0$, *as well as a unique constant* $c(g_-, g_+) \in \mathbb{R}$, *such that the linear functional* $F^{(g_-,g_+)} = F$ *from (1.26) satisfies*

$$GF^{(g_-,g_+)}(\mathcal{X}) = g_-\big(M_-(\mathcal{X})\big) + g_+\big(M_+(\mathcal{X})\big) - c(g_-, g_+). \tag{2.10}$$

The proof of Theorem 10 will be split into a number of lemmas.

Lemma 11 (Differential equation) *Assume (A3) and (A6), let* $g_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$, *and let* $w_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$ *satisfy* $w_\pm(I_\pm) = 0$. *Then the linear function* $F^{(w_-,w_+)}$ *associated with* (w_-, w_+) *satisfies (2.10) for some* $c(g_-, g_+) \in \mathbb{R}$ *if and only if*

$$\begin{aligned} &\text{(i)} \quad w_+\mathrm{d}\lambda_+ + \mathrm{d}(\lambda_+ w_-) = -\mathrm{d}g_-,\\ &\text{(ii)} \quad w_-\mathrm{d}\lambda_- + \mathrm{d}(\lambda_- w_+) = -\mathrm{d}g_+. \end{aligned} \tag{2.11}$$

Proof Defining functions $q_\pm$ as in (1.29), Lemma 7 tells us that (2.10) is satisfied for some $c(g_-, g_+) \in \mathbb{R}$ if and only if there exist real constants $c_\pm$ such that $q_\pm = g_\pm + c_\pm$, or equivalently, if there exist $c'_\pm \in \mathbb{R}$ such that

$$\begin{aligned} g_-(x) &= c'_- + \int_{[x,I_+)} \big\{w_+\mathrm{d}\lambda_+ + \mathrm{d}(w_-\lambda_+)\big\} &&\big(x \in [I_-, I_+)\big),\\ g_+(x) &= c'_+ - \int_{(I_-,x]} \big\{w_-\mathrm{d}\lambda_- + \mathrm{d}(w_+\lambda_-)\big\} &&\big(x \in (I_-, I_+]\big), \end{aligned} \tag{2.12}$$

which is equivalent to (2.11). ∎

We can integrate the differential equation (2.11) explicitly. Let $h_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$ be any pair of functions such that

$$\mathrm{d}h_\pm = -\lambda_\pm \mathrm{d}g_\pm, \tag{2.13}$$

i.e., $h_-(x) = c_- + \int_{[x,I_+)} \lambda_- \mathrm{d}g_-$ and similarly for h_+, where $c_\pm$ are some fixed, but otherwise arbitrary constants.

Lemma 12 (Integrated equation) *Assume (A3) and (A6), let* $g_\pm \in B^\pm_{\rm bv}(\overline{I})$ *be given and let* $h_\pm \in B^\pm_{\rm bv}(\overline{I})$ *be as in (2.13). Then a pair of functions* $w_\pm \in B^\pm_{\rm bv}(\overline{I})$ *satisfies (2.11) if and only if there exists a constant* $\kappa \in \mathbb{R}$ *such that*

$$\begin{aligned} &\text{(i)} && \mathrm{d}w_- = \frac{\kappa + h_+}{\lambda_-}\,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{1}{\lambda_-}\,\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big), \\ &\text{(ii)} && \mathrm{d}w_+ = \frac{\kappa + h_-}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) + \frac{1}{\lambda_+}\,\mathrm{d}\Big(\frac{h_+}{\lambda_-}\Big), \\ &\text{(iii)} && w_- + w_+ = \frac{\kappa + h_- + h_+}{\lambda_-\lambda_+}. \end{aligned} \tag{2.14}$$

Moreover, given (2.14) (iii), the equations (2.14) (i) and (ii) imply each other.

Proof Multiplying the equations (2.11) (i) and (ii) by λ_- and λ_+, respectively, and then adding both equations, using the product rule (2.5) and (2.13), we obtain

$$\mathrm{d}\big(\lambda_-(\lambda_+ w_-)\big) + \mathrm{d}\big(\lambda_+(\lambda_- w_+)\big) = \mathrm{d}h_- + \mathrm{d}h_+, \tag{2.15}$$

which shows that there exists a constant $\kappa \in \mathbb{R}$ such that (2.14) (iii) holds. Given (2.14) (iii), we can rewrite (2.11) (i) as

$$\mathrm{d}(\lambda_+ w_-) = -w_+ \mathrm{d}\lambda_+ + \frac{\mathrm{d}h_-}{\lambda_-} = \Big(w_- - \frac{\kappa + h_- + h_+}{\lambda_-\lambda_+}\Big)\mathrm{d}\lambda_+ + \frac{\mathrm{d}h_-}{\lambda_-}. \tag{2.16}$$

Dividing by λ_+ and reordering terms, this says that

$$\frac{\mathrm{d}(\lambda_+ w_-) - w_- \mathrm{d}\lambda_+}{\lambda_+} = -\frac{(\kappa + h_- + h_+)\,\mathrm{d}\lambda_+}{\lambda_-\lambda_+^2} + \frac{\mathrm{d}h_-}{\lambda_-\lambda_+}, \tag{2.17}$$

which using the product and chain rules (2.5)–(2.6) can be rewritten as (2.14) (i). In a similar way, we see that given (2.14) (iii), (1.11) (ii) is equivalent to (2.14) (ii). Differentiating (2.14) (iii), using the product rule, we obtain

$$\begin{aligned} &\mathrm{d}w_- + \mathrm{d}w_+ \\ &= \frac{\kappa}{\lambda_-}\,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{\kappa}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) + \frac{1}{\lambda_-}\,\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big) + \frac{h_-}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) + \frac{h_+}{\lambda_-}\,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{1}{\lambda_+}\,\mathrm{d}\Big(\frac{h_+}{\lambda_-}\Big), \end{aligned} \tag{2.18}$$

which is the same as we would obtain adding the equations (2.14) (i) and (2.14) (ii). We conclude that (2.14) (i) and (ii) are equivalent given (iii). ∎

For later use, assuming (A3) and (A6), we define a constant Γ by

$$\Gamma := \frac{1}{\lambda_-(I_+)\lambda_+(I_+)} - \int_{I_-}^{I_+} \frac{1}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) = \frac{1}{\lambda_-(I_-)\lambda_+(I_-)} + \int_{I_-}^{I_+} \frac{1}{\lambda_+}\mathrm{d}\Big(\frac{1}{\lambda_-}\Big), \tag{2.19}$$

where the equality of both formulas follows from the product rule (2.5) applied to the functions $1/\lambda_-$ and $1/\lambda_+$. Note that $\Gamma > 0$ since $\mathrm{d}(1/\lambda_-)$ is nonnegative while $\lambda_\pm$ are strictly positive by (A6).

Lemma 13 (Existence and uniqueness) *Assume (A3) and (A6). Then, for each pair of functions* $g_\pm \in B^\pm_{\rm bv}(\overline{I})$*, there exist unique functions* $w_\pm \in B^\pm_{\rm bv}(\overline{I})$ *that solve the differential* *equation (2.11) together with the boundary conditions $w_\pm(I_\pm) = 0$. These functions are given by*

$$
\begin{aligned}
&\text{(i)} \quad w_-(x) = \int_{[I_-,x)} \Big\{ \frac{\kappa + h_+}{\lambda_-} \,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{1}{\lambda_-} \,\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big) \Big\}, \\
&\text{(ii)} \quad w_+(x) = -\int_{(x,I_+]} \Big\{ \frac{\kappa + h_-}{\lambda_+} \,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) + \frac{1}{\lambda_+} \,\mathrm{d}\Big(\frac{h_+}{\lambda_-}\Big) \Big\},
\end{aligned} \tag{2.20}
$$

where $h_\pm$ are as in (2.13) and

$$
\kappa = \kappa(h_-, h_+) := \Gamma^{-1} \Big[\int_{\bar{I}} \Big\{ \frac{h_+}{\lambda_-} \,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{1}{\lambda_-} \,\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big) \Big\} - \frac{h_-(I_+) + h_+(I_+)}{\lambda_-(I_+)\lambda_+(I_+)} \Big], \tag{2.21}
$$

with $\Gamma > 0$ the constant from (2.19).

Proof By Lemma 12, $w_\pm$ solve the difference equation (2.11) together with the left boundary condition $w_-(I_-) = 0$ if and only if there exists a $\kappa \in \mathbb{R}$ such that (2.20) (i) and (2.14) (iii) hold. In view of the latter equation, $w_\pm$ also solves the right boundary condition $w_+(I_+) = 0$ if and only if

$$
w_-(I_+) + 0 = \frac{\kappa + h_- + h_+}{\lambda_-\lambda_+}(I_+). \tag{2.22}
$$

In view of (2.20) (i), this says that

$$
\int_{[I_-,I_+)} \Big\{ \frac{\kappa + h_+}{\lambda_-} \,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{1}{\lambda_-} \,\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big) \Big\} = \frac{\kappa + h_-(I_+) + h_+(I_+)}{\lambda_-(I_+)\lambda_+(I_+)}, \tag{2.23}
$$

or equivalently (note that since h_- is left-continuous, it has no jump at I_+)

$$
\begin{aligned}
&\int_{[I_-,I_+]} \Big\{ \frac{h_+}{\lambda_-} \,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{1}{\lambda_-} \,\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big) \Big\} - \frac{h_-(I_+) + h_+(I_+)}{\lambda_-(I_+)\lambda_+(I_+)} \\
&\qquad = \kappa \Big\{ \frac{1}{\lambda_-(I_+)\lambda_+(I_+)} - \int_{I_-}^{I_+} \frac{1}{\lambda_-} \,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) \Big\},
\end{aligned} \tag{2.24}
$$

which by the fact that the constant Γ from (2.19) is nonzero is equivalent to (2.21). ∎

Proof of Theorem 10 Immediate from Lemmas 11 and 13. ∎

2.3 Luckock's equation

In the present subsection we prove Theorem 8. We start by proving that Luckock's equation has a unique solution. By definition, a solution to *Luckcock's equation* is a pair functions (f_-, f_+) such that $f_\pm \in B^\mp_{\mathrm{bv}}(\bar{I})$ and (1.11) holds.

Lemma 14 (Luckock's equation) *Assume (A3) and (A6). Then Luckock's equation has a unique solution (f_-, f_+), which is given by*

$$
\begin{aligned}
&\text{(i)} \quad \Big(\frac{f_+}{\lambda_+}\Big)(x) = \frac{1}{\lambda_+(I_-)} + \kappa \int_{I_-}^{x} \frac{1}{\lambda_-} \,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big), \\
&\text{(ii)} \quad \Big(\frac{f_-}{\lambda_-}\Big)(x) = \frac{1}{\lambda_-(I_+)} - \kappa \int_{x}^{I_+} \frac{1}{\lambda_+} \,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big),
\end{aligned} \tag{2.25}
$$

where κ is given by

$$
\kappa = \kappa_{\mathrm{L}} := \Gamma^{-1} \Big(\frac{1}{\lambda_-(I_+)} + \frac{1}{\lambda_+(I_-)} \Big), \tag{2.26}
$$

and $\Gamma > 0$ is the constant from (2.19).

Proof Setting $v_+ := f_-/\lambda_-$ and $v_- := f_+/\lambda_+$ and dividing the equations (1.11) (i) and (ii) by λ_- and λ_+, respectively, we see that these equations are equivalent to $v_\pm \mathrm{d}\lambda_\pm = -\mathrm{d}(\lambda_\pm v_\mp)$, which is equation (2.11) with $w_\pm = v_\pm$ and $g_\pm = 0$. Now Lemma 12 tells us that (f_-, f_+) solves (1.11) (i) and (ii) if and only if there exists a constant $\kappa \in \mathbb{R}$ such that

$$\begin{aligned} &\text{(i)} \quad \mathrm{d}\Big(\frac{f_+}{\lambda_+}\Big) = \frac{\kappa}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big), \\ &\text{(ii)} \quad \mathrm{d}\Big(\frac{f_-}{\lambda_-}\Big) = \frac{\kappa}{\lambda_+}\mathrm{d}\Big(\frac{1}{\lambda_-}\Big), \\ &\text{(iii)} \quad \frac{f_+}{\lambda_+} + \frac{f_-}{\lambda_-} = \frac{\kappa}{\lambda_-\lambda_+}. \end{aligned} \tag{2.27}$$

Moreover, of these equations, the first two are equivalent given the third one.

It follows that (f_-, f_+) solves (1.11) (i) and (ii) together with the left boundary condition $f_+(I_-) = 1$ if and only if (2.25) (i) and (2.27) (iii) hold. In view of the latter equation, the right boundary condition $f_-(I_+) = 1$ is satisfied if and only if

$$\frac{f_+}{\lambda_+}(I_+) + \frac{1}{\lambda_-(I_+)} = \frac{\kappa}{\lambda_-\lambda_+}(I_+). \tag{2.28}$$

By (2.25) (i), this says that

$$\frac{1}{\lambda_+(I_-)} + \kappa\int_{I_-}^{I_+}\frac{1}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{1}{\lambda_-(I_+)} = \frac{\kappa}{\lambda_-(I_+)\lambda_+(I_+)}, \tag{2.29}$$

which is equivalent to (2.26). ∎

Proof of Theorem 8 Let $\mathcal{G}$ be the space of all pairs (g_-, g_+) with $g_\pm \in B^\pm_{\rm bv}(\overline{I})$ and set $\mathcal{W} := \{(w_-, w_+) \in \mathcal{G} : w_\pm(I_\pm) = 0\}$. We equip the spaces $B^\pm_{\rm bv}(\overline{I})$ with a topology as in Section 2.1, $\mathcal{G}$ with the product topology, and $\mathcal{W}$ with the induced topology. For any interval J, we let $\mathcal{M}(J)$ denote the space of finite signed measures on J, equipped with the topology of weak convergence, and we let $\mathcal{R} := \mathcal{M}[I_-, I_+) \times \mathcal{M}(I_-, I_+]$, equipped with the product topology.

Let $\psi : \mathcal{W} \to \mathcal{G}$ be the linear function that maps a pair $(w_-, w_+) \in \mathcal{W}$ into the pair $(q_-, q_+) \in \mathcal{G}$ defined in (1.29) and let $D : \mathcal{G} \to \mathcal{R}$ be the map

$$D(g_-, g_+) := \big(\mathrm{d}g_-, \mathrm{d}g_+\big). \tag{2.30}$$

Setting $\phi := D \circ \psi$, we see that

$$\phi(w_-, w_+) = -\big(w_+\mathrm{d}\lambda_+ + \mathrm{d}(\lambda_+ w_-), w_-\mathrm{d}\lambda_- + \mathrm{d}(\lambda_- w_+)\big), \tag{2.31}$$

so Lemma 13 tells us that $\phi : \mathcal{W} \to \mathcal{R}$ is a bijection.

We claim that the maps ψ, D, ϕ, and ϕ^{-1} are continuous with respect to the topologies on $\mathcal{W}$, $\mathcal{G}$, and $\mathcal{R}$. The continuity of D is immediate from the definition of the topologies on $\mathcal{G}$ and $\mathcal{R}$ and the continuity of ψ follows from (1.29). The continuity of ϕ is easily derived from (2.31), while the continuity of ϕ^{-1} follows from the explicit formulas in Lemma 13 and the continuity of the functions $h_\pm$ from (2.13) as a function of $g_\pm$, for a given choice of the boundary conditions.

Let $\psi(\mathcal{W})$ denote the image of $\mathcal{W}$ under ψ and define $\pi : \mathcal{G} \to \psi(\mathcal{W})$ by $\pi := \psi \circ \phi^{-1} \circ D$. Since $\pi \circ \psi = \psi \circ \phi^{-1} \circ (D \circ \psi) = \psi$, we see that π is the identity on $\psi(\mathcal{W})$. Since $D \circ \pi = (D \circ \psi) \circ \phi^{-1} \circ D = D$, we see that $\pi(g) = \pi(g')$ if and only if $D(g) = D(g')$. These facts imply that for each $g \in \mathcal{G}$, there exists a unique $q \in \psi(\mathcal{W})$, namely $q = \pi(g)$, such that $D(g) = D(q)$, i.e., for every $g \in \mathcal{G}$ there exists a unique $q \in \psi(\mathcal{W})$ and unique constants $c_\pm(g_-, g_+) \in \mathbb{R}$ such that

$$(g_-, g_+) = \big(q_- + c_-(g_-, g_+), q_+ + c_+(g_-, g_+)\big). \tag{2.32}$$

Since ψ, ϕ^{-1} and D are continuous, so is π and hence also the maps $c_\pm : \mathcal{G} \to \mathbb{R}$ are continuous. In fact, they are the unique continuous linear forms on $\mathcal{G}$ such that

$$\begin{array}{lllll} \text{(i)} & c_-(1,0) = 1, & c_-(0,1) = 0, & c_-\big(\psi(w_-, w_+)\big) = 0 & \forall (w_-, w_+) \in \mathcal{W}, \\ \text{(ii)} & c_+(1,0) = 0, & c_+(0,1) = 1, & c_+\big(\psi(w_-, w_+)\big) = 0 & \forall (w_-, w_+) \in \mathcal{W}. \end{array} \tag{2.33}$$

The map $(g_-, g_+) \mapsto c(g_-, g_+)$ from Theorem 10 is given by $c = c_- + c_+$, i.e., c is the unique continuous linear form on $\mathcal{G}$ such that

$$c(1,0) = 1, \quad c(0,1) = 1, \quad c\big(\psi(w_-, w_+)\big) = 0 \quad \forall (w_-, w_+) \in \mathcal{W}. \tag{2.34}$$

Let (f_-, f_+) be the unique solution to Luckock's equation, and observe from (2.25) that $f_\pm$ are continuous on $\overline{I}$. We claim that

$$c(g_-, g_+) = g_-(I_+) f_-(I_+) - \int_{\overline{I}} f_- \mathrm{d}g_- + g_+(I_-) f_+(I_-) + \int_{\overline{I}} f_+ \mathrm{d}g_+. \tag{2.35}$$

Clearly, (2.35) defines a continuous linear form on $\mathcal{G}$. We will show that this linear form satisfies (2.34). The boundary conditions (1.11) (iii) imply that $c(1,0) = 1 = c(0,1)$. Recall that for $(w_-, w_+) \in \mathcal{W}$, $\psi(w_-, w_+) = (q_-, q_+)$ is defined as in (1.29). Then

$$\begin{aligned} c\big(\psi(w_-, w_+)\big) = & -(w_-\lambda_+)(I_+) f_-(I_+) + \int_{\overline{I}} f_- \big\{ w_+ \mathrm{d}\lambda_+ + \mathrm{d}(w_-\lambda_+) \big\} \\ & -(w_+\lambda_-)(I_-) f_+(I_-) - \int_{\overline{I}} f_+ \big\{ w_- \mathrm{d}\lambda_- + \mathrm{d}(w_+\lambda_-) \big\}. \end{aligned} \tag{2.36}$$

By partial integration, using the continuity of $f_\pm$ and $\lambda_\pm$, as well as the boundary condition $w_-(I_-) = 0$, we have

$$-(w_-\lambda_+)(I_+) f_-(I_+) + \int_{\overline{I}} f_- \mathrm{d}(w_-\lambda_+) = -\int_{\overline{I}} (w_-\lambda_+) \mathrm{d}f_-, \tag{2.37}$$

Inserting this into the first line of (2.36) and treating the second line similarly, we find that

$$\begin{aligned} c\big(\psi(w_-, w_+)\big) &= \int_{\overline{I}} \big\{ f_- w_+ \mathrm{d}\lambda_+ - (w_-\lambda_+) \mathrm{d}f_- \big\} + \int_{\overline{I}} \big\{ - f_+ w_- \mathrm{d}\lambda_- + (w_+\lambda_-) \mathrm{d}f_+ \big\} \\ &= \int_{\overline{I}} w_+ \big\{ f_- \mathrm{d}\lambda_+ + \lambda_- \mathrm{d}f_+ \big\} - \int_{\overline{I}} w_- \big\{ f_+ \mathrm{d}\lambda_- + \lambda_+ \mathrm{d}f_- \big\} = 0, \end{aligned} \tag{2.38}$$

where we have used (1.11) (i) and (ii) in the last step. This completes the proof of (2.35).

In particular, formula (2.35) shows that

$$c(1_{[I_-, z]}, 0) = f_-(z) \quad \text{and} \quad c(0, 1_{[z, I_+]}) = f_+(z) \qquad (z \in \overline{I}), \tag{2.39}$$

which together with Theorem 10 implies Theorem 8. ∎

2.4 Some explicit formulas and conditions

In the present section, we prove Proposition 2 as well as two lemmas (Lemmas 15 and 16 below) giving explicit formulas for the weight functions of Theorem 8.

Proof of Proposition 2 The fact that Luckock's equation has a unique solution under the conditions (A3) and (A6) has already been proved in Lemma 14.

In order to prove (1.12), it suffices to prove part (i); the other part then follows by symmetry. Let $\Lambda_{+-} := \int_{I_-}^{I_+} \frac{1}{\lambda_+} \mathrm{d}\big(\frac{1}{\lambda_-}\big)$. Then (2.25) (ii) says that

$$f_-(I_-) = \lambda_-(I_-)\Big\{\frac{1}{\lambda_-(I_+)} - \kappa_L \Lambda_{+-}\Big\}. \tag{2.40}$$

Filling in the definition of κ_L in (2.26), we see that $f_-(I_-) > 0$ if and only if

$$\frac{1}{\lambda_-(I_+)} > \Gamma^{-1}\Big(\frac{1}{\lambda_-(I_+)} + \frac{1}{\lambda_+(I_-)}\Big)\Lambda_{+-}. \tag{2.41}$$

Using also formula (2.19) and the fact that $\Gamma > 0$, this can be rewritten as

$$\Big(\frac{1}{\lambda_-(I_-)\lambda_+(I_-)} + \Lambda_{+-}\Big)\frac{1}{\lambda_-(I_+)} > \Big(\frac{1}{\lambda_-(I_+)} + \frac{1}{\lambda_+(I_-)}\Big)\Lambda_{+-}, \tag{2.42}$$

which can be simplified to (1.12) (i). The same argument also works with all inequality signs reversed.

To complete the proof, we need to show (1.13). Consider the weight functions

$$w_- := -1_{(I_-,I_+]} \quad \text{and} \quad w_+ := 1_{[I_-,I_+)}, \tag{2.43}$$

which correspond through (1.26) to the linear function $F(\mathcal{X}) = \mathcal{X}(I)$. For these weight functions, the functions $q_\pm$ from (1.29) are given by

$$\begin{aligned} q_-(x) &= \lambda_+(I_+) - \lambda_+(I_-)1_{\{I_-\}}(x), \\ q_+(x) &= -\lambda_-(I_-) + \lambda_-(I_+)1_{\{I_+\}}(x), \end{aligned} \tag{2.44}$$

so Lemma 7 tells us that

$$GF(\mathcal{X}) = -\lambda_+(I_-)1_{\{M_-(\mathcal{X})=I_-\}} + \lambda_-(I_+)1_{\{M_+(\mathcal{X})=I_+\}} + \lambda_+(I_+) - \lambda_-(I_-). \tag{2.45}$$

By Theorem 10, the weight functions $w_\pm$ are in fact uniquely characterized by the requirement that

$$GF(\mathcal{X}) = -\lambda_+(I_-)1_{\{M_-(\mathcal{X})=I_-\}} + \lambda_-(I_+)1_{\{M_+(\mathcal{X})=I_+\}} + c \tag{2.46}$$

for some $c \in \mathbb{R}$. Defining weight functions $\tilde{w}_\pm$ by

$$\tilde{w}_- := -\lambda_+(I_-)w^{I_-,-} + \lambda_-(I_+)w^{I_+,+} \tag{2.47}$$

and denoting the corresponding linear function by $\tilde{F}$, we see from Theorem 8 that

$$G\tilde{F}(\mathcal{X}) = -\lambda_+(I_-)1_{\{M_-(\mathcal{X})=I_-\}} + \lambda_-(I_+)1_{\{M_+(\mathcal{X})=I_+\}} + \lambda_+(I_-)f_-(I_-) - \lambda_-(I_+)f_+(I_+). \tag{2.48}$$

We conclude from this that $w_\pm = \tilde{w}_\pm$ and the constant from (2.46) is given by

$$\lambda_+(I_+) - \lambda_-(I_-) = c = \lambda_+(I_-)f_-(I_-) - \lambda_-(I_+)f_+(I_+), \tag{2.49}$$

which proves (1.13). ∎

We next set out to derive explicit formulas for the weight functions $(w_-^{z,\pm}, w_+^{z,\pm})$ from Theorem 8. To state the result, we define functions $u_{-+}, u_{+-} : \overline{I} \to \mathbb{R}$ by

$$\begin{aligned} u_{-+}(x) &:= \Gamma^{-1}\Big\{\frac{1}{\lambda_-(I_+)\lambda_+(I_+)} - \int_x^{I_+} \frac{1}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big)\Big\},\\ u_{+-}(x) &:= \Gamma^{-1}\Big\{\frac{1}{\lambda_-(I_-)\lambda_+(I_-)} + \int_{I_-}^x \frac{1}{\lambda_+}\mathrm{d}\Big(\frac{1}{\lambda_-}\Big)\Big\}. \end{aligned} \tag{2.50}$$

In view of (2.19), we observe that $u_{-+}(I_-) = 1 = u_{+-}(I_+)$. Moreover, u_{-+} is nonincreasing with $u_{-+}(I_+) > 0$ while u_{+-} is nondecreasing with $u_{-+}(I_-) > 0$. By partial integration, our formulas for u_{-+} and u_{+-} can be rewritten as

$$\begin{aligned} u_{-+}(x) &:= \Gamma^{-1}\Big\{\frac{1}{\lambda_-(x)\lambda_+(x)} + \int_x^{I_+} \frac{1}{\lambda_+}\mathrm{d}\Big(\frac{1}{\lambda_-}\Big)\Big\},\\ u_{+-}(x) &:= \Gamma^{-1}\Big\{\frac{1}{\lambda_-(x)\lambda_+(x)} - \int_{I_-}^x \frac{1}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big)\Big\}. \end{aligned} \tag{2.51}$$

Combining this with our previous formulas and (2.19), we see that

$$u_{-+}(x) + u_{+-}(x) = \frac{\Gamma^{-1}}{\lambda_-(x)\lambda_+(x)} + 1. \tag{2.52}$$

Lemma 15 (Formulas for special weight functions) *The weight functions from Theorem 8 are given by*

$$\begin{aligned} &\text{(i)} && w_-^{z,-}(x) = \lambda_-(z)\Gamma\big(u_{+-}(z) - 1_{\{x\leq z\}}\big)\big(u_{-+}(x) - 1_{\{x\leq z\}}\big)\\ &\text{(ii)} && w_+^{z,-}(x) = \lambda_-(z)\Gamma\big[u_{+-}(x\vee z) - 1\big]u_{+-}(x\wedge z)\\ &\text{(iii)} && w_-^{z,+}(x) = \lambda_+(z)\Gamma\big[u_{-+}(x\wedge z) - 1\big]u_{-+}(x\vee z)\\ &\text{(iv)} && w_+^{z,+}(x) = \lambda_+(z)\Gamma\big(u_{-+}(z) - 1_{\{x\geq z\}}\big)\big(u_{+-}(x) - 1_{\{x\geq z\}}\big). \end{aligned} \tag{2.53}$$

Proof We start with formula (2.53) (ii). Since $w_+^{I_+,-} = 0$ which agrees with the right-hand side of (2.53) (ii), we assume from now on without loss of generality that $z \in [I_-, I_+)$. We apply Lemma 13 with $g_- = 1_{[I_-,z]}$ and $g_+ = 0$. For the functions $h_\pm$ from (2.13) we choose the boundary conditions $h_-(I_+) = 0 = h_+(I_-)$, which means that

$$h_-(x) = \int_{[x,I_+)} \lambda_- \mathrm{d}1_{[I_-,z]} = -\lambda_-(z)1_{[I_-,z]} \quad \text{and} \quad h_+ = 0. \tag{2.54}$$

Since $h_+ = 0$ and $h_-(I_+) = 0$, formulas (2.20) (ii) and (2.21) now simplify to

$$w_+(x) = -\int_{(x,I_+]} \frac{\kappa + h_-}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) \quad \text{with} \quad \kappa = \Gamma^{-1}\int_{\overline{I}} \frac{1}{\lambda_-}\,\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big). \tag{2.55}$$

Here, by (2.54),

$$\begin{aligned} \int_{\overline{I}} \frac{1}{\lambda_-}\,\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big) &= \Big[\frac{h_-}{\lambda_-\lambda_+}(I_+) - \frac{h_-}{\lambda_-\lambda_+}(I_-)\Big] - \int_{\overline{I}} \frac{h_-}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big)\\ &= \lambda_-(z)\Big\{\frac{1}{\lambda_-(I_-)\lambda_+(I_-)} - \int_{[I_-,z]} \frac{1}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big)\Big\}, \end{aligned} \tag{2.56}$$

which shows that

$$\kappa = \lambda_-(z)u_{+-}(z). \tag{2.57}$$

Using the fact that $u_{+-}(I_+) = 1$, it follows that

$$\begin{aligned} w_+(x) &= -\lambda_-(z)u_{+-}(z)\int_{(x,I_+]}\frac{1}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) + \lambda_-(z)1_{\{x<z\}}\int_{(x,z]}\frac{1}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) \\ &= -\lambda_-(z)\Gamma\Big\{u_{+-}(z)\big[1-u_{+-}(x)\big] - 1_{\{x<z\}}\big[u_{+-}(z)-u_{+-}(x)\big]\Big\}, \end{aligned} \tag{2.58}$$

which can be rewritten as (2.53) (ii).

We next prove (2.53) (i). By (2.14) (iii), (2.54), (2.57), and (2.53) (ii),

$$\begin{aligned} w_+(x) &= \frac{\kappa + h_- + h_+}{\lambda_-\lambda_+}(x) - w_-(x) \\ &= \lambda_-(z)\frac{u_{+-}(z) - 1_{[I_-,z]}}{\lambda_-\lambda_+} - \lambda_-(z)\Gamma\big(u_{+-}(x\vee z) - 1\big)u_{+-}(x\wedge z). \end{aligned} \tag{2.59}$$

For $x \leq z$, using (2.52), this yields

$$\begin{aligned} w_+(x) &= \lambda_-(z)\Gamma\Big\{\Gamma^{-1}\frac{u_{+-}(z)-1}{\lambda_-\lambda_+} - \big(u_{+-}(z)-1\big)u_{+-}(x)\Big\} \\ &= \lambda_-(z)\Gamma\big(u_{+-}(z)-1\big)\Big\{\frac{\Gamma^{-1}}{\lambda_-\lambda_+} - u_{+-}(x)\Big\} \\ &= \lambda_-(z)\Gamma\big(u_{+-}(z)-1\big)\big(u_{-+}(x)-1\big), \end{aligned} \tag{2.60}$$

while for $x > z$, again with the help of (2.52), we obtain

$$\begin{aligned} w_+(x) &= \lambda_-(z)\Gamma\Big\{\Gamma^{-1}\frac{u_{+-}(z)}{\lambda_-\lambda_+} - \big(u_{+-}(x)-1\big)u_{+-}(z)\Big\} \\ &= \lambda_-(z)\Gamma u_{+-}(z)\Big\{\frac{\Gamma^{-1}}{\lambda_-\lambda_+} - u_{+-}(x) + 1\Big\} \\ &= \lambda_-(z)\Gamma u_{+-}(z)u_{-+}(x). \end{aligned} \tag{2.61}$$

Combining the previous two formulas, we arrive at (2.53) (i).

Formulas (2.53) (iii) and (2.53) (iv) can be proved in exactly the same way. Alternatively, they can be derived from (2.53) (ii) and (2.53) (i) using the symmetry between buy and sell orders. ∎

Assume (A3) and (A6) and for $z \in \overline{I}$, let $F^{z,\pm}$ be linear functionals defined in terms of weight functions $(w_-^{z,\pm}, w_+^{z,\pm})$ as in Theorem 8. We will in particular be interested in the case $z = I_\pm$ and introduce the shorthands

$$w_-^{(\pm)} := w_-^{I_\pm,\pm} \quad w_+^{(\pm)} := w_+^{I_\pm,\pm}, \quad \text{and} \quad F^{(\pm)} := F^{I_\pm,\pm}. \tag{2.62}$$

We will prove Theorem 3 by constructing a Lyapunov function from $F^{(-)}$ and $F^{(+)}$, see formula (3.5) and Proposition 17 below. The next lemma prepares for the proof of Proposition 17.

Lemma 16 (Extremal weight functions) *Assume (A3) and (A6), let (f_-, f_+) denote the solution to Luckock's equation, and let $(w_-^{(\pm)}, w_+^{(\pm)})$ be defined as in (2.62). Then for $x \in I$, one has*

$$\begin{aligned} &\text{(i)}\quad w_-^{(-)}(x) = \frac{u_{-+}(x)}{\lambda_+(I_-)}, &&\text{(ii)}\quad w_+^{(-)}(x) = -\frac{1-u_{+-}(x)}{\lambda_+(I_-)}, \\ &\text{(iii)}\quad w_-^{(+)}(x) = -\frac{1-u_{-+}(x)}{\lambda_-(I_+)}, &&\text{(iv)}\quad w_+^{(+)}(x) = \frac{u_{+-}(x)}{\lambda_-(I_+)}. \end{aligned} \tag{2.63}$$

Moreover,

$$\begin{aligned} &\text{(i)} \quad f_+(I_+) > 0 \quad \Leftrightarrow \quad \inf_{x\in I}\big[w_-^{(-)}(x) + w_-^{(+)}(x)\big] > 0, \\ &\text{(ii)} \quad f_-(I_-) > 0 \quad \Leftrightarrow \quad \inf_{x\in I}\big[w_+^{(-)}(x) + w_+^{(+)}(x)\big] > 0. \end{aligned} \tag{2.64}$$

Both formulas also hold with the inequality signs reversed.

Proof We only prove (2.63) (i) and (ii) and (2.64) (i); the proof of the other formulas follows from the symmetry between buy and sell orders. By Lemma 15 and the facts that

$$u_{-+}(I_+) = \frac{\Gamma^{-1}}{\lambda_-(I_+)\lambda_+(I_+)} \quad \text{and} \quad u_{+-}(I_-) = \frac{\Gamma^{-1}}{\lambda_-(I_-)\lambda_+(I_-)}, \tag{2.65}$$

we have

$$\begin{aligned} &\text{(i)} \quad w_-^{(-)}(x) = \lambda_-(I_-)\Gamma\Big[\frac{\Gamma^{-1}}{\lambda_-(I_-)\lambda_+(I_-)} - 1_{\{x=I_-\}}\Big]\big(u_{-+}(x) - 1_{\{x=I_-\}}\big) \\ &\text{(ii)} \quad w_+^{(-)}(x) = \lambda_-(I_-)\Gamma\big[u_{+-}(x) - 1\big]\frac{\Gamma^{-1}}{\lambda_-(I_-)\lambda_+(I_-)}. \end{aligned} \tag{2.66}$$

For $x \neq I_-$, these formulas simplify to (2.63) (i) and (ii).

Adding formulas (2.63) (i) and (iii) yields

$$w_-^{(-)}(x) + w_+^{(-)}(x) = \Big[\frac{1}{\lambda_+(I_-)} + \frac{1}{\lambda_-(I_+)}\Big]u_{-+}(x) - \frac{1}{\lambda_-(I_+)}. \tag{2.67}$$

Since u_{-+} is nonincreasing and continuous, the infimum of this function over $x \in I$ is equal to the value in $x = I_+$, i.e.,

$$\inf_{x\in I}\big[w_-^{(-)}(x) + w_+^{(-)}(x)\big] = \Big[\frac{1}{\lambda_+(I_-)} + \frac{1}{\lambda_-(I_+)}\Big]\frac{\Gamma^{-1}}{\lambda_-(I_+)\lambda_+(I_+)} - \frac{1}{\lambda_-(I_+)}. \tag{2.68}$$

Using the fact that $\Gamma > 0$, we see that the expression in (2.68) is positive if and only if

$$\frac{1}{\lambda_+(I_-)\lambda_+(I_+)} + \frac{1}{\lambda_-(I_+)\lambda_+(I_+)} > \Gamma. \tag{2.69}$$

Taking into account (2.19) and (1.12) (ii) (which also holds with the equality signs reversed), this is equivalent to $f_+(I_+) > 0$. ∎

2.5 Restricted models

In the present subsection, we prove Proposition 5 as well as Lemmas 4 and 6 and formula (1.22).

Proof of Lemma 4 To prove that $J_- < \phi_+(J_-)$ for all $J_- \in I$, it suffices to show that $\Lambda_+(J_-, J_- + \varepsilon) > 0$ for $\varepsilon > 0$ sufficiently small. Here

$$\Lambda_+(J_-, J_- + \varepsilon) = \frac{1}{\lambda_+(J_-)\lambda_+(J_- + \varepsilon)} + \int_{J_-}^{J_-+\varepsilon} \frac{1}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) \tag{2.70}$$

By assumptions (A3) and (A5) and the fact that $J_- \in I$, the first term tends to a positive limit as $\varepsilon \downarrow 0$ while the second term tends to zero. By the symmetry between buy and sell orders, we see that also $\phi_-(J_+) < J_+$ for all $J_+ \in I$.

Using (A3), we see that for fixed J_-, the function $\Lambda_+(J_-, J_+)$ is nonincreasing as a function of J_+, and hence that $\Lambda_+(J_-, J_+) > 0$ if and only if $J_+ < \phi_+(J_-)$. Similarly, $\Lambda_-(J_-, J_+) > 0$ if and only if $\phi_-(J_+) < J_-$, so the second claim of the lemma follows from Theorem 3, where we use that for $I_- < J_- < J_+ < I_+$, the restricted model on (J_-, J_+) satisfies (A6) since the model on I satisfies (A5). ∎

Proof of formula (1.22) Let D be the set of all pairs $(J_-, J_+) \in \mathbb{R}^2$ such that $I_- \leq J_- \leq J_+ \leq I_+$ and let $T := \sup\{J_+ - J_- : (J_-, J_+) \in D,\ \lambda_-(J_-) = \lambda_+(J_+)\}$. We define a curve $\gamma : [0, T] \to D$ with $\gamma(t) = \big(\gamma_-(t), \gamma_+(t)\big)$ by

$$\gamma_-(t) := \inf\big\{J_- \in \overline{I} : \lambda_-(t) \leq \lambda_+(J_- + t)\big\} \quad \text{and} \quad \gamma_+(t) := \gamma_-(t) + t. \tag{2.71}$$

Using (A3), it is not hard to see that γ_- is nonincreasing, γ_+ is nondecreasing, and γ is Lipschitz continuous with Lipschitz constant 1. Using also (A4), we see that $\gamma_-(0) = (x_{\rm W}, x_{\rm W})$, where $x_{\rm W}$ is the Walrasian price from (1.17).

Let $D' := \{(J_-, J_+) : I_- < J_- \leq J_+ < I_+\}$. For any J_-, J'_-, J_+, J'_+ with $J_- \leq J'_-$ and $J_+ \leq J'_+$, we observe that

$$\begin{aligned}
\Lambda_+(J'_-, J_+) - \Lambda_+(J_-, J_+) &= \int_{J_-}^{J'_-} \Big\{\frac{1}{\lambda_+(J_+)} - \frac{1}{\lambda_-}\Big\}\,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big),\\
\Lambda_+(J_-, J'_+) - \Lambda_+(J_-, J_+) &= \int_{J_+}^{J'_+} \Big\{\frac{1}{\lambda_+(J_-)} + \frac{1}{\lambda_-}\Big\}\,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big).
\end{aligned} \tag{2.72}$$

Applying Lemma 9, we obtain that

$$\begin{aligned}
\Lambda_+\big(\gamma(t)\big) = \Lambda_+\big(\gamma(0)\big) &+ \int_0^t \Big\{\frac{1}{\lambda_+ \circ \gamma_+} - \frac{1}{\lambda_- \circ \gamma_-}\Big\}\,\mathrm{d}\Big(\frac{1}{\lambda_+ \circ \gamma_-}\Big)\\
&+ \int_0^t \Big\{\frac{1}{\lambda_+ \circ \gamma_-} + \frac{1}{\lambda_- \circ \gamma_+}\Big\}\,\mathrm{d}\Big(\frac{1}{\lambda_+ \circ \gamma_+}\Big)
\end{aligned} \tag{2.73}$$

for any $t \geq 0$ such that $\gamma(t) \in D'$. Since by construction $\lambda_-(\gamma_-(s)) = \lambda_+(\gamma_+(s))$ for all $s \in [0, t]$, the first integral in (2.73) is zero. Set $\psi := \lambda_+ \circ \gamma_+$ and $\psi^{-1}(V) := \inf\{t \geq 0 : \psi(t) \geq V\}$, and observe that $\gamma(\psi^{-1})(V) = \big(j_-(V), j_+(V)\big)$. Using the substitution of variables $W = \psi(t)$ (recall (2.7)) using also the fact that

$$\Lambda_+\big(\gamma(0)\big) = \Lambda_+(x_{\rm W}, x_{\rm W}) = \frac{1}{V_{\rm W}^2}, \tag{2.74}$$

we can rewrite (2.73) as

$$\Lambda_+\big(j_-(V), j_+(V)\big) = \frac{1}{V_{\rm W}^2} + \int_{V_{\rm W}}^{V} \Big\{\frac{1}{\lambda_+\big(j_-(W)\big)} + \frac{1}{\lambda_-\big(j_+(W)\big)}\Big\}\mathrm{d}\Big(\frac{1}{W}\Big), \tag{2.75}$$

which holds whevener $\big(j_-(V), j_+(V)\big) \in D'$.

The second inequality in (1.22) follows in the same way, or alternatively, one can use the fact that

$$\Lambda_+(J_-, J_+) - \Lambda_-(J_-, J_+) = \Big(\frac{1}{\lambda_+(J_+)} - \frac{1}{\lambda_-(J_-)}\Big)\Big(\frac{1}{\lambda_+(J_-)} + \frac{1}{\lambda_-(J_+)}\Big), \tag{2.76}$$

which follows from partial integration of the formulas in (1.12) and shows that $\Lambda_+(J_-, J_+) = \Lambda_-(J_-, J_+)$ whenever $\lambda_-(J_-) = \lambda_+(J_+)$. ∎

Proof of Proposition 5 Assume that a Stigler-Luckock model satisfying (A3)–(A5) and (A7) has a critical window J, and let (f_-, f_+) denote the solution to Luckock's equation on $\overline{J}$.

Since $f_-(\overline{J}) = 0 = f_+(J_+)$, by (1.13), we must have $\lambda_-(J_-) = \lambda_+(J_+)$. Calling this quantity V, formulas (1.12) and (1.22) tell us that $\Psi(V) = 0$. It is clear from (1.21) that Ψ is strictly decreasing, so V must equal Luckock's volume of trade $V_{\rm L}$ as defined in (1.23). Using also condition (ii) in the definition of a critical window, we conclude that $J = \big(j_-(V_{\rm L}), j_+(V_{\rm L})\big)$.

Conversely, if $\Psi(V_{\rm L}) = 0$ and $J := \big(j_-(V_{\rm L}), j_+(V_{\rm L})\big)$ satisfies $\overline{J} \subset I$, then the solution to Luckock's equation on $\overline{J}$ satisfies $f_-(\overline{J}) = 0 = f_+(J_+)$ by (1.12) and (1.22), while condition (ii) in the definition of a critical window is satisfied because of the way $j_\pm$ are defined. ∎

Proof of Lemma 6 For the uniform model with $\lambda_-(x) = 1 - x$ and $\lambda_+(x) = x$, we have

$$V_{\rm W} = \tfrac{1}{2}, \quad V_{\max} = 1, \quad j_-(V) = 1 - V, \quad \text{and} \quad j_+(V) = W. \tag{2.77}$$

It follows that the function Ψ from (1.21) is given by

$$\begin{aligned}\Psi(V) &= 4 + 2\int_{1/2}^{V} \frac{1}{1-W}\mathrm{d}\Big(\frac{1}{W}\Big) = 4 - 2\int_{1/V}^{2} \frac{1}{1-\frac{1}{y}}\,\mathrm{d}y = 4 - 2\int_{1/V}^{2} \big\{1 + \frac{1}{y-1}\big\}\,\mathrm{d}y \\ &= 4 - 2\Big|_{y=1/V}^{2} \big\{y + \log(y-1)\big\} = 2\big\{V^{-1} + \log(V^{-1} - 1)\big\}.\end{aligned} \tag{2.78}$$

Setting $\Psi(V) = 0$ gives

$$-V^{-1} = \log\big(V^{-1} - 1\big) \quad \Leftrightarrow \quad e^{-V^{-1}} = V^{-1} - 1, \tag{2.79}$$

which tells us that $V_{\rm L} = 1/z$ where z solves $f(z) := e^{-z} - z + 1 = 0$. Since the function f is continuous and strictly decreasing with $f(1) = e^{-1}$ and $f(z) \to -\infty$ for $z \to \infty$, the equation $f(z) = 0$ has a unique solution z, and this solution satisfies $z > 1$. ∎

3 Analysis of the Markov chain

3.1 A consequence of stationarity

In this subsection, we prove Theorem 1.

Proof of Theorem 1 Let ν be an invariant law, let X_0 be an $\mathcal{S}_{\rm ord}$-valued random variable with law ν, and let (U_1, σ_1) be independent of X_0 with law $\overline{\mu} := |\mu|^{-1}\mu$, as defined in Section 1.1. Then stationarity means that $X_1 := L_{U_1,\sigma_1}(X_0)$ has the same law as X_0, where $L_{u,\sigma}$ is the Luckock map from (1.9).

Set $M_\pm := M_\pm(X_0)$ and let $\mu_\pm$ be as in (1.1). We claim that

$$\begin{aligned}&\text{(i)} \quad \int_A \mathbb{P}[M_- < x]\,\mu_+(\mathrm{d}x) = \int_A \lambda_-(x)\,\mathbb{P}[M_+ \in \mathrm{d}x], \\ &\text{(ii)} \quad \int_A \mathbb{P}[M_+ > x]\,\mu_-(\mathrm{d}x) = \int_A \lambda_+(x)\,\mathbb{P}[M_- \in \mathrm{d}x]\end{aligned} \tag{3.1}$$

for each measurable $A \subset I$ that is contained in some compact subinterval $[J_-, J_+] \subset I$. Indeed, stationarity implies (see [Swa15, Lemma 10]) that for any measurable $A \subset I$, sell limit orders are added in A with the same frequency as they are removed, i.e.,

$$\mathbb{P}\big[X_1^+(A) = X_0^+(A) + 1\big] = \mathbb{P}\big[X_1^+(A) = X_0^+(A) - 1\big]. \tag{3.2}$$

Recalling the definition of the Luckock map in (1.9), we see that this means that

$$\mathbb{P}\big[\sigma_1 = +,\ U_1 \in A,\ M_- < U_1\big] = \mathbb{P}\big[\sigma_1 = -,\ M_+ \in A,\ M_+ \le U_1\big]. \tag{3.3}$$

Since (U_1, σ_1) has law $\overline{\mu}$ and is independent of $M_\pm$, it follows that

$$|\mu|^{-1} \int_A \mu_+(\mathrm{d}x)\mathbb{P}[M_- < x] = |\mu|^{-1} \int_A \mathbb{P}[M_+ \in \mathrm{d}x]\mu_-\big([x, I_+]\big), \tag{3.4}$$

which up to the factor $|\mu|^{-1}$ is (3.1) (i). Similarly, equation (3.1) (ii) follows from the requirement that buy limit orders are added in A with the same frequency as they are removed. Recalling the definitions of $\mu_\pm$ in (1.1), the integral equations (3.1) (i) and (ii) are just (1.11) (i) and (ii). Since $f_-(I_+) = \mathbb{P}[M_- \leq I_+] = 1$ and $f_+(I_-) = \mathbb{P}[M_+ \geq I_-] = 1$, the boundary conditions (1.11) (iii) also follow. ∎

3.2 A Lyapunov function

It follows from Theorem 1 that if a Stigler-Luckock model is positive recurrent, then the solution to Luckock's equation must satisfy $f_-(I_-) \wedge f_+(I_+) > 0$. Theorem 3 states that this condition is also sufficient. We will prove this by showing that

$$V(\mathcal{X}) := \sqrt{(F^{(-)}(\mathcal{X}) \vee 0)^2 + (F^{(+)}(\mathcal{X}) \vee 0)^2} \qquad \big(\mathcal{X} \in \mathcal{S}^{\mathrm{fin}}_{\mathrm{ord}}\big) \tag{3.5}$$

is a Lyapunov function. We note that this is the only place in the paper where we make use of a function of a Stigler-Luckock process that is not linear (namely V). In view of Theorem 10, we have fairly good control of linear functionals, which as in Theorem 1 (which depends on Theorem 8) allows us to more or less explicitly calculate the marginal distributions of the best buy and sell offers $M_-(\mathcal{X})$ and $M_+(\mathcal{X})$ in equilibrium. Proving that a Stigler-Luckock model is positive recurrent, however, always entails proving something about the joint distribution of $M_-(\mathcal{X})$ and $M_+(\mathcal{X})$. Indeed, the following proposition can be used to give a lower bound on the probability, in equilibrium, that the order book is empty, which corresponds to the event that $M_-(\mathcal{X}) = I_-$ and at the same time $M_+(\mathcal{X}) = I_+$, but thhis bound is not very explicit or sharp. It seems that such information cannot be obtained from linear functionals and indeed for no choice of weight functions (w_-, w_+) is a linear function of the form (1.26) a Lyapunov function.

Recall the definition (1.25) of the generator G of a Stigler-Luckock model. We have the following result.

Proposition 17 (Lyapunov function) *Assume (A3) and (A6), and that the unique solution (f_-, f_+) of Luckock's equation (1.11) satisfies $\varepsilon := f_-(I_-) \wedge f_+(I_+) > 0$. Then there exists a constant $K < \infty$ such that the function in (3.5) satisfies $GV(\mathcal{X}) \leq K$ for all $\mathcal{X} \in \mathcal{S}^{\mathrm{fin}}_{\mathrm{ord}}$. Moreover, for each $\varepsilon' < \varepsilon$, there exists an $N < \infty$ such that*

$$GV(\mathcal{X}) \leq -\varepsilon' \quad \textit{whenever} \quad |\mathcal{X}^-| + |\mathcal{X}^+| \geq N. \tag{3.6}$$

Proof Let us write $\vec{F}(\mathcal{X}) := \big(F^{(-)}(\mathcal{X}), F^{(+)}(\mathcal{X})\big)$ and let $|\cdot|$ denote the euclidean norm on $\mathbb{R}^2$. Let us also write $V(\mathcal{X}) = v(\vec{F}(\mathcal{X}))$ where $v : R^2 \to \mathbb{R}$ is the function

$$v(z_1, z_2) := \sqrt{(z_1 \vee 0)^2 + (z_2 \vee 0)^2}. \tag{3.7}$$

Set

$$W := \sup_{x \in I} \big|\big(w_-^{(-)}(x), w_-^{(+)}(x)\big)\big| \vee \sup_{x \in I} \big|\big(w_+^{(-)}(x), w_+^{(+)}(x)\big)\big|, \tag{3.8}$$

which is the maximal amount by which $\vec{F}(\mathcal{X})$ can change due to the addition or removal of a single limit order. Since the function v is Lipschitz continuous with Lipschitz constant 1, we can estimate

$$\begin{aligned} GV(\mathcal{X}) &= \int \big\{V\big(L_{u,\sigma}(\mathcal{X})\big) - V\big(\mathcal{X}\big)\big\}\,\mu\big(\mathrm{d}(u,\sigma)\big) \\ &\leq \int \big|\vec{F}\big(L_{u,\sigma}(\mathcal{X})\big) - \vec{F}(\mathcal{X})\big|\,\mu\big(\mathrm{d}(u,\sigma)\big) \\ &\leq W\big(\lambda_-(I_-) + \lambda_+(I_+)\big) =: K. \end{aligned} \tag{3.9}$$

Let

$$\delta := \inf_{x\in I}\big[w^{(-)}_-(x) + w^{(+)}_-(x)\big] \wedge \inf_{x\in I}\big[w^{(-)}_+(x) + w^{(+)}_+(x)\big], \tag{3.10}$$

which is positive by (2.64) and our assumption that $f_-(I_-)\wedge f_+(I_+) > 0$. Since adding a limit order to the order book always raises $F^{(-)} + F^{(+)}$ by at least δ,

$$F^{(-)}(\mathcal{X}) + F^{(+)}(\mathcal{X}) \geq \delta\big(|\mathcal{X}^-| + |\mathcal{X}^+|\big). \tag{3.11}$$

This shows that $\vec{F}(\mathcal{X})$ takes values in the half space $H := \{(z_1,z_2)\in\mathbb{R}^2 : z_1 + z_2 > 0\}$ as long as $\mathcal{X}\neq 0$, and moreover $\big|\vec{F}(\mathcal{X})\big|$ is large if $|\mathcal{X}^-| + |\mathcal{X}^+|$ is.

For any $z = (z_1,z_2)\in\mathbb{R}^2$ with $z_1 + z_2 > 0$, let us define

$$p_1(z) := \frac{z_1\vee 0}{\sqrt{(z_1\vee 0)^2 + (z_2\vee 0)^2}} \quad\text{and}\quad p_2(z) := \frac{z_2\vee 0}{\sqrt{(z_1\vee 0)^2 + (z_2\vee 0)^2}}. \tag{3.12}$$

Then, for any $y,z\in H$, we can write

$$v(z) = v(y) + p_1(y)(z_1 - y_1) + p_2(y)(z_2 - y_2) + R(y,z), \tag{3.13}$$

where for any y,z that differ at most by the constant W from (3.8), the error term $R(x,y)$ can be estimated as

$$R(y,z) \leq C|y|^{-1} \qquad \big(y,z\in H,\ |z-y|\leq W\big) \tag{3.14}$$

for some constant $C<\infty$. It follows that we can write

$$GV(\mathcal{X}) = p_1(\vec{F}(\mathcal{X}))GF^{(-)}(\mathcal{X}) + p_2(\vec{F}(\mathcal{X}))GF^{(+)}(\mathcal{X}) + E(\mathcal{X}), \tag{3.15}$$

where the error term can be estimated as

$$\big|E(\mathcal{X})\big| = \Big|\int R\big(\vec{F}(\mathcal{X}),\vec{F}(L_{u,\sigma}(\mathcal{X})\big)\,\mu\big(\mathrm{d}(u,\sigma)\big)\Big| \leq C\big(\lambda_-(I_-) + \lambda_+(I_+)\big)\big|\vec{F}(\mathcal{X})\big|^{-1}, \tag{3.16}$$

which in view of (3.11) can be made arbitrary small by choosing $|\mathcal{X}^-| + |\mathcal{X}^+|$ sufficiently large.

By Theorem 8 and the way we have defined the weight functions $w^{(-)}_\pm$ and $w^{(+)}_\pm$, one has

$$GF^{(-)}(\mathcal{X}) = -f_-(I_-) \text{ if } |\mathcal{X}^-|\neq 0 \quad\text{and}\quad GF^{(+)}(\mathcal{X}) = -f_+(I_+) \text{ if } |\mathcal{X}^+|\neq 0. \tag{3.17}$$

It follows from (2.63) and elementary properties of the functions in (2.50) that

$$w^{(-)}_- > 0, \quad w^{(-)}_+ < 0, \quad w^{(+)}_- < 0, \quad\text{and}\quad w^{(+)}_+ > 0 \quad\text{on } I. \tag{3.18}$$

In view of this, we have

$$\begin{aligned} |\mathcal{X}^-| = 0 \;\Rightarrow\; F^{(-)}(\mathcal{X}) \leq 0 \;\Rightarrow\; p_1(\vec{F}(\mathcal{X})) = 0, \\ |\mathcal{X}^+| = 0 \;\Rightarrow\; F^{(+)}(\mathcal{X}) \leq 0 \;\Rightarrow\; p_2(\vec{F}(\mathcal{X})) = 0. \end{aligned} \tag{3.19}$$

Combining this with (3.17), we obtain that

$$p_1(\vec{F}(\mathcal{X}))GF^{(-)}(\mathcal{X}) + p_2(\vec{F}(\mathcal{X}))GF^{(+)}(\mathcal{X}) \leq -\varepsilon \qquad (\mathcal{X}\neq 0). \tag{3.20}$$

Inserting this into (3.15), using our bound (3.16) on the error term, and using also (3.11), we see that by choosing $|\mathcal{X}^-| + |\mathcal{X}^+|$ large enough, we can make GV smaller than $-\varepsilon'$ for any $\varepsilon' < \varepsilon$. ∎

3.3 Positive recurrence

Proof of Theorem 3 If a Stigler-Luckock model is positive recurrent, then it it is possible to construct a stationary process $(X_k)_{k\in\mathbb{Z}}$ that makes i.i.d. excursions away from the empty state 0. In particular, positive recurrence implies the existence of an invariant law ν that is concentrated on $\mathcal{S}^{\rm fin}_{\rm ord}$ and satisfies $\nu(\{0\}) > 0$. By Theorem 1, it follows that Luckock's equation (1.11) has a solution (f_-, f_+) such that $f_-(I_-) \wedge f_+(I_+) \geq \nu(\{0\}) > 0$.

Conversely, assume (A3) and (A6) and that the (by Proposition 2 unique) solution to Luckock's equation satisfies $f_-(I_-) \wedge f_+(I_+) > 0$. Let P denote the transition kernel of the discrete-time process $(X_k)_{k\geq 0}$ and for any nonnegative measurable function $f : \mathcal{S}^{\rm fin}_{\rm ord} \to \mathbb{R}$ write $Pf(x) := \int P(x, \mathrm{d}y)f(y)$. Write

$$C_N := \{\mathcal{X} \in \mathcal{S}^{\rm fin}_{\rm ord} : |\mathcal{X}|^- + |\mathcal{X}|^+ < N\}. \tag{3.21}$$

Multiplying the Lyapunov function V of Proposition 17 by a suitable constant, we obtain a nonnegative function f and finite constants K, N such that

$$Pf - f \leq K1_{C_N} - 1. \tag{3.22}$$

Let $\tau_0 := \inf\{k > 0 : X_k = 0\}$ denote the first return time to the empty configuration. By assumption (A6), there exists a constant $\varepsilon > 0$ such that

$$\mathbb{P}^x[\tau_0 \leq N + 1] \geq \varepsilon \qquad (x \in C_N). \tag{3.23}$$

Moreover, (A6) guarantees that $\mathbb{P}^0[\tau_0 = 0]$, which shows that the model is aperiodic from 0. Applying Proposition 19 from Appendix A.2 in the appendix, we conclude that the Stigler-Luckock model under consideration is positive recurrent and (1.14) holds. ∎

A Appendix

A.1 The model in standard form

In this appendix, we show that the assumptions (A3) and (A4) from Section 1.1 can basically be made without loss of generality, since any Stigler-Luckock model satisfying (A1) and (A2) can be obtained as a function of a Stigler-Luckock model satisfying (A3) and, subject to mild additional conditions, also (A4).

Let μ be a finite nonnegative measure on $\overline{\mathbb{R}} := [-\infty, \infty]$ and let $\mathrm{supp}(\mu)$ denote its support. Then the complement of $\mathrm{supp}(\mu)$ is a countable union of disjoint open intervals. If for each left endpoint x_- of such an interval (x_-, x_+), we remove x_- from $\mathrm{supp}(\mu)$ if it carries no mass, then we obtain

$$\mathrm{supp}_+(\mu) = \big\{x \in \overline{\mathbb{R}} : \mu\big([x, y)\big) > 0 \ \forall y > x\big\}. \tag{A.1}$$

The set $\mathrm{supp}_+(\mu)$ is the support of μ with respect to the topology of convergence from the right, where a sequence x_n converges to a limit x if and only if $x_n \to x$ in the usual topology on $\overline{\mathbb{R}}$ and moreover $x_n \geq x$ for n large enough. A basis for this topology is formed by all sets of the form $[x, y)$ with $x < y$.

Below, if $\overline{I}' = [I'_-, I'_+]$ and $\overline{I} = [I_-, I_+]$ are closed intervals and $\psi : \overline{I}' \to \overline{I}$ is a nondecreasing map, then we let $\psi^{-1}_\pm$ denote the left and right continuous inverses of ψ, i.e.,

$$\psi^{-1}_-(y) := \sup\{x \in \overline{I}' : \psi(x) \leq y\} \quad \text{and} \quad \psi^{-1}_+(y) := \inf\{x \in \overline{I}' : \psi(x) \geq y\}, \tag{A.2}$$

with the conventions that $\sup\emptyset := I_+$ and $\inf\emptyset := I_-$. If f is a function on $\overline{I}'$, then we let $f \circ \psi_\pm^{-1}$ denote the composition of f with $\psi_\pm^{-1}$. If μ is a measure on $\overline{I}'$, then $\mu \circ \psi^{-1}$ denotes the image of μ under ψ, which is the same as the composition of μ with the inverse image map ψ^{-1}.

Proposition 18 (Standard form) *Let $I = (I_-, I_+)$ be a nonempty open interval and let $(\mathcal{X}_t)_{t\geq 0}$ be a Stigler-Luckock model with demand and supply functions $\lambda_\pm : \overline{I} \to [0,\infty)$ satisfying (A1) and (A2). Then there exists an open interval $I' = (I'_-, I'_+)$ and a Stigler-Luckock model $(\mathcal{X}'_t)_{t\geq 0}$ with demand and supply functions $\lambda'_\pm : \overline{I}' \to [0,\infty)$ satisfying (A3), as well as a nondecreasing function $\psi : \overline{I}' \to \overline{I}$ that maps I' into I and satisfies $\psi(I'_\pm) = I_\pm$, such that*

$$\lambda_\pm = \lambda_\pm \circ \psi_\pm^{-1} \quad \text{and} \quad \mathcal{X}_t = \mathcal{X}'_t \circ \psi^{-1} \qquad (t \geq 0). \tag{A.3}$$

Assume that $\lambda_\pm$ are not both constant and

$$\mathcal{X}_0 \text{ is concentrated on } \mathrm{supp}_+(\mathrm{d}\lambda_+ - \mathrm{d}\lambda_-). \tag{A.4}$$

Assume also that either $\mathcal{X}_0 \in \mathcal{S}^{\rm fin}_{\rm ord}$ or (A5) holds. Then we can choose $I'_\pm$, $\lambda'_\pm$, and ψ in such a way that ψ is right-continuous on I' and

$$\lambda'_+(x) - \lambda'_-(x) = x \qquad (x \in \overline{I}'). \tag{A.5}$$

Subject to these constraints, $I'_\pm$, $\lambda'_\pm$, and ψ are unique.

If the demand and supply functions $\lambda_\pm$ of a Stigler-Luckock model satisfy (A.5), then we say that the model is in *standard form.* Note that a model in standard form satisfies (A3) and (A4). For such a model, $\mathrm{d}\lambda_+ - \mathrm{d}\lambda_- = \mathrm{d}x$, the Lebesgue measure. A model in standard form is uniquely characterized by its interval (I_-, I_+) and the measurable function $p_+ : \overline{I} \to [0,1]$ defined by the Radon-Nikodym derivative

$$p_+ := \frac{\mathrm{d}\lambda_+}{\mathrm{d}\lambda_+ - \mathrm{d}\lambda_-}. \tag{A.6}$$

Proof of Proposition 18 Let $I' = (I'_-, I'_+)$ be a nonempty open interval and let $(\mathcal{X}'_t)_{t\geq 0}$ be a Stigler-Luckock model with demand and supply functions $\lambda'_\pm : \overline{I} \to [0,\infty)$ satisfying (A1) and (A2). Assume that $\psi : \overline{I}' \to \overline{I}$ is nondecreasing, maps I' into I, and satisfies $\psi(I'_\pm) = I_\pm$. Assume that

$$\begin{gathered}\psi(x_+) \leq \psi(x_-) \quad \Leftrightarrow \quad x_+ \leq x_- \\ \text{for a.e. } x_+ \text{ w.r.t. } \mathcal{X}'^{+}_0 + \mathrm{d}\lambda'_+ \text{ and a.e. } x_- \text{ w.r.t. } \mathcal{X}'^{-}_0 - \mathrm{d}\lambda'_-.\end{gathered} \tag{A.7}$$

Then we claim that the process $(\mathcal{X}_t)_{t\geq 0}$ defined as $\mathcal{X}_t := \mathcal{X}'_t \circ \psi^{-1}$ is a Stigler-Luckock model with demand and supply functions $\lambda_\pm : \overline{I} \to [0,\infty)$ given by

$$\lambda_\pm = \lambda'_\pm \circ \psi_\pm^{-1}. \tag{A.8}$$

To see this, construct the process $(\mathcal{X}'_t)_{t\geq 0}$ from a Poisson point process

$$\Pi' = \{(U'_k, \sigma_k, \tau_k) : k = 1, 2, \ldots\} \quad \text{with} \quad 0 < \tau_1 < \tau_2 < \cdots \tag{A.9}$$

as in (1.2), set $\mathcal{X}_0 := \mathcal{X}'_0 \circ \psi^{-1}$, and construct a Stigler-Luckock model $(\mathcal{X}_t)_{t\geq 0}$ with initial state $\mathcal{X}_0$ from the Poisson point process

$$\Pi := \{(U_k, \sigma_k, \tau_k) : k = 1, 2, \ldots\} \quad \text{with} \quad U_k := \psi(U'_k). \tag{A.10}$$

Then $(\mathcal{X}_t)_{t\geq 0}$ has demand and supply functions $\lambda_\pm$ as in (A.8). Moreover, since (A.7) guarantees that buy and sell orders in $\mathcal{X}'$ can be matched if and only if they can be matched in $\mathcal{X}$, we see that (A.3) holds.

Clearly, a sufficient condition for (A.7) is that ψ is strictly increasing. More precisely, we observe that (A.7) is equivalent to the condition:

$$\begin{aligned}&\text{If } \psi(x)=\psi(z) \text{ for some } x<z, \text{ then there exists an } y \text{ with } x\leq y\leq z\\&\text{such that } (\mathcal{X}'^{-}_0-\mathrm{d}\lambda'_-)\big([x,y)\big)=0 \text{ and } (\mathcal{X}'^{+}_0+\mathrm{d}\lambda'_+)\big((y,z]\big)=0.\end{aligned}\tag{A.11}$$

Assume for the moment that $\lambda_\pm$ are not both constant and that (A.4) holds, and assume also that either $\mathcal{X}_0\in\mathcal{S}^{\rm fin}_{\rm ord}$ or (A5) holds. Then the requirements that $\lambda_\pm=\lambda'_\pm\circ\psi^{-1}_\pm$ and $\lambda_+(x)-\lambda_-(x)=x$ force us to take $I'_\pm:=\lambda_+(I_\pm)-\lambda_-(I_\pm)$ and

$$\psi(y):=\inf\{x\in\overline{I}:\lambda_+(x)-\lambda_-(x)\geq y\}\qquad\big(y\in(I'_-,I'_+]\big).\tag{A.12}$$

Note that $I'_-<I'_+$ by our assumption that $\lambda_\pm$ are not both constant. Using conditions (A1) and (A2), we see that $\lambda_+-\lambda_-$ is continuous at $I_\pm$. As a result, ψ maps I' into I and $\psi(I'_+)=I_+$. We set $\psi(I'_-):=I_-$, which may result in ψ not being right continuous at I'_-.

By (A.4) and the fact that either $\mathcal{X}_0\in\mathcal{S}^{\rm fin}_{\rm ord}$ or (A5) holds, we can choose $\mathcal{X}'_0\in\mathcal{S}_{\rm ord}$ such that $\mathcal{X}_0=\mathcal{X}'_0\circ\psi^{-1}$ and

$$\text{If } \psi(x)=\psi(z) \text{ for some } x<z, \text{ then } \mathcal{X}^-\big([x,z)\big)=0 \text{ and } \mathcal{X}^+\big([x,z)\big)=0\tag{A.13}$$

In view of this, using also the fact that λ_- is left-continuous and λ_+ is right-continuous, (A.11) simplifies to

$$\begin{aligned}&\text{If } \psi(x)=\psi(z) \text{ for some } x<z, \text{ then there exists an } y \text{ with } x\leq y\leq z\\&\text{such that } \lambda'_-(x)=\lambda'_-(y) \text{ and } \lambda'_+(y)=\lambda'_+(z).\end{aligned}\tag{A.14}$$

The condition (A.8) determines $\lambda'_\pm$ uniquely except on intervals (x,z) such that $\psi(x)=\psi(z)$. We claim that on each such interval, $\lambda'_\pm$ are uniquely determined by (A.14). Indeed, if (x,z) is such an interval of maximal length and $u:=\psi(x)=\psi(z)$, then

$$(\lambda'_+-\lambda'_-)(z)-(\lambda'_+-\lambda'_-)(x)=\big(\lambda_+(u)-\lambda_+(u-)\big)+\big(\lambda_-(u)-\lambda_-(u+)\big),\tag{A.15}$$

and we need to choose the point y from (A.14) in such a way that

$$y-x=\lambda_+(u)-\lambda_+(u-)\quad\text{and}\quad z-y=\lambda_-(u)-\lambda_-(u+).\tag{A.16}$$

If we drop the assumption (A.4), allow for the case that $\lambda_\pm$ are both constant, and also drop the assumption that either $\mathcal{X}_0\in\mathcal{S}^{\rm fin}_{\rm ord}$ or (A5) holds, then the argument is similar if in (A.12) we replace $\lambda_+(x)-\lambda_-(x)$ by $\lambda_+(x)-\lambda_-(x)+x$. ∎

A.2 Ergodicity of Markov chains

Let $(E,\mathcal{E})$ be a measurable space and let P be a measurable probability kernel on E. For simplicity, we assume that the one-point sets are measurable, i.e., $\{x\}\in\mathcal{E}$ for all $x\in E$. It is known[1] [MT09, Thm 3.4.1] that for each probability measure μ on E there exists a Markov chain $X=(X_k)_{k\geq 0}$, unique in distribution, such that X_0 has law μ and the conditional law of X_{k+1} given $(X_0,\ldots,X_k)$ is given by P, for each $k\geq 0$.

[1] This statement is not quite as straightforward as it may sound since for general measurable spaces, Kolmogorov's extension theorem is not available.

We let P^k denote the k-th power of P. For any measurable real function $f : E \to [-\infty, \infty]$, we write $P^k f(x) := \int_E P^k(x, dy) f(y)$, as long as the integral is well-defined for all $x \in E$. For any probability measure μ on E we let $\mu P^k(A) := \int_E \mu(\mathrm{d}x) P^k(x, A)$ ($A \in \mathcal{E}$). Then μP^k is the law of X_k if X_0 has law μ. An invariant law of X is a probability measure ν such that $\nu P = \nu$. We let $\|\mu - \nu\|$ denote the total variation norm distance between two probability measures μ and ν.

For any point $x \in E$, let $\mathbb{P}^x$ denote the law of the Markov chain X started from $X_0 = x$. Let $\tau_x := \inf\{k > 0 : X_k = x\}$ denote the first return time to x. We say that the Markov chain X is *aperiodic from* x if the greatest common divisor of $\{k > 0 : \mathbb{P}^x[\tau_x = k]\}$ is one.

Markov chains satisfying the conditions (A.17) and (A.18) below behave in many ways like positive recurrent Markov chains with countable state space. In particular, (A.17) says that f is a Lyapunov function that guarantees that the return times to the set C have finite expectation, while (A.18) says that once the chain enters C, there is a uniformly positive probability of entering the atom 0 after a finite number of steps.

Proposition 19 (Ergodicity for positive point recurrent chain) *Fix a point* $0 \in E$. *Assume that there exists a measurable function* $f : E \to [0, \infty)$*, a measurable set* $0 \in C \subset E$*, and constants* $F, K < \infty$ *such that* $\sup_{x \in C} f(x) \leq F$ *and*

$$Pf - f \leq K 1_C - 1. \tag{A.17}$$

Assume moreover that there exist constants $\varepsilon > 0$ *and* $k \geq 0$ *such that*

$$\mathbb{P}^x[\tau_0 \leq k] \geq \varepsilon \qquad (x \in C). \tag{A.18}$$

Then $\mathbb{E}^x[\tau_0] < \infty$ *for all* $x \in E$*, and the Markov chain* X *has a unique invariant law* ν*. If moreover* X *is aperiodic from* 0*, then*

$$\|\mu P^n - \nu\| \underset{n \to \infty}{\longrightarrow} 0 \tag{A.19}$$

for each probability measure μ *on* E*.*

Proof Let $\tau_C := \inf\{k \geq 1 : X_k \in C\}$ denote the first entry time of C. Then [MT09, Thm 11.3.4] tells us that $\mathbb{E}^x[\tau_C] \leq f(x) + K 1_C(x)$ ($x \in E$). Since after each visit to C, by (A.18) there is a probability of at least ε to visit 0 in the next k steps, it is not hard to deduce that $\mathbb{E}^x[\tau_0] < \infty$ for all $x \in E$. Again by [MT09, Thm 11.3.4] and the fact that, in the light of (A.18), C is petite as defined in [MT09, Section 5.5.2], we have that X is positive Harris recurrent. In particular, by [MT09, Thm 10.0.1], X has a unique invariant law ν. Since $\mathbb{E}^x[\tau_0] < \infty$ for all $x \in E$, it is easy to see that $\nu(\{0\}) > 0$. By [MT09, Thm 10.4.9], ν is equivalent to the measure ψ from [MT09, Prop. 4.2.2], so aperiodicity from 0 as we have defined it implies ψ-aperiodicity as defined in [MT09, Section 5.4.3]. Now (A.19) follows from [MT09, Thm 13.3.3]. ∎

A.3 Discrete models

Often, it is natural to consider Stigler-Luckock models where the interval I is of the form $I = [0, n]$, with $n \geq 2$ an integer, and the measures $\mu_\pm$ that determine the rate at which orders arrive are supported on the set of integers $\{0, \ldots, n\}$. One motivation for this is that real prices take values that differ by a minimal amount, the so called tick size. Also, the numerical data for the uniform model shown in Figures 2 and 3 are based on approximation with discrete models with a high value of n.

Although, in the light of Appendix A.1, discrete Stigler-Luckock models are in principle included in our general analysis, in practise, when doing (numerical) calculations, it is more convenient to replace the differential equations for the general model by difference equations. It turns out that these difference equations can be solved explicitly much in the same way as the differential equations of the general model.

In the discrete setting, it is convenient to reparametrize the model somewhat. We replace the set $\{1, 2, \ldots, n\}$ of possible prices of buy orders by $\{4, 6, \ldots, 2n + 2\}$ and we let 2 (instead of 0) be the value of $M_-(\mathcal{X})$ that signifies that the order book contains no buy limit orders. Likewise, for sell orders or $M_+(\mathcal{X})$, we replace the set of possible prices $\{0, 1, \ldots, n\}$ by $\{1, 3, \ldots, 2n + 1\}$. Note that in this new parametrization, a buy and sell order that were previously both placed at the price k are now placed at the prices $2k + 2$ and $2k + 1$, respectively, and hence still match. We let $\mathcal{X}_t^-(2k+2)$ (resp. $\mathcal{X}_t^+(2k+1)$) denote the number of buy (resp. sell) limit orders in the order book at a given time and price. We define demand and supply functions

$$\lambda_- : \{3, 5, \ldots, 2n + 1\} \to \mathbb{R} \quad \text{and} \quad \lambda_+ : \{2, 4, \ldots, 2n\} \to \mathbb{R} \tag{A.20}$$

in such a way that $\lambda_-(2k + 1)$ (resp. $\lambda_+(2k)$) is the total rate at which buy (resp. sell) orders are placed at prices in $\{2k + 2, \ldots, 2n + 2\}$ (resp. $\{1, \ldots, 2k - 1\}$). In particular, $\lambda_-(2n + 1)$ and $\lambda_+(2)$ are the rates of buy and sell market orders, respectively.

For any function of the form $f : \{k, k + 2, \ldots, m\} \to \mathbb{R}$, we define a discrete derivative $\mathrm{d}f : \{k + 1, k + 3, \ldots, m - 1\} \to \mathbb{R}$ by

$$\mathrm{d}f(x) := f(x + 1) - f(x - 1). \tag{A.21}$$

For sums over sets of the form $\{k, k + 2, \ldots, m\}$, we use the shorthand

$$\sum_k^m g := \sum_{x\in\{k,k+2,\ldots,m\}} g(x), \tag{A.22}$$

and we define $\sum_k^{k-2} g := 0$. We let

$$f'(x) := f(x + 1) \quad \text{and} \quad f^*(x) := f(x - 1) \tag{A.23}$$

denote the function f shifted by one to the left or right, respectively. It is straightforward to prove the product rule

$$\mathrm{d}(fg) = f'\mathrm{d}g + g^*\mathrm{d}f. \tag{A.24}$$

We also have the following special case of the chain rule:

$$\mathrm{d}\Big(\frac{1}{f}\Big) = -\frac{\mathrm{d}f}{f'f^*}. \tag{A.25}$$

Let

$$w_- : \{2, 4, \ldots, 2n\} \to \mathbb{R} \quad \text{and} \quad w_+ : \{3, 5, \ldots, 2n + 1\} \to \mathbb{R} \tag{A.26}$$

be weight functions satisfying $w_-(2) := 0$ and $w_+(2n + 1) := 0$, and define a linear function F by

$$F(\mathcal{X}) := \sum_{x=4}^{2n} w_-(x)\mathcal{X}^-(x) + \sum_{x=3}^{2n-1} w_+(x)\mathcal{X}^+(x), \tag{A.27}$$

Then, in analogy with Lemma 7, one can check that

$$
\begin{aligned}
GF(\mathcal{X}) = & \sum_{M_-(\mathcal{X})+1}^{2n-1} w_+ \mathrm{d}\lambda_+ - w_-\big(M_-(\mathcal{X})\big)\lambda_+\big(M_-(\mathcal{X})\big) \\
& - \sum_{4}^{M_+(\mathcal{X})-1} w_- \mathrm{d}\lambda_- - w_+\big(M_+(\mathcal{X})\big)\lambda_-\big(M_+(\mathcal{X})\big).
\end{aligned}
\tag{A.28}
$$

In analogy with Theorem 8, one can show that if the rates of market orders $\lambda_-(2n+1)$ and $\lambda_+(2n)$ are both positive, then, for each $z \in \{2, 4, \dots, 2n\}$, there exist a unique pair of weight functions $(w_-^{z,+}, w_+^{z,+}) = (w_-, w_+)$ and a unique constant $f_+(z) \in \mathbb{R}$, such that the linear functional $F^{z,+} = F$ from (A.27) satisfies

$$
GF(\mathcal{X}) = 1_{\{M_+(\mathcal{X}) > z\}} - f_+(z). \tag{A.29}
$$

Also, for each $z \in \{3, 5, \dots, 2n+1\}$, there exist a unique pair of weight functions $(w_-^{z,-}, w_+^{z,-}) = (w_-, w_+)$ and constant $f_-(z)$ such that the linear functional $F^{z,-} = F$ from (A.27) satisfies

$$
GF(\mathcal{X}) = 1_{\{M_-(\mathcal{X}) < z\}} - f_-(z). \tag{A.30}
$$

It is possible to derive nice, explicit formulas for these weight functions. In analogy with (2.19), define

$$
\Gamma := \frac{1}{\lambda_-(2n+1)\lambda_+(2n)} - \sum_{3}^{2n-1} \frac{1}{\lambda_-} \mathrm{d}\Big(\frac{1}{\lambda_+}\Big) = \frac{1}{\lambda_-(3)\lambda_+(2)} + \sum_{4}^{2n} \frac{1}{\lambda_+} \mathrm{d}\Big(\frac{1}{\lambda_-}\Big), \tag{A.31}
$$

where the equality of both formulas follows from the product rule (A.24) applied to the functions $1/\lambda'_-$ and $1/\lambda_+$. Set $I_{\mathrm{even}} := \{2, 4, \dots, 2n\}$ and $I_{\mathrm{odd}} := \{3, 5, \dots, 2n+1\}$. In analogy with (2.50), define

$$
\begin{aligned}
u_{-+}(x) &:= \Gamma^{-1}\Big\{\frac{1}{\lambda_-(2n+1)\lambda_+(2n)} - \sum_{x+1}^{2n-1} \frac{1}{\lambda_-} \mathrm{d}\Big(\frac{1}{\lambda_+}\Big)\Big\} && \big(x \in I_{\mathrm{even}}\big), \\
u_{+-}(x) &:= \Gamma^{-1}\Big\{\frac{1}{\lambda_-(3)\lambda_+(2)} + \sum_{4}^{x-1} \frac{1}{\lambda_+} \mathrm{d}\Big(\frac{1}{\lambda_-}\Big)\Big\} && \big(x \in I_{\mathrm{odd}}\big).
\end{aligned}
\tag{A.32}
$$

Then, in analogy with Lemma 15, one has

$$
\begin{aligned}
w_+^{z,-}(x) &= \lambda_-(z)\Gamma\big[u_{+-}(x \vee z) - 1\big]u_{+-}(x \wedge z) && \big(x, z \in I_{\mathrm{odd}}\big) \\
w_-^{z,-}(x) &= \lambda_-(z)\Gamma\big(u_{+-}(z) - 1_{\{x<z\}}\big)\big(u_{-+}(x) - 1_{\{x<z\}}\big) && \big(x \in I_{\mathrm{even}},\ z \in I_{\mathrm{odd}}\big) \\
w_-^{z,+}(x) &= \lambda_+(z)\Gamma\big[u_{-+}(x \wedge z) - 1\big]u_{-+}(x \vee z) && \big(x, z \in I_{\mathrm{even}}\big) \\
w_+^{z,+}(x) &= \lambda_+(z)\Gamma\big(u_{-+}(z) - 1_{\{x>z\}}\big)\big(u_{+-}(x) - 1_{\{x>z\}}\big) && \big(x \in I_{\mathrm{odd}},\ z \in I_{\mathrm{even}}\big).
\end{aligned}
\tag{A.33}
$$

Moreover, the functions

$$
f_- : \{3, 5, \dots, 2n+1\} \to \mathbb{R} \quad \text{and} \quad f_+ : \{2, 6, \dots, 2n\} \to \mathbb{R} \tag{A.34}
$$

from (A.29) and (A.30) satisfy the discrete version of Luckock's equation, which reads

$$
\begin{aligned}
&\text{(i)} \quad f_- \mathrm{d}\lambda_+ = -\lambda_- \mathrm{d}f_+ && \text{on } \{3, 5, \dots, 2n-1\}, \\
&\text{(ii)} \quad f_+ \mathrm{d}\lambda_- = -\lambda_+ \mathrm{d}f_- && \text{on } \{4, 6, \dots, 2n\}, \\
&\text{(iii)} \quad f_+(2) = 1 = f_-(2n+1).
\end{aligned}
\tag{A.35}
$$

The solution to this equation can explicitly be written as

$$\begin{aligned}
&\text{(i)} \quad \Big(\frac{f_+}{\lambda_+}\Big)(x) = \frac{1}{\lambda_+(2)} + \sum_{3}^{x-1} \frac{\kappa}{\lambda_-}\,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) && \big(x \in \{2,4,\ldots,2n\}\big),\\
&\text{(ii)} \quad \Big(\frac{f_-}{\lambda_-}\Big)(x) = \frac{1}{\lambda_-(2n+1)} - \sum_{x+1}^{2n} \frac{\kappa}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) && \big(x \in \{3,5,\ldots,2n+1\}\big),
\end{aligned} \tag{A.36}$$

where κ is given by

$$\kappa = \kappa_{\mathrm{L}} := \Gamma^{-1}\Big(\frac{1}{\lambda_-(2n+1)} + \frac{1}{\lambda_+(2)}\Big), \tag{A.37}$$

and $\Gamma > 0$ is the constant from (A.31).

A.4 Suggestions for future work

Several problems concerning Stigler-Luckock models remain open. In particular, these include:

I. If a Stigler-Luckock model has a critical window (J_-, J_+), then show that in the long run, buy orders below J_- and sell orders above J_+ are never matched.

II. Show that all orders inside the critical window are eventually matched.

III. If the solution to Luckock's equation satisfies $f_-(I_-) \wedge f_+(I_+) \geq 0$, then show that there is an invariant law on $\mathcal{S}_{\mathrm{ord}}$.

Let $(X_k)_{k\geq 0}$ be a Stigler-Luckock model on an interval I and let $X_k\big|_Z$ be the restriction of X_k to a subinterval $Z = (z_-, z_+) \subset I$. (Note that this is not what we have called the restricted model on Z; in particular, the latter is a Markov chain, while $X_k\big|_Z$ is not.) To solve Problem I, one would need to show that if Z is slightly larger than the critical window, then the process $X_k\big|_Z$ is transient, in a suitable sense, while Problems II and III could be solved if one could show that if Z is slightly smaller than the critical window, then the process $X_k\big|_Z$ spends a positive time in the empty state, with some uniform bounds on the expected number of buy and sell orders in Z.

In this context, it is natural to look at linear functions F as in (1.26) such that the weight functions $w_\pm$ are supported on $\overline{Z}$ and $GF(\mathcal{X})$ depends only on the relative order of $M_\pm(\mathcal{X})$ and $z_\pm$. It appears that such weight functions exist and form a two-dimensional space. Using notation as in Theorem 8, let us define

$$\hat{w}_\pm := w^{z_-,-} + w^{z_+,+} \quad \text{and} \quad \dot{w}_\pm := w^{z_+,-} + w^{z_-,+}. \tag{A.38}$$

Then it appears that there exists a unique constant $c \in \mathbb{R}$ such that

$$\overline{w}_\pm := \hat{w}_\pm + c\dot{w}_\pm \tag{A.39}$$

are supported on $\overline{Z}$. Moreover, it seems that the two-dimensional space we just mentioned is spanned by the "symmetric" weight functions $(\overline{w}_-, \overline{w}_+)$ and the "asymmetric" weight functions (w^*_-, w^*_+) defined as

$$w^*_- := -1_{(z_-,z_+]} \quad \text{and} \quad w^*_+ := 1_{[z_-,z_+)}. \tag{A.40}$$

Letting $\overline{F}$ and F^* denote the corresponding linear functions, a natural way to attack Problem I is to show that if the critical window J satisfies $\overline{J} \subset Z$, then there exists a function $h(\overline{F}, F^*)$ that is subharmonic for the generator G in (1.25) and that shows that $\overline{F}(\mathcal{X}_t) \to \infty$ a.s. in such a way that $|F^*(\mathcal{X}_t)| \ll \overline{F}(\mathcal{X}_t)$.

Also, a natural way to attack Problems II and III is to find a “Lyapunov style” function V that depends on $\overline{F}$, F^*, and perhaps some other functions of the process, and that solves an inequality of the form (3.6).

To conclude the paper, we mention a few more open problems, for which we have nothing more concrete to say.

IV. Show that the invariant law from Problem III is unique and the long-time limit law started from any initial law.

V. Investigate existence and uniqueness of solutions to Luckock’s equation with assumption (A6) replaced by the weaker (A5) plus perhaps some conditions involving the function Ψ from (1.21)

VI. Investigate whether the restricted model on the critical window is null recurrent or transient.

VII. Prove a limit theorem for the shape of the stationary process near the boundary of the critical window, in the spirit of [FS15].

VIII. For the critical model, investigate the tail of the distribution of the time till a limit order is matched.

Acknowledgements

I would like to thank Martin Šmíd for drawing my attention to Luckock’s paper [Luc03], and Marco Formentin, Martin Ondreját, and Jan Seidler for useful discussions.

References

[BS93] P. Bak and K. Sneppen. Punctuated equilibrium and criticality in a simple model of evolution. *Phys. Rev. Lett.* 74 (1993), 4083–4086.

[CB00] M. Carter and B. van Brunt. *The Lebesgue-Stieltjes Integral. A Practical Introduction.* Springer, New York, 2000.

[CG09] A. Gabrielli and G. Caldarelli. Invasion percolation and the time scaling behavior of a queuing model of human dynamics. *J. Stat. Mech.* (2009), P02046 (10 pages).

[FS15] M. Formentin and J.M. Swart. The limiting shape of a full mailbox. Preprint (2015).

[Hoe77] G. Högnäs. Characterization of weak convergence of signed measures on $[0, 1]$. *Math. Scand.* 41 (1977), 175–184.

[Kru12] L. Kruk. Limiting distribution for a simple model of order book dynamics *Cent. Eur. J. Math.* 10(6) (2012), 2283–2295.

[LPW09] D.A. Levin, Y. Peres and E.L. Wilmer. *Markov Chains and Mixing Times.* AMS, Providence, 2009.

[Luc03] H. Luckock. A steady-state model of the continuous double auction. *Quantitative Finance* 3(5) (2003), 385–404.

[Mai13] P. Maillard. Speed and fluctuations of N-particle branching Brownian motion with spatial selection. Preprint (2013), 84 pages, ArXiv:1304.0562v3.

[Mas00] S. Maslov. Simple model of a limit order-driven market *Physica A* 278 (2000), 571–578.

[MS12] R. Meester and A. Sarkar. Rigorous self-organised criticality in the modified Bak-Sneppen model. *J. Stat. Phys.* 149 (2012), 964–968.

[MT09] S. Meyn and R.L. Tweedie. *Markov chains and stochastic stability. 2nd ed.* Cambridge University Press, 2009.

[Pla11] Jana Plačková. *Shluky volatility a dynamika poptávky a nabídky.* (In Czech) Master Thesis, MFF, Charles University Prague, 2011.

[Sla13] F. Slanina. *Essentials of Econophysics Modelling.* Oxford University Press, 2013.

[Smi12] M. Šmíd. Probabilistic properties of the continuous double auction. *Kybernetika* 48(1) (2012), 50–82.

[SRR16] E. Scalas, F. Rapallo, and T. Radivojević. Low-traffic limit and first-passage times for a simple model of the continuous double auction. Preprint (2016), ArXiv:1603.09666.

[Sti64] G.J. Stigler. Public regulation of the securities markets. *The Journal of Business* 37(2) (1964), 117–142.

[Swa15] J.M. Swart. A simple rank-based Markov chain with self-organized criticality. Preprint (2015), ArXiv:1405.3609v3.

[Wal74] L. Walras. *Éléments d' économie politique pure, ou Théorie de la richesse sociale.* First published 1874; published again in Paris by R. Pichon and R. Durand-Auzias and in Lausanne by F. Rouge, 1926.