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RIGOROUS RESULTS FOR THE STIGLER–LUCKOCK MODEL FOR THE EVOLUTION OF AN ORDER BOOK[1]

BY JAN M. SWART

The Czech Academy of Sciences, Institute of Information Theory and Automation

In 1964, G. J. Stigler introduced a stochastic model for the evolution of an order book on a stock market. This model was independently rediscovered and generalized by H. Luckock in 2003. In his formulation, traders place buy and sell limit orders of unit size according to independent Poisson processes with possibly different intensities. Newly arriving buy (sell) orders are either immediately matched to the best available matching sell (buy) order or stay in the order book until a matching order arrives. Assuming stationarity, Luckock showed that the distribution functions of the best buy and sell order in the order book solve a differential equation, from which he was able to calculate the position of two prices $J_-^c < J_+^c$ such that buy orders below J_-^c and sell orders above J_+^c stay in the order book forever while all other orders are eventually matched. We extend Luckock's model by adding market orders, that is, with a certain rate traders arrive at the market that take the best available buy or sell offer in the order book, if there is one, and do nothing otherwise. We give necessary and sufficient conditions for such an extended model to be positive recurrent and show how these conditions are related to the prices J_-^c and J_+^c of Luckock.

1. Introduction and results.

1.1. *Definition of the model.* We will be interested in a stochastic model for traders interacting through an order book as is commonly used on a stock market or commodity market. In the more theoretical economic literature, the sort of trading system we are interested in is also known as the continuous double auction. In our specific model of interest, traders arrive according to independent Poisson processes and place either a buy or sell limit order for exactly one item of a certain stock or commodity. If the order book already contains a suitable offer, then the new limit order is immediately matched with the best available offer, that is, a new buy limit order at a price x is cancelled against an existing sell limit order at the lowest possible price $x' \leq x$, if such a sell limit order exists, and vice versa for new sell limit orders. Orders that are not immediately matched stay in the order book until they are matched with a new incoming order, or, if such an order never

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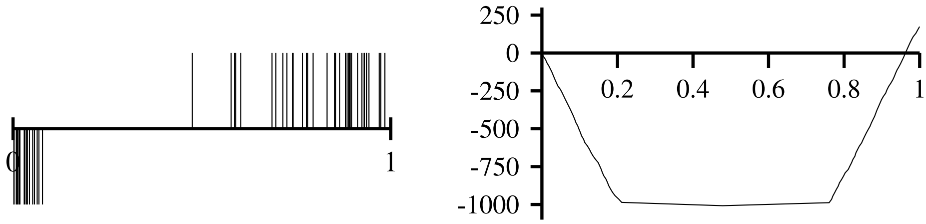

FIG. 1. *Simulation of the "uniform" Stigler–Luckock model with* $I = (0, 1)$, $\lambda_-(x) = 1 - x$, *and* $\lambda_+(x) = x$. *On the left*: *the state* X_{250} *of the order book after the arrival of* 250 *traders*, *starting from the empty initial state. On the right*: *the distribution function* $x \mapsto X_k([0, x])$ *of the random signed measure* X_k *after the arrival of* $k = 10{,}000$ *traders.*

comes, forever. See Figure 1 for a numerical simulation. This model, in discrete time and for a specific choice of the parameters, was invented by Stigler [23] and, in its full generality, independently by Luckock [13]. The model was subsequently again independently reinvented by Plačková in her master thesis [19] and by Yudovina in her Ph.D. thesis [27, 28]. We will generalize the model by also allowing market orders, that is, with a certain rate a trader arrives that takes the best available limit buy (sell) order in the order book, if such an order exists, and does nothing otherwise.

To formulate this model in more mathematical detail, let $I = (I_-, I_+) \subset \mathbb{R}$ be a nonempty open interval, modeling the possible prices of limit orders, and let $\overline{I} := [I_-, I_+] \subset [-\infty, \infty]$ denote its closure. Let $\lambda_\pm : \overline{I} \to [0, \infty)$ be functions such that:

(A1) λ_- is nonincreasing and left continuous, while λ_+ is nondecreasing and right continuous.

(A2) $\lim_{x \downarrow I_-} \lambda_-(x) = \lambda_-(I_-)$ and $\lim_{x \uparrow I_+} \lambda_+(x) = \lambda_+(I_+)$.

We interpret $\lambda_-(x)$ and $\lambda_+(x)$ as the *demand* and *supply* functions, which describe how many items per time unit traders are willing to buy or sell at the price level x. More precisely, let $\mu_\pm$ be finite measures on $\overline{I}$ such that

$$\mu_-\big([x, I_+]\big) = \lambda_-(x) \quad \text{and} \quad \mu_+\big([I_-, x]\big) = \lambda_+(x) \qquad (x \in \overline{I}). \tag{1.1}$$

Then the restriction of μ_- (resp., μ_+) to I will be the Poisson intensity at which traders place buy (resp., sell) limit orders at a given price, while $\mu_-(\{I_+\})$ [resp., $\mu_+(\{I_-\})$] will be the Poisson intensity at which traders place buy (resp., sell) market orders. Note that $\mu_-(\{I_-\}) = 0 = \mu_+(\{I_+\})$ by assumption (A2).

We let τ_k ($k \geq 1$) denote the time when the kth trader arrives at the market, we let $\sigma_k \in \{-, +\}$ be a random variable that indicates whether this trader wants to buy ($-$) or sell ($+$), and we let $U_k \in \overline{I}$ denote the price associated with this trader, where $U_k \in I$ for limit orders and $U_k = I_\pm$ for market orders. Then

$$\Pi = \big\{(U_k, \sigma_k, \tau_k) : k = 1, 2, \dots\big\} \qquad \text{with } 0 < \tau_1 < \tau_2 < \cdots \tag{1.2}$$

is a Poisson point process on $\overline{I} \times \{-,+\} \times [0,\infty)$ with intensity $\mu \otimes \ell$, where ℓ is the Lebesgue measure on $[0,\infty)$ and μ is the finite measure on $\overline{I} \times \{-,+\}$ given by $\mu(\{\sigma\} \times A) = \mu_\sigma(A)$ for all $\sigma \in \{-,+\}$ and measurable $A \subset \overline{I}$. We let

$$|\mu_\pm| := \mu_\pm(\overline{I}) \quad \text{and} \quad |\mu| := \mu\big(\overline{I} \times \{-,+\}\big) = |\mu_-| + |\mu_+| \tag{1.3}$$

denote the total masses of the measures $\mu_\pm$ and μ. To avoid trivialities, we assume that $|\mu| \neq 0$. Our assumption that the point process Π in (1.2) is Poisson with intensity $\mu \otimes \ell$ implies that $(\tau_k - \tau_{k-1})_{k\geq 1}$ are i.i.d. exponentially distributed with mean $1/|\mu|$. Moreover, the random variables $(U_k, \sigma_k)_{k\geq 1}$ are i.i.d. with law $\overline{\mu} := |\mu|^{-1}\mu$ and independent of $(\tau_k)_{k\geq 1}$.

By definition, a *counting measure* is a measure that can be written as a finite or countable sum of delta measures. We represent the state of the order book at a time $t \geq 0$ by a signed counting measure of the form

$$\mathcal{X} = \mathcal{X}^+ - \mathcal{X}^-, \tag{1.4}$$

where $\mathcal{X}^-$ and $\mathcal{X}^+$ are counting measures on I satisfying:

$$\begin{aligned} &\text{(i)} \quad \text{there are no } x, y \in I \text{ such that } x \leq y,\ \mathcal{X}^+\big(\{x\}\big) > 0,\ \mathcal{X}^-\big(\{y\}\big) > 0, \\ &\text{(ii)} \quad \mathcal{X}^-\big((x, I_+)\big) < \infty \text{ and } \mathcal{X}^+\big((I_-, x)\big) < \infty \text{ for all } x \in I. \end{aligned} \tag{1.5}$$

We interpret $\mathcal{X}^-(A)$ [resp., $\mathcal{X}^+(A)$] as the number of buy (resp., sell) limit orders in a measurable set $A \subset I$, and let $\mathcal{S}_{\text{ord}}$ denote the set of all signed measures that can be written in the form (1.4) with $\mathcal{X}^\pm$ satisfying (1.5). For any $\mathcal{X} \in \mathcal{S}_{\text{ord}}$, we let

$$\begin{aligned} M_-(\mathcal{X}) &:= \max\big(\{I_-\} \cup \big\{x \in I : \mathcal{X}\big(\{x\}\big) < 0\big\}\big), \\ M_+(\mathcal{X}) &:= \min\big(\{I_+\} \cup \big\{x \in I : \mathcal{X}\big(\{x\}\big) > 0\big\}\big), \end{aligned} \tag{1.6}$$

which can be interpreted as the highest bid and lowest ask price in the order book.

The state of our Markov process changes only at the times $\tau_1, \tau_2, \ldots$ and we denote the corresponding embedded Markov chain by

$$X_k := \mathcal{X}_{\tau_k} \qquad (k \geq 0) \text{ with } \tau_0 := 0. \tag{1.7}$$

Our previous informal description of the model then translates into the following definition. Given the initial state $X_0 \in \mathcal{S}_{\text{ord}}$, we inductively define $(X_k)_{k\geq 1}$ as

$$X_k := L_{U_k,\sigma_k}(X_{k-1}) \qquad (k \geq 1), \tag{1.8}$$

where for each $(u,\sigma) \in \overline{I} \times \{-,+\}$, we define a "Luckock map" $L_{u,\sigma} : \mathcal{S}_{\text{ord}} \to \mathcal{S}_{\text{ord}}$ by

$$L_{u,\sigma}(\mathcal{X}) := \begin{cases} \mathcal{X} - \delta_{u \wedge M_+(\mathcal{X})} & \text{if } \sigma = -, u \wedge M_+(\mathcal{X}) \in I, \\ \mathcal{X} + \delta_{u \vee M_-(\mathcal{X})} & \text{if } \sigma = +, u \vee M_-(\mathcal{X}) \in I, \\ \mathcal{X} & \text{otherwise.} \end{cases} \tag{1.9}$$

For example, if $\sigma = +$, then this says that a new sell limit order is added at the price u, unless the current best buy offer $M_-(\mathcal{X})$ is higher than u, in which case this offer is taken, which amounts to adding a delta measure at $M_-(\mathcal{X})$. The rules for sell market orders are the same, except that these are not added to the order book if no suitable buy offer exists.

It is easy to see that $(X_k)_{k\geq 0}$ is a Markov chain; in fact, using terminology from [12], we have just given a *random mapping representation* for it. We call the Markov chain in (1.8) or, more or less equivalently, the corresponding continuous-time Markov process $(\mathcal{X}_t)_{t\geq 0}$ the *Stigler–Luckock model* with parameters $\lambda_\pm$. In the special case that there are no market orders, this is the model introduced in [13]. The authors [19, 23] considered only the case that $\mu_- = \mu_+$ is the uniform distribution on a set of 10, respectively, 100 prices. In [8], infinite piles of limit orders are used as a construction that is mathematically equivalent to market orders. As we will see, the introduction of market orders is natural also from a mathematical point of view and helps us understand the model without market orders.

We let

$$V_{\max} := \lambda_-(I_-) \wedge \lambda_+(I_+) \quad \text{and} \quad V_{\mathrm{W}} := \sup_{x\in\overline{I}} \lambda_-(x) \wedge \lambda_+(x) \tag{1.10}$$

denote the maximal possible volume of trade and the maximal possible volume of trade at a fixed price level, respectively. We will sometimes need the following stronger conditions on our demand and supply functions:

- (A3) $\lambda_\pm$ are continuous on $\overline{I}$, λ_- is nonincreasing, and λ_+ is nondecreasing.
- (A4) The function $\lambda_+ - \lambda_-$ is strictly increasing on $\overline{I}$.
- (A5) The functions λ_- and λ_+ are > 0 on I.
- (A6) The rates $\lambda_+(I_-)$ and $\lambda_-(I_+)$ of market orders are both >0.
- (A7) $V_{\mathrm{W}} < V_{\max}$.

In particular, (A3) implies (A1) and (A2). As shown in Appendix A.1, (A3) and (A4) are not really a restriction, since every Stigler–Luckock model satisfying (A1) and (A2) can be obtained as a function of a Stigler–Luckock model satisfying (A3) and, under mild extra assumptions, also (A4). Condition (A5) also comes basically without loss of generality, since sell orders on the right of the first point x where $\lambda_-(x) = 0$ are trivially never matched, and similarly for buy orders at the other end of the interval. The conditions (A6) and (A7) are restrictive, of course.

Conditions (A3), (A4) and (A7) imply that there exists a unique point $x_{\mathrm{W}} \in I$ such that

$$\lambda_-(x_{\mathrm{W}}) = \lambda_+(x_{\mathrm{W}}). \tag{1.11}$$

Classical economic theory going back to Walras [26] says that in an infinitely liquid market, the equilibrium price is x_{W}, which is why we call x_{W} the *Walrasian price*. Note that $V_{\mathrm{W}} := \lambda_-(x_{\mathrm{W}}) = \lambda_+(x_{\mathrm{W}})$, which is why we call this the *Walrasian volume of trade*.

1.2. *Luckcock's differential equation.* The following theorem is essentially proved in [13], but for completeness we will provide a proof in the present setting. Below, if $f : \overline{I} \to \mathbb{R}$ is a continuous function of bounded variation, then we let $\mathrm{d}f$ denote the signed measure on $\overline{I}$ such that $\mathrm{d}f((x, y]) := f(y) - f(x)$. If $g : \overline{I} \to \mathbb{R}$ is a bounded measurable function, then $g\,\mathrm{d}f$ denotes the measure $\mathrm{d}f$ weighted with g, that is, $g\,\mathrm{d}f((x, y]) := \int_x^y g\,\mathrm{d}f$. We call any pair (f_-, f_+) of continuous functions of bounded variation such that (1.13) below holds a *solution to Luckock's equation*.

THEOREM 1 (Luckock's differential equation). *Consider a Stigler–Luckock model with supply and demand functions* $\lambda_\pm : \overline{I} \to [0, \infty)$ *satisfying* (A3) *and* (A5). *Assume that the model has an invariant law on* $\mathcal{S}_{\mathrm{ord}}$ *and let* $(X_k)_{k \geq 0}$ *denote the corresponding stationary process. Then the functions* $f_\pm : \overline{I} \to \mathbb{R}$ *defined by*

$$f_-(x) := \mathbb{P}\big[M_-(X_k) \leq x\big] \quad \text{and} \quad f_+(x) := \mathbb{P}\big[M_+(X_k) \geq x\big] \qquad (x \in \overline{I}) \tag{1.12}$$

are continuous and solve the equations:

$$\begin{aligned} &\text{(i)} \quad f_-\,\mathrm{d}\lambda_+ + \lambda_-\,\mathrm{d}f_+ = 0,\\ &\text{(ii)} \quad f_+\,\mathrm{d}\lambda_- + \lambda_+\,\mathrm{d}f_- = 0,\\ &\text{(iii)} \quad f_-(I_+) = 1 = f_+(I_-). \end{aligned} \tag{1.13}$$

We remark that although Theorem 1 shows that the equilibrium distributions of the best buy and sell order in the order book can more or less be solved explicitly [depending on our ability to solve (1.13)]; this does not automatically mean that Stigler–Luckock models as a whole are "solvable." For example, we do not know how to explicitly calculate the joint distribution of $M_-(\mathcal{X})$ and $M_+(\mathcal{X})$ (as opposed to its marginals). Also, it seems to be quite hard to get information about the equilibrium distribution of seemingly simple functions of the process like the number of sell (or buy) limit orders in a certain interval.

Theorem 1 motivates the study of solutions to Luckock's equation (1.13).

PROPOSITION 2 (Solutions to Luckock's equation). *Assume* (A3) *and* (A6). *Then Luckock's equation has a unique solution* (f_-, f_+). *One has*:

(1.14)

$$\text{(i)} \quad f_-(I_-) \geq 0 \quad \Leftrightarrow \quad \Lambda_- := \frac{1}{\lambda_-(I_-)\lambda_-(I_+)} - \int_{I_-}^{I_+} \frac{1}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) \geq 0,$$

$$\text{(ii)} \quad f_+(I_+) \geq 0 \quad \Leftrightarrow \quad \Lambda_+ := \frac{1}{\lambda_+(I_-)\lambda_+(I_+)} + \int_{I_-}^{I_+} \frac{1}{\lambda_-}\,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) \geq 0.$$

Both formulas also hold with the inequality signs reversed. The functions (f_-, f_+) *also satisfy*

$$\lambda_+(I_+) - f_-(I_-)\lambda_+(I_-) = \lambda_-(I_-) - f_+(I_+)\lambda_-(I_+). \tag{1.15}$$

If the solution (f_-, f_+) to Luckock's equation satisfies $f_-(I_-) \wedge f_+(I_+) \geq 0$, then we call such a solution *valid*. See Figure 3 for a plot of (f_-, f_+) for one particular model; in this particular example, (f_-, f_+) is not valid. By Theorem 1, a necessary condition for a Stigler–Luckock model to have an invariant law is that the solution to Luckock's equation is valid. We conjecture that this condition is also sufficient, but stop short of proving this. (Note however Theorem 3 below, which goes some way in this direction.)

If a Stigler–Luckock model has an invariant law, then the quantity in (1.15) can be interpreted as the *volume of trade*, that is, the expected number of orders (of either type) that are matched per unit of time. Indeed, since the process has an invariant law, sell limit orders, which arrive at rate $\lambda_+(I_+) - \lambda_+(I_-)$, are all eventually matched, while $1 - f_-(I_-) = \mathbb{P}[M_-(\mathcal{X}_t) > I_-]$ is the fraction of sell market orders that are matched, so $(\lambda_+(I_+) - \lambda_+(I_-)) + (1 - f_-(I_-))\lambda_+(I_-)$ is the total rate at which sell orders are matched, which equals the left-hand side of (1.15). The right-hand side of (1.15) has a similar interpretation in terms of buy orders.

1.3. *Positive recurrence.* Let $(X_k)_{k\geq 0}$ be a Stigler–Luckock model with discrete time [i.e., the embedded Markov chain from (1.7)], started in the empty initial state $X_0 = 0$, and let τ denote its first return time to 0, that is, $\tau := \inf\{k > 0 : X_k = 0\}$. We say that a Stigler–Luckock model is *positive recurrent* if $\mathbb{E}[\tau] < \infty$, *transient* if $\mathbb{P}[\tau = \infty] > 0$, and *null recurrent* in the remaining case. The main result of the present paper is the following result, that gives a more or less complete characterization of positive recurrent Stigler–Luckock models. Below and in what follows, we let $\mathcal{S}_{\mathrm{ord}}^{\mathrm{fin}}$ denote the set of all finite configurations $\mathcal{X} \in \mathcal{S}_{\mathrm{ord}}$, that is, those for which $\mathcal{X}^-$ and $\mathcal{X}^+$ are finite measures.

THEOREM 3 (Positive recurrence). *Assume* (A3) *and* (A6). *Then a Stigler–Luckock model is positive recurrent if and only if the unique solution (f_-, f_+) to Luckock's equation satisfies $f_-(I_-) \wedge f_+(I_+) > 0$. If a Stigler–Luckock model is positive recurrent, then it has an invariant law ν that is concentrated on $\mathcal{S}_{\mathrm{ord}}^{\mathrm{fin}}$. Moreover, the process started in any initial law that is concentrated on $\mathcal{S}_{\mathrm{ord}}^{\mathrm{fin}}$ satisfies*

$$\big\|\mathbb{P}[X_k \in \cdot\,] - \nu\big\| \underset{k\to\infty}{\longrightarrow} 0, \tag{1.16}$$

where $\|\cdot\|$ denotes the total variation norm.

Under rather restrictive additional assumptions, a version of Theorem 3 has also been proved in [8], Theorem 2.1 (3). We will prove Theorem 3 by constructing a Lyapunov function. Instead, the authors of [8], being unable to find a Lyapunov function, used fluid limit methods.

1.4. *Restricted models.* Assume that the demand and supply functions λ_- and λ_+ satisfy (A3) and (A5). Then, for each interval $J = (J_-, J_+)$ such that $\overline{J} = [J_-, J_+] \subset I$, the restrictions of λ_- and λ_+ to $\overline{J}$ satisfy (A6). We call the corresponding Stigler–Luckock model the *restricted model on* J. By Proposition 2, Luckock's equation has a unique solution for this restricted model, and by Theorem 3 we can read off from this solution whether the restricted model is positive recurrent. In the present section, for fixed I and $\lambda_\pm$, we investigate the set of all subintervals $\overline{J} \subset I$ for which the restricted model is positive recurrent.

We note that if $(X_k)_{k\geq 0}$ is a Stigler–Luckock model on I and $X_k|_J$ denotes the restriction of the random signed measure X_k to a subinterval $J \subset I$, then it is in general not true that $(X_k|_J)_{k\geq 0}$ is a Markov chain. In particular, this is not the same as the restricted model on J. Nevertheless, we will see that under suitable conditions, there exists a special subinterval $J \subset I$ (below, this is called the *competitive window*) such that in the long run, we expect the model on I to behave basically like the model restricted to J, with all buy limit orders on the left of J and all sell limit orders on the right of J never being matched and as a result staying in the order book forever.

Assume (A3) and (A5) and for $I_- < J_- < J_+ < I_+$, let $\Lambda_-(J_-, J_+)$ and $\Lambda_+(J_-, J_+)$ denote the expressions in (1.14), calculated for the process restricted to the subinterval $\overline{J}$. For fixed $J_- \in I$, respectively, $J_+ \in I$, we define

$$\begin{aligned} \phi_-(J_+) &:= \sup\{J_- \in (I_-, J_+) : \Lambda_-(J_-, J_+) \leq 0\}, \\ \phi_+(J_-) &:= \inf\{J_+ \in (J_-, I_+) : \Lambda_+(J_-, J_+) \leq 0\}, \end{aligned} \tag{1.17}$$

with the conventions $\sup \varnothing := I_-$ and $\inf \varnothing := I_+$. Let

$$\begin{aligned} R := \{&(J_-, J_+) \in I \times I : J_- < J_+ \text{ and the restricted} \\ &\text{model on } J \text{ is positive recurrent}\}. \end{aligned} \tag{1.18}$$

The following lemma says that the set R is bounded by the graphs of the functions $\phi_\pm$, as well as (trivially) the line $J_- = J_+$.

LEMMA 4 (Positive recurrence of restricted models). *Assume* (A3) *and* (A5). *Then* $\phi_-(J_+) < J_+$ *and* $J_- < \phi_+(J_-)$ *for all* $J_-, J_+ \in I$. *Moreover, a point* $(J_-, J_+) \in I \times I$ *belongs to the set* R *from* (1.18) *if and only* $\phi_-(J_+) < J_-$, $J_+ < \phi_+(J_-)$, *and* $J_- < J_+$.

In Figure 2, we have pictured the set R and the graphs of the functions $\phi_\pm$ for the "uniform" model with $I = [0, 1]$, $\lambda_-(x) = 1 - x$, and $\lambda_+(x) = x$. For this model, one can check that the solution of Luckock's equation for the restricted model on J satisfies $f_-(J_-) = 0$ if and only if $J_- = \phi_-(J_+)$, and likewise one has $f_+(J_+) = 0$ if and only if (J_-, J_+) lies on the graph $\{J_+ = \phi_+(J_-)\}$. The graphs of the functions $\phi_\pm$ intersect in a unique point, which in the light of (1.15) must satisfy $\lambda_-(J_-) = \lambda_+(J_+)$.

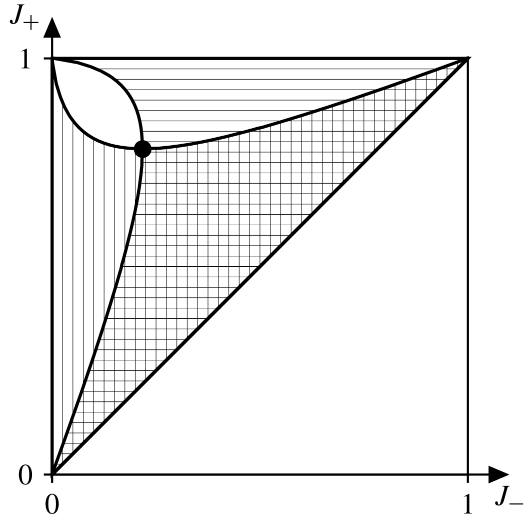

FIG. 2. *Restrictions of the uniform Stigler–Luckock model with $I = (0,1)$, $\lambda_-(x) = 1 - x$, and $\lambda_+(x) = x$ to a subinterval (J_-, J_+). The solution (f_-, f_+) to Luckock's equation for the restricted model satisfies $f_+(J_+) > 0$ in the vertically striped area and $f_-(J_-) > 0$ in the horizontally striped area. The intersection of these areas corresponds to the set R of subintervals on which the restricted model is positive recurrent. The intersection of the curves $J_- = \phi_-(J_+)$ and $J_+ = \phi_+(J_-)$, indicated with a dot, corresponds to the competitive window.*

1.5. *The competitive window.* In this subsection, we focus on subintervals $J = (J_-, J_+)$ that are "symmetric" in the sense that $\lambda_-(J_-) = \lambda_+(J_+)$. Let $\lambda_-^{-1} : [\lambda_-(I_+), \lambda_-(I_-)] \to \overline{I}$ and $\lambda_+^{-1} : [\lambda_+(I_-), \lambda_+(I_+)] \to \overline{I}$ denote the left-continuous inverses of the functions λ_- and λ_+, respectively, that is,

$$\begin{aligned} \lambda_-^{-1}(V) &:= \sup\{x \in \overline{I} : \lambda_-(x) \geq V\}, \\ \lambda_+^{-1}(V) &:= \inf\{x \in \overline{I} : \lambda_+(x) \geq V\}. \end{aligned} \tag{1.19}$$

Assuming (A5) and (A7), we define a function $\Phi : [V_{\mathrm{W}}, V_{\max}] \to [0, \infty]$ by

$$\Phi(V) := -\int_{V_{\mathrm{W}}}^{V} \left\{ \frac{1}{\lambda_+(\lambda_-^{-1}(W))} + \frac{1}{\lambda_-(\lambda_+^{-1}(W))} \right\} \mathrm{d}\left(\frac{1}{W}\right). \tag{1.20}$$

Note that Φ is increasing since $1/W$ is a decreasing function.

PROPOSITION 5 (Symmetric subintervals). *Assume* (A3), (A5) *and* (A7), *let $V \in (V_{\mathrm{W}}, V_{\max}]$, assume that $J = (J_-, J_+) := (\lambda_-^{-1}(V), \lambda_+^{-1}(V))$ satisfies $\overline{J} \subset I$, and let (f_-, f_+) be the unique solution to Luckock's equation on J. Then*:

(i) *If $\Phi(V) < V_{\mathrm{W}}^{-2}$, then $f_-(J_-) > 0$ and $f_+(J_+) > 0$.*
(ii) *If $\Phi(V) = V_{\mathrm{W}}^{-2}$, then $f_-(J_-) = 0$ and $f_+(J_+) = 0$.*
(iii) *If $\Phi(V) > V_{\mathrm{W}}^{-2}$, then $f_-(J_-) < 0$ and $f_+(J_+) < 0$.*

Theorem 3 and Proposition 5 motivate us to define *Luckock's volume of trade* as

$$V_{\mathrm{L}} := \sup\{V \in [V_{\mathrm{W}}, V_{\max}] : \Phi(V) \leq V_{\mathrm{W}}^{-2}\}, \tag{1.21}$$

and the *competitive window* as $J^{\mathrm{c}} = (J^{\mathrm{c}}_{-}, J^{\mathrm{c}}_{+}) := (\lambda_{-}^{-1}(V_{\mathrm{L}}), \lambda_{+}^{-1}(V_{\mathrm{L}}))$. Further motivation for this definition comes from the following observation.

LEMMA 6 (Competitive window). *Assume* (A3), (A5) *and* (A7) *and that* $V_{\mathrm{L}} < V_{\max}$. *Then there exists a unique subinterval* J *such that* $\overline{J} \subset I$ *and*:

(i) *The unique solution* (f_{-}, f_{+}) *to Luckock's equation on* J *satisfies* $f_{-}(J_{-}) = 0 = f_{+}(J_{+})$.

(ii) $\lambda_{-} < \lambda_{-}(J_{-})$ *on* $(J_{-}, J_{+}]$ *and* $\lambda_{+} < \lambda_{+}(J_{+})$ *on* $[J_{-}, J_{+})$.

This subinterval is given by $J = J^{\mathrm{c}}$.

We note that $V_{\mathrm{W}} < V_{\mathrm{L}}$ always, but it is possible the $V_{\mathrm{L}} = V_{\max}$. For example, this happens for the model with $\overline{I} = [0, 1]$, $\lambda_{-}(x) = (1 - x)^{\alpha}$, and $\lambda_{+}(x) = x^{\alpha}$ if $0 < \alpha \leq 1/2$. Another example are models of the form $\overline{I} = [0, 1 + \lambda]$, $\lambda_{-}(x) = (1 + \lambda - x) \wedge 1$ and $\lambda_{+}(x) = x \wedge 1$, where $\lambda \in [1 - 1/z, 1)$ with z as in Lemma 7 below; compare the discussion in [8], Section 5.1.

Since V_{L} us usually much larger than V_{W}, the Stigler–Luckock model is highly nonliquid. As such, it is not a realistic model of a real market, though it may be a useful first step towards building more realistic models. The special case where buy and sell limit orders are uniformly distributed on the unit interval is of some special interest. Numerically, the constant V_{L} from Lemma 7 is given by $V_{\mathrm{L}} \approx 0.78218829428020$.

LEMMA 7 (Uniform model). *The Stigler–Luckock model with* $\overline{I} = [0, 1]$, $\lambda_{-}(x) = 1 - x$ *and* $\lambda_{+}(x) = x$ *has a competitive window* $(J^{\mathrm{c}}_{-}, J^{\mathrm{c}}_{+})$ *which is given by* $1 - J^{\mathrm{c}}_{-} = J^{\mathrm{c}}_{+} = V_{\mathrm{L}}$, *where* $V_{\mathrm{L}} = 1/z$ *with* z *the unique solution of the equation* $e^{-z} - z + 1 = 0$.

1.6. *Discussion and open problems.* Kelly and Yudovina [8], Theorem 2.1 (1) and (2) have shown that there exists deterministic constants $J^{*}_{\pm}$ such that

$$\liminf_{k \to \infty} M_{-}(X_k) = J^{*}_{-} \quad \text{and} \quad \limsup_{k \to \infty} M_{+}(X_k) = J^{*}_{+} \qquad \text{a.s..} \tag{1.22}$$

For the uniform model of Lemma 7, they moreover show [8], Corollary 2.3 that $J^{*}_{\pm} = J^{\mathrm{c}}_{\pm}$, where $(J^{\mathrm{c}}_{-}, J^{\mathrm{c}}_{+})$ is the competitive window as defined below (1.21). Their methods apply more generally, although they need certain technical assumptions, in particular, that the measures $\mu_{\pm}$ from (1.1) have densities w.r.t. the Lebesgue measure that are bounded away from zero and infinity.

We conjecture that $\liminf_{k\to\infty} \lambda_\pm(M_\pm(X_k)) = V_{\mathrm{L}}$ a.s. holds generally under the assumptions (A3), (A5) and (A7). Assuming moreover that $V_{\mathrm{L}} < V_{\max}$, we conjecture that if $X_k|_{J^{\mathrm{c}}}$ denotes the restriction of X_k to J^{c}, then, starting from any finite initial state, the law of $X_k|_{J^{\mathrm{c}}}$ converges as $k \to \infty$ to a limit law that is concentrated on $\mathcal{S}_{\mathrm{ord}}$, and that this limit law is an invariant law for the restricted model on J^{c}. Indeed, (1.22) says that in the long run, the price of the best buy offer almost never drops below J^{c}_- and the price of the best sell offer almost never climbs above J^{c}_+, which allows us to treat limit sell orders at prices below J^{c}_- and limit buy orders above J^{c}_+ as market orders. Simulations (see Figure 1) seem to support these conjectures.

Proving these conjectures remains an open problem. Theorems 1 and 3 allow us to conclude, however, that if $\lambda_\pm(J_\pm) > V_{\mathrm{L}}$, then the restricted model on J does not have an invariant law while if $\lambda_\pm(J_\pm) < V_{\mathrm{L}}$, then the restricted model on J is positive recurrent. Further motivation for the conjectures comes from the study of similar models. In [24], a “one-sided canyon model” is studied that is in many ways similar to the Stigler–Luckock model except that there is only one type of points as opposed to the two types (buy and sell orders) of a Stigler–Luckock model. This “one-sided” model also has a competitive window that can be calculated explicitly and in fact the analogues of the conjectures above have all been proved for this model, mainly due to the hugely simplifying fact that for this model, restricting the process to a smaller interval does again yield a Markov chain.

In this context, we also mention a model for email communication due to Gabrielli and Caldarelli [6]. This model is even simpler than the previous one since not only is the restriction of the process to a subinterval Markovian, but even just counting the number of points in a subinterval already yields a Markov chain. For this model, it has been possible to solve subtle questions about the behavior of the stationary process near the boundary of the competitive window [5].

The models mentioned so far belong to a wider class of models that also includes the Bak Sneppen model [1] and its modified version from [16], as well as the branching Brownian motions with strong selection treated in [14]. All these models implement some version of the rule “kill the lowest particle” and seem to exhibit self-organized criticality, although this has been rigorously proved only for some of the models.

As mentioned before, the Stigler–Luckock model describes an extremely nonliquid market, and (mainly) for that reason is not a realistic model for a real market, although it may perhaps be used as a first step towards more realistic models. In recent years, there has been considerable activity in the search for simple, yet realistic models for an order book. An attempt to make the Stigler–Luckock model more realistic by introducing market makers is made in [18]. Often, authors assume that new orders arrive relative to the current best bid or ask price [4, 9–11, 15, 20]. For mathematical simplicity, it is sometimes assumed that the order book is “one-sided” in the sense that all buy orders are market orders and all sell orders are limit orders [10, 11, 25]. A very general model is formulated in [22]. We refer the reader to the overview article [3] or Chapter 4 of the book [21], for a more complete view on this topic.

1.7. *Methods*. The results in Sections 1.2, 1.4 and 1.5 are mainly a reworking of similar results already proved by Luckock in [13], although Proposition 5 is a significant improvement over [13], Proposition 4. Nevertheless, Luckock already derived the differential equation (1.13) and showed how it could be used to calculate the competitive window for a given model. Throughout his paper, however, he takes stationarity as a model assumption, where in fact, "stationarity" for him means the existence of two prices $J_- < J_+$ such that buy orders on the left of J_- and sell orders on the right of J_+ are never matched while the process inside (J_-, J_-) is stationary in law.

From a mathematical point of view, the existence of such a stationary process requires proof. Moreover, one would like to prove that the process started in an arbitrary finite initial state converges, in a suitable sense, to such a stationary state. For Luckock's original model, these problems remain open, but for positive recurrent processes with market orders, these questions are resolved by Theorem 3, which is the most important contribution of the present paper.

The proof of Theorem 3 is based on a Lyapunov function. We equip the space $\mathcal{S}^{\mathrm{fin}}_{\mathrm{ord}}$ with a topology such that $\mathcal{X}(n) \to \mathcal{X}$ if and only if $\mathcal{X}^{\pm}(n) \Rightarrow \mathcal{X}^{\pm}$, where $\Rightarrow$ denotes weak convergence. We also equip $\mathcal{S}^{\mathrm{fin}}_{\mathrm{ord}}$ with the Borel σ-algebra associated with this topology. We call a function $F : \mathcal{S}^{\mathrm{fin}}_{\mathrm{ord}} \to \mathbb{R}$ *Lipschitz* if there exists a constant L such that $|F(\mathcal{X} + \delta_x) - F(\mathcal{X})| \leq L$ for all $x \in I$. For any measurable Lipschitz function $F : \mathcal{S}^{\mathrm{fin}}_{\mathrm{ord}} \to \mathbb{R}$, write

$$GF(\mathcal{X}) := \int \{F\big(L_{u,\sigma}(\mathcal{X})\big) - F(\mathcal{X})\}\mu\big(\mathrm{d}(u, \sigma)\big), \tag{1.23}$$

where $L_{u,\sigma}$ is the Luckock map defined in (1.9) and μ is the measure defined below (1.2). Then G is the generator of the continuous-time Markov process $(\mathcal{X}_t)_{t \geq 0}$.

It turns out that there is a useful and explicit formula for GF when F is a "linear" function of the form

$$F(\mathcal{X}) := \int_I w_-(x)\mathcal{X}^-(\mathrm{d}x) + \int_I w_+(x)\mathcal{X}^+(\mathrm{d}x) \qquad \big(\mathcal{X} \in \mathcal{S}^{\mathrm{fin}}_{\mathrm{ord}}\big), \tag{1.24}$$

where $w_\pm : \overline{I} \to \mathbb{R}$ are bounded "weight" functions such that w_- is left continuous and w_+ is right continuous. The values of w_- and w_+ in the boundary points I_- and I_+ are irrelevant for (1.24), but for notational convenience, we define $w_-(I_+)$ and $w_+(I_-)$ by left, respectively, right continuity, and use the convention that

$$w_-(I_-) := 0 \quad \text{and} \quad w_+(I_+) := 0. \tag{1.25}$$

With this convention, the following lemma describes the action of the generator on linear functions of the form (1.24).

LEMMA 8 (Generator on linear functionals). *Assume* (A3). *Then, for functions of the form* (1.24), *one has*

$$GF(\mathcal{X}) = q_-\big(M_-(\mathcal{X})\big) + q_+\big(M_+(\mathcal{X})\big) \qquad \big(\mathcal{X} \in \mathcal{S}_{\mathrm{ord}}^{\mathrm{fin}}\big), \tag{1.26}$$

where $q_\pm : \overline{I} \to \mathbb{R}$ *are given by*

$$\begin{aligned} q_-(x) &:= \int_x^{I_+} w_+ \,\mathrm{d}\lambda_+ - w_-(x)\lambda_+(x), \\ q_+(x) &:= -\int_{I_-}^{x} w_- \,\mathrm{d}\lambda_- - w_+(x)\lambda_-(x). \end{aligned} \tag{1.27}$$

PROOF. We observe that $\int_{M_-(\mathcal{X})}^{I_+} w_+ \,\mathrm{d}\lambda_+$ is the rate at which $F(\mathcal{X})$ increases due to sell limit orders being added to the order book while the product of $w_-(M_-(\mathcal{X}))$ and $\lambda_+(M_-(\mathcal{X}))$ is the rate at which $F(\mathcal{X})$ decreases due to buy limit orders being removed from the order book. In view of our convention (1.25), the latter term is zero when the order book contains no buy limit orders. The two terms in $q_+(M_+(\mathcal{X}))$ have similar interpretations. $\square$

Formula (1.27) tells us how to calculate the functions $q_\pm$ from (1.26) from the weight functions $w_\pm$. It turns out that under the assumptions (A3) and (A6), one can uniquely solve the following inverse problem: if $q_\pm$ are given up to an additive constant, then find $w_\pm$ such that (1.27) holds. This is shown in Theorem 11 below and more specifically for indicator functions of the form $q_- = 1_{[I_-,z]}$ and $q_+ = 1_{[z,I_+]}$ in the following theorem, that moreover specifies the additive constant.

THEOREM 9 (Special weight functions). *Assume* (A3) *and* (A6). *Then, for each* $z \in \overline{I}$, *there exist a unique pair of bounded weight functions*

$$\big(w_-^{z,-}, w_+^{z,-}\big) = (w_-, w_+)$$

such that w_- *is left continuous and* w_+ *is right continuous, and the linear functional* $F^{z,-} = F$ *from* (1.24) *satisfies*

$$GF(\mathcal{X}) = 1_{\{M_-(\mathcal{X}) \le z\}} - f_-(z) \qquad \big(\mathcal{X} \in \mathcal{S}_{\mathrm{ord}}^{\mathrm{fin}}\big), \tag{1.28}$$

where (f_-, f_+) *is the unique solution to Luckock's equation* (1.13). *Likewise, there exist a unique pair of weight functions* $(w_-^{z,+}, w_+^{z,+}) = (w_-, w_+)$ *such that the linear functional* $F^{z,+} = F$ *from* (1.24) *satisfies*

$$GF(\mathcal{X}) = 1_{\{M_+(\mathcal{X}) \ge z\}} - f_+(z) \qquad \big(\mathcal{X} \in \mathcal{S}_{\mathrm{ord}}^{\mathrm{fin}}\big). \tag{1.29}$$

Figure 3 shows plots of weight functions as in Theorem 9 together with the solution of Luckock's equation, for one explicit example of a Stigler–Luckock model. Theorem 9 is closely related to Luckock's result Theorem 1. Indeed, if a Stigler–Luckock model has an invariant law that is concentrated on $\mathcal{S}_{\mathrm{ord}}^{\mathrm{fin}}$, then the

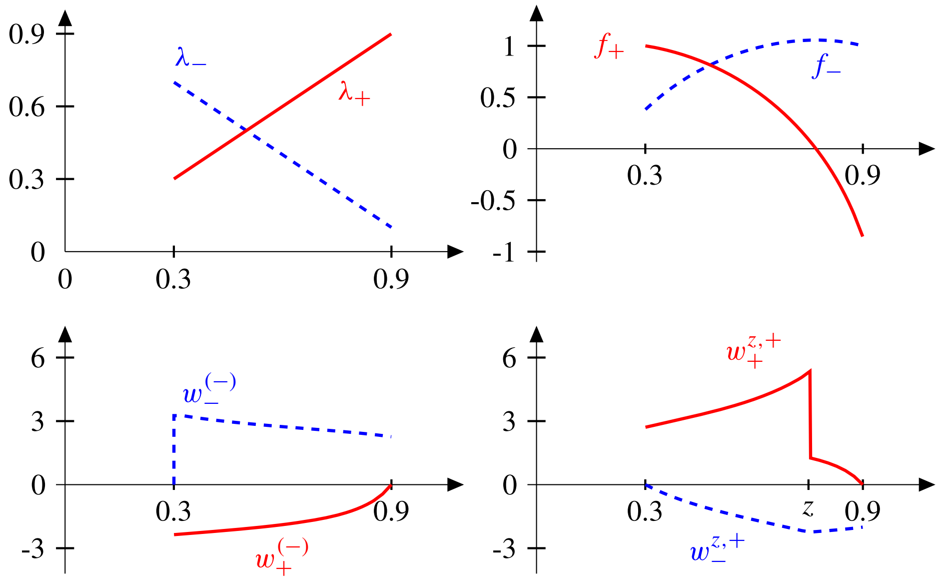

FIG. 3. *The solution* (f_-, f_+) *to Luckock's equation, as well as two examples of weight functions* (w_-, w_+) *as in Theorem* 9. *In this example,* $I = (0.3, 0.9)$, $\lambda_-(x) = 1 - x$ *and* $\lambda_+(x) = x$. *The lower left picture shows the weight functions* $w_\pm^{(-)} = w_\pm^{I_-,-}$ *while the lower right picture shows the weight functions* $w_\pm^{z,+}$ *for* $z = 0.75$.

fact that the functions in (1.12) are given by the solution to Luckock's equation follows from Theorem 9 and the equilibrium equation $\mathbb{E}[GF(\mathcal{X}_t)] = 0$.

Theorem 9 is more powerful that Theorem 1, however, since it gives an interpretation to the solution to Luckock's equation even if such a solution is not valid. Also, we have fairly explicit expressions for the weight functions $(w_-^{z,\pm}, w_+^{z,\pm})$ (see Lemma 16 below), and their associated linear functions $F^{z,\pm}$ are useful also in a nonstationary setting. In particular, we will prove Theorem 3 by constructing a Lyapunov function from the functions $F^{I_-,-}$ and $F^{I_+,+}$ [see formula (3.5) below].

We hope that the linear functions $F^{z,\pm}$ from Theorem 9 will also prove useful in future work aimed at resolving the open problems mentioned in Section 1.6. In Appendix A.4, we have recorded some concrete ideas on how the functions $F^{z,\pm}$ could possibly be used to attack the conjectures of Section 1.6.

1.8. *Outline.* In Section 2, we investigate two differential equations: Luckock's equation (1.13) and a differential equation that allows one to solve the weight functions $w_\pm$ in terms of the functions $q_\pm$ from (1.27). In particular, we prove Theorem 9 in Section 2.3, Proposition 2 in Section 2.4 and Proposition 5 and Lemmas 4, 6 and 7 in Section 2.5.

After the preparatory work on the differential equations in Section 2, the analysis of the Markov chain, which is contained in Section 3, is actually quite short. In particular, we prove Theorem 1 in Section 3.1 and Theorem 3 in Section 3.3.

The paper concludes with four appendices. In Appendix A.1, we show that the assumptions (A3) and (A4) can basically be made without loss of generality. Appendix A.2 collects some facts from the general theory of Markov chains needed to translate the properties of our Lyapunov function into properties of the Markov chain. In Appendix A.3, we have collected (without proof) some formulas for Stigler–Luckock models that take only finitely many values, and that are analogues to our integral formulas for continuous models but cannot easily be deduced from them. Appendix A.4 collects some concrete open problems with some ideas on how to approach them.

2. Analysis of the differential equations.

2.1. *Lebesgue–Stieltjes integrals.* For any interval $J \subset [-\infty, \infty]$ that can be either closed, open, or half open, with left and right boundaries $J_- < J_+$, we let $B(J)$ denote the space of bounded measurable functions $f : J \to \mathbb{R}$. If a function $f : J \to \mathbb{R}$ is of bounded variation, then the limits

$$\begin{aligned} f(x-) &:= \lim_{y \uparrow x} f(y) \qquad (x \neq J_-) \quad \text{and} \\ f(x+) &:= \lim_{y \downarrow x} f(y) \qquad (x \neq J_+) \end{aligned} \tag{2.1}$$

exist for all $x \in J \setminus \{J_-\}$ resp. $x \in J \setminus \{J_+\}$. For such functions, if $J_- \in J$, then we set $f(J_- -) := f(J_-)$, and we define $f(J_+ +)$ similarly. We let $B_{\mathrm{bv}}(J)$ denote the space of functions $f \in B(J)$ that are of bounded variation and satisfy $f(x) \in \{f(x-), f(x+)\}$ for each $x \in J$, and we let

$$B^{\pm}_{\mathrm{bv}}(J) := \{ f \in B_{\mathrm{bv}}(J) : f(x\pm) = f(x) \ \forall x \in J \} \tag{2.2}$$

denote the spaces of left $(-)$ and right $(+)$ continuous functions $f : J \to \mathbb{R}$ of bounded variation. Each $f \in B_{\mathrm{bv}}(J)$ defines a finite signed measure $\mathrm{d}f$ on J through the formula

$$\mathrm{d}f([x, y]) := f(y+) - f(x-) \qquad (x, y \in J, x \leq y). \tag{2.3}$$

The set of atoms of $\mathrm{d}f$ is the set $\mathcal{D}_f := \{x \in J : f(x-) \neq f(x+)\}$ of points of discontinuity of f. For each finite signed measure ρ on J, we can find functions $f \in B^-_{\mathrm{bv}}(J)$ and $g \in B^+_{\mathrm{bv}}(J)$ such that $\mathrm{d}f = \rho = \mathrm{d}g$, and these functions are unique up to an additive constant. We equip $B^{\pm}_{\mathrm{bv}}(J)$ with a topology such that $f_n \to f$ if and only if $\mathrm{d}f_n$ converges weakly to $\mathrm{d}f$ and $f_n(x) \to f(x)$ for at least one (and hence every) point $x \in J \setminus \mathcal{D}_f$. It is known ([7], page 182) that if J is a closed interval, then $f_n \to f$ in this topology if and only if:

$$
(2.4)\qquad
\begin{aligned}
&\text{(i)} && \sup_n \|\mathrm{d}f_n\| < \infty, \text{ where } \|\cdot\| \text{ denotes the total variation norm,}\\
&\text{(ii)} && \mathrm{d}f_n(J) \to \mathrm{d}f(J),\\
&\text{(iii)} && \int_J \big|f_n(x) - f(x)\big|\,\mathrm{d}x \to 0, \text{ that is, } f_n \to f \text{ in } L^1 \text{ norm}\\
& && \text{w.r.t. to the Lebesgue measure.}
\end{aligned}
$$

In line with earlier notation, we write $g\,\mathrm{d}f$ to denote the measure $\mathrm{d}f$ weighted with a bounded measurable function g. We will make use of the *product rule* which says that

$$
\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f \qquad \big(f, g \in B_{\mathrm{bv}}(J), \mathcal{D}_f \cap \mathcal{D}_g = \varnothing\big), \tag{2.5}
$$

and also of the *chain rule* which tells us that if $f \in B_{\mathrm{bv}}(J)$ takes values in a compact interval K and $F : K \to \mathbb{R}$ is continuously differentiable, then

$$
\mathrm{d}(F \circ f) = \big(F' \circ f\big)\,\mathrm{d}f \qquad \big(f \in B_{\mathrm{bv}}(J), \mathcal{D}_f = \varnothing\big), \tag{2.6}
$$

where $(F \circ f)(x) := F(f(x))$ denotes the composition of F and f. All our integrals will be of Lebesgue type, which coincides with the Riemann–Stieltjes integral if both functions involved are of bounded variation and do not share points of discontinuity.

If $g : [a, b] \to \mathbb{R}$ is a function of bounded variation and $\psi : [a, b] \to \mathbb{R}$ is nondecreasing, then we write $\mathrm{d}g \ll \mathrm{d}\psi$ if $\mathrm{d}g$ is absolutely continuous with respect to ψ, that is, if for each $s \le t$, $\psi(s-) = \psi(t+)$ implies $g(s-) = g(t+)$. We will sometimes use the *substitution of variables rule*, which says that

$$
\int_a^b f\,\mathrm{d}g = \int_{\psi(a)}^{\psi(b)} \big(f \circ \psi^{-1}\big)\,\mathrm{d}\big(g \circ \psi^{-1}\big) \qquad \big(f \in B[a, b], g \in B_{\mathrm{bv}}[a, b]\big), \tag{2.7}
$$

and which holds provided $\psi : [a, b] \to [-\infty, \infty]$ is a nondecreasing function such that $\mathrm{d}g \ll \mathrm{d}\psi$, and $\psi^{-1} : [\psi(a), \psi(b)] \to [a, b]$ is a right inverse of ψ.

As a general reference to these rules, we refer to [2], Section 6.2. In the substitution of variables rule, the condition $\mathrm{d}g \ll \mathrm{d}\psi$ guarantees that $f \circ \psi^{-1} \circ \psi$ differs from f only on a set of measure zero under $\mathrm{d}g$.

We will need one more result that we formulate as a lemma. The result holds in any dimension but since we only need the two-dimensional case, for ease of notation, we restrict to two dimensions.

LEMMA 10 (Integrals along curves). *Let $D \subset \mathbb{R}^2$ be a closed, convex set that is the closure of its interior. Let F, g_1, g_2, f_1, f_2 be continuous real functions on D such that f_1 and f_2 are moreover Lipschitz. Assume that for all x_1, x_1', x_2, x_2'*

such that $x_1 \leq x_1'$, $x_2 \leq x_2'$, *and* $(x_1, x_2), (x_1', x_2), (x_1, x_2') \in D$,

$$\begin{aligned} F(x_1', x_2) - F(x_1, x_2) &= \int_{x_1}^{x_1'} g_1(\cdot, x_2)\,\mathrm{d}f_1(\cdot, x_2), \\ F(x_1, x_2') - F(x_1, x_2) &= \int_{x_2}^{x_2'} g_2(x_1, \cdot)\,\mathrm{d}f_2(x_1, \cdot). \end{aligned} \tag{2.8}$$

Let $[t_-, t_+]$ *be a closed interval and let* $\gamma : [t_-, t_+] \to D$ *be a continuous function of bounded variation. Then*

$$F\big(\gamma(t_+)\big) - F\big(\gamma(t_-)\big) = \int_{t_-}^{t_+} \big\{(g_1 \circ \gamma)\,\mathrm{d}(f_1 \circ \gamma) + (g_2 \circ \gamma)\,\mathrm{d}(f_2 \circ \gamma)\big\}. \tag{2.9}$$

PROOF (SKETCH). Formula (2.8) shows that (2.9) holds for any continuous function $[t_-, t_+] \mapsto (\gamma_1(t), \gamma_2(t)) \in D$ of bounded variation such that moreover either γ_1 or γ_2 is constant. It follows that (2.9) also holds for any finite concatenation of such curves; call such curves simple. Then it is not hard to see that any $\gamma : [t_-, t_+] \to D$ that is continuous and of bounded variation can be approximated by simple curves $\gamma^{(n)}$ in such a way that $\gamma^{(n)}(t_-) \to \gamma(t_-)$ and $\mathrm{d}\gamma_i^{(n)}$ converges weakly to $\mathrm{d}\gamma_i$ for $i = 1, 2$. In particular, this implies that $\gamma^{(n)}$ converges uniformly to γ so by the continuity of g_i $(i = 1, 2)$, also $g_i \circ \gamma^{(n)}$ converges uniformly to $g_i \circ \gamma$. In view of (2.4), the Lipschitz continuity of f_i $(i = 1, 2)$ moreover implies that $\mathrm{d}(f_i \circ \gamma^{(n)})$ converges weakly to $\mathrm{d}(f_i \circ \gamma)$. Using this and the continuity of F, taking the limit in (2.8), which holds for $\gamma^{(n)}$, we obtain that the formula also holds for γ. □

2.2. *The inverse problem.* The main result of the present subsection is the following theorem.

THEOREM 11 (Inverse problem). *Assume* (A3) *and* (A6). *Then, for each pair of functions* (g_-, g_+) *with* $g_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$, *there exists a unique pair of functions* $(w_-^{(g_-,g_+)}, w_+^{(g_-,g_+)}) = (w_-, w_+)$ *with* $w_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$ *and* $w_\pm(I_\pm) = 0$, *as well as a unique constant* $c(g_-, g_+) \in \mathbb{R}$, *such that the linear functional* $F^{(g_-,g_+)} = F$ *from* (1.24) *satisfies*

$$GF^{(g_-,g_+)}(\mathcal{X}) = g_-\big(M_-(\mathcal{X})\big) + g_+\big(M_+(\mathcal{X})\big) - c(g_-, g_+). \tag{2.10}$$

The proof of Theorem 11 will be split into a number of lemmas.

LEMMA 12 (Differential equation). *Assume* (A3) *and* (A6), *let* $g_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$ *and let* $w_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$ *satisfy* $w_\pm(I_\pm) = 0$. *Then the linear function* $F^{(w_-,w_+)}$ *associated with* (w_-, w_+) *satisfies* (2.10) *for some* $c(g_-, g_+) \in \mathbb{R}$ *if and only if*:

$$\begin{aligned} &\text{(i)} \quad w_+\,\mathrm{d}\lambda_+ + \mathrm{d}(\lambda_+ w_-) = -\mathrm{d}g_-, \\ &\text{(ii)} \quad w_-\,\mathrm{d}\lambda_- + \mathrm{d}(\lambda_- w_+) = -\mathrm{d}g_+. \end{aligned} \tag{2.11}$$

PROOF. Defining functions $q_\pm$ as in (1.27), Lemma 8 tells us that (2.10) is satisfied for some $c(g_-, g_+) \in \mathbb{R}$ if and only if there exist real constants $c_\pm$ such that $q_\pm = g_\pm + c_\pm$, or equivalently, if there exist $c'_\pm \in \mathbb{R}$ such that

$$\begin{aligned} g_-(x) &= c'_- + \int_{[x,I_+)} \{w_+ \,\mathrm{d}\lambda_+ + \mathrm{d}(w_-\lambda_+)\} \qquad \big(x \in [I_-, I_+)\big),\\ g_+(x) &= c'_+ - \int_{(I_-,x]} \{w_- \,\mathrm{d}\lambda_- + \mathrm{d}(w_+\lambda_-)\} \qquad \big(x \in (I_-, I_+]\big), \end{aligned} \tag{2.12}$$

which is equivalent to (2.11). $\square$

We can integrate the differential equation (2.11) explicitly. Let $h_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$ be any pair of functions such that

$$\mathrm{d}h_\pm = -\lambda_\pm \,\mathrm{d}g_\pm, \tag{2.13}$$

that is, $h_-(x) = c_- + \int_{[x,I_+)} \lambda_- \,\mathrm{d}g_-$ and similarly for h_+, where $c_\pm$ are some fixed, but otherwise arbitrary constants.

LEMMA 13 (Integrated equation). *Assume* (A3) *and* (A6), *let $g_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$ be given and let $h_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$ be as in* (2.13). *Then a pair of functions $w_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$ satisfies* (2.11) *if and only if there exists a constant $\kappa \in \mathbb{R}$ such that*:

$$\begin{aligned} &\text{(i)} \quad \mathrm{d}w_- = \frac{\kappa + h_+}{\lambda_-} \mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{1}{\lambda_-} \mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big),\\ &\text{(ii)} \quad \mathrm{d}w_+ = \frac{\kappa + h_-}{\lambda_+} \mathrm{d}\Big(\frac{1}{\lambda_-}\Big) + \frac{1}{\lambda_+} \mathrm{d}\Big(\frac{h_+}{\lambda_-}\Big),\\ &\text{(iii)} \quad w_- + w_+ = \frac{\kappa + h_- + h_+}{\lambda_-\lambda_+}. \end{aligned} \tag{2.14}$$

Moreover, given (2.14) (iii), *the equations* (2.14) (i) *and* (ii) *imply each other.*

PROOF. Multiplying the equations (2.11) (i) and (ii) by λ_- and λ_+, respectively, and then adding both equations, using the product rule (2.5) and (2.13), we obtain

$$\mathrm{d}\big(\lambda_-(\lambda_+ w_-)\big) + \mathrm{d}\big(\lambda_+(\lambda_- w_+)\big) = \mathrm{d}h_- + \mathrm{d}h_+, \tag{2.15}$$

which shows that there exists a constant $\kappa \in \mathbb{R}$ such that (2.14) (iii) holds. Given (2.14) (iii), we can rewrite (2.11) (i) as

$$\mathrm{d}(\lambda_+ w_-) = -w_+ \,\mathrm{d}\lambda_+ + \frac{\mathrm{d}h_-}{\lambda_-} = \Big(w_- - \frac{\kappa + h_- + h_+}{\lambda_-\lambda_+}\Big) \mathrm{d}\lambda_+ + \frac{\mathrm{d}h_-}{\lambda_-}. \tag{2.16}$$

Dividing by λ_+ and reordering terms, this says that

$$\frac{\mathrm{d}(\lambda_+ w_-) - w_- \,\mathrm{d}\lambda_+}{\lambda_+} = -\frac{(\kappa + h_- + h_+)\,\mathrm{d}\lambda_+}{\lambda_-\lambda_+^2} + \frac{\mathrm{d}h_-}{\lambda_-\lambda_+}, \tag{2.17}$$

which using the product and chain rules (2.5)–(2.6) can be rewritten as (2.14) (i). In a similar way, we see that given (2.14) (iii), (1.13) (ii) is equivalent to (2.14) (ii). Differentiating (2.14) (iii), using the product rule, we obtain

$$\begin{aligned}\mathrm{d}w_- + \mathrm{d}w_+ = {} & \frac{\kappa}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{\kappa}{\lambda_+}\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) + \frac{1}{\lambda_-}\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big) \\ & + \frac{h_-}{\lambda_+}\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) + \frac{h_+}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{1}{\lambda_+}\mathrm{d}\Big(\frac{h_+}{\lambda_-}\Big),\end{aligned} \tag{2.18}$$

which is the same as we would obtain adding the equations (2.14) (i) and (2.14) (ii). We conclude that (2.14) (i) and (ii) are equivalent given (iii). □

For later use, assuming (A3) and (A6), we define a constant Γ by

$$\begin{aligned}\Gamma &:= \frac{1}{\lambda_-(I_+)\lambda_+(I_+)} - \int_{I_-}^{I_+}\frac{1}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) \\ &= \frac{1}{\lambda_-(I_-)\lambda_+(I_-)} + \int_{I_-}^{I_+}\frac{1}{\lambda_+}\mathrm{d}\Big(\frac{1}{\lambda_-}\Big),\end{aligned} \tag{2.19}$$

where the equality of both formulas follows from the product rule (2.5) applied to the functions $1/\lambda_-$ and $1/\lambda_+$. Note that $\Gamma > 0$ since $\mathrm{d}(1/\lambda_-)$ is nonnegative while $\lambda_\pm$ are strictly positive by (A6).

LEMMA 14 (Existence and uniqueness). *Assume* (A3) *and* (A6). *Then, for each pair of functions* $g_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$, *there exist unique functions* $w_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$ *that solve the differential equation* (2.11) *together with the boundary conditions* $w_\pm(I_\pm) = 0$. *These functions are given by*

$$\begin{aligned}&\text{(i)} \quad w_-(x) = \int_{[I_-,x)}\Big\{\frac{\kappa + h_+}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{1}{\lambda_-}\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big)\Big\}, \\ &\text{(ii)} \quad w_+(x) = -\int_{(x,I_+]}\Big\{\frac{\kappa + h_-}{\lambda_+}\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) + \frac{1}{\lambda_+}\mathrm{d}\Big(\frac{h_+}{\lambda_-}\Big)\Big\},\end{aligned} \tag{2.20}$$

where $h_\pm$ *are as in* (2.13) *and*

$$\kappa := \Gamma^{-1}\Big[\int_{\overline{I}}\Big\{\frac{h_+}{\lambda_-}\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{1}{\lambda_-}\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big)\Big\} - \frac{h_-(I_+) + h_+(I_+)}{\lambda_-(I_+)\lambda_+(I_+)}\Big], \tag{2.21}$$

with $\Gamma > 0$ *the constant from* (2.19).

PROOF. By Lemma 13, $w_\pm$ solve the difference equation (2.11) together with the left boundary condition $w_-(I_-) = 0$ if and only if there exists a $\kappa \in \mathbb{R}$ such that (2.20) (i) and (2.14) (iii) hold. In view of the latter equation, $w_\pm$ also solves the right boundary condition $w_+(I_+) = 0$ if and only if

$$w_-(I_+) + 0 = \frac{\kappa + h_- + h_+}{\lambda_-\lambda_+}(I_+). \tag{2.22}$$

In view of (2.20) (i), this says that

$$\int_{[I_-,I_+)}\left\{\frac{\kappa+h_+}{\lambda_-}\,\mathrm{d}\left(\frac{1}{\lambda_+}\right)+\frac{1}{\lambda_-}\,\mathrm{d}\left(\frac{h_-}{\lambda_+}\right)\right\}=\frac{\kappa+h_-(I_+)+h_+(I_+)}{\lambda_-(I_+)\lambda_+(I_+)}, \tag{2.23}$$

or equivalently (note that since h_- is left continuous, it has no jump at I_+)

$$\begin{aligned}&\int_{[I_-,I_+]}\left\{\frac{h_+}{\lambda_-}\,\mathrm{d}\left(\frac{1}{\lambda_+}\right)+\frac{1}{\lambda_-}\,\mathrm{d}\left(\frac{h_-}{\lambda_+}\right)\right\}-\frac{h_-(I_+)+h_+(I_+)}{\lambda_-(I_+)\lambda_+(I_+)}\\&\quad=\kappa\left\{\frac{1}{\lambda_-(I_+)\lambda_+(I_+)}-\int_{I_-}^{I_+}\frac{1}{\lambda_-}\,\mathrm{d}\left(\frac{1}{\lambda_+}\right)\right\},\end{aligned} \tag{2.24}$$

which by the fact that the constant Γ from (2.19) is nonzero is equivalent to (2.21). □

PROOF OF THEOREM 11. Immediate from Lemmas 12 and 14. □

2.3. *Luckock's equation.* In the present subsection, we prove Theorem 9. We start by proving that Luckock's equation has a unique solution. By definition, a solution to *Luckcock's equation* is a pair of functions (f_-, f_+) such that $f_\pm \in B^{\mp}_{\mathrm{bv}}(\overline{I})$ and (1.13) holds.

LEMMA 15 (Luckock's equation). *Assume* (A3) *and* (A6). *Then Luckock's equation has a unique solution* (f_-, f_+), *which is given by*:

$$\begin{aligned}&\text{(i)}\quad \left(\frac{f_+}{\lambda_+}\right)(x)=\frac{1}{\lambda_+(I_-)}+\kappa\int_{I_-}^{x}\frac{1}{\lambda_-}\,\mathrm{d}\left(\frac{1}{\lambda_+}\right),\\&\text{(ii)}\quad \left(\frac{f_-}{\lambda_-}\right)(x)=\frac{1}{\lambda_-(I_+)}-\kappa\int_{x}^{I_+}\frac{1}{\lambda_+}\,\mathrm{d}\left(\frac{1}{\lambda_-}\right),\end{aligned} \tag{2.25}$$

where κ *is given by*

$$\kappa=\kappa_{\mathrm{L}}:=\Gamma^{-1}\left(\frac{1}{\lambda_-(I_+)}+\frac{1}{\lambda_+(I_-)}\right), \tag{2.26}$$

and $\Gamma>0$ *is the constant from* (2.19).

PROOF. Setting $v_+ := f_-/\lambda_-$ and $v_- := f_+/\lambda_+$ and dividing the equations (1.13) (i) and (ii) by λ_- and λ_+, respectively, we see that these equations are equivalent to $v_\pm\,\mathrm{d}\lambda_\pm = -\mathrm{d}(\lambda_\pm v_\mp)$, which is equation (2.11) with $w_\pm = v_\pm$ and $g_\pm = 0$. Now Lemma 13 tells us that (f_-, f_+) solves (1.13) (i) and (ii) if and only if there exists a constant $\kappa \in \mathbb{R}$ such that:

$$\text{(2.27)}\qquad \begin{aligned} &\text{(i)} && \mathrm{d}\Big(\frac{f_+}{\lambda_+}\Big) = \frac{\kappa}{\lambda_-}\,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big),\\ &\text{(ii)} && \mathrm{d}\Big(\frac{f_-}{\lambda_-}\Big) = \frac{\kappa}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big),\\ &\text{(iii)} && \frac{f_+}{\lambda_+} + \frac{f_-}{\lambda_-} = \frac{\kappa}{\lambda_-\lambda_+}. \end{aligned}$$

Moreover, of these equations, the first two are equivalent given the third one.

It follows that (f_-, f_+) solves (1.13) (i) and (ii) together with the left boundary condition $f_+(I_-) = 1$ if and only if (2.25) (i) and (2.27) (iii) hold. In view of the latter equation, the right boundary condition $f_-(I_+) = 1$ is satisfied if and only if

$$\text{(2.28)}\qquad \frac{f_+}{\lambda_+}(I_+) + \frac{1}{\lambda_-(I_+)} = \frac{\kappa}{\lambda_-\lambda_+}(I_+).$$

By (2.25) (i), this says that

$$\text{(2.29)}\qquad \frac{1}{\lambda_+(I_-)} + \kappa\int_{I_-}^{I_+} \frac{1}{\lambda_-}\,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big) + \frac{1}{\lambda_-(I_+)} = \frac{\kappa}{\lambda_-(I_+)\lambda_+(I_+)},$$

which is equivalent to (2.26). $\square$

PROOF OF THEOREM 9. Let $\mathcal{G}$ be the space of all pairs (g_-, g_+) with $g_\pm \in B^\pm_{\mathrm{bv}}(\overline{I})$ and set $\mathcal{W} := \{(w_-, w_+) \in \mathcal{G} : w_\pm(I_\pm) = 0\}$. We equip the spaces $B^\pm_{\mathrm{bv}}(\overline{I})$ with a topology as in Section 2.1, $\mathcal{G}$ with the product topology, and $\mathcal{W}$ with the induced topology. For any interval J, we let $\mathcal{M}(J)$ denote the space of finite signed measures on J, equipped with the topology of weak convergence, and we let $\mathcal{R} := \mathcal{M}[I_-, I_+) \times \mathcal{M}(I_-, I_+]$, equipped with the product topology.

Let $\psi : \mathcal{W} \to \mathcal{G}$ be the linear function that maps a pair $(w_-, w_+) \in \mathcal{W}$ into the pair $(q_-, q_+) \in \mathcal{G}$ defined in (1.27) and let $D : \mathcal{G} \to \mathcal{R}$ be the map

$$\text{(2.30)}\qquad D(g_-, g_+) := (\mathrm{d}g_-, \mathrm{d}g_+).$$

Setting $\phi := D \circ \psi$, we see that

$$\text{(2.31)}\qquad \phi(w_-, w_+) = -\big(w_+\,\mathrm{d}\lambda_+ + \mathrm{d}(\lambda_+ w_-),\, w_-\,\mathrm{d}\lambda_- + \mathrm{d}(\lambda_- w_+)\big),$$

so Lemma 14 tells us that $\phi : \mathcal{W} \to \mathcal{R}$ is a bijection.

We claim that the maps ψ, D, ϕ and ϕ^{-1} are continuous with respect to the topologies on $\mathcal{W}$, $\mathcal{G}$ and $\mathcal{R}$. The continuity of D is immediate from the definition of the topologies on $\mathcal{G}$ and $\mathcal{R}$ and the continuity of ψ follows from (1.27). The continuity of ϕ is easily derived from (2.31), while the continuity of ϕ^{-1} follows from the explicit formulas in Lemma 14 and the continuity of the functions $h_\pm$ from (2.13) as a function of $g_\pm$, for a given choice of the boundary conditions.

Let $\psi(\mathcal{W})$ denote the image of $\mathcal{W}$ under ψ and define $\pi : \mathcal{G} \to \psi(\mathcal{W})$ by $\pi := \psi \circ \phi^{-1} \circ D$. Since $\pi \circ \psi = \psi \circ \phi^{-1} \circ (D \circ \psi) = \psi$, we see that π is the identity on $\psi(\mathcal{W})$. Since $D \circ \pi = (D \circ \psi) \circ \phi^{-1} \circ D = D$, we see that $\pi(g) = \pi(g')$ if and only if $D(g) = D(g')$. These facts imply that for each $g \in \mathcal{G}$, there exists a unique $q \in \psi(\mathcal{W})$, namely $q = \pi(g)$, such that $D(g) = D(q)$, that is, for every $g \in \mathcal{G}$ there exists a unique $q \in \psi(\mathcal{W})$ and unique constants $c_\pm(g_-, g_+) \in \mathbb{R}$ such that

$$(g_-, g_+) = \big(q_- + c_-(g_-, g_+), q_+ + c_+(g_-, g_+)\big). \tag{2.32}$$

Since ψ, ϕ^{-1} and D are continuous, so is π, and hence also the maps $c_\pm : \mathcal{G} \to \mathbb{R}$ are continuous. In fact, they are the unique continuous linear forms on $\mathcal{G}$ such that

$$\begin{aligned} &\text{(i)} \quad c_-(1, 0) = 1, \qquad c_-(0, 1) = 0, \qquad c_-\big(\psi(w_-, w_+)\big) = 0 \\ &\qquad \forall (w_-, w_+) \in \mathcal{W}, \\ &\text{(ii)} \quad c_+(1, 0) = 0, \qquad c_+(0, 1) = 1, \qquad c_+\big(\psi(w_-, w_+)\big) = 0 \\ &\qquad \forall (w_-, w_+) \in \mathcal{W}. \end{aligned} \tag{2.33}$$

The map $(g_-, g_+) \mapsto c(g_-, g_+)$ from Theorem 11 is given by $c = c_- + c_+$, that is, c is the unique continuous linear form on $\mathcal{G}$ such that

$$c(1, 0) = 1, \qquad c(0, 1) = 1, \qquad c\big(\psi(w_-, w_+)\big) = 0 \quad \forall (w_-, w_+) \in \mathcal{W}. \tag{2.34}$$

Let (f_-, f_+) be the unique solution to Luckock's equation, and observe from (2.25) that $f_\pm$ are continuous on $\overline{I}$. We claim that

$$c(g_-, g_+) = g_-(I_+) f_-(I_+) - \int_{\overline{I}} f_- \,\mathrm{d}g_- + g_+(I_-) f_+(I_-) + \int_{\overline{I}} f_+ \,\mathrm{d}g_+. \tag{2.35}$$

Clearly, (2.35) defines a continuous linear form on $\mathcal{G}$. We will show that this linear form satisfies (2.34). The boundary conditions (1.13) (iii) imply that $c(1, 0) = 1 = c(0, 1)$. Recall that for $(w_-, w_+) \in \mathcal{W}$, $\psi(w_-, w_+) = (q_-, q_+)$ is defined as in (1.27). Then

$$\begin{aligned} c\big(\psi(w_-, w_+)\big) = {} & -(w_-\lambda_+)(I_+) f_-(I_+) + \int_{\overline{I}} f_- \big\{ w_+ \,\mathrm{d}\lambda_+ + \mathrm{d}(w_-\lambda_+) \big\} \\ & - (w_+\lambda_-)(I_-) f_+(I_-) - \int_{\overline{I}} f_+ \big\{ w_- \,\mathrm{d}\lambda_- + \mathrm{d}(w_+\lambda_-) \big\}. \end{aligned} \tag{2.36}$$

By partial integration, using the continuity of $f_\pm$ and $\lambda_\pm$, as well as the boundary condition $w_-(I_-) = 0$, we have

$$-(w_-\lambda_+)(I_+) f_-(I_+) + \int_{\overline{I}} f_- \,\mathrm{d}(w_-\lambda_+) = -\int_{\overline{I}} (w_-\lambda_+) \,\mathrm{d}f_-. \tag{2.37}$$

Inserting this into the first line of (2.36) and treating the second line similarly, we find that $c(\psi(w_-, w_+))$ equals

$$\begin{aligned}&\int_{\overline{I}}\{f_- w_+ \,\mathrm{d}\lambda_+ - (w_-\lambda_+)\,\mathrm{d}f_-\} + \int_{\overline{I}}\{-f_+ w_- \,\mathrm{d}\lambda_- + (w_+\lambda_-)\,\mathrm{d}f_+\}\\ &\qquad = \int_{\overline{I}} w_+\{f_- \,\mathrm{d}\lambda_+ + \lambda_- \,\mathrm{d}f_+\} - \int_{\overline{I}} w_-\{f_+ \,\mathrm{d}\lambda_- + \lambda_+ \,\mathrm{d}f_-\} = 0,\end{aligned} \tag{2.38}$$

where we have used (1.13) (i) and (ii) in the last step. This completes the proof of (2.35).

In particular, formula (2.35) shows that

$$c(1_{[I_-,z]}, 0) = f_-(z) \quad \text{and} \quad c(0, 1_{[z,I_+]}) = f_+(z) \qquad (z \in \overline{I}), \tag{2.39}$$

which together with Theorem 11 implies Theorem 9. □

2.4. *Some explicit formulas and conditions*. In the present section, we prove Proposition 2 as well as two lemmas (Lemmas 16 and 17 below) giving explicit formulas for the weight functions of Theorem 9.

PROOF OF PROPOSITION 2. The fact that under the conditions (A3) and (A6), Luckock's equation has a unique solution has already been proved in Lemma 15.

In order to prove (1.14), it suffices to prove part (i); the other part then follows by symmetry. Let $\Lambda_{+-} := \int_{I_-}^{I_+} \frac{1}{\lambda_+} \mathrm{d}(\frac{1}{\lambda_-})$. Then (2.25) (ii) says that

$$f_-(I_-) = \lambda_-(I_-)\left\{\frac{1}{\lambda_-(I_+)} - \kappa_L \Lambda_{+-}\right\}. \tag{2.40}$$

Filling in the definition of κ_L in (2.26), we see that $f_-(I_-) > 0$ if and only if

$$\frac{1}{\lambda_-(I_+)} > \Gamma^{-1}\left(\frac{1}{\lambda_-(I_+)} + \frac{1}{\lambda_+(I_-)}\right)\Lambda_{+-}. \tag{2.41}$$

Using also formula (2.19) and the fact that $\Gamma > 0$, this can be rewritten as

$$\left(\frac{1}{\lambda_-(I_-)\lambda_+(I_-)} + \Lambda_{+-}\right)\frac{1}{\lambda_-(I_+)} > \left(\frac{1}{\lambda_-(I_+)} + \frac{1}{\lambda_+(I_-)}\right)\Lambda_{+-}, \tag{2.42}$$

which can be simplified to (1.14) (i). The same argument also works with all inequality signs reversed.

To complete the proof, we need to show (1.15). Consider the weight functions

$$w_- := -1_{(I_-,I_+]} \quad \text{and} \quad w_+ := 1_{[I_-,I_+)}, \tag{2.43}$$

which correspond through (1.24) to the linear function $F(\mathcal{X}) = \mathcal{X}(I)$. For these weight functions, the functions $q_\pm$ from (1.27) are given by

$$\begin{aligned} q_-(x) &= \lambda_+(I_+) - \lambda_+(I_-)1_{\{I_-\}}(x),\\ q_+(x) &= -\lambda_-(I_-) + \lambda_-(I_+)1_{\{I_+\}}(x),\end{aligned} \tag{2.44}$$

so Lemma 8 tells us that

$$GF(\mathcal{X}) = -\lambda_+(I_-)1_{\{M_-(\mathcal{X})=I_-\}} + \lambda_-(I_+)1_{\{M_+(\mathcal{X})=I_+\}} + \lambda_+(I_+) - \lambda_-(I_-). \tag{2.45}$$

By Theorem 11, the weight functions $w_\pm$ are in fact uniquely characterized by the requirement that

$$GF(\mathcal{X}) = -\lambda_+(I_-)1_{\{M_-(\mathcal{X})=I_-\}} + \lambda_-(I_+)1_{\{M_+(\mathcal{X})=I_+\}} + c \tag{2.46}$$

for some $c \in \mathbb{R}$. Defining weight functions $\tilde{w}_\pm$ by

$$\tilde{w}_\pm := -\lambda_+(I_-)w_\pm^{I_-,-} + \lambda_-(I_+)w_\pm^{I_+,+} \tag{2.47}$$

and denoting the corresponding linear function by $\tilde{F}$, we see from Theorem 9 that

$$G\tilde{F}(\mathcal{X}) = -\lambda_+(I_-)1_{\{M_-(\mathcal{X})=I_-\}} + \lambda_-(I_+)1_{\{M_+(\mathcal{X})=I_+\}} + \lambda_+(I_-)f_-(I_-) - \lambda_-(I_+)f_+(I_+). \tag{2.48}$$

We conclude from this that $w_\pm = \tilde{w}_\pm$ and the constant from (2.46) is given by

$$\lambda_+(I_+) - \lambda_-(I_-) = c = \lambda_+(I_-)f_-(I_-) - \lambda_-(I_+)f_+(I_+), \tag{2.49}$$

which proves (1.15). $\square$

We next set out to derive explicit formulas for the weight functions $(w_-^{z,\pm}, w_+^{z,\pm})$ from Theorem 9. To state the result, we define functions $u_{\pm,\mp} : \overline{I} \to \mathbb{R}$ by

$$\begin{aligned} u_{-+}(x) &:= \Gamma^{-1}\left\{\frac{1}{\lambda_-(I_+)\lambda_+(I_+)} - \int_x^{I_+} \frac{1}{\lambda_-}\,\mathrm{d}\left(\frac{1}{\lambda_+}\right)\right\}, \\ u_{+-}(x) &:= \Gamma^{-1}\left\{\frac{1}{\lambda_-(I_-)\lambda_+(I_-)} + \int_{I_-}^x \frac{1}{\lambda_+}\,\mathrm{d}\left(\frac{1}{\lambda_-}\right)\right\}. \end{aligned} \tag{2.50}$$

In view of (2.19), we observe that $u_{-+}(I_-) = 1 = u_{+-}(I_+)$. Moreover, u_{-+} is nonincreasing with $u_{-+}(I_+) > 0$ while u_{+-} is nondecreasing with $u_{-+}(I_-) > 0$. By partial integration, our formulas for u_{-+} and u_{+-} can be rewritten as

$$\begin{aligned} u_{-+}(x) &:= \Gamma^{-1}\left\{\frac{1}{\lambda_-(x)\lambda_+(x)} + \int_x^{I_+} \frac{1}{\lambda_+}\,\mathrm{d}\left(\frac{1}{\lambda_-}\right)\right\}, \\ u_{+-}(x) &:= \Gamma^{-1}\left\{\frac{1}{\lambda_-(x)\lambda_+(x)} - \int_{I_-}^x \frac{1}{\lambda_-}\,\mathrm{d}\left(\frac{1}{\lambda_+}\right)\right\}. \end{aligned} \tag{2.51}$$

Combining this with our previous formulas and (2.19), we see that

$$u_{-+}(x) + u_{+-}(x) = \frac{\Gamma^{-1}}{\lambda_-(x)\lambda_+(x)} + 1. \tag{2.52}$$

LEMMA 16 (Formulas for special weight functions). *The weight functions from Theorem 9 are given by*:

$$
\begin{aligned}
&\text{(i)} \quad w_-^{z,-}(x) = \lambda_-(z)\Gamma\big(u_{+-}(z) - 1_{\{x\le z\}}\big)\big(u_{-+}(x) - 1_{\{x\le z\}}\big),\\
&\text{(ii)} \quad w_+^{z,-}(x) = \lambda_-(z)\Gamma\big[u_{+-}(x\vee z) - 1\big]u_{+-}(x\wedge z),\\
&\text{(iii)} \quad w_-^{z,+}(x) = \lambda_+(z)\Gamma\big[u_{-+}(x\wedge z) - 1\big]u_{-+}(x\vee z),\\
&\text{(iv)} \quad w_+^{z,+}(x) = \lambda_+(z)\Gamma\big(u_{-+}(z) - 1_{\{x\ge z\}}\big)\big(u_{+-}(x) - 1_{\{x\ge z\}}\big).
\end{aligned}
\tag{2.53}
$$

PROOF. We start with formula (2.53) (ii). Since $w_+^{I_+,-} = 0$ which agrees with the right-hand side of (2.53) (ii), we assume from now on without loss of generality that $z \in [I_-, I_+)$. We apply Lemma 14 with $g_- = 1_{[I_-,z]}$ and $g_+ = 0$. For the functions $h_\pm$ from (2.13), we choose the boundary conditions $h_-(I_+) = 0 = h_+(I_-)$, which means that

$$
h_-(x) = \int_{[x,I_+)} \lambda_- \,\mathrm{d}1_{[I_-,z]} = -\lambda_-(z)1_{[I_-,z]}(x) \quad \text{and} \quad h_+ = 0. \tag{2.54}
$$

Since $h_+ = 0$ and $h_-(I_+) = 0$, formulas (2.20) (ii) and (2.21) now simplify to

$$
w_+(x) = -\int_{(x,I_+]} \frac{\kappa + h_-}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big) \quad \text{with } \kappa = \Gamma^{-1}\int_{\overline{I}} \frac{1}{\lambda_-}\,\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big). \tag{2.55}
$$

Here, by (2.54),

$$
\begin{aligned}
\int_{\overline{I}} \frac{1}{\lambda_-}\,\mathrm{d}\Big(\frac{h_-}{\lambda_+}\Big) &= \Big[\frac{h_-}{\lambda_-\lambda_+}(I_+) - \frac{h_-}{\lambda_-\lambda_+}(I_-)\Big] - \int_{\overline{I}} \frac{h_-}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big)\\
&= \lambda_-(z)\Big\{\frac{1}{\lambda_-(I_-)\lambda_+(I_-)} - \int_{[I_-,z]} \frac{1}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big)\Big\},
\end{aligned}
\tag{2.56}
$$

which shows that

$$
\kappa = \lambda_-(z)u_{+-}(z). \tag{2.57}
$$

Using the fact that $u_{+-}(I_+) = 1$, it follows that

$$
\begin{aligned}
w_+(x) &= -\lambda_-(z)u_{+-}(z)\int_{(x,I_+]} \frac{1}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big)\\
&\qquad + \lambda_-(z)1_{\{x<z\}}\int_{(x,z]} \frac{1}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big)\\
&= -\lambda_-(z)\Gamma\big\{u_{+-}(z)\big[1 - u_{+-}(x)\big] - 1_{\{x<z\}}\big[u_{+-}(z) - u_{+-}(x)\big]\big\},
\end{aligned}
\tag{2.58}
$$

which can be rewritten as (2.53) (ii).

We next prove (2.53) (i). By (2.14) (iii), (2.54), (2.57) and (2.53) (ii),

$$
\begin{aligned}
w_+(x) &= \frac{\kappa + h_- + h_+}{\lambda_- \lambda_+}(x) - w_-(x) \\
&= \lambda_-(z)\frac{u_{+-}(z) - 1_{[I_-,z]}}{\lambda_-\lambda_+} - \lambda_-(z)\Gamma\big(u_{+-}(x \vee z) - 1\big)u_{+-}(x \wedge z).
\end{aligned} \tag{2.59}
$$

For $x \le z$, using (2.52), this yields

$$
\begin{aligned}
w_+(x) &= \lambda_-(z)\Gamma\left\{\Gamma^{-1}\frac{u_{+-}(z) - 1}{\lambda_-\lambda_+} - \big(u_{+-}(z) - 1\big)u_{+-}(x)\right\} \\
&= \lambda_-(z)\Gamma\big(u_{+-}(z) - 1\big)\left\{\frac{\Gamma^{-1}}{\lambda_-\lambda_+} - u_{+-}(x)\right\} \\
&= \lambda_-(z)\Gamma\big(u_{+-}(z) - 1\big)\big(u_{-+}(x) - 1\big),
\end{aligned} \tag{2.60}
$$

while for $x > z$, again with the help of (2.52), we obtain

$$
\begin{aligned}
w_+(x) &= \lambda_-(z)\Gamma\left\{\Gamma^{-1}\frac{u_{+-}(z)}{\lambda_-\lambda_+} - \big(u_{+-}(x) - 1\big)u_{+-}(z)\right\} \\
&= \lambda_-(z)\Gamma u_{+-}(z)\left\{\frac{\Gamma^{-1}}{\lambda_-\lambda_+} - u_{+-}(x) + 1\right\} \\
&= \lambda_-(z)\Gamma u_{+-}(z)u_{-+}(x).
\end{aligned} \tag{2.61}
$$

Combining the previous two formulas, we arrive at (2.53) (i).

Formulas (2.53) (iii) and (2.53) (iv) can be proved in exactly the same way. Alternatively, they can be derived from (2.53) (ii) and (2.53) (i) using the symmetry between buy and sell orders. □

Assume (A3) and (A6) and for $z \in \overline{I}$, let $F^{z,\pm}$ be linear functionals defined in terms of weight functions $(w_-^{z,\pm}, w_+^{z,\pm})$ as in Theorem 9. We will in particular be interested in the case $z = I_\pm$ and introduce the shorthand

$$
w_-^{(\pm)} := w_-^{I_\pm,\pm}, \qquad w_+^{(\pm)} := w_+^{I_\pm,\pm} \quad \text{and} \quad F^{(\pm)} := F^{I_\pm,\pm}. \tag{2.62}
$$

We will prove Theorem 3 by constructing a Lyapunov function from $F^{(-)}$ and $F^{(+)}$; see formula (3.5) and Proposition 18 below. The next lemma prepares for the proof of Proposition 18.

LEMMA 17 (Extremal weight functions). *Assume* (A3) *and* (A6), *let* (f_-, f_+) *denote the solution to Luckock's equation and let* $(w_-^{(\pm)}, w_+^{(\pm)})$ *be defined as in* (2.62). *Then for* $x \in I$, *one has*

$$
\begin{aligned}
&\text{(i)} \quad w_-^{(-)}(x) = \frac{u_{-+}(x)}{\lambda_+(I_-)}, \qquad &&\text{(ii)} \quad w_+^{(-)}(x) = -\frac{1 - u_{+-}(x)}{\lambda_+(I_-)}, \\
&\text{(iii)} \quad w_-^{(+)}(x) = -\frac{1 - u_{-+}(x)}{\lambda_-(I_+)}, \qquad &&\text{(iv)} \quad w_+^{(+)}(x) = \frac{u_{+-}(x)}{\lambda_-(I_+)}.
\end{aligned} \tag{2.63}
$$

Moreover,

$$
\text{(2.64)}\qquad
\begin{aligned}
&\text{(i)}\quad f_+(I_+) > 0 \quad\Leftrightarrow\quad \inf_{x\in I}\bigl[w_-^{(-)}(x) + w_-^{(+)}(x)\bigr] > 0,\\
&\text{(ii)}\quad f_-(I_-) > 0 \quad\Leftrightarrow\quad \inf_{x\in I}\bigl[w_+^{(-)}(x) + w_+^{(+)}(x)\bigr] > 0.
\end{aligned}
$$

Both formulas also hold with the inequality signs reversed.

PROOF. We only prove (2.63) (i) and (ii) and (2.64) (i); the proof of the other formulas follows from the symmetry between buy and sell orders. By Lemma 16 and the facts that

$$
\text{(2.65)}\qquad u_{-+}(I_+) = \frac{\Gamma^{-1}}{\lambda_-(I_+)\lambda_+(I_+)} \quad\text{and}\quad u_{+-}(I_-) = \frac{\Gamma^{-1}}{\lambda_-(I_-)\lambda_+(I_-)},
$$

we have:

$$
\text{(2.66)}\qquad
\begin{aligned}
&\text{(i)}\quad w_-^{(-)}(x) = \lambda_-(I_-)\Gamma\biggl[\frac{\Gamma^{-1}}{\lambda_-(I_-)\lambda_+(I_-)} - 1_{\{x=I_-\}}\biggr]\\
&\qquad\qquad\qquad \times \bigl(u_{-+}(x) - 1_{\{x=I_-\}}\bigr)\\
&\text{(ii)}\quad w_+^{(-)}(x) = \lambda_-(I_-)\Gamma\bigl[u_{+-}(x) - 1\bigr]\frac{\Gamma^{-1}}{\lambda_-(I_-)\lambda_+(I_-)}.
\end{aligned}
$$

For $x \neq I_-$, these formulas simplify to (2.63) (i) and (ii).

Adding formulas (2.63) (i) and (iii) yields

$$
\text{(2.67)}\qquad w_-^{(-)}(x) + w_-^{(+)}(x) = \biggl[\frac{1}{\lambda_+(I_-)} + \frac{1}{\lambda_-(I_+)}\biggr]u_{-+}(x) - \frac{1}{\lambda_-(I_+)}.
$$

Since u_{-+} is nonincreasing and continuous, the infimum of this function over $x \in I$ is equal to the value in $x = I_+$, that is,

$$
\text{(2.68)}\qquad
\begin{aligned}
&\inf_{x\in I}\bigl[w_-^{(-)}(x) + w_-^{(+)}(x)\bigr]\\
&\qquad = \biggl[\frac{1}{\lambda_+(I_-)} + \frac{1}{\lambda_-(I_+)}\biggr]\frac{\Gamma^{-1}}{\lambda_-(I_+)\lambda_+(I_+)} - \frac{1}{\lambda_-(I_+)}.
\end{aligned}
$$

Using the fact that $\Gamma > 0$, we see that the expression in (2.68) is positive if and only if

$$
\text{(2.69)}\qquad \frac{1}{\lambda_+(I_-)\lambda_+(I_+)} + \frac{1}{\lambda_-(I_+)\lambda_+(I_+)} > \Gamma.
$$

Taking into account (2.19) and (1.14) (ii) (which also holds with the equality signs reversed), this is equivalent to $f_+(I_+) > 0$. $\square$

2.5. *Restricted models.* In the present subsection, we prove Proposition 5 as well as Lemmas 4, 6 and 7.

PROOF OF LEMMA 4. To prove that $J_- < \phi_+(J_-)$ for all $J_- \in I$, it suffices to show that $\Lambda_+(J_-, J_- + \varepsilon) > 0$ for $\varepsilon > 0$ sufficiently small. Here,

$$\Lambda_+(J_-, J_- + \varepsilon) = \frac{1}{\lambda_+(J_-)\lambda_+(J_- + \varepsilon)} + \int_{J_-}^{J_- + \varepsilon} \frac{1}{\lambda_-} \mathrm{d}\Big(\frac{1}{\lambda_+}\Big). \tag{2.70}$$

By assumptions (A3) and (A5) and the fact that $J_- \in I$, the first term tends to a positive limit as $\varepsilon \downarrow 0$ while the second term tends to zero. By the symmetry between buy and sell orders, we see that also $\phi_-(J_+) < J_+$ for all $J_+ \in I$.

Using (A3), we see that for fixed J_-, the function $\Lambda_+(J_-, J_+)$ is nonincreasing as a function of J_+, and hence that $\Lambda_+(J_-, J_+) > 0$ if and only if $J_+ < \phi_+(J_-)$. Similarly, $\Lambda_-(J_-, J_+) > 0$ if and only if $\phi_-(J_+) < J_-$, so the second claim of the lemma follows from Theorem 3, where we use that for $I_- < J_- < J_+ < I_+$, the restricted model on (J_-, J_+) satisfies (A6) since the model on I satisfies (A5). $\square$

PROOF OF PROPOSITION 5. Recall from Section 1.4 that for any subinterval J such that $\overline{J} \subset I$, $\Lambda_\pm(J_-, J_+)$ denote the expressions in (1.14), calculated for the process restricted to the subinterval $\overline{J}$. We will prove that

$$V_{\mathrm{W}}^{-2} - \Phi(V) = \Lambda_-\big(\lambda_-^{-1}(V), \lambda_+^{-1}(V)\big) = \Lambda_+\big(\lambda_-^{-1}(V), \lambda_+^{-1}(V)\big) \tag{2.71}$$

for any $V \in (V_{\mathrm{W}}, V_{\max}]$ such that $I_- < \lambda_-^{-1}(V) < \lambda_+^{-1}(V) < I_+$. By Proposition 2, this then implies Proposition 5.

We will first prove (2.71) for Stigler–Luckock models in standard form (see Appendix A.1). Let D be the set of all pairs $(J_-, J_+) \in \mathbb{R}^2$ such that $I_- \le J_- \le J_+ \le I_+$ and $\lambda_-(J_-) = \lambda_+(J_+)$, and let $T := \sup\{J_+ - J_- : (J_-, J_+) \in D\}$. Note that $D \neq \varnothing$ and $T > 0$ by (A3) and (A7). We define a curve $\gamma : [0, T] \to D$ with $\gamma(t) = (\gamma_-(t), \gamma_+(t))$ by

$$\begin{aligned} &\gamma_-(t) := \inf\big\{J_- \in \overline{I} : \lambda_-(J_-) \le \lambda_+(J_- + t)\big\} \quad \text{and} \\ &\gamma_+(t) := \gamma_-(t) + t. \end{aligned} \tag{2.72}$$

Using (A3), it is not hard to see that γ_- is nonincreasing, γ_+ is nondecreasing, and γ is Lipschitz continuous with Lipschitz constant 1. Using also (A4), we see that $\gamma(0) = (x_{\mathrm{W}}, x_{\mathrm{W}})$, where x_{W} is the Walrasian price where in fact $x_{\mathrm{W}} = 0$ since we are assuming $\lambda_\pm$ are in standard form.

For any J_-, J_-', J_+, J_+' with $J_- \le J_-'$ and $J_+ \le J_+'$, we observe that

$$\begin{aligned} \Lambda_+(J_-', J_+) - \Lambda_+(J_-, J_+) &= \int_{J_-}^{J_-'} \Big\{\frac{1}{\lambda_+(J_+)} - \frac{1}{\lambda_-}\Big\} \mathrm{d}\Big(\frac{1}{\lambda_+}\Big), \\ \Lambda_+(J_-, J_+') - \Lambda_+(J_-, J_+) &= \int_{J_+}^{J_+'} \Big\{\frac{1}{\lambda_+(J_-)} + \frac{1}{\lambda_-}\Big\} \mathrm{d}\Big(\frac{1}{\lambda_+}\Big). \end{aligned} \tag{2.73}$$

Since the model is in standard form, the functions $\lambda_\pm$ are Lipschitz continuous with Lipschitz constant one. Since we are also assuming (A5), it follows that $\lambda_\pm^{-1}$ are locally Lipschitz on $\{(J_-,J_+):\overline{J}\subset I\}$. Therefore, we can apply Lemma 10 to conclude that

$$\begin{aligned}\Lambda_+\big(\gamma(t)\big)=\Lambda_+\big(\gamma(0)\big)&+\int_0^t\Big\{\frac{1}{\lambda_+\circ\gamma_+}-\frac{1}{\lambda_-\circ\gamma_-}\Big\}\,\mathrm{d}\Big(\frac{1}{\lambda_+\circ\gamma_-}\Big)\\&+\int_0^t\Big\{\frac{1}{\lambda_+\circ\gamma_-}+\frac{1}{\lambda_-\circ\gamma_+}\Big\}\,\mathrm{d}\Big(\frac{1}{\lambda_+\circ\gamma_+}\Big)\end{aligned}\tag{2.74}$$

for any $t\in[0,T')$, where $T':=\sup\{t\in[0,T]:I_-<\gamma_-(t)<\gamma_+(t)<I_+\}$.

Since $\lambda_-(\gamma_-(s))=\lambda_+(\gamma_+(s))$ for all $s\in[0,t]$, the first integral in (2.74) is zero. Set $\phi:=\lambda_+\circ\gamma_+$ and $\phi^{-1}(V):=\inf\{t\geq 0:\phi(t)\geq V\}$, and observe that $\gamma(\phi^{-1})(V)=(\lambda_-^{-1}(V),\lambda_+^{-1}(V))$ since the latter is the smallest interval J such that $\lambda_-(J_-)=V=\lambda_+(J_+)$, and ϕ^{-1} is left continuous. Using the substitution of variables $W=\phi(t)$ [recall (2.7)], using also the fact that

$$\Lambda_+\big(\gamma(0)\big)=\Lambda_+(x_\mathrm{W},x_\mathrm{W})=\frac{1}{V_\mathrm{W}^2},\tag{2.75}$$

we can rewrite (2.74) as

$$\begin{aligned}&\Lambda_+\big(\lambda_-^{-1}(V),\lambda_+^{-1}(V)\big)\\&\qquad=\frac{1}{V_\mathrm{W}^2}+\int_{V_\mathrm{W}}^V\Big\{\frac{1}{\lambda_+(\lambda_-^{-1}(W))}+\frac{1}{\lambda_-(\lambda_+^{-1}(W))}\Big\}\,\mathrm{d}\Big(\frac{1}{W}\Big),\end{aligned}\tag{2.76}$$

which holds whenever $V<V_{\max}$. This proves (2.71) for Λ_+. The equality for Λ_- follows in the same way, or alternatively, one can use the fact that

$$\begin{aligned}&\Lambda_+(J_-,J_+)-\Lambda_-(J_-,J_+)\\&\qquad=\Big(\frac{1}{\lambda_+(J_+)}-\frac{1}{\lambda_-(J_-)}\Big)\Big(\frac{1}{\lambda_+(J_-)}+\frac{1}{\lambda_-(J_+)}\Big),\end{aligned}\tag{2.77}$$

which follows from partial integration of the formulas in (1.14) and shows that $\Lambda_+(J_-,J_+)=\Lambda_-(J_-,J_+)$ whenever $\lambda_-(J_-)=\lambda_+(J_+)$.

This proves (2.71) for models in standard form. Since by Proposition 20 in the Appendix, any Stigler–Luckock model can be brought in standard form, to prove (2.71) more generally it suffices to show that the quantities V_W, $\Lambda_\pm(\lambda_-^{-1}(V),\lambda_+^{-1}(V))$ and $\Phi(V)$ do not change when we bring a model in standard form.

By Proposition 20, for any demand and supply functions $\lambda'_\pm$ satisfying (A1) and (A2) on some interval I', there exists demand and supply functions $\lambda_\pm$ in standard form on some interval I, together with a nondecreasing function $\psi:\overline{I}\to\overline{I}'$ that satisfies $\psi(I_\pm)=I'_\pm$ and that is right continuous on I, such that $\mu'_\pm=\mu_\pm\circ\psi^{-1}$

and (A.7) holds. It is not hard to see that as a result of (A.7), the supply and demand functions $\lambda_\pm$ and $\lambda'_\pm$ have the same Walrasian volume of trade.

In fact, we are only interested here in the case that $\lambda'_\pm$ satisfy moreover (A3), (A5) and (A7). In particular, (A3) says that $\mu'_\pm$ have no atoms in I', and hence ψ is strictly increasing, so (A.7) is trivial in our setting. Since ψ is strictly increasing, it has an inverse $\psi^{-1}:\overline{I}' \to I$ which is a continuous, nondecreasing function. Since $\mu'_\pm = \mu_\pm \circ \psi^{-1}$, the functions $\lambda'_\pm$ are given in terms of $\lambda_\pm$ by

$$\lambda'_\pm(y) = \lambda_\pm(\psi^{-1}(y)) \qquad (y \in \overline{I}'). \tag{2.78}$$

Since

$$\lambda_\pm(\lambda_\mp^{-1}(V)) = \sup\{\lambda_\pm(x) : x \in \overline{I}, \lambda_\mp(x) \geq V\}, \tag{2.79}$$

these quantities are the same for the model in standard form and the transformed model, and hence the same is true for $\Phi(V)$. By the substitution of variables formula [see (2.7)], also

$$\int_{\lambda_-^{-1}(V)}^{\lambda_+^{-1}(V)} \frac{1}{\lambda_+}\,\mathrm{d}\left(\frac{1}{\lambda_-}\right) \quad \text{and} \quad \int_{\lambda_-^{-1}(V)}^{\lambda_+^{-1}(V)} \frac{1}{\lambda_-}\,\mathrm{d}\left(\frac{1}{\lambda_+}\right) \tag{2.80}$$

are the same for the transformed model, and hence this is also true for $\Lambda_\pm(\lambda_-^{-1}(V), \lambda_+^{-1}(V))$. $\square$

PROOF OF LEMMA 6. Assume that $J \subset I$ satisfies $\overline{J} \subset I$ and (i) and (ii). Since $f_-(J_-) = 0 = f_+(J_+)$, by (1.15), we must have $\lambda_-(J_-) = \lambda_+(J_+)$. Call this quantity V. If $V = V_{\mathrm{W}}$, then it is easy to check that $\Lambda_\pm(J_-, J_+) = V_{\mathrm{W}}^{-2} > 0$, which by Proposition 2 contradicts $f_\pm(J_\pm) = 0$, so we must have $V > V_{\mathrm{W}}$. Assumption (ii) implies that $J = (\lambda_-^{-1}(V), \lambda_+^{-1}(V))$. Now Proposition 5 tells us that $\Phi(V) = V_{\mathrm{W}}^{-2}$, and hence $V = V_{\mathrm{L}}$ and $J = J^{\mathrm{c}}$.

Conversely, if $V_{\mathrm{L}} < V_{\max}$, then $J^{\mathrm{c}} := (\lambda_-^{-1}(V_{\mathrm{L}}), \lambda_+^{-1}(V_{\mathrm{L}}))$ satisfies $\overline{J} \subset I$ and Proposition 5 tells us that the solution to Luckock's equation on $\overline{J}$ satisfies $f_-(J_-) = 0 = f_+(J_+)$ while (ii) holds because of the way $\lambda_\pm^{-1}$ have been defined. $\square$

PROOF OF LEMMA 7. For the uniform model with $\lambda_-(x) = 1 - x$ and $\lambda_+(x) = x$, we have

$$V_{\mathrm{W}} = \frac{1}{2}, \qquad V_{\max} = 1, \qquad \lambda_-^{-1}(V) = 1 - V, \quad \text{and} \quad \lambda_+^{-1}(V) = V. \tag{2.81}$$

It follows that the function Φ from (1.20) is given by

$$\begin{aligned}\Phi(V) &= -2\int_{1/2}^{V} \frac{1}{1-W}\,\mathrm{d}\left(\frac{1}{W}\right) = 2\int_{1/V}^{2} \frac{1}{1-\frac{1}{y}}\,\mathrm{d}y \\ &= 2\int_{1/V}^{2}\left\{1 + \frac{1}{y-1}\right\}\mathrm{d}y = 4 - 2\{V^{-1} + \log(V^{-1} - 1)\}.\end{aligned} \tag{2.82}$$

Setting $\Phi(V) = V_{\mathrm{W}}^{-2}$ gives

$$(2.83) \qquad -V^{-1} = \log\bigl(V^{-1} - 1\bigr) \quad \Leftrightarrow \quad e^{-V^{-1}} = V^{-1} - 1,$$

which tells us that $V_{\mathrm{L}} = 1/z$ where z solves $f(z) := e^{-z} - z + 1 = 0$. Since the function f is continuous and strictly decreasing with $f(1) = e^{-1}$ and $f(z) \to -\infty$ for $z \to \infty$, the equation $f(z) = 0$ has a unique solution z, and this solution satisfies $z > 1$. $\square$

3. Analysis of the Markov chain.

3.1. *A consequence of stationarity.* In this subsection, we prove Theorem 1.

PROOF OF THEOREM 1. Let ν be an invariant law, let X_0 be an $\mathcal{S}_{\mathrm{ord}}$-valued random variable with law ν and let (U_1, σ_1) be independent of X_0 with law $\overline{\mu} := |\mu|^{-1}\mu$, as defined in Section 1.1. Then stationarity means that $X_1 := L_{U_1,\sigma_1}(X_0)$ has the same law as X_0, where $L_{u,\sigma}$ is the Luckock map from (1.9).

Set $M_\pm := M_\pm(X_0)$ and let $\mu_\pm$ be as in (1.1). We claim that:

$$(3.1) \qquad \begin{aligned} &\text{(i)} \quad \int_A \mathbb{P}[M_- < x]\mu_+(\mathrm{d}x) = \int_A \lambda_-(x)\mathbb{P}[M_+ \in \mathrm{d}x], \\ &\text{(ii)} \quad \int_A \mathbb{P}[M_+ > x]\mu_-(\mathrm{d}x) = \int_A \lambda_+(x)\mathbb{P}[M_- \in \mathrm{d}x] \end{aligned}$$

for each measurable $A \subset I$ that is contained in some compact subinterval $[J_-, J_+] \subset I$. Indeed, stationarity implies (see [24], Lemma 10) that for any measurable $A \subset I$, sell limit orders are added in A with the same frequency as they are removed, that is,

$$(3.2) \qquad \mathbb{P}\bigl[X_1^+(A) = X_0^+(A) + 1\bigr] = \mathbb{P}\bigl[X_1^+(A) = X_0^+(A) - 1\bigr].$$

Recalling the definition of the Luckock map in (1.9), we see that this means that

$$(3.3) \qquad \mathbb{P}[\sigma_1 = +, U_1 \in A, M_- < U_1] = \mathbb{P}[\sigma_1 = -, M_+ \in A, M_+ \le U_1].$$

Since (U_1, σ_1) has law $\overline{\mu}$ and is independent of $M_\pm$, it follows that

$$(3.4) \qquad |\mu|^{-1} \int_A \mu_+(\mathrm{d}x)\mathbb{P}[M_- < x] = |\mu|^{-1} \int_A \mathbb{P}[M_+ \in \mathrm{d}x]\mu_-\bigl([x, I_+]\bigr),$$

which up to the factor $|\mu|^{-1}$ is (3.1) (i). Similarly, equation (3.1) (ii) follows from the requirement that buy limit orders are added in A with the same frequency as they are removed.

By (A3), the measures $\mu_\pm$ do not have atoms in I, and hence by (A5) and (3.1), the same is true for the laws of $M_\pm$. It follows that $\mathbb{P}[M_- < x] = f_-(x)$ and $\mathbb{P}[M_+ > x] = f_+(x)$ $(x \in I)$ and $f_\pm$ are continuous. Since by (1.1), $-\mathrm{d}\lambda_-$ and $\mathrm{d}\lambda_+$ are the restrictions of the measures μ_- and μ_+ to I, respectively, we can rewrite (3.1) (i) and (ii) as (1.13) (i) and (ii). Since $f_-(I_+) = \mathbb{P}[M_- \le I_+] = 1$ and $f_+(I_-) = \mathbb{P}[M_+ \ge I_-] = 1$, the boundary conditions (1.13) (iii) also follow. $\square$

3.2. *A Lyapunov function*. It follows from Theorem 1 that if a Stigler–Luckock model is positive recurrent, then the solution to Luckock's equation must satisfy $f_-(I_-) \wedge f_+(I_+) > 0$. Theorem 3 states that this condition is also sufficient. We will prove this by showing that

$$V(\mathcal{X}) := \sqrt{\big(F^{(-)}(\mathcal{X}) \vee 0\big)^2 + \big(F^{(+)}(\mathcal{X}) \vee 0\big)^2} \qquad (\mathcal{X} \in \mathcal{S}^{\mathrm{fin}}_{\mathrm{ord}}) \tag{3.5}$$

is a Lyapunov function. We note that this is the only place in the paper where we make use of a function of a Stigler–Luckock process that is not linear (namely V). In view of Theorem 11, we have fairly good control of linear functionals, which as in Theorem 1 (which depends on Theorem 9) allows us to more or less explicitly calculate the marginal distributions of the best buy and sell offers $M_-(\mathcal{X})$ and $M_+(\mathcal{X})$ in equilibrium. Proving that a Stigler–Luckock model is positive recurrent, however, always entails proving something about the joint distribution of $M_-(\mathcal{X})$ and $M_+(\mathcal{X})$. Indeed, the following proposition can be used to give a lower bound on the probability, in equilibrium, that the order book is empty, which corresponds to the event that $M_-(\mathcal{X}) = I_-$ and at the same time $M_+(\mathcal{X}) = I_+$, but this bound is not very explicit or sharp. It seems that such information cannot be obtained from linear functionals and indeed for no choice of weight functions (w_-, w_+) is a linear function of the form (1.24) a Lyapunov function.

Recall the definition (1.23) of the generator G of a Stigler–Luckock model. We have the following result.

PROPOSITION 18 (Lyapunov function). *Assume* (A3) *and* (A6), *and that the unique solution* (f_-, f_+) *of Luckock's equation* (1.13) *satisfies* $\varepsilon := f_-(I_-) \wedge f_+(I_+) > 0$. *Then there exists a constant* $K < \infty$ *such that the function in* (3.5) *satisfies* $GV(\mathcal{X}) \le K$ *for all* $\mathcal{X} \in \mathcal{S}^{\mathrm{fin}}_{\mathrm{ord}}$. *Moreover, for each* $\varepsilon' < \varepsilon$, *there exists an* $N < \infty$ *such that*

$$GV(\mathcal{X}) \le -\varepsilon' \qquad \textit{whenever } |\mathcal{X}^-| + |\mathcal{X}^+| \ge N. \tag{3.6}$$

PROOF. Let us write $\vec{F}(\mathcal{X}) := (F^{(-)}(\mathcal{X}), F^{(+)}(\mathcal{X}))$ and let $|\cdot|$ denote the euclidean norm on $\mathbb{R}^2$. Let us also write $V(\mathcal{X}) = v(\vec{F}(\mathcal{X}))$ where $v : R^2 \to \mathbb{R}$ is the function

$$v(z_1, z_2) := \sqrt{(z_1 \vee 0)^2 + (z_2 \vee 0)^2}. \tag{3.7}$$

Set

$$W := \sup_{x \in I} \big|\big(w_-^{(-)}(x), w_-^{(+)}(x)\big)\big| \vee \sup_{x \in I} \big|\big(w_+^{(-)}(x), w_+^{(+)}(x)\big)\big|, \tag{3.8}$$

which is the maximal amount by which $\vec{F}(\mathcal{X})$ can change due to the addition or removal of a single limit order. Since the function v is Lipschitz continuous with

Lipschitz constant 1, we can estimate

$$\begin{aligned} GV(\mathcal{X}) &= \int \{V(L_{u,\sigma}(\mathcal{X})) - V(\mathcal{X})\}\mu(\mathrm{d}(u,\sigma)) \\ &\leq \int |\vec{F}(L_{u,\sigma}(\mathcal{X})) - \vec{F}(\mathcal{X})|\mu(\mathrm{d}(u,\sigma)) \\ &\leq W(\lambda_-(I_-) + \lambda_+(I_+)) =: K. \end{aligned} \tag{3.9}$$

Let

$$\delta := \inf_{x\in I}[w_-^{(-)}(x) + w_-^{(+)}(x)] \wedge \inf_{x\in I}[w_+^{(-)}(x) + w_+^{(+)}(x)], \tag{3.10}$$

which is positive by (2.64) and our assumption that $f_-(I_-) \wedge f_+(I_+) > 0$. Since adding a limit order to the order book always raises $F^{(-)} + F^{(+)}$ by at least δ,

$$F^{(-)}(\mathcal{X}) + F^{(+)}(\mathcal{X}) \geq \delta(|\mathcal{X}^-| + |\mathcal{X}^+|). \tag{3.11}$$

This shows that $\vec{F}(\mathcal{X})$ takes values in the half space $H := \{(z_1, z_2) \in \mathbb{R}^2 : z_1 + z_2 > 0\}$ as long as $\mathcal{X} \neq 0$, and moreover $|\vec{F}(\mathcal{X})|$ is large if $|\mathcal{X}^-| + |\mathcal{X}^+|$ is.

For any $z = (z_1, z_2) \in \mathbb{R}^2$ with $z_1 + z_2 > 0$, let us define

$$\begin{aligned} p_1(z) &:= \frac{z_1 \vee 0}{\sqrt{(z_1 \vee 0)^2 + (z_2 \vee 0)^2}} \quad \text{and} \\ p_2(z) &:= \frac{z_2 \vee 0}{\sqrt{(z_1 \vee 0)^2 + (z_2 \vee 0)^2}}. \end{aligned} \tag{3.12}$$

Then, for any $y, z \in H$, we can write

$$v(z) = v(y) + p_1(y)(z_1 - y_1) + p_2(y)(z_2 - y_2) + R(y, z), \tag{3.13}$$

where for any y, z that differ at most by the constant W from (3.8), the error term $R(x, y)$ can be estimated as

$$R(y, z) \leq C|y|^{-1} \qquad (y, z \in H, |z - y| \leq W) \tag{3.14}$$

for some constant $C < \infty$. It follows that we can write

$$GV(\mathcal{X}) = p_1(\vec{F}(\mathcal{X}))GF^{(-)}(\mathcal{X}) + p_2(\vec{F}(\mathcal{X}))GF^{(+)}(\mathcal{X}) + E(\mathcal{X}), \tag{3.15}$$

where the error term can be estimated as

$$\begin{aligned} |E(\mathcal{X})| &= \left| \int R(\vec{F}(\mathcal{X}), \vec{F}(L_{u,\sigma}(\mathcal{X}))\mu(\mathrm{d}(u,\sigma)) \right| \\ &\leq C(\lambda_-(I_-) + \lambda_+(I_+))|\vec{F}(\mathcal{X})|^{-1}, \end{aligned} \tag{3.16}$$

which in view of (3.11) can be made arbitrary small by choosing $|\mathcal{X}^-| + |\mathcal{X}^+|$ sufficiently large.

By Theorem 9 and the way we have defined the weight functions $w_\pm^{(-)}$ and $w_\pm^{(+)}$, one has

$$GF^{(-)}(\mathcal{X}) = -f_-(I_-) \qquad \text{if } |\mathcal{X}^-| \neq 0, \tag{3.17}$$
$$GF^{(+)}(\mathcal{X}) = -f_+(I_+) \qquad \text{if } |\mathcal{X}^+| \neq 0.$$

It follows from (2.63) and elementary properties of the functions in (2.50) that

$$w_-^{(-)} > 0, \qquad w_+^{(-)} < 0, \qquad w_-^{(+)} < 0, \quad \text{and} \quad w_+^{(+)} > 0 \quad \text{on } I. \tag{3.18}$$

In view of this, we have

$$|\mathcal{X}^-| = 0 \quad \Rightarrow \quad F^{(-)}(\mathcal{X}) \leq 0 \quad \Rightarrow \quad p_1(\vec{F}(\mathcal{X})) = 0, \tag{3.19}$$
$$|\mathcal{X}^+| = 0 \quad \Rightarrow \quad F^{(+)}(\mathcal{X}) \leq 0 \quad \Rightarrow \quad p_2(\vec{F}(\mathcal{X})) = 0.$$

Combining this with (3.17), we obtain that

$$p_1(\vec{F}(\mathcal{X}))GF^{(-)}(\mathcal{X}) + p_2(\vec{F}(\mathcal{X}))GF^{(+)}(\mathcal{X}) \leq -\varepsilon \qquad (\mathcal{X} \neq 0). \tag{3.20}$$

Inserting this into (3.15), using our bound (3.16) on the error term, and using also (3.11), we see that by choosing $|\mathcal{X}^-| + |\mathcal{X}^+|$ large enough, we can make GV smaller than $-\varepsilon'$ for any $\varepsilon' < \varepsilon$. □

3.3. *Positive recurrence.*

PROOF OF THEOREM 3. If a Stigler–Luckock model is positive recurrent, then it it is possible to construct a stationary process $(X_k)_{k\in\mathbb{Z}}$ that makes i.i.d. excursions away from the empty state 0. In particular, positive recurrence implies the existence of an invariant law ν that is concentrated on $\mathcal{S}_{\mathrm{ord}}^{\mathrm{fin}}$ and satisfies $\nu(\{0\}) > 0$. By Theorem 1, it follows that Luckock's equation (1.13) has a solution (f_-, f_+) such that $f_-(I_-) \wedge f_+(I_+) \geq \nu(\{0\}) > 0$.

Conversely, assume (A3) and (A6) and that the (by Proposition 2 unique) solution to Luckock's equation satisfies $f_-(I_-) \wedge f_+(I_+) > 0$. Let P denote the transition kernel of the discrete-time process $(X_k)_{k\geq 0}$ and for any nonnegative measurable function $f : \mathcal{S}_{\mathrm{ord}}^{\mathrm{fin}} \to \mathbb{R}$ write $Pf(x) := \int P(x, \mathrm{d}y) f(y)$. Write

$$C_N := \{\mathcal{X} \in \mathcal{S}_{\mathrm{ord}}^{\mathrm{fin}} : |\mathcal{X}|^- + |\mathcal{X}|^+ < N\}. \tag{3.21}$$

Multiplying the Lyapunov function V of Proposition 18 by a suitable constant, we obtain a nonnegative function f and finite constants K, N such that

$$Pf - f \leq K 1_{C_N} - 1. \tag{3.22}$$

Let $\tau_0 := \inf\{k > 0 : X_k = 0\}$ denote the first return time to the empty configuration. By assumption (A6), there exists a constant $\varepsilon > 0$ such that

$$\mathbb{P}^x[\tau_0 \leq N + 1] \geq \varepsilon \qquad (x \in C_N). \tag{3.23}$$

Moreover, (A6) guarantees that $\mathbb{P}^0[\tau_0 = a] > 0$, which shows that the model is aperiodic from 0. Applying Proposition 21 from Appendix A.2 in the Appendix, we conclude that the Stigler–Luckock model under consideration is positive recurrent and (1.16) holds. $\square$

APPENDIX

A.1. The model in standard form. By definition, we say that a Stigler–Luckock model is in *standard form* if

$$\lambda_+(x) - \lambda_-(x) = x \qquad (x \in \overline{I}). \tag{A.1}$$

Note that for a model in standard form, $\mathrm{d}\lambda_+ - \mathrm{d}\lambda_-$, which is the total rate at which limit orders arrive in I, is the Lebesgue measure. Clearly, a model in standard form always satisfies (A3) and (A4). Let $p_+ : \overline{I} \to [0, 1]$ be the Radon–Nikodym derivative of $\mathrm{d}\lambda_+$ w.r.t. the Lebesgue measure and let $p_- := 1 - p_+$. Then

$$\begin{aligned} &\lambda_-(x) = \lambda_-(I_+) + \int_x^{I_+} p_-(x)\,\mathrm{d}x \quad \text{and} \\ &\lambda_+(x) = \lambda_+(I_-) + \int_{I_-}^{x} p_+(x)\,\mathrm{d}x. \end{aligned} \tag{A.2}$$

In particular, a model in standard form is uniquely characterized by its interval I, rates of market order $\lambda_\pm(I_\mp)$, and the function p_+.

In the present section, we will show that under mild additional assumptions, each Stigler–Luckock model satisfying (A1) and (A2) can be brought in standard form. To demonstrate the idea on a concrete example, consider a Stigler–Luckock model $(\mathcal{X}_t)_{t\geq 0}$ with $I = (0, 2n)$ where $n \geq 1$ is some integer, there are no market orders, and

$$p_+(x) = 1_{\{\lfloor x \rfloor \text{ is even}\}} \qquad (x \in I). \tag{A.3}$$

Assume that $\mathcal{X}_0 = 0$ (the empty initial state). Define a function $\psi : I \to I' := (0, n+1)$ by $\psi(x) := 1 + \lfloor x/2 \rfloor$, and let $\mathcal{X}'_t := \mathcal{X}_t \circ \psi^{-1}$ $(t \geq 0)$ denote the image of $\mathcal{X}_t$ under ψ. Then $(\mathcal{X}'_t)_{t\geq 0}$ is a Stigler–Luckock model where limit orders are placed at discrete prices only. More precisely, for each $k \in \{1, \ldots, n\}$, in the original model, buy and sell limit orders arrive in $(2k-2, 2k)$ with rate one each, in such a way that sell limit orders arrive on the left of buy limit orders. After applying the map ψ, all these orders arrive at the price k, where they still match.

Our aim is to show that this construction works quite generally, that is, given a fairly general Stigler–Luckock model $(\mathcal{X}'_t)_{t\geq 0}$ with parameters I' and $\lambda'_\pm$, we can find a Stigler–Luckock model $(\mathcal{X}_t)_{t\geq 0}$ in standard form with parameters I and $\lambda_\pm$, as well as a nondecreasing right-continuous function $\psi : I \to I'$, such that $\mathcal{X}'_t = \mathcal{X}_t \circ \psi^{-1}$ $(t \geq 0)$. We will see that moreover, the parameters of $(\mathcal{X}'_t)_{t\geq 0}$ determine those of the model in standard form as well as the function ψ uniquely.

We first investigate which functions ψ have the property that the image of a Stigler–Luckock model under ψ is again a Stigler–Luckock model. Clearly, it suffices if ψ is strictly increasing, but as our previous example showed, this condition can be weakened.

LEMMA 19 (Transformed Stigler–Luckock model). *Let $(\mathcal{X}_t)_{t\geq 0}$ be a Stigler–Luckock model on some nonempty open interval $I \subset \mathbb{R}$ with demand and supply functions $\lambda_\pm$ satisfying* (A1) *and* (A2). *Let $I' \subset \mathbb{R}$ be an open interval, let $\psi : I \to I'$ be nondecreasing, and assume that*

$$\int_I \nu_-(\mathrm{d}x) \int_I \nu_+(\mathrm{d}y) 1_{\{x<y \text{ and } \psi(x)\geq\psi(y)\}} = 0, \tag{A.4}$$

where $\nu_\pm := \mu_\pm + \mathcal{X}_0^\pm$ with $\mu_\pm$ defined in (1.1). *Extend ψ to $\overline{I}$ by setting $\psi(I_\pm) := I'_\pm$. Then $\mathcal{X}'_t := \mathcal{X}_t \circ \psi^{-1}$ $(t \geq 0)$ is a Stigler–Luckock model with demand and supply functions $\lambda'_\pm$ and related measures $\mu'_\pm$ as in* (1.1) *given by $\mu'_\pm := \mu_\pm \circ \psi^{-1}$.*

PROOF. The condition (A.4) guarantees that a.s., of all the orders that are at time zero in the order book or that arrive at later times, a buy and a sell order match in the original model $\mathcal{X}$ if and only if they match in the transformed model $\mathcal{X}'$. $\square$

We wish to show that a general Stigler–Luckock model $(\mathcal{X}'_t)_{t\geq 0}$ can be obtained as a function of a Stigler–Luckock model $(\mathcal{X}_t)_{t\geq 0}$ in standard form. We will need a weak assumption on the initial state of $(\mathcal{X}'_t)_{t\geq 0}$. To formulate this properly, we need some definitions. Let μ be a finite nonnegative measure on $\overline{\mathbb{R}} := [-\infty, \infty]$ and let $\mathrm{supp}(\mu)$ denote its support. Then the complement of $\mathrm{supp}(\mu)$ is a countable union of disjoint open intervals. If for each left endpoint x_- of such an interval (x_-, x_+), we remove x_- from $\mathrm{supp}(\mu)$ if it carries no mass, then we obtain

$$\mathrm{supp}_+(\mu) = \{x \in \overline{\mathbb{R}} : \mu([x, y)) > 0\ \forall y > x\}. \tag{A.5}$$

The set $\mathrm{supp}_+(\mu)$ is the support of μ with respect to the topology of convergence from the right, where a sequence x_n converges to a limit x if and only if $x_n \to x$ in the usual topology on $\overline{\mathbb{R}}$, and moreover, $x_n \geq x$ for n large enough. A basis for this topology is formed by all sets of the form $[x, y)$ with $x < y$.

PROPOSITION 20 (Standard form). *Let $(\mathcal{X}'_t)_{t\geq 0}$ be a Stigler–Luckock model with demand and supply functions $\lambda'_\pm : \overline{I}' \to [0, \infty)$ satisfying* (A1) *and* (A2). *Assume that $\lambda'_\pm$ are not both constant and that*

$$\mathcal{X}'_0 \text{ is concentrated on } \mathrm{supp}_+(\mu'_- + \mu'_+), \tag{A.6}$$

where $\mu'_\pm$ are defined in terms of $\lambda'_\pm$ as in (1.1). *Then there exist demand and supply functions $\lambda_\pm$ in standard form on some interval I, as well as a nondecreasing* *function* $\psi : \overline{I} \to \overline{I}'$ *that maps* I *into* I', *satisfies* $\psi(I_\pm) = I'_\pm$, *and that is right continuous on* I, *such that* $\mu'_\pm = \mu_\pm \circ \psi^{-1}$ *and*

$$\int_I \mu_-(\mathrm{d}x) \int_I \mu_+(\mathrm{d}y) 1_{\{x<y \text{ and } \psi(x) \geq \psi(y)\}} = 0. \tag{A.7}$$

Moreover, these conditions determine $I_\pm$, $\lambda_\pm$, *and* ψ *uniquely. Finally, there exists a Stigler–Luckock model* $(\mathcal{X}_t)_{t\geq 0}$ *with demand and supply functions* $\lambda_\pm$ *such that* (A.4) *holds and the process* $\mathcal{X}''_t := \mathcal{X}_t \circ \psi^{-1}$ $(t \geq 0)$ *is equal in law to* $(\mathcal{X}'_t)_{t\geq 0}$.

PROOF. Let $\chi(y) := \lambda'_+(y) - \lambda'_-(y)$ $(y \in \overline{I}')$. Since $\lambda'_\pm$ satisfy (A1) and (A2), the function χ is nondecreasing on $\overline{I}'$, and continuous at $I'_\pm$. We claim that our conditions imply that we must choose

$$\text{(i)} \quad I_\pm = \chi(I'_\pm), \qquad \text{(ii)} \quad \psi(x) = \sup\{y \in I' : \chi(y) \leq x\} \qquad (x \in I). \tag{A.8}$$

Indeed, $\chi(I'_-) = \lambda'_+(I'_-) - \lambda'_-(I'_-)$ is the rate of sell market orders minus the total rate of buy orders for the process $\mathcal{X}'$. Since ψ maps I into I' and $\mu'_\pm = \mu_\pm \circ \psi^{-1}$, this must equal the same quantity for the process $\mathcal{X}$, that is, $\lambda_+(I_-) - \lambda_-(I_-)$, which equals I_- by the assumption that $\mathcal{X}$ is in standard form. The same argument shows that $I_+ = \chi(I'_+)$. Let ρ be the restriction to I of $\mu_+ + \mu_-$ and let ρ' be similarly defined for the process $\mathcal{X}'$. Then ρ is the Lebesgue measure on I by the assumption that $\mathcal{X}$ is in standard form. Define $\psi^{-1} : I' \to [I_-, I_+)$ by

$$\psi^{-1}(y) := \inf\{x \in I : \psi(x) \geq y\} \qquad (y \in I'), \tag{A.9}$$

where the infimum of the empty set is $:= I_-$. Then, for $y \in I'$,

$$\psi^{-1}\big((I'_-, y)\big) := \{x \in I : \psi(x) \in (I'_-, y)\} = \big(I_-, \psi^{-1}(y)\big). \tag{A.10}$$

Letting $\chi(y-)$ denote the left-continuous modification of χ, using (A.8) (i) and the fact that $\rho' = \rho \circ \psi^{-1}$, it follows that

$$\chi(y-) = \chi(I'_-) + \rho'\big((I'_-, y)\big) = I_- + \rho\big((I'_-, \psi^{-1}(y))\big) = \psi^{-1}(y), \tag{A.11}$$

where in the last step we have used that ρ is the Legesgue measure. This proves that $\chi(y-)$ is the left-continuous inverse of ψ, and hence that ψ is the right-continuous inverse of χ, completing the proof of (A.8) (ii).

In particular, this shows that I and ψ are uniquely determined by our conditions. Conversely, choosing I and ψ as in (A.8), our arguments show that $\rho' = \rho \circ \psi^{-1}$ and that $\mathcal{X}$ and $\mathcal{X}'$ have the same rates of buy and sell market orders. Let $\rho_\pm$ denote the restrictions of $\mu_\pm$ to I, and let $\rho'_\pm$ be defined similarly for the process $\mathcal{X}'$. To complete the proof, it suffices to show that we can satisfy $\rho'_\pm = \rho_\pm \circ \psi^{-1}$ and (A.7). Let $p_\pm : I \to [0, 1]$ and $p'_\pm : I' \to [0, 1]$ be defined as the Radon–Nikodym derivatives

$$p_\pm := \frac{\mathrm{d}\rho_\pm}{\mathrm{d}\rho} \quad \text{and} \quad p'_\pm := \frac{\mathrm{d}\rho'_\pm}{\mathrm{d}\rho'}. \tag{A.12}$$

To specify $\rho_\pm$ in terms of ρ, it suffices to specify $p_\pm$. Letting $\chi(y-)$ and $\chi(y+)$ denote the left- and right-continuous modifications of χ, we have, for $y \in I$,

$$\chi(y+) - \chi(y) = \mu'_-(\{y\}) \quad \text{and} \quad \chi(y) - \chi(y-) = \mu'_+(\{y\}). \tag{A.13}$$

We will choose

$$p_\pm(x) := p'_\pm(y) \qquad \text{if } \psi(x) = y, \rho(\{y\}) = 0, \tag{A.14}$$

and

$$\left.\begin{array}{l} p_-(x) := 1_{(\chi(y), \chi(y+))} \\ p_+(x) := 1_{(\chi(y-), \chi(y))} \end{array}\right\} \qquad \text{if } \psi(x) = y, \rho(\{y\}) > 0. \tag{A.15}$$

In view of (A.13), choosing $p_\pm$ in this way ensures that $\rho'_\pm = \rho_\pm \circ \psi^{-1}$. Moreover, (A.15) guarantees that if ψ is constant on an interval of the form $[\chi(y-), \chi(y+))$, then buy orders are placed on the right of sell orders, so that (A.7) is satisfied. It is not hard to see that this is the only way to choose $\rho_\pm$ such that $\rho'_\pm = \rho_\pm \circ \psi^{-1}$ and (A.7) is satisfied.

Note that (A.4) simplifies to (A.7) if $\mathcal{X}_0$ is the empty initial state. In view of this and Lemma 19, our proof is complete if $\mathcal{X}'$ is started in the empty initial state. To prove the statement for more general initial states, we must show that if (A.6) holds, then we can choose $\mathcal{X}_0$ such that $\mathcal{X}_0 \circ \psi^{-1} = \mathcal{X}'_0$ and (A.4) is satisfied. We choose

$$\mathcal{X}_0^\pm := \mathcal{X}_0^{\pm'} \circ \chi_\mp^{-1}, \tag{A.16}$$

where χ_- and χ_+ denote the left- and right-continuous modifications of the function χ, respectively, that is, a buy (resp., sell) limit order of $\mathcal{X}'_0$ at the price y is represented in $\mathcal{X}_0$ by a buy (resp., sell) limit order at $\chi_+(y)$ [resp., $\chi_-(y)$].

Let $\psi(I)$ denote the image of I under ψ. Since $\chi_\pm$ are the left- and right-continuous inverses of ψ, we have $\chi_\pm(y) \in I$ and $\psi(\chi_\pm(y)) = y$ for all $y \in \psi(I)$ We claim that $\psi(I) = \mathrm{supp}_+(\mu'_- + \mu'_+) \cap I'$. Indeed, for any $y_1, y_2 \in I'$ with $y_1 < y_2$, one has $\psi(I) \cap [y_1, y_2) = \varnothing$ if and only if $\chi_-(y_1) = \chi_-(y_2)$. By (A.10) and (A.11),

$$\rho'([y_1, y_2)) = \rho([\chi(y_1-), \chi(y_2-))) = \chi(y_2-) - \chi(y_1-), \tag{A.17}$$

so $\psi(I) \cap [y_1, y_2) = \varnothing$ if and only if $\rho'([y_1, y_2)) = 0$. In view of this, (A.6) guarantees that $\mathcal{X}'_0 = \mathcal{X}_0 \circ \psi^{-1}$. [Indeed, the only reason why we need (A.6) is that the image of $\mathcal{X}_0$ under ψ can only contain limit orders in $\psi(I)$, which is $\mathrm{supp}_+(\mu'_- + \mu'_+) \cap I'$.]

To complete the proof, we must show that (A.7) and our choice of $\mathcal{X}_0$ imply (A.4). The only way (A.4) can fail is that there exists some $y \in I'$ such that $\chi_-(y) < \chi_+(y)$, and hence ψ is constant on the interval $J := [\chi_-(y), \chi_+(y))$, while

$$\int_J \nu_-(\mathrm{d}x) \int_J \nu_+(\mathrm{d}y) 1_{\{x<y\}} > 0. \tag{A.18}$$

Since we represent a buy (resp., sell) limit order of $\mathcal{X}_0'$ at the price y by a buy (resp., sell) limit order of $\mathcal{X}_0$ at $\chi_+(y)$ [resp., $\chi_-(y)$], the only atoms of $\mathcal{X}_0^+$ in the interval J must be located at $\chi_-(y)$ while $\mathcal{X}_0^-$ cannot have atoms in J. In view of this, (A.7) implies that (A.18) cannot hold. $\square$

A.2. Ergodicity of Markov chains. Let $(E, \mathcal{E})$ be a measurable space and let P be a measurable probability kernel on E. For simplicity, we assume that the one-point sets are measurable, that is., $\{x\} \in \mathcal{E}$ for all $x \in E$. It is known [17], Theorem 3.4.1 that for each probability measure μ on E there exists a Markov chain $X = (X_k)_{k\geq 0}$, unique in distribution, such that X_0 has law μ and the conditional law of X_{k+1} given $(X_0, \ldots, X_k)$ is given by P, for each $k \geq 0$. (This statement is not quite as straightforward as it may sound since for general measurable spaces, Kolmogorov's extension theorem is not available.)

We let P^k denote the kth power of P. For any measurable real function $f : E \to [-\infty, \infty]$, we write $P^k f(x) := \int_E P^k(x, \mathrm{d}y) f(y)$, as long as the integral is well defined for all $x \in E$. For any probability measure μ on E we let $\mu P^k(A) := \int_E \mu(\mathrm{d}x) P^k(x, A)$ $(A \in \mathcal{E})$. Then μP^k is the law of X_k if X_0 has law μ. An invariant law of X is a probability measure ν such that $\nu P = \nu$. We let $\|\mu - \nu\|$ denote the total variation norm distance between two probability measures μ and ν.

For any point $x \in E$, let $\mathbb{P}^x$ denote the law of the Markov chain X started from $X_0 = x$. Let $\tau_x := \inf\{k > 0 : X_k = x\}$ denote the first return time to x. We say that the Markov chain X is *aperiodic from* x if the greatest common divisor of $\{k > 0 : \mathbb{P}^x[\tau_x = k] > 0\}$ is one.

Markov chains satisfying the conditions (A.19) and (A.20) below behave in many ways like positive recurrent Markov chains with countable state space. In particular, (A.19) says that f is a Lyapunov function that guarantees that the return times to the set C have finite expectation, while (A.20) says that once the chain enters C, there is a uniformly positive probability of entering the atom 0 after a finite number of steps.

PROPOSITION 21 (Ergodicity for positive point recurrent chain). *Fix a point* $0 \in E$. *Assume that there exists a measurable function* $f : E \to [0, \infty)$, *a measurable set* $0 \in C \subset E$ *and constants* $F, K < \infty$ *such that* $\sup_{x\in C} f(x) \leq F$ *and*

$$Pf - f \leq K 1_C - 1. \tag{A.19}$$

Assume moreover that there exist constants $\varepsilon > 0$ *and* $k \geq 0$ *such that*

$$\mathbb{P}^x[\tau_0 \leq k] \geq \varepsilon \qquad (x \in C). \tag{A.20}$$

Then $\mathbb{E}^x[\tau_0] < \infty$ *for all* $x \in E$, *and the Markov chain* X *has a unique invariant law* ν. *If moreover* X *is aperiodic from* 0, *then*

$$\|\mu P^n - \nu\| \underset{n\to\infty}{\longrightarrow} 0 \tag{A.21}$$

for each probability measure μ *on* E.

PROOF. Let $\tau_C := \inf\{k \geq 1 : X_k \in C\}$ denote the first entry time of C. Then [17], Theorem 11.3.4 tells us that $\mathbb{E}^x[\tau_C] \leq f(x) + K 1_C(x)$ $(x \in E)$. Since after each visit to C, by (A.20) there is a probability of at least ε to visit 0 in the next k steps, it is not hard to deduce that $\mathbb{E}^x[\tau_0] < \infty$ for all $x \in E$. Again by [17], Theorem 11.3.4 and the fact that, in the light of (A.20), C is petite as defined in [17], Section 5.5.2, we have that X is positive Harris recurrent. In particular, by [17], Theorem 10.0.1, X has a unique invariant law ν. Since $\mathbb{E}^x[\tau_0] < \infty$ for all $x \in E$, it is easy to see that $\nu(\{0\}) > 0$. By [17], Theorem 10.4.9, ν is equivalent to the measure ψ from [17], Proposition 4.2.2, so aperiodicity from 0 as we have defined it implies ψ-aperiodicity as defined in [17], Section 5.4.3. Now (A.21) follows from [17], Theorem 13.3.3. □

A.3. Discrete models. Often, it is natural to consider Stigler–Luckock models where the interval I is of the form $I = [0, n]$, with $n \geq 2$ an integer, and the measures $\mu_\pm$ that determine the rate at which orders arrive are supported on the set of integers $\{0, \dots, n\}$. One motivation for this is that real prices take values that differ by a minimal amount, the so-called tick size. Also, the numerical data for the uniform model shown in Figures 2 and 3 are based on approximation with discrete models with a high value of n.

Although, in the light of Appendix A.1, discrete Stigler–Luckock models are in principle included in our general analysis, in practice, when doing (numerical) calculations, it is more convenient to replace the differential equations for the general model by difference equations. It turns out that these difference equations can be solved explicitly much in the same way as the differential equations of the general model.

In the discrete setting, it is convenient to reparametrize the model somewhat. We replace the set $\{1, 2, \dots, n\}$ of possible prices of buy orders by $\{4, 6, \dots, 2n + 2\}$ and we let 2 (instead of 0) be the value of $M_-(\mathcal{X})$ that signifies that the order book contains no buy limit orders. Likewise, for sell orders or $M_+(\mathcal{X})$, we replace the set of possible prices $\{0, 1, \dots, n\}$ by $\{1, 3, \dots, 2n + 1\}$. Note that in this new parametrization, a buy and sell order that were previously both placed at the price k are now placed at the prices $2k + 2$ and $2k + 1$, respectively, and hence still match. We let $\mathcal{X}_t^-(2k + 2)$ [resp., $\mathcal{X}_t^+(2k + 1)$] denote the number of buy (resp., sell) limit orders in the order book at a given time and price. We define demand and supply functions

$$\text{(A.22)} \qquad \lambda_- : \{3, 5, \dots, 2n + 1\} \to \mathbb{R} \quad \text{and} \quad \lambda_+ : \{2, 4, \dots, 2n\} \to \mathbb{R}$$

in such a way that $\lambda_-(2k + 1)$ [resp., $\lambda_+(2k)$] is the total rate at which buy (resp., sell) orders are placed at prices in $\{2k + 2, \dots, 2n + 2\}$ (resp., $\{1, \dots, 2k - 1\}$). In particular, $\lambda_-(2n + 1)$ and $\lambda_+(2)$ are the rates of buy and sell market orders, respectively.

For any function of the form $f : \{k, k+2, \dots, m\} \to \mathbb{R}$, we define a discrete derivative $\mathrm{d}f : \{k+1, k+3, \dots, m-1\} \to \mathbb{R}$ by

$$\mathrm{d}f(x) := f(x+1) - f(x-1). \tag{A.23}$$

For sums over sets of the form $\{k, k+2, \dots, m\}$, we use the shorthand

$$\sum_k^m g := \sum_{x \in \{k, k+2, \dots, m\}} g(x), \tag{A.24}$$

and we define $\sum_k^{k-2} g := 0$. We let

$$f'(x) := f(x+1) \quad \text{and} \quad f^*(x) := f(x-1) \tag{A.25}$$

denote the function f shifted by one to the left or right, respectively. It is straightforward to prove the product rule

$$\mathrm{d}(fg) = f' \,\mathrm{d}g + g^* \,\mathrm{d}f. \tag{A.26}$$

We also have the following special case of the chain rule:

$$\mathrm{d}\Big(\frac{1}{f}\Big) = -\frac{\mathrm{d}f}{f' f^*}. \tag{A.27}$$

Let

$$w_- : \{2, 4, \dots, 2n\} \to \mathbb{R} \quad \text{and} \quad w_+ : \{3, 5, \dots, 2n+1\} \to \mathbb{R} \tag{A.28}$$

be weight functions satisfying $w_-(2) := 0$ and $w_+(2n+1) := 0$, and define a linear function F by

$$F(\mathcal{X}) := \sum_{x=4}^{2n} w_-(x)\mathcal{X}^-(x) + \sum_{x=3}^{2n-1} w_+(x)\mathcal{X}^+(x). \tag{A.29}$$

Then, in analogy with Lemma 8, one can check that

$$\begin{aligned} GF(\mathcal{X}) = {} & \sum_{M_-(\mathcal{X})+1}^{2n-1} w_+ \,\mathrm{d}\lambda_+ - w_-\big(M_-(\mathcal{X})\big)\lambda_+\big(M_-(\mathcal{X})\big) \\ & - \sum_4^{M_+(\mathcal{X})-1} w_- \,\mathrm{d}\lambda_- - w_+\big(M_+(\mathcal{X})\big)\lambda_-\big(M_+(\mathcal{X})\big). \end{aligned} \tag{A.30}$$

In analogy with Theorem 9, one can show that if the rates of market orders $\lambda_-(2n+1)$ and $\lambda_+(2)$ are both positive, then, for each $z \in \{2, 4, \dots, 2n\}$, there exist a unique pair of weight functions $(w_-^{z,+}, w_+^{z,+}) = (w_-, w_+)$ and a unique constant $f_+(z) \in \mathbb{R}$, such that the linear functional $F^{z,+} = F$ from (A.29) satisfies

$$GF(\mathcal{X}) = 1_{\{M_+(\mathcal{X}) > z\}} - f_+(z). \tag{A.31}$$

Also, for each $z \in \{3,5,\dots,2n+1\}$, there exist a unique pair of weight functions $(w_-^{z,-}, w_+^{z,-}) = (w_-, w_+)$ and constant $f_-(z)$ such that the linear functional $F^{z,-} = F$ from (A.29) satisfies

$$GF(\mathcal{X}) = 1_{\{M_-(\mathcal{X})<z\}} - f_-(z). \tag{A.32}$$

It is possible to derive nice, explicit formulas for these weight functions. In analogy with (2.19), define

$$\begin{aligned}\Gamma &:= \frac{1}{\lambda_-(2n+1)\lambda_+(2n)} - \sum_{3}^{2n-1} \frac{1}{\lambda_-}\,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big)\\ &= \frac{1}{\lambda_-(3)\lambda_+(2)} + \sum_{4}^{2n} \frac{1}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big),\end{aligned} \tag{A.33}$$

where the equality of both formulas follows from the product rule (A.26) applied to the functions $1/\lambda'_-$ and $1/\lambda_+$. Set $I_0 := \{2,4,\dots,2n\}$ and $I_1 := \{3,5,\dots,2n+1\}$. In analogy with (2.50), define

$$\begin{aligned} u_{-+}(x) &:= \Gamma^{-1}\Big\{\frac{1}{\lambda_-(2n+1)\lambda_+(2n)} - \sum_{x+1}^{2n-1} \frac{1}{\lambda_-}\,\mathrm{d}\Big(\frac{1}{\lambda_+}\Big)\Big\} \qquad (x \in I_0),\\ u_{+-}(x) &:= \Gamma^{-1}\Big\{\frac{1}{\lambda_-(3)\lambda_+(2)} + \sum_{4}^{x-1} \frac{1}{\lambda_+}\,\mathrm{d}\Big(\frac{1}{\lambda_-}\Big)\Big\} \qquad (x \in I_1).\end{aligned} \tag{A.34}$$

Then, in analogy with Lemma 16, one has

$$\begin{aligned} w_+^{z,-}(x) &= \lambda_-(z)\Gamma\big[u_{+-}(x \vee z) - 1\big]u_{+-}(x \wedge z) \qquad (x,z \in I_1),\\ w_-^{z,-}(x) &= \lambda_-(z)\Gamma\big(u_{+-}(z) - 1_{\{x<z\}}\big)\big(u_{-+}(x) - 1_{\{x<z\}}\big)\\ &\qquad (x \in I_0, z \in I_1),\\ w_-^{z,+}(x) &= \lambda_+(z)\Gamma\big[u_{-+}(x \wedge z) - 1\big]u_{-+}(x \vee z) \qquad (x,z \in I_0),\\ w_+^{z,+}(x) &= \lambda_+(z)\Gamma\big(u_{-+}(z) - 1_{\{x>z\}}\big)\big(u_{+-}(x) - 1_{\{x>z\}}\big)\\ &\qquad (x \in I_1, z \in I_0).\end{aligned} \tag{A.35}$$

Moreover, the functions

$$f_- : \{3,5,\dots,2n+1\} \to \mathbb{R} \quad \text{and} \quad f_+ : \{2,6,\dots,2n\} \to \mathbb{R} \tag{A.36}$$

from (A.31) and (A.32) satisfy the discrete version of Luckock's equation, which reads

$$\begin{aligned} &\text{(i)} \quad f_-\,\mathrm{d}\lambda_+ = -\lambda_-\,\mathrm{d}f_+ \qquad \text{on } \{3,5,\dots,2n-1\},\\ &\text{(ii)} \quad f_+\,\mathrm{d}\lambda_- = -\lambda_+\,\mathrm{d}f_- \qquad \text{on } \{4,6,\dots,2n\},\\ &\text{(iii)} \quad f_+(2) = 1 = f_-(2n+1).\end{aligned} \tag{A.37}$$

The solution to this equation can explicitly be written as

$$
\text{(A.38)}\qquad
\begin{aligned}
&\text{(i)} && \Big(\frac{f_+}{\lambda_+}\Big)(x) = \frac{1}{\lambda_+(2)} + \sum_{3}^{x-1} \frac{\kappa}{\lambda_-} \mathrm{d}\Big(\frac{1}{\lambda_+}\Big) && (x \in I_0),\\
&\text{(ii)} && \Big(\frac{f_-}{\lambda_-}\Big)(x) = \frac{1}{\lambda_-(2n+1)} - \sum_{x+1}^{2n} \frac{\kappa}{\lambda_+} \mathrm{d}\Big(\frac{1}{\lambda_-}\Big) && (x \in I_1),
\end{aligned}
$$

where κ is given by

$$
\kappa = \kappa_{\mathrm{L}} := \Gamma^{-1}\Big(\frac{1}{\lambda_-(2n+1)} + \frac{1}{\lambda_+(2)}\Big), \tag{A.39}
$$

and $\Gamma > 0$ is the constant from (A.33).

A.4. Suggestions for future work. Several open problems concerning Stigler–Luckock models remain. In particular, these include:

I. If a Stigler–Luckock model has a competitive window $(J_-^{\mathrm{c}}, J_+^{\mathrm{c}})$, then show that in the long run, buy orders below J_-^{c} and sell orders above J_+^{c} are never matched.

II. Show that all orders inside the competitive window are eventually matched.

III. Assuming (A3) and (A6), if the solution to Luckock's equation satisfies $f_-(I_-) \wedge f_+(I_+) = 0$, then show that there is an invariant law on $\mathcal{S}_{\mathrm{ord}}$.

Under certain technical assumptions, Problems I and II have been solved in [8], but Problem III remains completely open. In view of this, and also since the methods of [8] fail for certain extensions of the model such as the one discussed in [18], let us investigate how our methods could possibly be applied to these problems.

Let $(X_k)_{k\geq 0}$ be a Stigler–Luckock model on an interval I and let $X_k|_Z$ be the restriction of X_k to a subinterval $Z = (z_-, z_+) \subset I$. (Note that this is not what we have called the restricted model on Z; in particular, the latter is a Markov chain, while $X_k|_Z$ is not.) To solve Problem I, one would need to show that if Z is slightly larger than the competitive window, then the process $X_k|_Z$ is transient, in a suitable sense, while Problems II and III could be solved if one could show that if Z is slightly smaller than the competitive window, then the process $X_k|_Z$ spends a positive time in the empty state, with some uniform bounds on the expected number of buy and sell orders in Z.

In this context, it is natural to look at linear functions F as in (1.24) such that the weight functions $w_\pm$ are supported on $\overline{Z}$ and $GF(\mathcal{X})$ depends only on the relative order of $M_\pm(\mathcal{X})$ and $z_\pm$. It appears that such weight functions exist and form a two-dimensional space. Using notation as in Theorem 9, let us define

$$
\hat{w}_\pm := w^{z_-,-} + w^{z_+,+} \quad \text{and} \quad \dot{w}_\pm := w^{z_+,-} + w^{z_-,+}. \tag{A.40}
$$

Then it appears that there exists a unique constant $c \in \mathbb{R}$ such that

$$\overline{w}_\pm := \hat{w}_\pm + c\dot{w}_\pm \tag{A.41}$$

are supported on $\overline{Z}$. Moreover, it seems that the two-dimensional space we just mentioned is spanned by the "symmetric" weight functions $(\overline{w}_-, \overline{w}_+)$ and the "asymmetric" weight functions (w^*_-, w^*_+) defined as

$$w^*_- := -1_{(z_-, z_+]} \quad \text{and} \quad w^*_+ := 1_{[z_-, z_+)}. \tag{A.42}$$

Letting $\overline{F}$ and F^* denote the corresponding linear functions, a natural way to attack Problem I is to show that if the competitive window J^c satisfies $\overline{J}^c \subset Z$, then there exists a function $h(\overline{F}, F^*)$ that is subharmonic for the generator G in (1.23) and that shows that $\overline{F}(\mathcal{X}_t) \to \infty$ a.s. in such a way that $|F^*(\mathcal{X}_t)| \ll \overline{F}(\mathcal{X}_t)$.

Also, a natural way to attack Problems II and III is to find a "Lyapunov style" function V that depends on $\overline{F}$, F^*, and perhaps some other functions of the process, and that solves an inequality of the form (3.6).

To conclude the paper, we mention a few more open problems, for which we have nothing more concrete to say.

IV. Show that the invariant law from Problem III is unique and the long-time limit law started from any initial law.

V. Investigate existence and uniqueness of solutions to Luckock's equation with assumption (A6) replaced by the weaker (A5) plus perhaps some conditions involving the function Φ from (1.20).

VI. Investigate whether the restricted model on the competitive window is null recurrent or transient.

VII. Prove a limit theorem for the shape of the stationary process near the boundary of the competitive window, in the spirit of [5].

VIII. For the model on the competitive window, investigate the tail of the distribution of the time till a limit order is matched.

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ÚTIA AV ČR
POD VODÁRENSKOU VĚŽÍ 4
18208 PRAHA 8
CZECH REPUBLIC
E-MAIL: swart@utia.cas.cz