# Two self-reinforced processes with self-organized criticality

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**Abstract**

We introduce two self-reinforced point processes on the unit interval that appear to exhibit self-organized criticality, somewhat reminiscent of the well-known Bak Sneppen model. The first process takes values in the finite subsets of the unit interval and evolves according to the following rules. In each time step, a particle is added at a uniformly chosen position, independent of the particles that are already present. If there are any particles to the left of the newly arrived particle, then the left-most of these is removed. We show that all particles arriving to the left of $p_{\rm c} = 1 - e^{-1}$ are a.s. eventually removed, while for large enough time, particles arriving to the right of $p_{\rm c}$ stay in the system forever. Our second process of interest has particles of two types and models traders placing buy and sell limit orders at a stock market. The behavior of this process appears to be similar to the previous one, but we can prove only partial results for it.

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## Contents

# 1 Introduction and results

## 1.1 A model for canyon formation

Let $(U_k)_{k\geq 1}$ be an i.i.d. collection of uniformly distributed $[0,1]$-valued random variables. For each finite subset $x$ of $[0,1]$, we inductively define a sequence $X^x = (X^x_k)_{k\geq 0}$ of random finite subsets of $[0,1]$ by $X^x_0 := x$, $M^x_{k-1} := \min(X^x_{k-1} \cup \{1\})$ and

$$X^x_k := \begin{cases} X^x_{k-1} \cup \{U_k\} & \text{if } U_k < M^x_{k-1}, \\ (X^x_{k-1} \cup \{U_k\}) \backslash \{M^x_{k-1}\} & \text{if } U_k > M^x_{k-1}. \end{cases} \qquad (k \geq 1). \tag{1}$$

In words, this says that $M^x_{k-1}$ is the minimal element of $X^x_{k-1}$ and that the set $X^x_k$ is constructed from $X^x_{k-1}$ by adding $U_k$, and in case that $M^x_{k-1} < U_k$, removing $M^x_{k-1}$ from $X^x_{k-1}$. Since the $(U_k)_{k\geq 1}$ are i.i.d. and $X^x_k$ is a function of $X^x_{k-1}$ and $U_k$, it is clear that $X^x$ is a Markov chain. (In fact, we have just given a *random mapping representation* for it.) The state space of $X^x$ is the set $\mathcal{P}_{\rm fin}[0,1]$ of all finite subsets of $[0,1]$, which is naturally isomorphic to the space of all simple counting measures on $[0,1]$ (i.e., $\mathbb{N}$-valued measures $\nu$ such that $\nu(\{x\}) \leq 1$ for all $x \in [0,1]$). We equip this space with the topology of weak convergence and the associated Borel-$\sigma$-algebra.

The process $X^x$ is an example of a Markov process with self-reinforcement (compare [Pem07]), since the number of particles in the system can grow without bounds and influences the fate of newly arrived particles. As we will see in a moment, it also appears to exhibit self-organized criticality in a way that is reminiscent of the well-known Bak Sneppen model. The empirical distribution function $F^x_k(q) := \big|X^x_k \cap [0,q]\big|$ can loosely be interpreted as the profile of a canyon being cut out by a river. If $U_k < M^x_{k-1}$, then the river cuts deeper into the rock. If $U_k > M^x_{k-1}$, then the slope of the canyon between $U_k$ and the river is eroded one step down.

Our first result says that particles arriving on the left of the critical point $p_{\rm c} := 1 - e^{-1}$ are eventually removed from the system, but for large enough time, particles arriving on the right of $p_{\rm c}$ stay in the system forever.

**Theorem 1 (A.s. behavior of the minimum)** *For any finite $x \subset [0,1]$, one has*

$$\limsup_{k\to\infty} M^x_k = 1 - e^{-1} \quad \text{a.s.} \tag{2}$$

To understand Theorem 1 better, note that for each $0 \leq q \leq 1$, the restriction $X^x_k \cap [0,q]$ of $X^x_k$ to $[0,q]$ is a Markov chain. Indeed, particles arriving on the right of $q$ just have the effect that in each time step, with probability $1-q$, the minimal element of $X^x_k \cap [0,q]$, if there is one, is removed, while no new particles are added inside $[0,q]$. Theorem 1 says that this Markov chain is recurrent for $q < p_{\rm c}$ and transient for $q > p_{\rm c}$. For any $q \in [0,1]$, let

$$\tau^\emptyset_q := \inf\{k > 0 : X^\emptyset_k \cap [0,q] = \emptyset\} \tag{3}$$

be the first time the restricted process $X^\emptyset_k \cap [0,q]$ returns to the empty set. Our next theorem shows that for $q < p_{\rm c}$, the restricted chain is positively recurrent and ergodic. Below, we call a subset of $[0,p_{\rm c})$ *locally finite* if its intersection with any compact subset of $[0,p_{\rm c})$ is finite.

**Theorem 2 (Ergodicity of restricted process)** *Let $p_{\rm c} := 1 - e^{-1}$. Then*

$$\mathbb{E}[\tau^\emptyset_q] = \big(1 + \log(1-q)\big)^{-1} \quad (q < p_{\rm c}) \quad \textit{and} \quad \mathbb{P}[\tau^\emptyset_q = \infty] > 0 \quad (q > p_{\rm c}). \tag{4}$$

*Moreover, there exists a random, locally finite subset $X_\infty \subset [0, p_c)$ such that, regardless of the initial state $x$,*

$$\mathbb{P}\big[X_k^x \cap [0,q] \in \cdot\,\big] \underset{k\to\infty}{\longrightarrow} \mathbb{P}\big[X_\infty \cap [0,q] \in \cdot\,\big] \qquad (0 < q < p_c), \tag{5}$$

*where $\to$ denotes convergence of probability measures in total variation norm distance. The random point set $X_\infty$ a.s. consists of infinitely many points.*

Numerical simulations strongly suggest that at $q = p_c$, the restricted chain $X_k^x \cap [0,q]$ is null recurrent and, starting from a state with no particles on the left on $q$, the probability that one has to wait longer than $k$ steps before the area on the left of $q$ is again empty decays as $k^{-1/2}$, but we have no proof for this. Note that such a proof would establish self-organized criticality for our process. Our process is self-organized in the sense that it finds the transition point $p_c$ by itself. In particular, one does not have to tune a parameter of the model to exactly the right value to see the (presumed) power-law critical behavior at $p_c$.

### 1.2 Plačková's model

As mentioned in the abstract, our second process of interest models traders placing buy and sell limit orders at a stock market. This process was developed (in a discrete-space setting) by Jana Plačková in her master thesis [Pla11]. Its formulation is as follows. Traders arrive one after the other at a stock market. Each trader with equal probabilities either wants to buy or sell one item of a certain good and has a uniformly distributed price that is the maximum price she wants to spend or the minimum price she wants to obtain for the item. If the trader sees a suitable offer in the order book, she takes the best available order. Otherwise, she places a buy or sell limit order at her price in the order book.

Mathematically, this translates into a Markov chain $(L_k, R_k)_{k\geq 0}$ taking values in the space of all pairs $(L, R)$ of finite subsets of $[0, 1]$ with the property that $x < y$ for all $x \in L$ and $y \in R$. Let $(U_k)_{k\geq 0}$ be i.i.d. and uniformly distributed on $[0, 1]$ and let $(B_k)_{k\geq 1}$ be i.i.d. and uniformly distributed on $\{-1, +1\}$. Starting with an initial state $(L_0, R_0)$ in the space we have just described, the Markov chain $(L_k, R_k)_{k\geq 0}$ is inductively defined by the following rules.

$$(L_k, R_k) := \begin{cases} (L_{k-1} \cup \{U_k\}, R_{k-1}) & \text{if } B_k = -1,\ U_k < M_k^R, \\ (L_{k-1}, R_{k-1}\backslash\{M_k^R\}) & \text{if } B_k = -1,\ M_k^R < U_k, \\ (L_{k-1}\backslash\{M_k^L\}, R_{k-1}) & \text{if } B_k = +1,\ U_k < M_k^L, \\ (L_{k-1}, R_{k-1} \cup \{U_k\}) & \text{if } B_k = +1,\ M_k^L < U_k. \end{cases} \qquad (k \geq 1), \tag{6}$$

where $M_k^L := \sup(L_k \cup \{0\})$ and $M_k^R := \inf(R_k \cup \{1\})$ $(k \geq 0)$. In the interpretation above, $L_k$ and $R_k$ describe the buy and sell orders present in the order book after the $k$-th trader has arrived, and $M_k^L$ and $M_k^R$ are the highest buy and lowest ask price, respectively. Note that our rules preserve the property that no point of $L$ lies on the right of a point of $R$.

Numerical simulations suggest that there exists a critical point $q_c \approx 0.2177(2)$ such that regardless of the initial state,

$$\liminf_{k\to\infty} M_k^L = \liminf_{k\to\infty} M_k^R = q_c \quad \text{and} \quad \limsup_{k\to\infty} M_k^L = \limsup_{k\to\infty} M_k^R = 1 - q_c \quad \text{a.s.}, \tag{7}$$

while for each $q_c < q < 1 - q_c$, both $M_k^L$ and $M_k^R$ spend a positive fraction of time on either side of $q$. While we cannot prove this, we can prove a theorem that features a constant $q_c$ that is in good numerical agreement with the constant from (7) and that we conjecture to be the same.

To explain this, we need to introduce a cut-off process. The difficulty of Plačková's model compared to the canyon model from the previous section is that its restriction to a subinterval of $[0, 1]$ is no longer Markovian. Nevertheless, it is possible to define a cut-off process that plays in many ways a similar role.

We fix $0 \leq q_- < q_+ \leq 1$ and define a Markov chain taking values in the space of all pairs $(L, R)$ of subsets of $[q_-, q_+]$ with the properties:

$$\begin{array}{llll} \text{(i)} & q_- \in L, & \text{(ii)} & L \cap (q_-, q_+] \text{ is locally finite,} \\ \text{(iii)} & q_+ \in R, & \text{(iv)} & R \cap [q_-, q_+) \text{ is locally finite,} \\ \text{(iv)} & x < y \text{ for all } x \in L \text{ and } y \in R. & & \end{array} \tag{8}$$

Starting with any $(L_0, R_0)$ in this space, letting $(U_k)_{k\geq 0}$ and $(B_k)_{k\geq 1}$ be as before, the cut-off Markov chain $(L_k, R_k)_{k\geq 0}$ is inductively defined for $k \geq 1$ by the following rules (compare (6)).

$$(L_k, R_k) := \begin{cases} \big((L_{k-1} \cup \{U_k\})\backslash[0, q_-), R_{k-1}\big) & \text{if } B_k = -1,\ U_k < M_k^R, \\ \big(L_{k-1}, (R_{k-1}\backslash\{M_k^R\}) \cup \{q_+\}\big) & \text{if } B_k = -1,\ M_k^R < U_k, \\ \big((L_{k-1}\backslash\{M_k^L\}) \cup \{q_-\}, R_{k-1}\big) & \text{if } B_k = +1,\ U_k < M_k^L, \\ \big(L_{k-1}, (R_{k-1} \cup \{U_k\})\backslash(q_+, 1]\big) & \text{if } B_k = +1,\ M_k^L < U_k. \end{cases} \tag{9}$$

where $M_k^L := \sup(L_k)$ and $M_k^R := \inf(R_k)$ $(k \geq 0)$. Note that $(L_{k-1} \cup \{U_k\})\backslash[0, q_-)$ means that we *first* add $\{U_k\}$ to the set $L_{k-1}$, and *then* subtract $[0, q_-)$; the effect of this is that if $U_k < q_-$, then we do nothing. Likewise, $(L_{k-1}\backslash\{M_k^L\}) \cup \{q_-\}$ means that we *first* subtract $\{M_k^L\}$ and *then* add $\{q_-\}$; the effect of this is that if $M_k^L = q_-$, then we do nothing.

In particular, if $q_- = 0$ and $q_+ = 1$, then this is the same process as in (6), except that we have added the point 0 to each $L_k$ and the point 1 to each $R_k$. In general, we can think of the points $q_- \in L_k$ and $q_+ \in R_k$ as "immortal" points that cannot be removed. Contrary to the canyon model from the previous section, there seems to be no easy way to compare such a cut-off process with the original process.

We observe that if $(L_k, R_k)_{k\geq 0}$ are defined in terms of $(U_k)_{k\geq 1}$ and $(B_k)_{k\geq 1}$ as in (9), then the joint process $(L_k, R_k, U_k, B_k)_{k\geq 1}$ is in fact a Markov chain. We will be interested in invariant laws of this Markov chain.

**Theorem 3 (Stationary cut-off process)** *Assume that for some $0 \leq q_- < q_+ \leq 1$ the cut-off process in (9) has an invariant law, and let $(L_k, R_k, U_k, B_k)_{k\in\mathbb{Z}}$ be the corresponding stationary process. Assume moreover that for this stationary process*

$$\mathbb{P}[M_k^L = q_-] = 0 \quad \textit{and} \quad \mathbb{P}[M_k^R = q_+] = 0 \qquad (k \in \mathbb{Z}). \tag{10}$$

*Then $q_- = q_\mathrm{c}$ and $q_+ = 1 - q_\mathrm{c}$, where $q_\mathrm{c} := 1 + 1/z$ with $z$ the unique solution of the equation $1 + z + e^z = 0$.*

Numerically, the constant $q_\mathrm{c}$ from Theorem 3 is given by $q_\mathrm{c} \approx 0.21781170571980$, in good agreement with the numerical value for the constant in (7). The conditions (10) say that the boundaries $q_-$ and $q_+$ are natural in the sense that the process never "tries to remove" the immortal particles at $q_-$ and $q_+$, i.e., the difference between the cut-off rules (9) and the original rules (6) never comes into play. Note that if one believes in formula (7) for *some* value of $q_c$, then it is quite natural to expect that in analogy with Theorem 2, the restriction of the process to $(q_c, 1 - q_c)$ converges to an invariant law for a cut-off process with such natural boundaries.

### 1.3 Discussion

Our models, and the first one in particular, are similar to the well-known Bak Sneppen model [BS93], which is one of the best-known models exhibiting self-organized criticality, although this is only been fully rigorously established for a simplified version of the model [MS12]. Like our first process, the Bak Sneppen model (and its modifications) is also based on the principle that the particle with the lowest value is killed. This rule alone, however, is not enough to see interesting behavior.

In our first process, we add particles one by one and also kill the particle with the lowest value, but only if this is not the newly arrived particle. In this way, the total population is allowed to grow and the process takes the limit of large population size by itself, so to say. The second process is similar, except that particles heap up at both ends of the unit interval.

In the Bak Sneppen model, the total number of particles is fixed, and when a particle is killed, not only this particle, but also some of its neighbors (according to some additional structure) are killed, and the killed particles are replaced by new particles with uniformly chosen values. The original Bak Sneppen model and its modifications differ in the way these "neighbors" are chosen. In the original model, the particles are numbered $0, \ldots, N-1$ and their neighbors are those with neighboring numbers (modulo $N$). In the modified model from [MS12], the neighbors are uniformly chosen from the population.

Closely related to the Bak Sneppen model is the Barabási queueing system introduced in [Bar05], which has so far been studied only in the physics literature. Exact results for this model have been derived in [Vaz05, Ant09]. In the original model, items in a queue have a priority taking values in a continuous interval. In each time step, with probability $p$ close to one, the item with the highest priority is served, and with the remaining probability a random item is removed from the list. At the end of each step, a new item is added so that the length of the queue remains constant.

In [CG09], this latter assumption is dropped and the number of items added in each time step is assumed to be larger than one, with the result that some items never get served and the length of the queue grows without bounds, in a way that is very similar to our first model. They show that their model can be mapped to invasion percolation on a tree. A similar mapping also exists for Barabási's original model [CG07]. Contrary to our models, the critical point for the model in [CG09] is trivial since the number of items added and removed in each step is known.

Somewhat similar in spirit to these models is also the model [GMS11], which is basically a supercitical branching process in which fitnesses are assigned to the particles and those killed have the lowest fitness.

We note that in the construction of all these processes and in particular also ours, only the relative order of the points (priorities) matters, so replacing the uniformly distribution on $[0, 1]$ by any other atomless law on $\mathbb{R}$ yields the same model up to a continuous transformation of space. Starting from the empty initial state, adding points one by one, one arrives after $k$ steps at a random permutation of $k$ elements. Our quantities of interest may thus be described as functions of such a random permutation. This is somewhat reminiscent of the way the authors of [AD99] use what they call Hammersley's process to study the longest increasing subsequence of a random permutation. There is an extensive literature on functions of random permutations, but none of those studied so far seem relevant for our processes.

# 2 Main idea of the proofs

## 2.1 The canyon model

In the present section, we describe the main idea of the proof of Theorems 1 and 2, and in Section 2.2 below, we explain the main idea behind Theorem 3.

As already mentioned in Section 1.3, by a simple transformation of space, we may replace the uniformly distributed random variables $(U_k)_{k\geq 1}$ by real random variables having any non-atomic distribution. At present, it will be more convenient to work with exponentially distributed random variables with mean one, so we transform the unit interval $[0,1]$ into the closed halfline $[0,\infty]$ with the transformation $q \mapsto f(q) := -\log(1-q)$, set $\sigma_k := f(U_k)$ $(k\geq 1)$, and, concentrating for the moment on the process started in the empty initial state, we let $Y_k := f(X^{\emptyset}_k)$ denote the image of $X^{\emptyset}_k$ under $f$. Then

$$Y_k := \begin{cases} Y_{k-1}\cup\{\sigma_k\} & \text{if } \sigma_k < N_{k-1}, \\ (Y_{k-1}\cup\{\sigma_k\})\backslash\{N_{k-1}\} & \text{if } \sigma_k > N_{k-1}. \end{cases} \qquad (k\geq 1), \tag{11}$$

where $N_k := \min(Y_k\cup\{\infty\})$. Let

$$F_t(k) := \big|Y^{(t)}_k\big| \quad \text{with} \quad Y^{(t)}_k := Y_k\cap[0,t] \qquad (k\geq 0,\ t\geq 0) \tag{12}$$

denote the number of points on the left of $t$.

We claim that the function-valued process $(F_t)_{t\geq 0}$ with $F_t = (F_t(k))_{k\geq 0}$ is a continuous-time Markov processes, where the parameter $t$ plays the role of time. Indeed, at each time $t=\sigma_k$, let $F_{t-} := \big|Y_k\cap[0,t)\big|$ denote the state immediately prior to $t$ and let

$$\kappa_t(k) := \inf\{k' > k : F_{t-}(k'-1) = 0 \text{ and } \sigma_{k'} \geq t\}, \tag{13}$$

with the convention that $\inf\emptyset := \infty$. Then at the time $t=\sigma_k$, the function $F_t$ changes as

$$F_t(k') = \begin{cases} F_{t-}(k')+1 & \text{if } k\leq k' < \kappa_t(k), \\ F_{t-}(k') & \text{otherwise} \end{cases} \qquad (k'\geq 0). \tag{14}$$

In the language of self-organized criticality, we may call such a move an avelange. In analogy with (3), let

$$\tilde\tau_t := \inf\{k>0 : Y^{(t)}_k = \emptyset\} \tag{15}$$

denote the first time the restricted process $Y^{(t)}$ returns to the empty set. At each (deterministic) $t\geq 0$, the function $F_t$ starts in $F_t(0)=0$ and makes i.i.d. excursions away from 0 whose length is distributed as $\tilde\tau_t$.

We will be interested in the quantity

$$\Delta F_t(k) := \begin{cases} \underline{0} & \text{if } F_t(k) = F_t(k-1) = 0, \\ \overline{0} & \text{if } F_t(k) = F_t(k-1) > 0, \\ -1 & \text{if } F_t(k) = F_t(k-1)-1, \\ +1 & \text{if } F_t(k) = F_t(k-1)+1. \end{cases} \qquad (k\geq 1,\ t\geq 0). \tag{16}$$

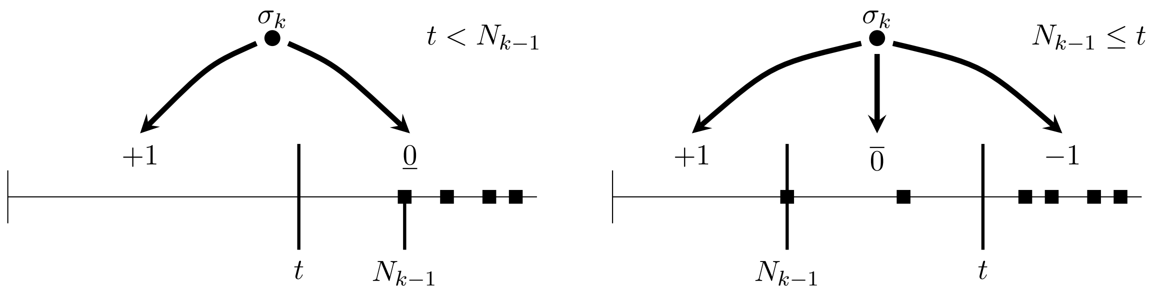


Figure 1: Illustration of the quantity $\Delta F_t(k)$ from (16). The value of $\Delta F_t(k)$ depends on the relative order of $t$, $N_{k-1}$, and $\sigma_k$.

It follows from (14) that $\Delta F_t$, too, evolves in a Markovian way as a function of $t$. For each $k \geq 1$, at time $t = \sigma_k$, one has $\Delta F_{t-}(k) \in \{\underline{0}, -1\}$ immediately prior to $t$, and the function $\Delta F_t$ changes at time $t$ according to the following rules.

(i) If $\Delta F_t(k) = \underline{0}$ prior to $\sigma_k$, then $\Delta F_t(k)$ becomes $+1$ at time $\sigma_k$.

(ii) If $\Delta F_t(k) = -1$ prior to $\sigma_k$, then $\Delta F_t(k)$ becomes $\overline{0}$ at time $\sigma_k$.

(iii) In both previous cases, the next $\underline{0}$ to the right of $k$, if there is one, becomes a $-1$.

These rules are further illustrated in Figure 1. Note that in these pictures, moving the level $t$ up across the value of $\sigma_k$, the value of $\Delta F_t(k)$ changes either from $\underline{0}$ to $+1$ or from $-1$ to $\overline{0}$.

We observe, similar to what we did for the cut-off version of Plačková's model in Section 1.2, that if $(Y_k)_{k\geq 0}$ is defined in terms of $(\sigma_k)_{k\geq 1}$ as in (11) and $(Y_k^{(t)})_{k\geq 0}$ is the restricted process as in (12), then the joint process $(Y_k^{(t)}, \sigma_k)_{k\geq 1}$ is a Markov chain. If for some $t_+ > 0$, the return time $\tilde{\tau}_{t_+}$ from (15) has finite expectation, then it is not hard to see that this Markov chain (with $t = t_+$) is ergodic, so it is possible to construct a stationary process $(Y_k^{(t_+)}, \sigma_k)_{k\in\mathbb{Z}}$, and such a process is unique in law. Setting

$$Y_k^{(t)} := Y_k^{(t_+)} \cap [0, t] \qquad (0 \leq t \leq t_+) \tag{17}$$

we also obtain stationary Markov chains $(Y_k^{(t)}, \sigma_k)_{k\in\mathbb{Z}}$ for all $0 \leq t \leq t_+$, and associated functions $(F_t(k))_{k\in\mathbb{Z}}$. We claim that for the stationary process, the densities of $\underline{0}$'s and $-1$'s satisfy the following differential equations as a function of $t$, for $0 \leq t \leq t_+$:

$$\begin{aligned} \tfrac{\partial}{\partial t}\mathbb{P}[\Delta F_t(k) = \underline{0}] &= -2\mathbb{P}[\Delta F_t(k) = \underline{0}] - \mathbb{P}[\Delta F_t(k) = -1], \\ \tfrac{\partial}{\partial t}\mathbb{P}[\Delta F_t(k) = -1] &= \mathbb{P}[\Delta F_t(k) = \underline{0}]. \end{aligned} \tag{18}$$

To see this, note that at a per site rate that is proportional to the density of $\underline{0}$'s, rules (i) and (iii) come into effect, leading to the disappearance of two $\underline{0}$'s and the creation of one $-1$. Similarly, at a per site rate that is proportional to the density of $-1$'s, rules (ii) and (iii) come into effect, leading to the disappearance of one $\underline{0}$ and no net change in the number of $-1$'s. We can solve (18) with the initial condition $\mathbb{P}[\Delta F_t(1) = \underline{0}] = 1$, $\mathbb{P}[\Delta F_t(1) = -1] = 0$ explicitly to find

$$\mathbb{P}[\Delta F_t(1) = \underline{0}] = (1-t)e^{-t} \quad \text{and} \quad \mathbb{P}[\Delta F_t(1) = -1] = te^{-t} \qquad (0 \leq t \leq t_+). \tag{19}$$

Since the density of $\underline{0}$'s must be a nonnegative number, we see that no stationary process $(Y_k^{(t_+)}, \sigma_k)_{k\in\mathbb{Z}}$ can exist for $t_+ > 1$. We will prove that on the other hand, for each $t_+ \leq 1$, a stationary process exists, and $\tilde{\tau}_t$ has finite expectation for $t < 1$. Since the function $F_t$ makes i.i.d. excursions away from 0 whose length is distributed as $\tilde{\tau}_t$, we can solve the expectation of $\tilde{\tau}_t$ from the density of $\underline{0}$'s. Indeed, by a simple renewal argument, for each $t < 1$,

$$\mathbb{P}\big[\Delta F_t(k) = \underline{0}\big] = \mathbb{E}[\tilde{\tau}_t]^{-1}\mathbb{P}[\tilde{\tau} = 1] = e^{-t}\mathbb{E}[\tilde{\tau}_t]^{-1}. \tag{20}$$

Combining this with (19), we find that

$$\mathbb{E}[\tilde{\tau}_t] = (1-t)^{-1} \qquad (t < 1). \tag{21}$$

Taking into account the transformation of variables $t = f(q) := -\log(1-q)$, this yields the formula for $\mathbb{E}[\tau_q^{\emptyset}]$ in (4).

### 2.2 Plačková's model

In the present section, we descibe the main idea of the proof of Theorem 3. The idea is similar to what we did in the previous section. Fix $0 \leq q_- < q_+ \leq 1$, assume that for these values of $q_-$ and $q_+$, the Markov chain defined by (9) has an invariant law, and let $(L_k, R_k, U_k, B_k)_{k\in\mathbb{Z}}$ be the corresponding stationary process. We will for each $q \in [q_-, q_+]$ be interested in a stationary process $(\Delta_q(k))_{k\in\mathbb{Z}}$ that is similar to the process $(\Delta F_t(k))_{k\in\mathbb{Z}}$ considered in the previous section, and derive a differential equation for the law of $\Delta_q(k)$ as a function of $q$.

The random variable $\Delta_q(k)$ takes values in

$$\{L\rightarrow, L\uparrow, L*, L\downarrow, R\rightarrow, R\uparrow, R*, R\downarrow\}, \tag{22}$$

and is defined as

$$\Delta_q(k) := \begin{cases} L\rightarrow & \text{if } B_k = -1,\ U_k < q \wedge M^R_{k-1}, \\ L\uparrow & \text{if } B_k = -1,\ q < U_k < M^R_{k-1}, \\ R* & \text{if } B_k = -1,\ q \leq M^R_{k-1} < U_k, \\ R\downarrow & \text{if } B_k = -1,\ M^R_{k-1} < q \wedge U_k, \\ R\rightarrow & \text{if } B_k = +1,\ q \vee M^L_{k-1} < U_k, \\ R\uparrow & \text{if } B_k = +1,\ M^L_{k-1} < U_k < q, \\ L* & \text{if } B_k = +1,\ U_k < M^L_{k-1} \leq q, \\ L\downarrow & \text{if } B_k = +1,\ q \vee U_k < M^L_{k-1}. \end{cases} \tag{23}$$

The definition of $\Delta_q(k)$ is further illustrated in Figure 2. Note that the event that $M^R_{k-1} = q$ only has positive probability if $q = q_+$, since in this case it can happen that $M^R_{k-1}$ is the location of the "immortal" particle at $q_+$. For each (deterministic) $q \in [q_-, q_+)$, the probability that $M^R_{k-1} = q$ is zero. Similar statements hold for $M^L_{k-1}$. Also, the events $M^R_{k-1} = U_k$, $M^L_{k-1} = U_k$, and $U_k = q$ have probability zero.

In Section 4.1 below, we will prove the following result. We note that in Proposition 18 of Section 4.2, we will moreover show that solutions $(p_L, p_R, g_L, g_R)$ to (25) and (26), if they exist, are unique.

**Theorem 4 (Differential equation)** *Let $0 \leq q_- < q_+ \leq 1$ and let $(L_k, R_k, U_k, B_k)_{k\in\mathbb{Z}}$ be a stationary cut-off process as in (9). Let*

$$p_L := \mathbb{E}[M^L_k] \quad \text{and} \quad p_R := \mathbb{E}[M^R_k]. \tag{24}$$

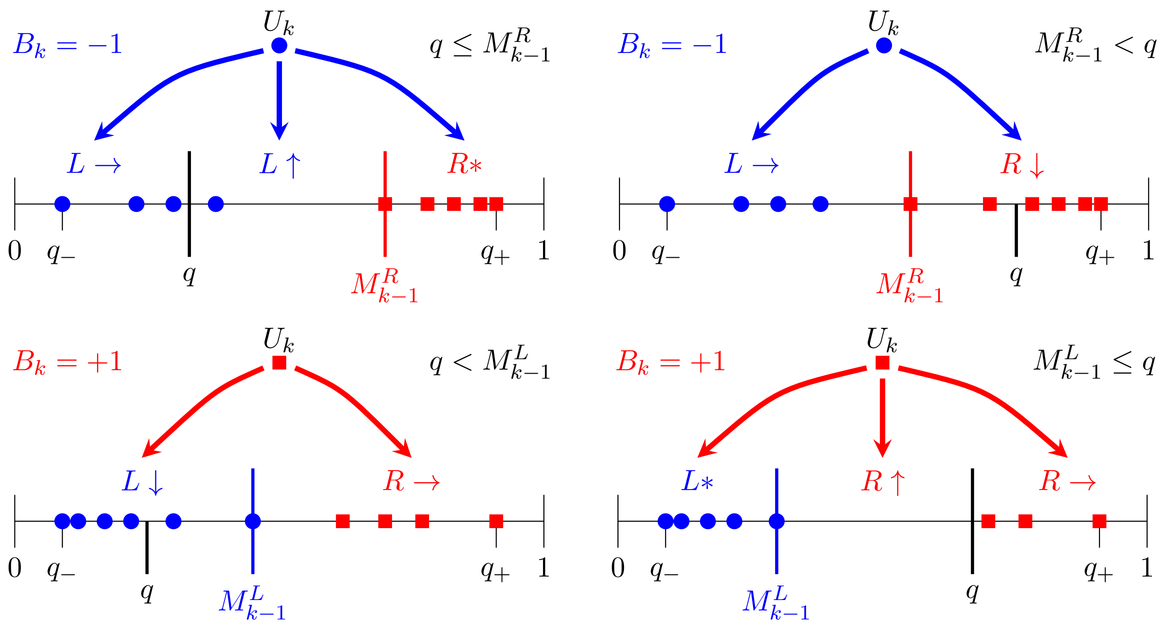


Figure 2: Illustration of the quantity $\Delta_q(k)$ from (23). If $B_k = -1$, then the quantity $\Delta_q(k)$ takes one of the values $L \rightarrow, L \uparrow, R*$, and $R \downarrow$ depending on the relative order of $U_k, q$, and $M^R_{k-1}$. If $B_k = +1$, then the quantity $\Delta_q(k)$ takes one of the values $R \rightarrow, R \uparrow, L*$, and $L \downarrow$ depending on the relative order of $U_k, q$, and $M^L_{k-1}$.

*Then there exists continuous real functions $g_L, g_R$ on $[q_-, q_+]$ that are continuously differentiable on $(q_-, q_+)$ and solve the differential equations*

$$\left.\begin{aligned} \tfrac{\partial}{\partial q} g_L(q) &= -(1-q)^{-1}\big[\tfrac{1}{2}(1-p_R) + g_L(q) - g_R(q)\big] \\ \tfrac{\partial}{\partial q} g_R(q) &= q^{-1}\big[\tfrac{1}{2}p_L + g_R(q) - g_L(q)\big] \end{aligned}\right\} \quad \big(q \in (q_-, q_+)\big), \tag{25}$$

*with the boundary conditions*

$$\begin{aligned} g_L(q_-) &= \tfrac{1}{2}(p_R - q_-) \qquad & g_L(q_+) &= 0, \\ g_R(q_-) &= 0 & g_R(q_+) &= \tfrac{1}{2}(q_+ - p_L). \end{aligned} \tag{26}$$

*Moreover, for each $q \in [q_-, q_+]$ and $k \in \mathbb{Z}$, the distribution of the random variable $\Delta_q(k)$ from (23) is given by*

$$\begin{aligned} \mathbb{P}[\Delta_q(k) = L \rightarrow] &= \tfrac{1}{2}p_R - g_L(q) \qquad & \mathbb{P}[\Delta_q(k) = R \rightarrow] &= \tfrac{1}{2}(1-p_L) - g_R(q), \\ \mathbb{P}[\Delta_q(k) = L \uparrow] &= g_L(q) & \mathbb{P}[\Delta_q(k) = R \uparrow] &= g_R(q), \\ \mathbb{P}[\Delta_q(k) = L*] &= \tfrac{1}{2}p_L - g_L(q) & \mathbb{P}[\Delta_q(k) = R*] &= \tfrac{1}{2}(1-p_R) - g_R(q), \\ \mathbb{P}[\Delta_q(k) = L \downarrow] &= g_L(q) & \mathbb{P}[\Delta_q(k) = R \downarrow] &= g_R(q). \end{aligned} \tag{27}$$

We observe that by the definition of $\Delta_q(k)$ in (23),

$$\begin{aligned} &\mathbb{P}[\Delta_{q_+}(k) = R*] = \mathbb{P}[B_k = -1,\ U_k > M^R_{k-1} \ge q_+] \\ &\quad = \mathbb{P}[B_k = -1,\ U_k > q_+,\ M^R_{k-1} = q_+] = \tfrac{1}{2}(1-q_+)\mathbb{P}[M^R_{k-1} = q_+], \end{aligned} \tag{28}$$

while on the other hand, by (27) and (26),

$$\begin{aligned}&\mathbb{P}[\Delta_{q_+}(k)=R*]=\tfrac{1}{2}(1-p_R)-g_R(q_+)\\&=\tfrac{1}{2}(1-p_R)-\tfrac{1}{2}(q_+-p_L)=\tfrac{1}{2}\big[(1-q_+)-(p_R-p_L)\big].\end{aligned}\tag{29}$$

It follows that

$$(1-q_+)\mathbb{P}[M^R_{k-1}<q_+]=p_R-p_L.\tag{30}$$

In a similar way, we obtain that

$$q_-\mathbb{P}[M^L_{k-1}<q_-]=p_R-p_L.\tag{31}$$

These equations can also be understood by requiring that the average number of particles of $R$ and $L$ added between $q_-$ and $q_+$ in each time step equals the average number of particles of each type that are removed. Combining (30) and (31), we see that the requirement that $\mathbb{P}[M^L_{k-1}=q_-]=0=\mathbb{P}[M^R_{k-1}=q_+]$ implies $q_-=1-q_+$. Using symmetry, one can now derive a differential equation for the difference $g_\Delta:=g_R-g_L$ that can be explicitly solved, and one arrives at Theorem 3.

# 3 Proofs for the canyon model

## 3.1 The lower invariant process

To make the arguments in Section 2.1 precise, we must show that the process $(\Delta F_t(k))_{k\in\mathbb{Z}}$ in (18) is well-defined and that its one-dimensional distributions satisfy the differential equation (18) up to the first time that $\mathbb{P}[\Delta F_t(k)=0]$ hits zero. It is tempting to view $(\Delta F_t)_{t\geq 0}$ as an interacting particle system, but since its jump rates do not satisfy the summability conditions of [Lig85, Thm. I.3.9], standard theory cannot be applied and we have to proceed differently.

Our first lemma says that solutions to the inductive formula (11) are monotone in the starting configuration.

**Lemma 5 (First comparison lemma)** *Let $y$ and $\tilde{y}$ be finite subsets of $[0,1]$ and let $(Y_k)_{k\geq 0}$ and $(\tilde{Y}_k)_{k\geq 0}$ be defined by the inductive relation (11) with $Y_0=y$ and $\tilde{Y}_0=\tilde{y}$. Then $y\subset\tilde{y}$ implies that $Y_k\subset\tilde{Y}_k$ for all $k\geq 0$.*

**Proof** It suffices to show that $Y_{k-1}\subset\tilde{Y}_{k-1}$ implies $Y_k\subset\tilde{Y}_k$. Adding the point $\sigma_k$ to both $Y_{k-1}$ and $\tilde{Y}_{k-1}$ obviously preserves the order of inclusion, as does simultaneously removing the minimal elements $N_{k-1}$ from $Y_{k-1}$ and $\tilde{N}_{k-1}$ from $\tilde{Y}_{k-1}$. Since $Y_k\subset\tilde{Y}_k$ we have $N_{k-1}\geq\tilde{N}_{k-1}$ and it may happen that $\tilde{N}_{k-1}<\sigma_k\leq N_{k-1}$, in which case we remove $\tilde{N}_{k-1}$ from $\tilde{Y}_{k-1}$ but not $N_{k-1}$ from $Y_{k-1}$, but in this case $\tilde{N}_{k-1}$ is not an element of $Y_{k-1}$ so again the order is preserved. ∎

We will be interested in stationary solutions to the inductive relation (11). To this aim, we consider a two-way infinite sequence $(\sigma_k)_{k\in\mathbb{Z}}$ of i.i.d. exponentially distributed random variables with mean one. For each $m\in\mathbb{Z}$, we let $(Y_{m,k})_{k\geq m}$ denote the solution to the inductive relation (11) started in $Y_{m,m}:=\emptyset$. Since $Y_{m-1,m}\supset\emptyset=Y_{m,m}$, we see by Lemma 5, that $Y_{m-1,k}\supset Y_{m,k}$ for all $k\geq m$, so there exists a collection $(Y_k)_{k\in\mathbb{Z}}$ of countable subsets of $[0,\infty)$ such that

$$Y_{m,k}\uparrow Y_k\quad\text{as } m\downarrow-\infty.\tag{32}$$

We call the limit process $(Y_k)_{k\in\mathbb{Z}}$ from (32) the *lower invariant process.* We set

$$F_t(k) := \big|Y_k \cap [0,t]\big| \qquad (t \in [0,\infty),\ k \in \mathbb{Z}). \tag{33}$$

The following theorem is the main result of the present subsection.

**Theorem 6 (Lower invariant process)** *For all $k \in \mathbb{Z}$, one has*

$$F_t(k) \begin{cases} < \infty & \text{a.s.} \qquad \text{if } t \in [0,1), \\ = \infty & \text{a.s.} \qquad \text{if } t \in [1,\infty). \end{cases} \tag{34}$$

*For each $k$, the set $Y_k$ a.s. has a minimal element $N_k := \min(Y_k)$ and $(Y_k)_{k\in\mathbb{Z}}$ solves the inductive relation (11) for all $k \in \mathbb{Z}$. Finally, one has*

$$\begin{aligned} \mathbb{P}[\Delta F_t(k) = \underline{0}] &= (1-t)e^{-t}, \\ \mathbb{P}[\Delta F_t(k) = \overline{0}] &= 1-(1+t)e^{-t}, \\ \mathbb{P}[\Delta F_t(k) = -1] &= te^{-t}, \\ \mathbb{P}[\Delta F_t(k) = +1] &= te^{-t}, \end{aligned} \qquad (t \in [0,1),\ k \in \mathbb{Z}), \tag{35}$$

*where $\Delta F_t(k)$ is defined as in (16).*

As a first step towards the proof of Theorem 6, for $t \in [0,\infty)$, we look at the restricted lower invariant process

$$Y_k^{(t)} := Y_k \cap [0,t] \qquad (k \in \mathbb{Z}). \tag{36}$$

We define $N_k^{(t)} := \inf(Y_k^{(t)} \cup \{t\})$ $(k \in \mathbb{Z})$.

**Lemma 7 (Restricted process)** *For each $t \in [0,\infty)$, one of the following two alternatives occurs:*

(i) $|Y_k^{(t)}| = \infty$ *a.s. for all $k \in \mathbb{Z}$.*

(ii) $|Y_k^{(t)}| < \infty$ *a.s. for all $k \in \mathbb{Z}$ and $(Y_k^{(t)})_{k\in\mathbb{Z}}$ solves the inductive relation*

$$Y_k^{(t)} := \begin{cases} Y_{k-1}^{(t)} \cup \{\sigma_k\} & \text{if } \sigma_k \le N_{k-1}^{(t)}, \\ \big(Y_{k-1}^{(t)} \cup \{\sigma_k\}\big)\backslash\{N_{k-1}^{(t)}\} & \text{if } N_{k-1}^{(t)} < \sigma_k \le t, \\ Y_{k-1}^{(t)}\backslash\{N_{k-1}^{(t)}\} & \text{if } t < \sigma_k, \end{cases} \qquad (k \in \mathbb{Z}). \tag{37}$$

**Proof** Set $Y_{m,k}^{(t)} := Y_{m,k} \cap [0,t]$. Since $|Y_{m,k-1}^{(t)}|$ and $|Y_{m,k}^{(t)}|$ differ at most by one, letting $m \to -\infty$, we see that the event

$$\big\{|Y_{m,k}^{(t)}| = \infty\ \forall k \in \mathbb{Z}\big\} \cup \big\{|Y_{m,k}^{(t)}| < \infty\ \forall k \in \mathbb{Z}\big\} \tag{38}$$

has probability one. Since the indicators of both events occuring in this expression are translation invariant functions of the ergodic random variables $(\sigma_k)_{k\in\mathbb{Z}}$, it follows that exactly one of these events has probability one.

For each $m \le k-1$, the sets $Y_{m,k-1}$ and $Y_{m,k-1}$ are related as in (11), which is easily seen to imply that $Y_{m,k-1}^{(t)}$ and $Y_{m,k}^{(t)}$ are related as in (37). If $|Y_{k-1}^{(t)}| < \infty$, then, since $Y_{k-1}^{(t)}$ is the increasing limit of $Y_{m,k-1}^{(t)}$ as $m \to -\infty$, there exists some $m_0$ such that $Y_{k-1}^{(t)} = Y_{m,k-1}^{(t)}$ for all $m \le m_0$. Now also $Y_k^{(t)} = Y_{m,k}^{(t)}$ for all $m \le m_0$ and hence $Y_k^{(t)}$ and $Y_{k-1}^{(t)}$ are related as in (37). ∎

**Lemma 8 (Minimality)** *Let $t \in [0,\infty)$ be such that case (ii) of Lemma 7 holds, let $(Y^{(t)}_k)_{k\in\mathbb{Z}}$ be the restricted lower invariant process defined in (36) and let $(\tilde Y^{(t)}_k)_{k\in\mathbb{Z}}$ be any other solution of the two-way infinite inductive relation (37), taking values in the finite subsets of $[0,t]$. Then $Y^{(t)}_k \subset \tilde Y^{(t)}_k$ for all $k \in \mathbb{Z}$ a.s.*

**Proof** Set $Y^{(t)}_{m,k} := Y_{m,k} \cap [0,t]$ as in the previous proof. Then $(Y^{(t)}_{m,k})_{k\geq m}$ solves the inductive relation (37) for $k \geq m$ and $Y^{(t)}_{m,m} = \emptyset \subset \tilde Y^{(t)}_m$. In analogy with Lemma 5, it is easy to prove that solutions to (37) are monotone in the initial state, so it follows that $Y^{(t)}_{m,k} \subset \tilde Y^{(t)}_k$ for all $k \geq m$. Letting $m \downarrow -\infty$ for fixed $k$ now proves the statement. ∎

Since a.s. $\sigma_k \neq 0$ for all $k \in \mathbb{Z}$, it is clear from the definition that $Y_k \cap \{0\} = \emptyset$ and hence $\Delta F_0(k) = \underline{0}$ for all $k \in \mathbb{Z}$. The next key proposition shows that $F_t(k)$ is also a.s. finite for $t$ small enough.

**Proposition 9 (Finite regime)** *Let $s \in [0,\infty)$ and assume that a.s. $F_s(k) < \infty$ for all $k \in \mathbb{Z}$. Then there exists some $\varepsilon$ such that*

$$\mathbb{P}[\Delta F_s(k) = \underline{0}] = \varepsilon > 0 \qquad (k \in \mathbb{Z}). \tag{39}$$

*Moreover, for all $t \geq s$ such that $2(e^{-s} - e^{-t}) < \varepsilon$, one has $F_t(k) < \infty$ for all $k \in \mathbb{Z}$ a.s. and*

$$\mathbb{P}[\Delta F_t(k) = \underline{0}] \geq \varepsilon - 2(e^{-s} - e^{-t}) \qquad (k \in \mathbb{Z}). \tag{40}$$

**Proof** By assumption, $F_s(k)$ is a.s. finite, so $\mathbb{P}[F_s(k) \leq m] > 0$ for some $m < \infty$. But then, by stationarity,

$$\begin{aligned}\mathbb{P}\big[\Delta F_s(k) = \underline{0}\big] &\geq \mathbb{P}\big[F_s(k-m-1) \leq m,\ \sigma_{k'} > s\ \forall k-m-1 < k' \leq k\big]\\ &= \mathbb{P}\big[F_s(k-m-1) \leq m\big] \cdot \mathbb{P}\big[\sigma_{k'} > s\ \forall k-m-1 < k' \leq k\big] > 0,\end{aligned} \tag{41}$$

where we have used that $F_s(k-m-1)$ is a function of $(\sigma_{k'})_{k' \leq k-m-1}$ and hence independent of $(\sigma_{k'})_{k-m-1<k'\leq k}$. This proves (39).

The idea of the proof of (40) is easily explained. For each $k$ such that $s < \sigma_k \leq t$, at most two $\underline{0}$'s are destroyed, so the density of $\underline{0}$'s can at most decrease by two times the density of such $k$'s, i.e., by $2(e^{-s} - e^{-t})$. To make this precise, let us define

$$\Delta G_t(k) := 2 1_{\{\sigma_k \in (s,t]\}} - 1_{\{\Delta F_s(k) = \underline{0}\}} \qquad (k \in \mathbb{Z}), \tag{42}$$

and let $G_t : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$G_t(0) = 0 \quad \text{and} \quad G_t(k) - G_t(k-1) = \Delta G_t(k) \qquad (k \in \mathbb{Z}). \tag{43}$$

We say that $k \in \mathbb{Z}$ is a *point of decrease* of the function $G_t$ if

$$G_t(k') > G_t(k) \qquad \forall k' < k. \tag{44}$$

Let $t \geq s$ be such that $2(e^{-s} - e^{-t}) < \varepsilon$. We will prove that

$$\mathbb{P}\big[k \text{ is a point of decrease of } G_t\big] = \varepsilon - 2(e^{-s} - e^{-t}), \tag{45}$$

and there exists a solution $(\tilde{Y}^{(t)}_k)_{k\in\mathbb{Z}}$ of the two-way infinite inductive relation (37) taking values in the finite subsets of $[0,t]$ such that the associated function $\tilde{F}_t(k) := |\tilde{Y}^{(t)}_k|$ satisfies

$$\Delta\tilde{F}_t(k) = \underline{0} \text{ for each point of decrease } k \text{ of } G_t. \tag{46}$$

By Lemma 8, we have $F_t \leq \tilde{F}_t$, so it follows that $F_t(k)$ is a.s. finite for each $k \in \mathbb{Z}$, and by (45) we see that

$$\mathbb{P}[\Delta F_t(k) = \underline{0}] \geq \mathbb{P}[\Delta\tilde{F}_t(k) = \underline{0}] \geq \mathbb{P}\big[k \text{ is a point of decrease of } G_t\big] = \varepsilon - 2(e^{-s} - e^{-t}), \tag{47}$$

proving (40).

We are left with the task of proving (45) and (46). We start with the latter. By (45) and the ergodicity of the random variables $(\sigma_k)_{k\in\mathbb{Z}}$, points of decrease of $G_t$ occur with spatial density $\varepsilon - 2(e^{-s} - e^{-t})$, which is positive by assumption. In particular, there exists sequences of such points tending to $-\infty$ and $+\infty$.

Let $m$ and $n$ be points of decrease of $G_t$ that are consecutive in the sense that $m < n$ and $\{m_1, \ldots, n-1\}$ does not contain any points of decrease of $G_t$. Let $(\tilde{Y}^{(t)}_k)_{k\geq m}$ be defined by the inductive relation (37) with $\tilde{Y}^{(t)}_m = \emptyset$. We claim that

$$\big|\tilde{Y}^{(t)}_k \cap (s,t]\big| \leq G_t(k) - G_t(m) \qquad (m \leq k < n). \tag{48}$$

Indeed, in a given step $k$, the left-hand side of this equation can only increase by one if $\sigma_k \in (s,t]$, but in this case $G_t$ also increases by one. The right-hand side decreases by one if $\Delta F_s(k) = 0$ and $\sigma_k \notin (s,t]$, but in this case either the left-hand case also decreases by one, or it is already zero. Since $m$ and $n$ are consecutive points of decrease of $G_t$, we have $G_t(k) - G_t(m) \geq 0$ for all $m \leq k < n$, so also in this case the order is preserved.

Since $n$ is a point of decrease of $G_t$, we have $\Delta F_s(n) = 0$ and $\sigma_n \notin (s,t]$, so (48) proves that $\tilde{F}_t(k) := |\tilde{Y}^{(t)}_k|$ satisfies $\Delta\tilde{F}_t(n) = \underline{0}$. Pasting together solutions of (37) between consecutive points of decrease of $G_t$, we obtain a two-way infinite solution satisfying (46).

We are left with the task of proving (45). By the ergodicity of the $(\sigma_k)_{k\in\mathbb{Z}}$,

$$\lim_{k\to\infty} \frac{1}{k} G_t(-k) = -\mathbb{E}[\Delta G_t(0)] = \varepsilon - 2(e^{-s} - e^{-t}), \tag{49}$$

which is by assumption positive. It follows that there are infinitely many points of decrease of $G_t$ on the left of the origin and hence such points must occur with a positive density. For each $k \in \mathbb{Z}$, let

$$\lambda(k) := \min\big\{k' \geq k : k' \text{ is a point of decrease of } G_t\big\}. \tag{50}$$

Set $f(k,k') := \mathbb{E}[\Delta G_t(k) 1_{\{\lambda(k)=k'\}}]$. Then $f(k+m, k'+m) = f(k,k')$ and hence, by the mass transport principle,

$$\begin{aligned} \mathbb{E}[\Delta G_t(0)] &= \sum_{k'\in\mathbb{Z}} f(0,k') = \sum_{k\in\mathbb{Z}} f(k,0) \\ &= \mathbb{E}\Big[\sum_{k:\,\lambda(k)=0} \Delta G_t(k)\Big] = -\mathbb{P}[0 \text{ is a point of decrease of } G_t], \end{aligned} \tag{51}$$

where we have used that $\sum_{k:\,\lambda(k)=0} \Delta G_t(k) = -1$ if 0 is a point of decrease and zero otherwise. ∎

Our next aim is to derive the differential equation (18). We will actually derive the somewhat different equation (52) below, but it is easy to check that the two equations have the same solution (19) with the initial conditions $\mathbb{P}[\Delta F_t(0) = \underline{0}] = 1$, $\mathbb{P}[\Delta F_t(0) = -1] = 0$.

**Proposition 10 (Differential equation)** *Let $F_t$ be the as in (33) and assume that $u > 0$ is such that $F_u(k) < \infty$ a.s. for all $k \in \mathbb{Z}$. For $t \in [0, u]$ define $\Delta F_t(k)$ as in (16). Then the functions $t \mapsto \mathbb{P}[\Delta F_t(0) = \underline{0}]$ and $t \mapsto \mathbb{P}[\Delta F_t(0) = -1]$ are continuously differentiable on $[0, u]$ and satisfy the differential equations*

$$\begin{aligned} \frac{\partial}{\partial t}\mathbb{P}[\Delta F_t(k) = \underline{0}] &= -\mathbb{P}[\Delta F_t(k) = \underline{0}] - e^{-t}, \\ \frac{\partial}{\partial t}\mathbb{P}[\Delta F_t(k) = -1] &= -\mathbb{P}[\Delta F_t(k) = -1] + e^{-t}. \end{aligned} \tag{52}$$

**Proof** For all $t \in [0, u]$ and $k \in \mathbb{Z}$, define $\kappa_t(k)$ as in (13), i.e.,

$$\kappa_t(k) := \inf\{k' > k : \Delta F_{t-}(k') = \underline{0}\}. \tag{53}$$

Since the set $\{k \in \mathbb{Z} : \Delta F_t(k) = \underline{0}\}$ is nonincreasing in $t$, we see that for each $k \in \mathbb{Z}$ there is at most one $t \in [0, u]$ such that

$$\text{at time } t, \text{ one has } t = \sigma_{k'} \text{ for some } k' < k \text{ and } \kappa_t(k') = k. \tag{54}$$

Note that at such a time, $\Delta F_{t-}(k) = \underline{0}$ and $\Delta F_t = -1$. If such a time exists, we denote it by $\tau_k$. For definiteness, we set $\tau_k := \infty$ if there exists no $t \in [0, u]$ such that (54) holds.

For all $0 \leq s < t \leq u$, we observe that

$$\begin{aligned} \mathbb{P}[\Delta F_t(k) = \underline{0}] &= \mathbb{P}\big[\Delta F_s(k) = \underline{0},\ \{\sigma_k, \tau_k\} \cap (s, t] = \emptyset\big], \\ \mathbb{P}[\Delta F_t(k) = -1] &= \mathbb{P}\big[\Delta F_s(k) = -1,\ \sigma_k \notin (s, t]\big] + \mathbb{P}\big[\tau_k \in (s, t],\ \sigma_k > t\big]. \end{aligned} \tag{55}$$

Since $\Delta F_s(k) = -1$ implies that $\sigma_k > s$ and $\Delta F_s$ depends only on information about the times $(\sigma_{k'})_{k' \in \mathbb{Z}}$ until time $s$, we have

$$\mathbb{P}\big[\Delta F_s(k) = -1,\ \sigma_k \notin (s, t]\big] = \mathbb{P}[\Delta F_s(k) = -1]e^{-(t-s)}. \tag{56}$$

Using translation invariance and changing the summation order, we see that

$$\begin{aligned} &\mathbb{P}\big[\tau_k \in (s, t]\big] = \sum_{k' \in \mathbb{Z}} \mathbb{P}\big[\sigma_{k'} \in (s, t],\ \kappa_{\sigma_{k'}}(k') = k\big] \\ &\quad = \sum_{k' \in \mathbb{Z}} \mathbb{P}\big[\sigma_0 \in (s, t],\ \kappa_{\sigma_0}(0) = k - k'\big] = \mathbb{P}\big[\sigma_0 \in (s, t]\big] = e^{-s} - e^{-t} = e^{-s}(1 - e^{-(t-s)}). \end{aligned} \tag{57}$$

Finally, we observe that $\tau_k \in (s, t]$ implies $\tau_k < \sigma_k$ and we use this to estimate

$$\mathbb{P}\big[\tau_k \in (s, t] \text{ and } \sigma_k \in (s, t]\big] \leq \mathbb{P}\big[\tau_k \in (s, t] \text{ and } \sigma_k - \tau_k \leq (t - s)\big] = e^{-s}\big(1 - e^{-(t-s)}\big)^2, \tag{58}$$

where in the last step we have used (57) and the fact that by the memoryless property of te exponential distribution, conditional on $\tau_k \in (s, t]$, the random variable $\sigma_k - \tau_k$ is exponentially distributed.

We rewrite the right-hand side of the first equation in (55) as

$$\begin{aligned} &\mathbb{P}\big[\Delta F_s(k) = \underline{0},\ \{\sigma_k, \tau_k\} \cap (s, t] = \emptyset\big] \\ &\quad = \mathbb{P}\big[\Delta F_s(k) = \underline{0}\big] - \mathbb{P}\big[\Delta F_s(k) = \underline{0},\ \sigma_k \in (s, t]\big] - \mathbb{P}\big[\Delta F_s(k) = \underline{0},\ \tau_k \in (s, t]\big] \\ &\qquad + \mathbb{P}\big[\Delta F_s(k) = \underline{0},\ \sigma_k \in (s, t] \text{ and } \tau_k \in (s, t]\big]. \end{aligned} \tag{59}$$

Here, arguing as in (56) and using (57)

$$\begin{aligned}\mathbb{P}\big[\Delta F_s(k)=\underline{0},\ \sigma_k\in(s,t]\big]&=\mathbb{P}[\Delta F_s(k)=\underline{0}]\big(1-e^{-(t-s)}\big),\\ \mathbb{P}\big[\Delta F_s(k)=\underline{0},\ \tau_k\in(s,t]\big]&=\mathbb{P}\big[\tau_k\in(s,t]\big]=1-e^{-(t-s)}.\end{aligned}\tag{60}$$

Using also (58), it follows that for any $0\leq s<s+\varepsilon\leq u$,

$$\mathbb{P}\big[\Delta F_{s+\varepsilon}(k)=\underline{0}\big]-\mathbb{P}\big[\Delta F_s(k)=\underline{0}\big]=-\varepsilon\mathbb{P}[\Delta F_s(k)=\underline{0}]-\varepsilon e^{-s}+O(\varepsilon^2),\tag{61}$$

where $O(\varepsilon^2)$ is a term that can uniformly be estimated as $|O(\varepsilon^2)|\leq K\varepsilon^2$ for some fixed $K<\infty$. Treating the second equation in (55) in a similar manner, using (56), (57) and (58), we obtain

$$\mathbb{P}\big[\Delta F_{s+\varepsilon}(k)=-1\big]-\mathbb{P}\big[\Delta F_s(k)=-1\big]=-\varepsilon\mathbb{P}\big[\Delta F_s(k)=-1\big]+\varepsilon e^{-s}+O(\varepsilon^2).\tag{62}$$

Letting $\varepsilon\downarrow 0$, we arrive at (52). ∎

**Proof of Theorem 6** Let $I$ be the set of all $t\in[0,\infty)$ such that case (i) of Lemma 7) holds, i.e., $F_t(k)<\infty$ a.s. for all $k\in\mathbb{Z}$. As remarked above Proposition 9, we have $0\in I$. It is immediate from the definition of $F_t(k)$ in (33) that $t\mapsto F_t(k)$ is a.s. nondecreasing, so $I$ is a subinterval of $[0,\infty)$ containing 0. It follows from Proposition 9 that $I$ is of the form $I=[0,t_{\rm c})$ for some $t_{\rm c}\in(0,\infty]$. Moreover, $\mathbb{P}[\Delta F_t(k)=\underline{0}]>0$ for $t\in[0,t_{\rm c})$ and if $t_{\rm c}<\infty$, then we must have

$$\lim_{t\uparrow t_{\rm c}}\mathbb{P}[\Delta F_t(k)=\underline{0}]=0.\tag{63}$$

Solving the differential equation of Proposition 10, we see that

$$\begin{aligned}\mathbb{P}[\Delta F_t(k)=\underline{0}]&=(1-t)e^{-t},\\ \mathbb{P}[\Delta F_t(k)=-1]&=te^{-t},\end{aligned}\qquad (t\in[0,t_{\rm c}),\ k\in\mathbb{Z}),\tag{64}$$

which together with (63) allows us to conclude that $t_{\rm c}=1$.

Since $F_1(k)=\infty$ a.s., it follows that $N_k:=\inf(Y_k)<1$ a.s., and since $F_t(k)<\infty$ a.s. for all $t<1$, this infimum is in fact a minimum. Since for each $t<1$, the restricted process $Y^{(t)}$ solves the inductive relation (37) and since for each $k$, there exists a $t<1$ such that $N^{(t)}_k<t$, it is easy to see that $(Y_k)_{k\in\mathbb{Z}}$ solves the inductive relation (11).

To conclude the proof, we must show that

$$\mathbb{P}[\Delta F_t(k)=+1]=\mathbb{P}[\Delta F_t(k)=-1]\qquad(0\leq t<1),\tag{65}$$

which toghether with (64) and the requirement that the total probability is one then yields (35). Let us write

$$F^M_t(k):=F_t(k)\wedge M\quad\text{and}\quad\Delta F^M_t(k):=F^M_t(k)-F^M_t(k-1)\qquad(k\in\mathbb{Z},\ M\in\mathbb{N}).\tag{66}$$

By stationarity $\mathbb{E}[F^M_t(k)]=\mathbb{E}[F^M_t(k-1)]$, so letting $M\uparrow\infty$ we conclude that $\mathbb{E}[\Delta F_t(k)]=0$ and hence the two probabilities in (65) are equal. ∎

**Remark** Although, by Theorem 6, the function $F_t$ is finite only for $t<1$, it is possible to give a sensible definition of $\Delta F_t$ also for $t\geq 1$ by setting (compare (16) and Figure 1)

$$\Delta F_t(k):=\begin{cases}\underline{0} & \text{if } t<N_{k-1}<\sigma_k,\\ \overline{0} & \text{if } N_{k-1}<\sigma_k\leq t\\ -1 & \text{if } N_{k-1}\leq t<\sigma_k,\\ +1 & \text{if } \sigma_k\leq t\wedge N_{k-1}.\end{cases}\qquad\big(k\in\mathbb{Z},\ t\in[0,1]\big),\tag{67}$$

where $N_k:=\min(Y_k)$ $(k\in\mathbb{Z})$ and $(Y_k)_{k\in\mathbb{Z}}$ is the lower invariant process.

### 3.2 Ergodicity

In the present section, we use Theorem 6 to derive Theorems 1 and 2. Most of the work is already done. The remaining arguments are for a large part standard Markov chain theory.

Let $Y_0 = y$ be any finite subset of $[0,\infty)$ and let $(Y_k)_{k\geq 0}$ be the Markov chain with initial state $Y_0$ defined by the inductive relation (11). We have already seen that for each $t \in [0,\infty)$, the restricted process $Y_k^{(t)} := Y_k \cap [0,t]$ is a Markov chain and in fact given by the inductive relation (37). When we need to specify the initial state $y$, we make this explicit in our notation by writing $(Y_k^y)_{k\geq 0}$ for the original process and $(Y_k^{y\,(t)})_{k\geq 0}$ for the restricted process. As in (3), we define

$$\tilde{\tau}_t^y := \inf\{k > 0 : Y_k^{y\,(t)} = \emptyset\}. \tag{68}$$

In particular, if $y = \emptyset$, then this is the first return time of the restricted process to the empty configuration.

**Lemma 11 (Expected return time)** *One has $\mathbb{E}[\tilde{\tau}_t^{\emptyset}] = (1-t)^{-1}$ for each $t \in [0,1)$.*

**Proof** Let $(Y_k)_{k\in\mathbb{Z}}$ be the lower invariant process from (32) and let $F_t$ and $\Delta F_t$ be defined as in (33) and (16). Then, by Theorem 6, for $t \in [0,1)$ one has

$$(1-t)e^{-t} = \mathbb{P}[\Delta F_t(k) = \underline{0}] = \mathbb{P}[F_t(k-1) = 0]\mathbb{P}[\sigma_k > t] = \mathbb{P}[F_t(k-1) = 0]e^{-t}, \tag{69}$$

which proves that the restricted lower invariant process satisfies $\mathbb{P}[Y_k^{(t)} = \emptyset] = 1-t$. Let $\lambda_t(k) := \sup\{k' < k : Y_{k'}^{(t)} = \emptyset\}$ $(k \in \mathbb{Z})$. Then, for $t \in [0,1)$, by the mass transport principle,

$$1 = \sum_{k\in\mathbb{Z}} \mathbb{P}[\lambda_t(0) = -k] = \sum_{k\in\mathbb{Z}} \mathbb{P}[\lambda_t(k) = 0] = \sum_{k\in\mathbb{Z}} \mathbb{P}[Y_0^{(t)} = \emptyset,\ 0 < \tilde{\tau}_t \leq k] = (1-t)\mathbb{E}[\tilde{\tau}_t^{\emptyset}]. \tag{70}$$

∎

**Lemma 12 (Transience)** *One has $\mathbb{P}[\tilde{\tau}_t^{\emptyset} = \infty] > 0$ for each $t \in (1,\infty)$.*

**Proof** Let $F_t(k) := \left|Y_k^{\emptyset} \cap [0,t]\right|$ $(k \geq 0)$ and for $k \geq 1$ define $\Delta F_t(k)$ as in (16). We observe that for any $0 \leq s < t$,

$$F_t(n) \geq \sum_{k=1}^{n} 1_{\{s < \sigma_k < t\}} - \sum_{k=1}^{n} 1_{\{F_s(k-1) = 0\}}, \tag{71}$$

where we have used that $\Delta F_t(k) \geq 0$ whenever $s < \sigma_k < t$, and $\Delta F_t(k) = +1$ if moreover $F_s(k-1) \neq 0$. Since the process $(F_s(k))_{k\geq 0}$ makes i.i.d. excursions from 0 with length distributed as $\tilde{\tau}_s^{\emptyset}$, by Lemma 11 and the strong law of large numbers, we have for each $s \in [0,1)$

$$n^{-1}\sum_{k=1}^{n} 1_{\{s < \sigma_k < t\}} \underset{n\to\infty}{\longrightarrow} (e^{-s} - e^{-t}) \quad \text{and} \quad n^{-1}\sum_{k=1}^{n} 1_{\{F_s(k-1) = 0\}} \underset{n\to\infty}{\longrightarrow} 1-s \quad \text{a.s.} \tag{72}$$

Choosing $s$ close enough to 1 such that $1 - s < e^{-s} - e^{-t}$, we see that $F_t(n) \to \infty$ a.s. Since $(F_t(k))_{k\geq 0}$ makes i.i.d. excursions from 0 with length distributed as $\tilde{\tau}_t^{\emptyset}$, it follows that $\mathbb{P}[\tilde{\tau}_t^{\emptyset} = \infty] > 0$. ∎

Lemmas 11 and 12 show that the restricted process $Y_k^{(t)} = Y_k \cap [0,t]$ started from the empty configuration returns to the empty configuration in finite expected time if $t < 1$ and has a positive probability never to return to the empty configuration if $t > 1$. We would like to conclude from this that the process, started in an arbitrary initial state, is "positively recurrent" for $t < 1$ and "transient" for $t > 1$. Since the state space of $Y^{(t)}$ is uncountable, we have to specify in exactly which meaning we use these terms. Recall from (68) that $\tilde{\tau}_t^y$ is the first time after time zero that the restricted process $Y^{y\,(t)}$ started in the initial state $y$ is in the empty state. We will say that $Y^{(t)}$ is *positive recurrent*, *null recurrent*, or *transient* depending on whether case (i), (ii), or (iii) of the following lemma occurs.

**Lemma 13 (Recurrence versus transience)** *For each $t > 0$, exactly one of the following three possibilities occurs.*

(i) *For all finite $y \subset [0, \infty)$, one has $\mathbb{E}[\tilde{\tau}_t^y] < \infty$.*

(ii) *For all finite $y \subset [0, \infty)$, one has $\tilde{\tau}_t^y < \infty$ a.s. and $\mathbb{E}[\tilde{\tau}_t^y] = \infty$.*

(iii) *For all finite $y \subset [0, \infty)$, one has $\mathbb{P}[\tilde{\tau}_t^y = \infty] > 0$.*

We first need a preparatory result. Instead of using general Markov chain techniques (such as the theory of Harris recurrence) to prove Lemma 13, we will rely on monotonicity arguments that are special to our model. The next lemma, which is of some interest in its own right, may loosely be described as saying that for $Y^{(t)}$ to avoid becoming the empty set, it is good to have many particles that are situated as far as possible to right in the interval $[0, t]$.

**Lemma 14 (Second comparison lemma)** *For each $0 \leq s \leq t$ and finite $y \subset [0, \infty)$, let $F_{s,t}^y(k) := \big|Y_k^y \cap [s, t]\big|$ $(k \geq 0)$. Fix $t > 0$ and let $x, y \subset [0, \infty)$ be finite. Then*

$$F_{s,t}^x(0) \leq F_{s,t}^y(0)\ \forall s \in [0, t] \quad \textit{implies} \quad F_{s,t}^x(k) \leq F_{s,t}^y(k)\ \forall s \in [0, t],\ k \geq 0. \tag{73}$$

**Proof** It suffices to prove (73) for $k = 1$; the general statement follows by induction. Without loss of generality, we may also assume that $x$ and $y$ are subsets of $[0, t]$. Order the elements of $x$ and $y$ as $x = \{x_1, \ldots, x_n\}$ and $y = \{y_1, \ldots, y_m\}$ with $x_n < \cdots < x_1$ (in this order!) and $y_m < \cdots < y_1$. Then the assumption that $F_{s,t}^x(0) \leq F_{s,t}^y(0)$ $\forall s \in [0, t]$ is equivalent to the statement that $m \geq n$ and $x_i \leq y_i$ for all $i = 1, \ldots, n$. We must show that we can order the elements of $\tilde{x} := Y_1^x \cap [0, t]$ and $\tilde{y} := Y_1^y \cap [0, t]$ in the same way. We distinguish three different cases.

Case I: $\sigma_1 < x_n$. In this case, no points are removed from $x$ while $\tilde{x}_{n+1} := \sigma_1$ is added as the $(n+1)$-th element. Since $x_n \leq y_n$, the elements $y_1, \ldots, y_n$ remain unchanged while $\tilde{y}_{n+1}$ is the minimal element of $\{\sigma_1\} \cup \{y_m, \ldots, y_{n+1}\}$, which lies on the right of $\tilde{x}_{n+1} = \sigma_1$.

Case II: $x_n < \sigma_1 < t$. In this case, $x_n$ is removed from $x$ and there exist $1 \leq n' \leq n$ and $n' \leq m' \leq m + 1$ such that $\sigma_1$ is inserted into $x$ between the $n'$-th and $(n'-1)$-th element and into $y$ between the $m'$-th and $(m'-1)$-th element, where we allow for the cases that $n' = 1$ ($\sigma_1$ is added at the right end of $x$ and possibly also of $y$) and $m' = m + 1$ ($\sigma_1$ is added at the left end of $y$). The elements of the new sets $\tilde{x}$ and $\tilde{y}$, ordered from low to high, are now

$$\begin{aligned} \{x_{n-1}, \ldots, x_{m'}, x_{m'-1}, \ldots, x_{n'}, \sigma_1, x_{n'-1}, \ldots, x_1\} &= \tilde{x}, \\ \{y_{m-1}, \ldots, y_n, y_{n-1}, \ldots, y_{m'},\ \sigma_1, y_{m'-1}, \ldots, y_{n'},\ y_{n'-1}, \ldots, y_1\} &= \tilde{y}. \end{aligned} \tag{74}$$

Here $x_{n-1}, \ldots, x_{m'}$ lie on the left of $y_{n-1}, \ldots, y_{m'}$, and likewise $x_{n'-1}, \ldots, x_1$ lie on the left of $y_{n'-1}, \ldots, y_1$, respectively. Since moreover

$$x_{m'-1} < \cdots < x_{n'} < \sigma_1 < y_{m'-1} < \cdots < y_{n'}, \tag{75}$$

these elements are ordered in the right way too.

Case III: $t < \sigma_1$. In this case, the lowest elements of $x$ and $y$ are removed while no new elements are added, which obviously also preserves the order. ∎

**Proof of Lemma 13** It suffices to show that for any finite $y \subset [0,\infty)$, one has $\tilde{\tau}^y_t < \infty$ a.s. if and only if $\tilde{\tau}^{\emptyset}_t < \infty$ a.s., and likewise, $\mathbb{E}[\tilde{\tau}^y_t] = \infty$ if and only if $\mathbb{E}[\tilde{\tau}^{\emptyset}_t] = \infty$. By the first comparison Lemma 5, $\tilde{\tau}^{\emptyset}_t \leq \tilde{\tau}^y_t$ which immediately gives us the implications in one direction. To complete the proof, we must show that $\mathbb{P}[\tilde{\tau}^y_t = \infty] > 0$ implies $\mathbb{P}[\tilde{\tau}^{\emptyset}_t = \infty] > 0$ and $\mathbb{E}[\tilde{\tau}^y_t] = \infty$ implies $\mathbb{E}[\tilde{\tau}^{\emptyset}_t] = \infty$.

Recalling notation introduced in the second comparison Lemma 14, we observe that for any finite $y \subset [0,\infty)$ there is a $k \geq 1$ such that

$$\mathbb{P}\big[\tilde{\tau}^{\emptyset}_t > k \text{ and } F^{\emptyset}_{s,t}(k) \geq F^y_{s,t}(0)\ \forall s \in [0,t]\big] > 0. \tag{76}$$

Using this, Lemma 14, and the Markov property, we see that

$$\begin{aligned}\mathbb{P}[\tilde{\tau}^{\emptyset}_t = \infty] &\geq \mathbb{P}\big[\tilde{\tau}^{\emptyset}_t > k \text{ and } F^{\emptyset}_{s,t}(k) \geq F^y_{s,t}(0)\ \forall s \in [0,t]\big]\mathbb{P}[\tilde{\tau}^y_t = \infty],\\ \mathbb{E}[\tilde{\tau}^{\emptyset}_t] &\geq \mathbb{P}\big[\tilde{\tau}^{\emptyset}_t > k \text{ and } F^{\emptyset}_{s,t}(k) \geq F^y_{s,t}(0)\ \forall s \in [0,t]\big]\big(k + \mathbb{E}[\tilde{\tau}^y_t]\big),\end{aligned} \tag{77}$$

which together with (76) gives us the desired implications. ∎

Positive recurrence in the sense of Lemma 13 suffices to prove ergodicity of the restricted process $Y^{(t)}$. In fact, the following general result applies.

**Proposition 15 (Markov chain with an atom)** *Let $P$ be a measurable probability kernel on a Polish space $E$ and for each $x \in E$, let $(X^x_k)_{k\geq 0}$ denote the Markov chain with initial state $x$ and transition kernel $P$. Let $z \in E$ be fixed and let*

$$\tau^x := \inf\{k > 0 : X^x_k = z\} \qquad (x \in E). \tag{78}$$

*Assume that $\mathbb{E}[\tau^z] < \infty$, $\mathbb{P}[\tau^x < \infty] = 1$ for all $x \in E$, and that the greatest common divisor of $\{k > 0 : \mathbb{P}[\tau^z = k] > 0\}$ is one. Then there exists a unique invariant law $\nu$ for $P$ and*

$$\big\|\nu - \mathbb{P}\big[X^x_k \in \cdot\,\big]\big\| \underset{k\to\infty}{\longrightarrow} 0, \tag{79}$$

*where $\|\cdot\|$ denotes the total variation norm.*

**Proof** This follows from standard arguments, so we only sketch the proof. First, one can check that

$$\nu := \mathbb{E}[\tau^z]^{-1} \sum_{k=1}^{\infty} \mathbb{P}[\tau^z \leq k \text{ and } X^z \in \cdot\,] \tag{80}$$

is an invariant law for $P$. We can couple the corresponding stationary process $(X_k)_{k\geq 0}$ and the process $(X^x_k)_{k\geq 0}$ started in a deterministic initial state $x$ in such a way that they evolve independently until the time $\sigma := \inf\{k \geq 0 : X^x_k = z = X_k\}$. Since $\mathbb{P}[\tau^x < \infty] = 1$ for all $x \in E$, both processes reach $z$ in a finite random time and after that make i.i.d. excursions away from $z$ whose length has finite mean $\mathbb{E}[\tau^z]$. Using also the aperiodicity assumption, it follows that $\sigma < \infty$ a.s. so the coupling is successful. ∎

**Remark 1** The assumption that the state space is Polish guarantees that Kolmogorov's extension theorem can be applied to construct the process from its finite dimensional distributions.

This assumption can certainly be relaxed; see [MT09, Section 3.1] for a "general" set-up which is, however, so general that singletons $\{z\}$ may fail to be measurable. When we apply Proposition 15 below to the restricted process $Y^{(t)}$, the state space is the set of all simple counting measures on $[0,t]$, equipped with the topology of weak convergence. This space is Polish because of the following facts: 1. for any Polish space $E$, the space $\mathcal{M}(E)$ of finite measures on $E$, equipped with the topology of weak convergence, is Polish, 2. the set $\mathcal{N}[0,t]$ of all counting measures on $[0,t]$ is a closed subset of $\mathcal{M}[0,t]$, 3. the set of all simple counting measures is a $G_\delta$-subset of $\mathcal{N}[0,t]$, 4. a $G_\delta$-subset of a Polish space is Polish.

**Remark 2** The fact that formula (80) defines an invariant law follows from [MT09, Theorem 10.1.2 (i)]. The fact that our coupling is successful follows from [Woe09, Lemma 3.46]. The latter is written down for Markov chains with countable state space only, but this applies generally since any $\mathbb{N}_+$-valued random variable with finite mean is the law of the return time of a suitably constructed positively recurrent Markov chain with countable state space.

**Proof of Theorem 1** By Lemma 11, for each $t<1$, the restricted process $Y^{y\,(t)}_k = Y^y_k \cap [0,t]$ is positively recurrent in the sense of Lemma 13, case (i), so a.s. $Y^{y\,(t)}_k = \emptyset$ for infinitely many $k$, proving that

$$\limsup_{k\to\infty} N^{y\,(t)}_k \geq t \quad \text{a.s.} \qquad (t<1), \tag{81}$$

where $N^{(t)}_k := \inf(Y^{y\,(t)}_k \cup \{t\})$ $(k\in\mathbb{Z})$. On the other hand, by Lemma 12, $Y^{y\,(t)}$ is transient in the sense of Lemma 13, case (i), for all $t>1$, so

$$\mathbb{P}\big[\exists n \text{ s.t. } Y^y_k \cap [0,t] \neq \emptyset \ \forall k\geq n\big] = 1. \tag{82}$$

Combining this with (81) we see that

$$\limsup_{k\to\infty} N^y_k = 1 \quad \text{a.s.,} \tag{83}$$

where $N^y_k := \min(Y^y_k \cup \{\infty\})$ $(k\geq 0)$. Translating this to the process $X$ through the transformaton $q = 1-e^{-t}$ as discussed in Section 2.1 yields Theorem 1. ∎

**Proof of Theorem 2** Formula (4) is just the translation of Lemmas 11 and 12 to the process $X$ through the transformaton $t = -\log(1-q)$ as discussed in Section 2.1.

Let $(Y_k)_{k\in\mathbb{Z}}$ denote the lower invariant process from (32). By Theorem 6, for each $t<1$, setting $\nu := \mathbb{P}[Y_0 \cap [0,t] \in \cdot\,]$ defines an invariant law for the process restricted to $[0,t]$. By Lemma 11, this process is positively recurrent in the sense of Lemma 13, case (i). Since $\mathbb{P}[\tilde\tau^{\emptyset}_t = 1] = \mathbb{P}[\sigma_1 > t] = e^{-t}$, this process is ergodic in the sense of Proposition 15, i.e., $\nu$ is its unique invariant law and the long-time limit law (w.r.t. the total variation norm) started from any initial state. Translated for the process $X$, this yields (5). It has been proved in Theorem 6 that $Y_0 \cap [0,1)$ is a.s. an infinite set, so the same is true for the set $X_\infty$ which is the image of $Y_0\cap[0,1)$ under the map $t\mapsto 1-e^{-t}$. ∎

# 4 Proofs for Plačková's model

## 4.1 The differential equation

Fix $0 \leq q_- < q_+ \leq 1$, assume that for these values of $q_-$ and $q_+$, the Markov chain defined by (9) has an invariant law, and let $(L_k, R_k, U_k, B_k)_{k\in\mathbb{Z}}$ denote the corresponding stationary process, where $(L_k, R_k)$ takes values in the space defined in (8).

**Lemma 16 (Stationary process)** *Any stationary process $(L_k, R_k, U_k, B_k)_{k\in\mathbb{Z}}$ as above has the following properties.*

(i) *$(U_k)_{k\in\mathbb{Z}}$ and $(B_k)_{k\in\mathbb{Z}}$ are independent i.i.d. sequences such that $U_k$ is uniformly distributed on $[0,1]$ and $B_k$ is uniformly distributed on $\{-1,+1\}$.*

(ii) *For each $k \in \mathbb{Z}$, the pair $(L_{k-1}, R_{k-1})$ is independent of $(U_{k'}, B_{k'})_{k'\geq k}$.*

**Proof** These may seem like tautologies but we have defined $(L_k, R_k, U_k, B_k)_{k\in\mathbb{Z}}$ as a stationary Markov process with certain transition probabilities, so the statements above are not a priori part of its definition. However, the transition probabilities of $(L_k, R_k, U_k, B_k)_{k\in\mathbb{Z}}$ are such that in each time step, $(U_k, B_k)$ are independent of $(L_{k-1}, R_{k-1})$, and then $(L_k, R_k)$ are given in terms of $(U_k, B_k)$ and $(L_{k-1}, R_{k-1})$ as in (9). From this, the statements follow easily. ∎

**Lemma 17 (Frequencies of events)** *Let $(L_k, R_k, U_k, B_k)_{k\in\mathbb{Z}}$ be a stationary process as in Lemma 16 and let $\Delta_q(k)$ be defined as in (23). Then, for each $q \in [q_-, q_+]$,*

$$\begin{aligned}
\mathbb{P}[\Delta_q(k) = L \to] &= \tfrac{1}{2}\mathbb{E}[q \wedge M^R_{k-1}] & \mathbb{P}[\Delta_q(k) = R \to] &= \tfrac{1}{2}\mathbb{E}[1 - q \vee M^L_{k-1}],\\
\mathbb{P}[\Delta_q(k) = L \uparrow] &= \tfrac{1}{2}\mathbb{E}[(M^R_{k-1} - q) \vee 0] & \mathbb{P}[\Delta_q(k) = R \uparrow] &= \tfrac{1}{2}\mathbb{E}[(q - M^L_{k-1}) \vee 0],\\
\mathbb{P}[\Delta_q(k) = L*] &= \tfrac{1}{2}\mathbb{E}[1_{\{M^L_{k-1}\leq q\}} M^L_{k-1}] & \mathbb{P}[\Delta_q(k) = R*] &= \tfrac{1}{2}\mathbb{E}[1_{\{q\leq M^R_{k-1}\}}(1 - M^R_{k-1})],\\
\mathbb{P}[\Delta_q(k) = L \downarrow] &= \tfrac{1}{2}\mathbb{E}[1_{\{q<M^L_{k-1}\}} M^L_{k-1}] & \mathbb{P}[\Delta_q(k) = R \downarrow] &= \tfrac{1}{2}\mathbb{E}[1_{\{M^R_{k-1}<q\}}(1 - M^R_{k-1})].
\end{aligned} \tag{84}$$

**Proof** Immediate from Lemma 16 and the definition of $\Delta_q(k)$ in (23). ∎

**Proof of Theorem 4** By symmetry, it suffices to prove the statements for the function $g_L$ only. Using Lemma 17, we observe that

$$\mathbb{P}\big[\Delta_q(k) \in \{L \to, L \uparrow\}\big] = \tfrac{1}{2}\mathbb{E}[q \wedge M^R_{k-1}] + \tfrac{1}{2}\mathbb{E}[(M^R_{k-1} - q) \vee 0] = \tfrac{1}{2}\mathbb{E}[M^R_{k-1}] = \tfrac{1}{2}p_R \tag{85}$$

and

$$\mathbb{P}\big[\Delta_q(k) \in \{L \downarrow, L*\}\big] = \tfrac{1}{2}\mathbb{E}[1_{\{M^L_{k-1}\leq q\}} M^L_{k-1}] + \mathbb{E}[1_{\{q<M^L_{k-1}\}} M^L_{k-1}] = \tfrac{1}{2}\mathbb{E}[M^L_{k-1}] = \tfrac{1}{2}p_L. \tag{86}$$

Let $F^L_q(k) := |L_k \cap [q, q_+]|$, which by assumption (see (8) is a.s. finite for each $q > q_-$. Then

$$\mathbb{E}\big[F^L_q(k) - F^L_q(k-1)\big] = \mathbb{P}\big[\Delta_q(k) = L \uparrow\big] - \mathbb{P}\big[\Delta_q(k) = L \downarrow\big]. \tag{87}$$

Using stationarity, arguing as in (66), we see that this expression is zero, so

$$\mathbb{P}\big[\Delta_q(k) = L \uparrow\big] = \mathbb{P}\big[\Delta_q(k) = L \downarrow\big] \tag{88}$$

for all $q \in (q_-, q_+]$. Using monotone convergence and Lemma 17, we see that the left-hand side of (88) is continuous as a function of $q$ on $[q_-, q_+]$ while the right-hand side is right-continuous as a function of $q$. We conclude that (88) holds also at $q = q_-$ and that

$$g_L(q) := \mathbb{P}\big[\Delta_q(k) = L \uparrow\big] \qquad \big(q \in [q_-, q_+]\big) \tag{89}$$

is a continuous function. This also shows that $\mathbb{P}\big[\Delta_q(k) = L \downarrow\big]$ is in fact also left-continuous as a function of $q$, which implies that

$$\mathbb{P}\big[M^L_{k-1} = q\big] = 0 \quad \big(q \in (q_-, q_+]\big), \tag{90}$$

i.e., the law of $M^L_k$ is at $q_-$ has no atoms except possibly at $q = q_-$. Note that by Lemma 17, $g_L$ satisfies the boundary conditions in (26). In view of this, (85), (86), and (88), to complete the proof, it sufficies to show that the function $g_L$ defined in (89) is continuously differentiable on $(q_-, q_+)$ and satisfies the differential equation in (25).

For any $q_- < q < q' < q_+$,

$$\begin{aligned}&\mathbb{P}\big[\Delta_q(k) = L\uparrow,\ \Delta_{q'}(k) \neq L\uparrow\big] = \mathbb{P}\big[B_k = -1,\ U_k \in (q, q'),\ U_k < M^R_{k-1}\big] \\ &\quad = \tfrac{1}{2}\mathbb{P}\big[U_k \in (q, q'),\ U_k < M^R_{k-1}\big],\end{aligned} \tag{91}$$

where

$$\mathbb{P}\big[U_k \in (q, q'),\ q' < M^R_{k-1}\big] \leq \mathbb{P}\big[U_k \in (q, q'),\ U_k < M^R_{k-1}\big] \leq P\big[U_k \in (q, q'),\ q < M^R_{k-1}\big] \tag{92}$$

By Lemma 16 (ii), the left- and right-hand sides of this equation equal

$$(q' - q)\mathbb{P}\big[q < M^R_{k-1}\big] \quad \text{and} \quad (q' - q)\mathbb{P}\big[q' < M^R_{k-1}\big], \tag{93}$$

respectively, so for any $q_- < q < q + \varepsilon < q_+$, we obtain that

$$\mathbb{P}\big[\Delta_{q+\varepsilon}(k) = L\uparrow\big] = \mathbb{P}\big[\Delta_q(k) = L\uparrow\big] + \varepsilon\tfrac{1}{2}\mathbb{P}\big[q < M^R_{k-1}\big] + o(\varepsilon), \tag{94}$$

where

$$|o(\varepsilon)| \leq \varepsilon\tfrac{1}{2}\mathbb{P}\big[M^R_{k-1} \in (q, q + \varepsilon)\big], \tag{95}$$

which by the symmetric analogue of (90) for $M^R_{k-1}$ is a small o or $\varepsilon$ as $\varepsilon \downarrow 0$.

By Lemma 17,

$$\begin{aligned}&\mathbb{P}[\Delta_q(k) = L\uparrow] + \mathbb{P}[\Delta_q(k) = R*] \\ &\quad = \tfrac{1}{2}\mathbb{E}[1_{\{q \leq M^R_{k-1}\}}(M^R_{k-1} - q)] + \tfrac{1}{2}\mathbb{E}[1_{\{q \leq M^R_{k-1}\}}(1 - M^R_{k-1})] = \tfrac{1}{2}(1 - q)\mathbb{P}\big[q \leq M^R_{k-1}\big].\end{aligned} \tag{96}$$

By the symmetric analogues of (86) and (88),

$$\begin{aligned}&\mathbb{P}[\Delta_q(k) = R*] = \tfrac{1}{2}(1 - p_R) - \mathbb{P}[\Delta_q(k) = R\downarrow] \\ &\quad = \tfrac{1}{2}(1 - p_R) - \mathbb{P}[\Delta_q(k) = R\uparrow] = \tfrac{1}{2}(1 - p_R) - g_R(q).\end{aligned} \tag{97}$$

Combining this with (94) and (96), we see that $g_L$ is continuously differentiable on $(q_-, q_+)$ and

$$\begin{aligned}&\tfrac{\partial}{\partial q} g_L(q) = \tfrac{1}{2}\mathbb{P}\big[q < M^R_{k-1}\big] = (1 - q)^{-1}\big(\mathbb{P}[\Delta_q(k) = L\uparrow] + \mathbb{P}[\Delta_q(k) = R*]\big) \\ &\quad = (1 - q)^{-1}\big(g_L(q) + \tfrac{1}{2}(1 - p_R) - g_R(q)\big),\end{aligned} \tag{98}$$

in agreement with (25). ∎

### 4.2 The critical point

**Proposition 18 (Uniqueness of solutions)** *For given $0 < q_- < q_+ < 1$, there exists at most one quadruple $(p_L, p_R, g_L, g_R)$ such that $p_L, p_R$ are real constants satisfying $q_- \leq p_L \leq p_R \leq q_+$ and $g_L, g_R$ are continuous real functions on $[q_-, q_+]$ that are continuously differentiable on $(q_-, q_+)$ and solve the differential equation (25) with the boundary conditions (26).*

**Proof** Setting

$$\hat{g}_L(q) := g_L(q) - \tfrac{1}{2}(p_R - q_-), \tag{99}$$

we can rewrite (25) as

$$\left.\begin{aligned} \tfrac{\partial}{\partial q}\hat{g}_L(q) &= -(1-q)^{-1}\big[\tfrac{1}{2}(1-q_-) + \hat{g}_L(q) - g_R(q)\big] \\ \tfrac{\partial}{\partial q}g_R(q) &= q^{-1}\big[\tfrac{1}{2}q_- - \tfrac{1}{2}(p_R - p_L) + g_R(q) - \hat{g}_L(q)\big] \end{aligned}\right\} \quad \big(q \in (q_-, q_+)\big), \tag{100}$$

while the boundary conditions (26) transform into

$$\begin{aligned} \hat{g}_L(q_-) &= 0 \qquad & \hat{g}_L(q_+) &= -\tfrac{1}{2}(p_R - q_-), \\ g_R(q_-) &= 0 \qquad & g_R(q_+) &= \tfrac{1}{2}(q_+ - p_L). \end{aligned} \tag{101}$$

Setting

$$p_\Delta := p_R - p_L \quad \text{and} \quad h(q) := g_R(q) - \hat{g}_L(q) \quad \big(q \in [q_-, q_+]\big), \tag{102}$$

we see that $h$ solves the differential equation

$$\tfrac{\partial}{\partial q}h(q) = \tfrac{1}{2}\Big(\frac{q_- - p_\Delta}{q} + \frac{1-q_-}{1-q}\Big) + \big\{q^{-1} - (1-q)^{-1}\big\}h(q) \tag{103}$$

with the boundary conditions

$$h(q_-) = 0 \quad \text{and} \quad h(q_+) = \tfrac{1}{2}(q_+ - q_- + p_\Delta). \tag{104}$$

For given values of $q_-$ and $p_\Delta$, the equation (103) has a unique solution $h$ with the initial condition $h(q_-) = 0$, and this solution is defined on all of $[q_-, 1)$. Since the equation (103) is an inhomogeneous linear differential equation of which the first, inhomogeneous term in is strictly decreasing in $p_\Delta$, we see that making $p_\Delta$ larger makes the solution $h$ strictly smaller on $(q_-, 1)$. Since the value of $h$ in $q_+$ is a decreasing function of $p_\Delta$ while the boundary condition at $q_+$ in (104) is an increasing function of $p_\Delta$, there can at most be one value of $p_\Delta$ for which (103) and (104) are satisfied.

Given this value of $p_\Delta$ and $q_-$, the equation (100) also has a unique solution on $[q_-, 1)$ subject to the left boundary conditions $\big(\hat{g}_L(q_-), g_R(q_-)\big) = (0,0)$ from (101). Given $q_+$, we can read off $p_R$ and $p_L$ from the right boundary conditions in (101), which by (99) also tells us what $g_L$ is. ∎

**Proof of Theorem 3** In Section 2.2, we have already seen that by (30) and (31), the requirement (10) implies $q_- = 1 - q_+$. Also, since the constants in (24) clearly satisfy $p_L < p_R$, by (30) and (31), we must have $0 < q_-$ and $q_+ < 1$. By Proposition 18, solutions to $(p_L, p_R, g_L, g_R)$ (25) and (26), if they exist, are unique, so whenever $q_- = 1 - q_+$, by symmetry, we must have

$$p_R = 1 - p_L \quad \text{and} \quad g_L(q) = g_R(1-q) \quad \big(q \in [q_-, q_+]\big). \tag{105}$$

Let

$$p_\Delta := p_R - p_L \quad \text{and} \quad g_\Delta(q) := g_R(q) - g_L(q) \quad \big(q \in [q_-, q_+]\big). \tag{106}$$

Symmetry (105) implies that

$$p_L = \tfrac{1}{2}(1 - p_\Delta), \quad p_R = \tfrac{1}{2}(1 + p_\Delta), \quad \text{and} \quad g_L(\tfrac{1}{2}) = g_R(\tfrac{1}{2}). \tag{107}$$

Using this, we see from (25) and (26) that $g_\Delta$ solves the equation

$$\tfrac{\partial}{\partial q} g_\Delta(q) = \tfrac{1}{4}(1-p_\Delta)\big\{q^{-1}+(1-q)^{-1}\big\} + \big\{q^{-1}-(1-q)^{-1}\big\} g_\Delta(q) \tag{108}$$

with the boundary conditions

$$g_\Delta(0)=0 \quad \text{and} \quad g_\Delta(q_+) = \tfrac{1}{2}q_+ - \tfrac{1}{4}(1-p_\Delta). \tag{109}$$

We can solve (108) with the boundary condition $g_\Delta(0)=0$ explicitly to obtain

$$g_\Delta(q) = \tfrac{1}{4}(1-p_\Delta)q(1-q)\big\{1/(1-q) - 1/q - 2\log(1-q) + 2\log(q)\big\}. \tag{110}$$

By (30) and our assumption that $\mathbb{P}[M^R_{k-1} < q_+] = 1$, we see that $q_+ = 1-p_\Delta$, so in this case the right boundary condition in (109) reads $g_\Delta(q_+) = \frac{1}{4}q_+$. Using the fact that $q_+ = 1-p_\Delta$ and the explicit solution (110), this yields the equation

$$\tfrac{1}{4}q_+^2(1-q_+)\big\{1/(1-q_+) - 1/q_+ - 2\log(1-q_+) + 2\log(q_+)\big\} = \tfrac{1}{4}q_+. \tag{111}$$

Using the fact that $\frac{1}{2} \leq q_+ < 1$, we can rewrite this as

$$\frac{q_+ - (1-q_+)}{q_+(1-q_+)} - 2\log(1-q_+) + 2\log(q_+) = \frac{1}{q_+(1-q_+)}, \tag{112}$$

or equivalently

$$-\frac{2(1-q_+)}{q_+(1-q_+)} = 2\log\Big(\frac{1-q_+}{q_+}\Big). \tag{113}$$

Setting $z = -1/q_+$, this can be rewritten as

$$z = \log(-z-1) \quad \Leftrightarrow \quad e^z = -z-1. \tag{114}$$

∎


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