A phase transition between endogeny and nonendogeny

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Abstract

The Marked Binary Branching Tree (MBBT) is the family tree of a rate one binary branching process, on which points have been generated according to a rate one Poisson point process, with i.i.d. uniformly distributed activation times assigned to the points. In frozen percolation on the MBBT, initially, all points are closed, but as time progresses points can become either frozen or open. Points become open at their activation times provided they have not become frozen before. Open points connect the parts of the tree below and above it and one says that a point percolates if the tree above it is infinite. We consider a version of frozen percolation on the MBBT in which at times of the form θ^n, all points that percolate are frozen. The limiting model for $\theta \to 1$, in which points freeze as soon as they percolate, has been studied before by Ráth, Swart, and Terpai. We extend their results by showing that there exists a $0 < \theta^* < 1$ such that the model is endogenous for $\theta \leq \theta^*$ but not for $\theta > \theta^*$. This means that for $\theta \leq \theta^*$, frozen percolation is a.s. determined by the MBBT but for $\theta > \theta^*$ one needs additional randomness to describe it.

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1 Introduction and main results

1.1 Introduction

The concept of frozen percolation was introduced by Aldous [Ald00]. In it, i.i.d. activation times that are uniformly distributed on $[0,1]$ are assigned to the edges of an infinite, unoriented graph. Initially, all edges are closed. At its activation time, an edge opens, provided it is not frozen. Here, by definition, an edge freezes as soon as one of its endvertices becomes part of an infinite open cluster. For general graphs, the existence of a process satisfying this description is not obvious. Indeed, Benjamini and Schramm observed that on the square lattice, frozen percolation does not exist (see [BT01, Section 3] for an account of the argument).

On the other hand, Aldous [Ald00] showed that frozen percolation on the infinite 3-regular tree does exist. Under natural additional assumptions, such a process is even unique in law. This was partially already observed in [Ald00] and made more precise in [RST19, Thm 2]. The problem of almost sure uniqueness stayed open for 19 years, but has recently been solved negatively in [RST19, Thm 3], where it is shown that the question whether a given edge freezes cannot be decided only by looking at the activation times of all edges.

The proof of [RST19, Thm 3] depends on detailed calculations that are specific to the details of the model. As a result, the question of almost sure uniqueness is still open for frozen percolation on n-regular trees with $n > 3$. This raises the question whether model specific calculations are necessary, or whether the absence of almost sure uniqueness can alternatively be demonstrated by more general, "soft" arguments that have so far been overlooked.

The results in the present paper suggest that this is not the case and model specific calculations are, to some degree, unavoidable. We look at a modified model in which edges can freeze only at a certain countable set of times. For the resulting model, which depends on a parameter $0 < \theta < 1$, we show that under the same natural additional assumptions that guarantee uniquess in law, there exists a nontrivial critical value θ^* such that almost sure uniqueness holds for $\theta \leq \theta^*$ but not for $\theta > \theta^*$.

It turns out that it is mathematically simpler to formulate our results for frozen percolation on a certain oriented tree, the Marked Binary Branching Tree (MBBT), a random oriented continuum tree introduced in [RST19]. Using methods of Section 3 of that paper, our results can also be translated into results for the unoriented 3-regular tree. For brevity, we omit the details of the latter step and stick for the remainder of the paper to the oriented (rather than the unoriented) setting on the MBBT (rather than the 3-regular tree).

1.2 Frozen percolation on the MBBT

Let $\mathbb{T}$ be the set of all finite words $\mathbf{i} = i_1 \cdots i_n$ $(n \geq 0)$ made up from the alphabet $\{1, 2\}$. We call $|\mathbf{i}| := n$ the length of the word $\mathbf{i}$ and denote the word of length zero by $\varnothing$, which we distinguish notationally from the empty set $\emptyset$. The concatenation of two words $\mathbf{i} = i_1 \cdots i_n$ and $\mathbf{j} = j_1 \cdots j_m$ is denoted by $\mathbf{ij} := i_1 \cdots i_n j_1 \cdots j_m$. We view $\mathbb{T}$ as an oriented tree with root $\varnothing$, in which each point $\mathbf{i}$ has two offspring $\mathbf{i}1$ and $\mathbf{i}2$, and each point $\mathbf{i} = i_1 \cdots i_n$ except for the root has one parent $\overleftarrow{\mathbf{i}} := i_1 \cdots i_{n-1}$. In pictures, we draw the root at the bottom and we draw the descendants of a point above their predecessor. By definition, a *rooted subtree* of $\mathbb{T}$ is a subset $\mathbb{U} \subset \mathbb{T}$ such that $\overleftarrow{\mathbf{i}} \in \mathbb{U}$ for all $\mathbf{i} \in \mathbb{U} \backslash \{\varnothing\}$. We call $\partial \mathbb{U} := \{\mathbf{i} \in \mathbb{T} \backslash \mathbb{U} : \overleftarrow{\mathbf{i}} \in \mathbb{U}\}$ the *boundary* of $\mathbb{U}$, and we use the convention that $\partial \mathbb{U} = \{\varnothing\}$ if $\mathbb{U} = \emptyset$.

Let $(\tau_{\mathbf{i}}, \kappa_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ be i.i.d. uniformly distributed on $[0,1] \times \{1, 2\}$. We interpret $\tau_{\mathbf{i}}$ as the *activation time* of $\mathbf{i}$ and $\kappa_{\mathbf{i}}$ as its *number of legal offspring*. If $\kappa_{\mathbf{i}} = 1$, then we call $\mathbf{i}1$ and $\mathbf{i}2$ the legal and illegal offspring of $\mathbf{i}$, respectively. Points $\mathbf{i} \in \mathbb{T}$ with $\kappa_{\mathbf{i}} = 1$ or $= 2$ are called *internal points* and *branching points*, respectively. We denote the corresponding sets as $\mathbb{I} := \{\mathbf{i} \in \mathbb{T} : \kappa_{\mathbf{i}} = 1\}$ and $\mathbb{B} := \{\mathbf{i} \in \mathbb{T} : \kappa_{\mathbf{i}} = 2\}$. Only activation times of internal points matter; activation times of branching points will not be used. For any $\mathbf{i} \in \mathbb{T}$ and $A \subset \mathbb{T}$, we

write $\mathbf{i} \xrightarrow{A} \infty$ if there exist $(j_k)_{k \geq 1}$ such that

$$\text{(i)} \quad j_{k+1} \leq \kappa_{\mathbf{i}j_1 \cdots j_k} \quad \text{and} \quad \text{(ii)} \quad \mathbf{i}j_1 \cdots j_k \in A \quad \text{for all } k \geq 0. \tag{1.1}$$

In words, this says that there is an infinite open upwards path through A starting at $\mathbf{i}$ such that each next point is a legal offspring of its parent.

We will be interested in frozen percolation on $\mathbb{T}$ with the following informal description. At any time, points can be *closed*, *frozen*, or *open*. Once a point is frozen or open, it stays that way. Initially, all branching points are open and all internal points are closed. Branching points stay open for all time. An internal point $\mathbf{i}$ becomes open at its activation time $\tau_{\mathbf{i}}$ provided that, by this time, it has not yet become frozen. The rules for freezing points are as follows. We fix a set $\Xi \subset (0,1]$ that is closed w.r.t. the relative topology of $(0,1]$. Letting $\mathbb{O}^t$ denote the set of open points at time t, we decree that up to its activation time, a closed internal point $\mathbf{i}$ becomes frozen at the first time in Ξ when its legal offspring percolates, i.e., when $\mathbf{i}1 \xrightarrow{\mathbb{O}^t} \infty$.

Let

$$\mathbb{T}^t := \{\mathbf{i} \in \mathbb{I} : \tau_{\mathbf{i}} \leq t\} \cup \mathbb{B} \qquad (0 \leq t \leq 1) \tag{1.2}$$

denote the set of all points at time t that are either an internal point that has already been activated or a branching point. Let $\mathbb{F}$ denote the set of internal points that eventually become frozen. Since once a point opens or freezes, it stays open or frozen for the remaining time, the set of open points at time t is given by $\mathbb{O}^t = \mathbb{T}^t \backslash \mathbb{F}$. In view of this, we make our informal description precise by saying that a random subset $\mathbb{F}$ of $\mathbb{T}$ *solves the frozen percolation equation* for the *set of possible freezing times* Ξ if

$$\mathbf{i} \in \mathbb{F} \text{ if and only if } \kappa_{\mathbf{i}} = 1 \text{ and } \mathbf{i}1 \xrightarrow{\mathbb{T}^t \backslash \mathbb{F}} \infty \text{ for some } t \in \Xi \cap (0, \tau_{\mathbf{i}}], \tag{1.3}$$

which says that the points that eventually become frozen are those internal points $\mathbf{i}$ for which $\mathbf{i}1$ percolates at some time in Ξ before or at the activation time of $\mathbf{i}$.

It turns out that solutions to (1.3) always exist, but the question of uniqueness is more subtle. To get at least uniqueness in law, we impose additional conditions. We write $\omega_{\mathbf{i}} := (\tau_{\mathbf{i}}, \kappa_{\mathbf{i}})$ $(\mathbf{i} \in \mathbb{T})$ and for any $\mathbf{j} \in \mathbb{T}$, we let

$$\Omega_{\mathbf{j}} := (\omega_{\mathbf{j}\mathbf{i}})_{\mathbf{i} \in \mathbb{T}} \tag{1.4}$$

denote the i.i.d. randomness that resides in the subtree of $\mathbb{T}$ rooted at $\mathbf{j}$. In particular, we write $\Omega := \Omega_\varnothing$. If $\mathbb{F}$ is a solution to the frozen percolation equation, then for each $\mathbf{j} \in \mathbb{T}$, we define a random subset $\mathbb{F}_{\mathbf{j}}$ of $\mathbb{T}$ by

$$\mathbb{F}_{\mathbf{j}} := \{\mathbf{i} \in \mathbb{T} : \mathbf{j}\mathbf{i} \in \mathbb{F}\}. \tag{1.5}$$

We say that a solution $\mathbb{F}$ to the frozen percolation equation (1.3) is *stationary* if the law of $(\Omega_{\mathbf{j}}, \mathbb{F}_{\mathbf{j}})$ does not depend on $\mathbf{j} \in \mathbb{T}$. We say that $\mathbb{F}$ is *adapted* if for each finite rooted subtree $\mathbb{U} \subset \mathbb{T}$, the collection of random variables $(\Omega_{\mathbf{j}}, \mathbb{F}_{\mathbf{j}})_{\mathbf{j} \in \partial \mathbb{U}}$ is independent of $(\omega_{\mathbf{i}})_{\mathbf{i} \in \mathbb{U}}$. Finally, we say that $\mathbb{F}$ *respects the tree structure* if $(\Omega_{\mathbf{j}}, \mathbb{F}_{\mathbf{j}})_{\mathbf{j} \in \partial \mathbb{U}}$ is a collection of independent random variables for each finite rooted subtree $\mathbb{U} \subset \mathbb{T}$.

With these definitions, we can formulate our first result about existence and uniqueness in law of solutions to the frozen percolation equation (1.3). In the special case that $\Xi = (0,1]$, the following theorem has been proved before in (in a somewhat different guise) in [RST19, Thm 2].

Theorem 1 (Uniqueness in law of frozen percolation) *Let Ξ be a closed subset of $(0,1]$ (w.r.t. the relative topology). Then there exists a solution $\mathbb{F}$ of the frozen percolation equation (1.3). This solution can be chosen so that it is stationary, adapted, and respects the tree structure. Subject to these additional conditions, the joint law of Ω and $\mathbb{F}$ is uniquely determined.*

We will prove Theorem 1 in Subsection 2.1. As we will see in the coming subsections, the question of almost sure uniqueness of solutions to the frozen percolation equation is subtle and the answer depends on the choice of the closed set Ξ.

In the remainder of the present subsection, which can be skipped at a first reading, we explain how our set-up relates to the definition of the Marked Binary Branching Tree (MBBT) introduced in [RST19]. Let

$$\mathbb{S} := \big\{ i_1 \cdots i_n \in \mathbb{T} : i_m \leq \kappa_{i_1 \cdots i_{m-1}} \ \forall 1 \leq m \leq n \big\}. \tag{1.6}$$

denote the random rooted subtree of $\mathbb{T}$ consisting of all legal descendants of the root. Then $\mathbb{S}$ is the family tree of a branching process in which each individual has one or two offspring, with equal probabilities. For any rooted subtree $\mathbb{U} \subset \mathbb{S}$, we call

$$\nabla \mathbb{U} := \partial \mathbb{U} \cap \mathbb{S} \tag{1.7}$$

the *boundary* of $\mathbb{U}$ *relative to* $\mathbb{S}$.

Let $(\ell_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ be i.i.d. exponentially distributed random variables with mean $1/2$, independent of Ω. We interpret $\ell_{\mathbf{i}}$ as the lifetime of the individual $\mathbf{i}$ and let

$$b_{i_1 \cdots i_n} := \sum_{k=0}^{n-1} \ell_{i_1 \cdots i_k} \quad \text{and} \quad d_{i_1 \cdots i_n} := \sum_{k=0}^{n} \ell_{i_1 \cdots i_k} \tag{1.8}$$

with $b_{\varnothing} := 0$ and $d_{\varnothing} := \ell_{\varnothing}$ denote the birth and death times of $i_1 \cdots i_n \in \mathbb{T}$. For $h \geq 0$, we let

$$\begin{aligned}
\mathbb{T}_h &:= \big\{ \mathbf{i} \in \mathbb{T} : d_{\mathbf{i}} \leq h \big\}, & \partial \mathbb{T}_h &= \big\{ \mathbf{i} \in \mathbb{T} : b_{\mathbf{i}} \leq h < d_{\mathbf{i}} \big\}, \\
\mathbb{S}_h &:= \mathbb{T}_h \cap \mathbb{S}, & \nabla \mathbb{S}_h &= \partial \mathbb{T}_h \cap \mathbb{S}
\end{aligned} \tag{1.9}$$

denote the sets of individuals in $\mathbb{T}$ or $\mathbb{S}$ that have died by time h and those that are alive at time h, respectively. Note that the former are a.s. finite rooted subtrees of $\mathbb{T}$ and $\mathbb{S}$, respectively, and the latter are their boundaries relative to $\mathbb{T}$ or $\mathbb{S}$. Now

$$(\nabla \mathbb{S}_h)_{h \geq 0} \tag{1.10}$$

is a continuous-time branching process subject to the following dymanics:

- each individual $\mathbf{i}$ is with rate 1 replaced by two new individuals $\mathbf{i}1$ and $\mathbf{i}2$,

- each individual $\mathbf{i}$ is with rate 1 replaced by one new individual $\mathbf{i}1$.

Let $(\nabla \mathbb{S}_{h-})_{h \geq 0}$ denote the left-continuous modification of the branching process in (1.10) and let

$$\mathcal{T} := \big\{ (\mathbf{i}, h) : \mathbf{i} \in \nabla \mathbb{S}_{h-}, h \geq 0 \big\}. \tag{1.11}$$

As in [RST19, Subsection 1.5], we can equip $\mathcal{T}$ with a natural metric, making in into a random continuum tree. We can think of $\mathcal{T}$ as the family tree of a rate one binary branching process. Recall that $\mathbb{I} = \{ \mathbf{i} \in \mathbb{T} : \kappa_{\mathbf{i}} = 1 \}$ denotes the set of internal points of $\mathbb{T}$. It is not hard to see that conditionally on $\mathcal{T}$, the set

$$\Pi := \big\{ (\mathbf{i}, d_{\mathbf{i}}, \tau_{\mathbf{i}}) : \mathbf{i} \in \mathbb{I} \cap \mathbb{S} \big\} \tag{1.12}$$

is a Poisson point process of intensity one on $\mathcal{T} \times [0,1]$. In particular, $\Pi_0 := \big\{ (\mathbf{i}, d_{\mathbf{i}}) : \mathbf{i} \in \mathbb{I} \cap \mathbb{S} \big\}$ is a Poisson point process of intensity one on $\mathcal{T}$ and conditionally on $(\mathcal{T}, \Pi_0)$, there is an independent, uniformly distributed activation time $\tau_{\mathbf{i}}$ attached to each point $(\mathbf{i}, d_{\mathbf{i}}) \in \Pi_0$. The pair $(\mathcal{T}, \Pi)$ is called the *Marked Binary Branching Tree* (MBBT), see [RST19, Subsection 1.5].

As explained in [RST19, Subsection 1.7], the MBBT naturally arises as the near-critical scaling limit of percolation on a wide class of oriented trees.

Frozen percolation on the MBBT has been introduced in [RST19, Subsection 1.6]. Our earlier definitions, translated into the language of the MBBT, result in a process with the following informal description. Initially, all points $(\mathbf{i}, d_{\mathbf{i}}) \in \Pi_0$ are closed. Such points open at their activation time $\tau_{\mathbf{i}}$, provided that by this time they have not yet become frozen. A point $(\mathbf{i}, d_{\mathbf{i}}) \in \Pi_0$ freezes at the first time in Ξ before or at its activation time when the open component of $\mathcal{T}$ that sits just above the point has infinite size.

1.3 Burning times

Let $\Xi \subset (0,1]$ be a relatively closed set of possible freezing times and let $\mathbb{F}$ be a solution to the frozen percolation equation (1.3). We define the *burning time* of a point $\mathbf{i} \in \mathbb{T}$ as

$$Y_{\mathbf{i}} := \inf \left\{ t \in \Xi : \mathbf{i} \overset{\mathbb{T}^t \setminus \mathbb{F}}{\longrightarrow} \infty \right\} \qquad (\mathbf{i} \in \mathbb{T}), \tag{1.13}$$

with the convention that $\inf \emptyset := \infty$. The choice of the term "burning time" is motivated by a certain analogy with forest fire models. The following lemma implies that if $Y_{\mathbf{i}} \leq 1$, then the infimum in (1.13) is in fact a minimum.

Lemma 2 (Percolation times) *For any random subset $\mathbb{A} \subset \mathbb{T}$ and $\mathbf{i} \in \mathbb{T}$, the set $\{t \in [0,1] : \mathbf{i} \overset{\mathbb{T}^t \setminus \mathbb{A}}{\longrightarrow} \infty\}$ is closed.*

We will prove Lemma 2 and Lemma 3 below in Section 2.1. By formula (1.13), the burning times $(Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ are a.s. uniquely determined by the set $\mathbb{F}$ and the i.i.d. randomness Ω. The following lemma shows that conversely, given Ω and $(Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$, one can recover $\mathbb{F}$.

Lemma 3 (Frozen points) *Let $\mathbb{F}$ be a solution to the frozen percolation equation (1.3) and let $(Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ be defined by (1.13). Then*

$$\mathbb{F} = \left\{ \mathbf{i} \in \mathbb{I} : Y_{\mathbf{i}1} \leq \tau_{\mathbf{i}} \right\}. \tag{1.14}$$

Remark 4 *If $\mathbb{F}$ is adapted, then $\mathbb{P}[Y_{\mathbf{i}1} = \tau_{\mathbf{i}}] = 0$ for each $\mathbf{i} \in \mathbb{T}$. According to our definitions, the point $\mathbf{i}$ freezes when $Y_{\mathbf{i}1} = \tau_{\mathbf{i}}$, but as long as we only discuss adapted solutions, it in fact does not matter how things are defined in this case.*

Let $I := [0,1] \cup \{\infty\}$. If $\mathbb{F}$ is a solution to the frozen percolation equation (1.3), then it is not hard to see that the burning times $(Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ satisfy the inductive relation

$$Y_{\mathbf{i}} = \chi[\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}](Y_{\mathbf{i}1}, Y_{\mathbf{i}2}), \tag{1.15}$$

where $\chi : [0,1] \times \{1,2\} \times I^2 \to I$ is the function

$$\chi[\tau, \kappa](x, y) := \begin{cases} x & \text{if } \kappa = 1, \ x > \tau, \\ \infty & \text{if } \kappa = 1, \ x \leq \tau, \\ x \wedge y & \text{if } \kappa = 2. \end{cases} \tag{1.16}$$

Assume that $\mathbb{F}$ is stationary, adapted, and respects the tree structure. Then law of $Y_{\varnothing}$ satisfies the Recursive Distributional Equation (RDE)

$$Y_{\varnothing} \overset{\mathrm{d}}{=} \chi[\omega](Y_1, Y_2), \tag{1.17}$$

where $\overset{\mathrm{d}}{=}$ denotes equality in distribution, Y_1, Y_2 are i.i.d. copies of $Y_{\varnothing}$, and ω is an independent uniformly distributed random variable on $[0,1] \times \{1,2\}$. Proposition 37 of [RST19] classifies all solutions of the RDE (1.17). Expanding on that result, we can prove the following lemma, which is the basis of our proof of Theorem 1.

Lemma 5 (Law of burning times) *For each set $\Xi \subset (0,1]$ that is closed w.r.t. the relative topology of $(0,1]$, there exists a unique probability measure ρ_Ξ on I such that*

(i) *ρ_Ξ solves the RDE (1.17),*

(ii) *ρ_Ξ is concentrated on $\Xi \cup \{\infty\}$,*

(iii) *$\rho_\Xi\big([0,t]\big) \geq \frac{1}{2}t$ for all $t \in \Xi$.*

Assume that $\mathbb{F}$ solves the frozen percolation equation (1.3) for the set of possible freezing times Ξ and that $\mathbb{F}$ is stationary, adapted, and respects the tree structure. Then the burning time of the root $Y_\varnothing$, defined in (1.13), has law ρ_Ξ.

We will prove Lemma 5 together with Lemma 6 below in Section 2.1. The following lemma shows that every solution of the RDE (1.17) is of the form ρ_Ξ for some closed set $\Xi \subset (0,1]$. Below, $\mathrm{supp}(\mu)$ denotes the support of a measure μ.

Lemma 6 (General solutions to the RDE) *If ρ solves the RDE (1.17), then $\rho = \rho_\Xi$ with $\Xi := (0,1] \cap \mathrm{supp}(\rho)$.*

By condition (i) of Lemma 5, for a general closed subset $\Xi \subset (0,1]$, we have $(0,1] \cap \mathrm{supp}(\rho_\Xi) \subset \Xi$. This inclusion may be strict, however, so the correspondence between solutions of the RDE (1.17) and sets of possible freezing times is not one-to-one.

1.4 Almost sure uniqueness

Recall from (1.4) that $\Omega = (\omega_\mathbf{i})_{\mathbf{i} \in \mathbb{T}}$ with $\omega_\mathbf{i} = (\tau_\mathbf{i}, \kappa_\mathbf{i})$. For a given set $\Xi \subset (0,1]$ of possible freezing times, we say that solutions to the frozen percolation equation (1.3) are *almost surely unique* if, whenever $\mathbb{F}$ and $\mathbb{F}'$ solve (1.3) relative to the same Ω, one has $\mathbb{F} = \mathbb{F}'$ a.s.

Let us first note that it is easy to show that if Ξ is a finite subset of $(0,1]$ then the solutions of (1.3) are almost surely unique. Indeed, if $\Xi = \{t_1, ..., t_n\}$ with $0 < t_1 < \cdots < t_n \leq 1$ then one proves by induction on $k = 1, \ldots, n$ that the set of vertices that burn at time t_k is determined by Ω. This implies that the burning time $Y_\mathbf{i}$ of each vertex $\mathbf{i} \in \mathbb{T}$ is determined by Ω, hence the set $\mathbb{F}$ is also determined by Ω using (1.14).

For the remainder of the paper, we will mostly focus our attention on a one-parameter family of sets of possible burning times. For $0 < \theta < 1$, we define $\Xi_\theta := \{\theta^n : n \in \mathbb{N}\}$ (with $\mathbb{N} := \{0, 1, 2, \ldots\}$) and we set $\Xi_1 := (0,1]$, which can naturally be viewed as the limit of Ξ_θ as $\theta \to 1$. As a straightforward application of [RST19, Prop 37], one can check that for these sets, the probability laws ρ_Ξ from Lemma 5 are given by

$$
\begin{aligned}
\rho_{\Xi_\theta}(\mathrm{d}t) &= \frac{1-\theta}{1+\theta} \sum_{k=0}^\infty \theta^k \delta_{\theta^k}(\mathrm{d}t) + \frac{\theta}{1+\theta}\delta_\infty(\mathrm{d}t) \qquad (0 < \theta < 1), \\
\rho_{\Xi_1}(\mathrm{d}t) &= \tfrac{1}{2}\mathrm{d}t + \tfrac{1}{2}\delta_\infty(\mathrm{d}t).
\end{aligned}
\tag{1.18}
$$

It is not hard to see that ρ_{Ξ_θ} converges weakly to ρ_{Ξ_1} as $\theta \to 1$.

We conjecture that for the sets Ξ_θ with $0 < \theta < \frac{1}{2}$, solutions to the frozen percolation equation are almost surely unique. We have not been able to prove this, but we can prove that there exists a $\frac{1}{2} < \theta^* < 1$ such that almost sure uniqueness does not hold for $\theta > \theta^*$ and almost sure uniqueness holds under additional assumptions for $\theta \leq \theta^*$.

To explain this in more detail, fix $\Omega = (\omega_\mathbf{i})_{\mathbf{i} \in \mathbb{T}}$, let $\mathbb{F}$ be a solution to the frozen percolation equation (1.3) that is stationary, adapted, and respects the tree structure, and let $(Y_\mathbf{i})_{\mathbf{i} \in \mathbb{T}}$ be the burning times defined in (1.13). Then the random variables $(\omega_\mathbf{i}, Y_\mathbf{i})_{\mathbf{i} \in \mathbb{T}}$ form a Recursive Tree Process (RTP) as defined in [AB05]. Following a definition from [AB05], one says that such an RTP is *endogenous* if $Y_\varnothing$ is measurable w.r.t. the σ-field generated by the collection of random variables $\Omega = (\omega_\mathbf{i})_{\mathbf{i} \in \mathbb{T}}$. We make the following observation.

Lemma 7 (Endogeny and almost sure uniqueness) *The following claims are equivalent:*

(i) *The RTP $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is endogenous.*

(ii) *If $\mathbb{F}$ and $\mathbb{F}'$ solve (1.3) relative to the same Ω, and moreover $\mathbb{F}$ and $\mathbb{F}'$ are stationary, adapted, and respect the tree structure, then $\mathbb{F} = \mathbb{F}'$ a.s.*

Proof Let $(\Omega, \mathbb{F})$ be as in Theorem 1 and let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the corresponding RTP. Condition (i) says that the law of $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ given Ω is a delta-measure, while condition (ii) says that the law of $\mathbb{F}$ given Ω is a delta-measure. Now the claim follows from the fact that by (1.13) and (1.14), given Ω, the set $\mathbb{F}$ and collection of random variables $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ determine each other a.s. uniquely. ∎

It follows from Lemma 7 that if the RTP $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is nonendogenous, then solutions to the frozen percolation equation (1.3) are not almost surely unique. We pose the converse implication as an open problem:

> **Question 1** Does endogeny of the RTP $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ imply almost sure uniqueness of solutions to the frozen percolation equation (1.3)?

In other words, Question 1 asks whether in part (ii) of Lemma 7, one can remove the conditions that $\mathbb{F}$ and $\mathbb{F}'$ are stationary, adapted, and respect the tree structure.

We now address the question of endogeny. To state our main result, we need one technical lemma, which introduces a parameter θ^*. Numerically, we find that $\theta^* \approx 0.636$.

Lemma 8 (The critical parameter) *Let $g : (0,1) \to \mathbb{R}$ be defined as*

$$g(\theta) := 2(1 + \theta) - \sum_{\ell=0}^{\infty} \frac{\theta^{2\ell}(1 - \theta^2)}{2/(1+\theta) - \theta^{\ell}}. \tag{1.19}$$

Then g is a strictly decreasing continuous function that changes sign at a point $\theta^* \in (\frac{1}{2}, 1)$.

The following theorem is the main result of our paper. For $\theta = 1$, the result has been proved in [RST19, Thm 12] but the result is new in the regime $0 < \theta < 1$.

Theorem 9 (Endogeny) *Let θ^* be as in Lemma 8. Let $0 < \theta \leq 1$, and for the set of possible freezing times Ξ_θ, let $(\Omega, \mathbb{F})$ be defined as in Theorem 1. Let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the corresponding RTP of burning times defined in (1.13). This RTP is endogenous for $0 < \theta \leq \theta^*$ but not for $\theta^* < \theta \leq 1$.*

In the Subsections 1.5 and 1.6 below, we eleborate a bit on our methods for proving Theorem 9. We use the remainder of the present subsection to make a few additional comments on Question 1 posed above.

Set $\mathbb{F}_0 := \emptyset$ and define inductively for $k \geq 1$

$$Y_{\mathbf{i}}^k := \inf\left\{t \in \Xi : \mathbf{i} \overset{\mathbb{T}^t \backslash \mathbb{F}_{k-1}}{\longrightarrow} \infty\right\} \quad (\mathbf{i} \in \mathbb{T}) \quad \text{and} \quad \mathbb{F}_k := \left\{\mathbf{i} \in \mathbb{I} : Y_{\mathbf{i}1}^k \leq \tau_{\mathbf{i}}\right\}, \tag{1.20}$$

with the usual convention that $\inf \emptyset := \infty$. Then it is not hard to see that

$$
\begin{array}{ll}
\text{(i) } \mathbb{F}_{2n} \subset \mathbb{F}_{2n+1} & \text{(ii) } \mathbb{F}_{2n+1} \supset \mathbb{F}_{2n+2}, \\
\text{(iii) } \mathbb{F}_{2n} \subset \mathbb{F}_{2n+2} & \text{(iv) } \mathbb{F}_{2n+1} \supset \mathbb{F}_{2n+3},
\end{array}
\quad (n \in \mathbb{N}). \tag{1.21}
$$

Moreover, if $\mathbb{F}$ solves the frozen percolation equation (1.3), then

$$\mathbb{F}_{2n} \subset \mathbb{F} \subset \mathbb{F}_{2n+1} \qquad (n \in \mathbb{N}). \tag{1.22}$$

For the sets of possible burning times Ξ_θ with $0 < \theta \leq 1$, it is possible to verify by calculation that $\mathbb{F}_2 = \emptyset$ a.s. if and only if $\theta \geq 1/2$. In particular, if $\theta < 1/2$, then there are points that must freeze in any solution to the frozen percolation equation (1.3). We conjecture that in fact, for any $\theta < 1/2$, the sets $\bigcup_{n \in \mathbb{N}} \mathbb{F}_{2n}$ and $\bigcap_{n \in \mathbb{N}} \mathbb{F}_{2n+1}$ are a.s. equal and as a result, solutions to the frozen percolation equation (1.3) are a.s. unique for all $\theta < 1/2$. Note that even if this conjecture is correct, it does not not fully settle Question 1, since the parameter θ^* from Lemma 8 is strictly larger than $1/2$.

1.5 Scale invariance

We fix a set of possible burning times Ξ, construct a frozen percolation process $(\Omega, \mathbb{F})$ as in Theorem 1 and let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ be the corresponding RTP of burning times defined in (1.13). Conditional on $\Omega = (\omega_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$, let $(Y'_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ be an independent copy of $(Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$. Then endogeny is equivalent to the statement that $Y_\varnothing = Y'_\varnothing$ a.s. It is easy to see that the joint law of $(Y_\varnothing, Y'_\varnothing)$ solves the *bivariate RDE*

$$(Y_\varnothing, Y'_\varnothing) \overset{\mathrm{d}}{=} \big(\chi[\omega](Y_1, Y_2), \chi[\omega](Y'_1, Y'_2)\big), \tag{1.23}$$

where (Y_1, Y'_1) and (Y_2, Y'_2) are independent copies of $(Y_\varnothing, Y'_\varnothing)$ and ω is an independent uniformly distributed random variable on $[0, 1] \times \{1, 2\}$. We define probability laws on I^2 by

$$\underline{\rho}^{(2)}_\Xi := \mathbb{P}\big[(Y_\varnothing, Y'_\varnothing) \in \,\cdot\,\big] \quad \text{and} \quad \overline{\rho}^{(2)}_\Xi := \mathbb{P}\big[(Y_\varnothing, Y_\varnothing) \in \,\cdot\,\big]. \tag{1.24}$$

Then endogeny is equivalent to the statement that $\underline{\rho}^{(2)}_\Xi = \overline{\rho}^{(2)}_\Xi$. General theory for RTPs developed in [AB05, Thm 11] (see also [MSS18, Thm 1]) yields the following:

Proposition 10 (Bivariate uniqueness) *The following statements are equivalent:*

(i) *The RTP $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ is endogenous.*

(ii) *The measure $\overline{\rho}^{(2)}_\Xi$ is the only solution of the bivariate RDE (1.23) in the space of symmetric probability measures on I^2 with marginals given by ρ_Ξ.*

Proposition 10 is our main tool for proving Theorem 9, but in order to be able to successfully apply Proposition 10, we need one more idea. For a general set of possible burning times Ξ, it is difficult to find all solutions of the bivariate RDE (1.23) in the space of symmetric probability measures with marginals given by ρ_Ξ. For the special sets Ξ_θ with $0 < \theta \leq 1$, however, it turns out to be sufficient to look only at scale invariant solutions of the bivariate RDE. As we will explain below, this leads to a significant simplification of the problem, which allows us to prove Theorem 9 for the sets Ξ_θ, but not for general Ξ.

It has been proved in [RST19, Prop 9] that the law of the MBBT is invariant under a certain scaling relation. This is ultimately the consequence of the fact that the MBBT is itself the scaling limit of near-critical percolation on trees of finite degree. We will not repeat the scaling property of the MBBT here but instead formulate scaling properties of solutions to the RDE (1.17) and bivariate RDE (1.23) that are consequences of the scaling of the MBBT.

For each $t > 0$, we define a scaling map $\psi_t : I \to I$ by

$$\psi_t(y) := \begin{cases} t^{-1}y & \text{if } y \leq t, \\ \infty & \text{otherwise.} \end{cases} \tag{1.25}$$

We let $\mathcal{M}^{(1)}$ denote the space of all probability measures ρ on $I = [0, 1] \cup \{\infty\}$ that satisfy $\rho\big([0, t]\big) \leq t$ for all $0 \leq t \leq 1$, and we define scaling maps Γ_t by

$$\Gamma_t \rho := t^{-1} \rho \circ \psi_t^{-1} + (1 - t^{-1})\delta_\infty \qquad (\rho \in \mathcal{M}^{(1)}, \; t > 0), \tag{1.26}$$

where δ_∞ denotes the delta-measure at ∞. It is not hard to see that Γ_t maps the space $\mathcal{M}^{(1)}$ into itself. In particular, for $0 < t < 1$, the assumption $\rho([0,t]) \le t$ guarantees that $\Gamma_t \rho$ puts nonnegative mass at ∞. The following lemma says that the set of solutions to the RDE (1.17) is invariant under the scaling maps Γ_t. Below, ρ_Ξ denotes the measure defined in Lemma 5.

Lemma 11 (Scale invariance of the RDE) *Let ρ be a solution to the RDE (1.17). Then $\rho \in \mathcal{M}^{(1)}$, and for each $t > 0$, the measure $\Gamma_t \rho$ is also a solution to the RDE (1.17). In particular, if Ξ is a relatively closed subset of $(0,1]$, then*

$$\Gamma_t \rho_\Xi = \rho_{\Xi'} \quad with \quad \Xi' := \{t^{-1}y : y \in \Xi\} \cap [0,1] \qquad (t > 0). \tag{1.27}$$

For the bivariate RDE, a result similar to Lemma 11 holds, which we formulate now. We say that a probability measure on I^2 is *symmetric* if it is invariant under the map $(y_1, y_2) \mapsto (y_2, y_1)$. Let $\mathcal{M}^{(2)}$ denote the space of all symmetric probability measures $\rho^{(2)}$ on I^2 that satisfy

$$\rho^{(2)}\big([0,t] \times I \cup I \times [0,t]\big) \le t \qquad \forall 0 \le t \le 1. \tag{1.28}$$

We define $\psi_t^{(2)} : I^2 \to I^2$ by $\psi_t^{(2)}(y, y') := \big(\psi_t(y), \psi_t(y')\big)$ and we define $\Gamma_t^{(2)} : \mathcal{M}^{(2)} \to \mathcal{M}^{(2)}$ by

$$\Gamma_t^{(2)} \rho := t^{-1} \rho \circ (\psi_t^{(2)})^{-1} + (1 - t^{-1})\delta_{(\infty,\infty)} \qquad (\rho \in \mathcal{M}^{(2)}, \ t > 0). \tag{1.29}$$

We will prove that $\rho \in \mathcal{M}^{(2)}$ indeed implies $\Gamma_t^{(2)} \rho \in \mathcal{M}^{(2)}$ in Section 2.2. With the above definitions, we have the following lemmas, which are analogous to Lemma 11. The measures $\underline{\rho}_\Xi^{(2)}$ and $\overline{\rho}_\Xi^{(2)}$ that occur in Lemma 13 are defined in (1.24).

Lemma 12 (Scale invariance of bivariate RDE) *Let $\rho^{(2)}$ be a symmetric solution to the bivariate RDE (1.23). Then $\rho^{(2)} \in \mathcal{M}^{(2)}$, and for each $t > 0$, the measure $\Gamma_t^{(2)} \rho^{(2)}$ is also a solution to (1.23).*

Lemma 13 (Scale invariance of special solutions) *Let $\Xi \subset (0,1]$ be relatively closed. Then, for each $t > 0$,*

$$\Gamma_t^{(2)} \underline{\rho}_\Xi^{(2)} = \underline{\rho}_{\Xi'}^{(2)} \quad and \quad \Gamma_t^{(2)} \overline{\rho}_\Xi^{(2)} = \overline{\rho}_{\Xi'}^{(2)} \quad with \quad \Xi' := \{t^{-1}y : y \in \Xi\} \cap [0,1]. \tag{1.30}$$

1.6 Scale invariant solutions to the bivariate RDE

In the present subsection, we explain how scale invariance helps us prove our main result Theorem 9. It follows from Lemma 11, and can also easily be checked by direct calculation using formula (1.18), that the measures ρ_{Ξ_θ} are invariant under scaling by θ, and hence also by θ^n for each $n \ge 0$. Likewise, ρ_{Ξ_1} is invariant under scaling by any $0 < t \le 1$, so we have

$$\Gamma_t \rho_{\Xi_\theta} = \rho_{\Xi_\theta} \qquad \big(0 < \theta \le 1, \ t \in \Xi_\theta\big). \tag{1.31}$$

Motivated by this, for $0 < \theta \le 1$, we let $\mathcal{M}_\theta^{(2)}$ denote the space of probability measures $\rho^{(2)}$ on I^2 such that:

(i) $\rho^{(2)} \in \mathcal{M}^{(2)}$,

(ii) the marginals of $\rho^{(2)}$ are given by ρ_{Ξ_θ},

(iii) $\Gamma_t^{(2)} \rho^{(2)} = \rho^{(2)}$ for all $t \in \Xi_\theta$.

Let $0 < \theta \le 1$, and for the set of possible freezing times Ξ_θ, let $(\Omega, \mathbb{F})$ be defined as in Theorem 1. Let $(\omega_\mathbf{i}, Y_\mathbf{i})_{\mathbf{i} \in \mathbb{T}}$ be the corresponding RTP of burning times defined in (1.13). It follows from Proposition 10 and Lemma 13 that the RTP $(\omega_\mathbf{i}, Y_\mathbf{i})_{\mathbf{i} \in \mathbb{T}}$ is endogenous if and only if $\overline{\rho}_{\Xi_\theta}^{(2)}$ is the only solution of the bivariate RDE (1.23) in the space $\mathcal{M}_\theta^{(2)}$. In view of this, Theorem 9 is implied by the following theorem.

Theorem 14 (Scale invariant solutions of the bivariate RDE) *Let θ^* be as in Lemma 8 and let $0 < \theta \leq 1$. Then:*

(i) *If $\theta \leq \theta^*$ then $\overline{\rho}^{(2)}_{\Xi_\theta}$ is the only solution of the bivariate RDE (1.23) in the space $\mathcal{M}^{(2)}_\theta$.*

(ii) *If $\theta^* < \theta$, then there exists a measure $\hat{\rho}^{(2)} \in \mathcal{M}^{(2)}_\theta$ with $\hat{\rho}^{(2)} \neq \overline{\rho}^{(2)}_{\Xi_\theta}$ that solves (1.23).*

We call $\overline{\rho}^{(2)}_{\Xi_\theta}$ the *diagonal* solution of the bivariate RDE since it is concentrated on $\{(y, y) : y \in I\}$ (see (1.24)). In the special case $\theta = 1$, Theorem 14 has been proved in [RST19, Thm 12], where it is moreover shown that the bivariate RDE (1.23) has precisely two solutions in the space $\mathcal{M}^{(2)}_1$. We conjecture that this holds more generally. In Remark 68 below, we present numerical evidence for the following conjecture.

Conjecture 15 (Uniqueness of the nondiagonal solution) *For all $\theta^* < \theta \leq 1$, the measures $\overline{\rho}^{(2)}_{\Xi_\theta}$ and $\underline{\rho}^{(2)}_{\Xi_\theta}$ defined in (1.24) are the only solutions of the bivariate RDE (1.23) in the space $\mathcal{M}^{(2)}_\theta$.*

The main advantage of scale invariance is that it reduces the number of parameters. In general, we can characterise a measure on $[0, 1]^2$ by its distribution function, which is a real function of two variables. However, using scale invariance, we can characterise a measure $\rho^{(2)} \in \mathcal{M}^{(2)}_\theta$ using a real function of one variable only, see Definition 37 below. This significantly simplifies the calculations.

We can in fact be a little more general. Generalizing the definition above (1.18), for $0 < \theta < 1$ and $0 < \alpha \leq 1$, let us define $\Xi_{\theta,\alpha} := \{\alpha\theta^n : n \in \mathbb{N}\}$. Then Lemma 11 implies that $\rho_{\Xi_{\theta,\alpha}} = \Gamma_{1/\alpha}\rho_{\Xi_\theta}$. Moreover, Proposition 10 and Lemma 13 imply that the RTP corresponding to $\rho_{\Xi_{\theta,\alpha}}$ is endogenous if and only if the RTP corresponding to ρ_{Ξ_θ} is endogenous. Since this does not conceptually add anything new, for simplicity, we have formulated our main results only for the set of possible burning times Ξ_θ.

2 Frozen percolation on the MBBT

2.1 Existence and uniqueness in law

In this subsection, we prove Theorem 1 and Lemmas 2, 3, 5, and 6.

Proof of Lemma 2 It suffices to prove the claim for $\mathbf{i} = \varnothing$. Let $P := \{t \in [0, 1] : \varnothing \xrightarrow{\mathbb{T}^t \setminus \mathbb{A}} \infty\}$. Similar to the definition in (1.1), for any $\mathbf{i}, \mathbf{k} \in \mathbb{T}$ and $A \subset \mathbb{T}$, we write $\mathbf{i} \xrightarrow{A} \mathbf{k}$ if there exist a $\mathbf{j} = j_1 \cdots j_n \in \mathbb{T}$ such that $\mathbf{k} = \mathbf{ij}$ and

$$\text{(i)} \quad j_{k+1} \leq \kappa_{\mathbf{i}j_1\cdots j_k} \quad (0 \leq k < n) \quad \text{and} \quad \text{(ii)} \quad \mathbf{i}j_1\cdots j_k \in A \quad (0 \leq k \leq n). \tag{2.1}$$

Then for each finite n, the set

$$P_n := \left\{t \in [0, 1] : \varnothing \xrightarrow{\mathbb{T}^t \setminus \mathbb{A}} \mathbf{j} \text{ for some } \mathbf{j} \in \mathbb{T} \text{ with } |\mathbf{j}| = n\right\} \tag{2.2}$$

is closed, and hence the same is true for $P = \bigcap_{n \geq 0} P_n$. ∎

Proof of Lemma 3 Let $\mathbf{i} \in \mathbb{I}$. If $Y_{\mathbf{i}1} > \tau_{\mathbf{i}}$, then clearly there exists no $t \in (0, \tau_{\mathbf{i}}]$ such that $\mathbf{i}1 \xrightarrow{\mathbb{T}^t \setminus \mathbb{F}} \infty$, and hence by (1.3) $\mathbf{i} \notin \mathbb{F}$. On the other hand, if $Y_{\mathbf{i}1} \leq \tau_{\mathbf{i}}$, then by Lemma 2 one has $\mathbf{i}1 \xrightarrow{\mathbb{T}^t \setminus \mathbb{F}} \infty$ for $t = Y_{\mathbf{i}1}$. Since $\Xi \subset (0, 1]$ is closed w.r.t. the relative topology and $Y_\varnothing > 0$ holds a.s. (this follows from Lemma 29, see below), we have $Y_{\mathbf{i}1} \in \Xi$ and hence by (1.3) $\mathbf{i} \in \mathbb{F}$. ∎

To prepare for the proof of Lemma 5, we need a bit of theory. Let BV denote the space of functions $F : \mathbb{R} \to \mathbb{R}$ that are locally of bounded variation. For each $F \in \mathrm{BV}$, the right and

left limits $F(t+) := \lim_{s \downarrow t} F(s)$ and $F(t-) := \lim_{s \uparrow t} F(s)$ exist for each $t \in \mathbb{R}$, and F defines a signed measure $\mathrm{d}F$ on $\mathbb{R}$ by any of the equivalent formulas

$$\mathrm{d}F\big((s,t]\big) = F(t+) - F(s+) \quad (s < t) \quad \text{and} \quad \mathrm{d}F\big([s,t)\big) = F(t-) - F(s-) \quad (s < t). \quad (2.3)$$

For $G, F \in \mathrm{BV}$, we let $G\mathrm{d}F$ denote the signed measure obtained by weighting $\mathrm{d}F$ with the density G. For $F \in \mathrm{BV}$, we define

$$\overline{F}(t) := \tfrac{1}{2}\big(F(t-) + F(t+)\big) \qquad (t \in \mathbb{R}). \quad (2.4)$$

It is well-known that a right-continuous function with left limits makes at most countably many jumps, and hence $\overline{F}(t) \neq F(t)$ for at most countably many values of t. We will need the following simple fact.

Lemma 16 (Product rule) *For $F, G \in \mathrm{BV}$, one has $FG \in \mathrm{BV}$ and $\mathrm{d}(FG) = \overline{F}\mathrm{d}G + \overline{G}\mathrm{d}F$.*

Proof The statement is well-known if F and G are continuous. Therefore, since our formula is linear in F and G and since each measure can be decomposed into an atomic and nonatomic part, it suffices to prove the statement only when $\mathrm{d}F$ and $\mathrm{d}G$ are purely atomic. Using again linearity and a simple limit argument, it suffices to prove the statement only in the case that $\mathrm{d}F = \delta_s$ and $\mathrm{d}G = \delta_t$ for some $s, t \in \mathbb{R}$. If $s \neq t$, the statement is trivial. If $s = t$, then the statement follows from the observation that

$$\begin{aligned} F(t+)G(t+) - F(t-)G(t-) &= \tfrac{1}{2}\big(F(t+) + F(t-)\big)\big(G(t+) - G(t-)\big) \\ &\quad + \tfrac{1}{2}\big(G(t+) + G(t-)\big)\big(F(t+) - F(t-)\big). \end{aligned} \quad (2.5)$$

∎

We cite the following lemma from [RST19, Lemma 38].

Lemma 17 (Integral formulation of RDE) *A probability measure ρ on I solves the RDE (1.17) if and only if*

$$\int_{[0,t]} \rho(\mathrm{d}s)s = \rho\big([0,t]\big)^2 \qquad \big(t \in [0,1]\big). \quad (2.6)$$

The following lemma is just a simple rewrite of the previous one. Below, we let $\mu\big|_A$ denote the restriction of a (signed) measure μ to a measurable set A, defined as $\mu\big|_A(B) := \mu(A \cap B)$.

Lemma 18 (Differential formulation of RDE) *Let T denote the identity function $T(t) := t$ $(t \in \mathbb{R})$. Assume that $F \in \mathrm{BV}$ is right-continuous and nondecreasing and satisfies $F(t) = 0$ $(t < 0)$, $F(t) = F(1)$ $(t > 1)$, and*

$$T\mathrm{d}F = 2\overline{F}\mathrm{d}F. \quad (2.7)$$

Then there exists a unique solution ρ to the RDE (1.17) such that

$$\rho\big([0,t]\big) = F(t) \qquad \big(t \in [0,1]\big), \quad (2.8)$$

and each solution ρ to the RDE (1.17) arises in this way.

Proof Let ρ be a solution of the RDE (1.17) and let $F : [0,1] \to \mathbb{R}$ be defined as in (2.8). Extend F to a function in BV by setting $F(t) := 0$ for $t < 0$ and $F(t) := F(1)$ for $t > 1$. Then by Lemma 17, $\int_{(0,t]} T\mathrm{d}F = F(t)^2$ $(t \in [0,1])$, which by Lemma 16 implies that F solves (2.7).

Assume, conversely, that $F \in \mathrm{BV}$ is right-continuous and nondecreasing and satisfies $F(t) = 0$ $(t < 0)$ and (2.7). Then clearly $F \geq 0$. Formula (2.7) implies that for a.e. t w.r.t. $\mathrm{d}F$, we have $\overline{F}(t) = \tfrac{1}{2}t$, which by the fact that $F \geq 0$ implies $F(t) \leq t$. It follows that setting $\rho\big([0,t]\big) := F(t)$ $(t \in [0,1])$ defines a subprobability measure on $[0,1]$, which can uniquely be

extended to a probability measure on $[0,1] \cup \{\infty\}$. Lemma 16 implies that $\int_{(0,t]} T \mathrm{d}F = F(t)^2$ ($t \in [0,1]$), so using the fact that $\rho(\{0\}) = F(0) - F(0-) = 0$ and Lemma 17, we conclude that ρ solves the RDE (1.17). ∎

Let $T \in \mathrm{BV}$ denote the identity function $T(t) := t$ ($t \in \mathbb{R}$). For a given closed set $\Xi \subset \mathbb{R}$, we will be interested in right-continuous functions $F \in \mathrm{BV}$ that solve the differential equation

$$\text{(i) } T\mathrm{d}F = 2\overline{F}\mathrm{d}F, \quad \text{(ii) } \mathrm{d}F\big|_{\Xi} = \mathrm{d}F \quad \text{(iii) } F(t) \geq \tfrac{1}{2}t \quad (t \in \Xi). \tag{2.9}$$

Note that condition (ii) says that the signed measure $\mathrm{d}F$ is concentrated on Ξ. Our first lemma says that the distance between two solutions of (2.9) is nonincreasing function of time.

Lemma 19 (Distance between two solutions) *Let $\Xi \subset \mathbb{R}$ be closed and for $i = 1, 2$, let $F_i \in \mathrm{BV}$ be right-continuous solutions to the differential equation (2.9). Then $\big|F_1(t) - F_2(t)\big| \leq \big|F_1(s) - F_2(s)\big|$ ($s \leq t$).*

Proof We observe that by (i), we have $\overline{F}_i(t) = \tfrac{1}{2}t$ for a.e. t w.r.t. $\mathrm{d}F_i$. In particular, $\overline{F}_i(t) = \tfrac{1}{2}t$ whenever $F_i(t-) \neq F_i(t)$, which we can combine with conditions (ii) and (iii) to get

$$\text{(iii)' } \overline{F}_i(t) \geq \tfrac{1}{2}t \quad (t \in \Xi).$$

We now use Lemma 16 to calculate

$$\begin{aligned}
\tfrac{1}{2}\mathrm{d}(F_1 - F_2)^2 &= (\overline{F}_1 - \overline{F}_2)(\mathrm{d}F_1 - \mathrm{d}F_2) \\
&= \overline{F}_1\mathrm{d}F_1 - \overline{F}_1\mathrm{d}F_2 - \overline{F}_2\mathrm{d}F_1 + \overline{F}_2\mathrm{d}F_2 = (\tfrac{1}{2}T - \overline{F}_2)\mathrm{d}F_1 + (\tfrac{1}{2}T - \overline{F}_1)\mathrm{d}F_2,
\end{aligned} \tag{2.10}$$

where in the last step we have used (i). Using moreover (ii) and (iii)', we see that the right-hand side of (2.10) is nonpositive, so the claim of the lemma follows by integration. ∎

Let $\mathcal{F}$ denote the space of all right-continuous, nondecreasing functions $F : \mathbb{R} \to \mathbb{R}$ that satisfy $0 \leq F(t) \leq 0 \vee t$ ($t \in \mathbb{R}$). In other words, these are the distribution functions of nonnegative measures $\mathrm{d}F$ on $[0, \infty)$ that satisfy $\mathrm{d}F([0, t]) \leq t$ for all $t \geq 0$. We equip $\mathcal{F}$ with a topology that corresponds to vague convergence of the measures $\mathrm{d}F$. Then $\mathcal{F}$ is a compact, metrisable space and $F_n \to F$ in the topology on $\mathcal{F}$ if and only if $F_n(t) \to F(t)$ for each continuity point t of F. Our aim is to prove that each closed set $\Xi \subset [0, \infty)$, there exists a unique $F \in \mathcal{F}$ that solves (2.9). Uniqueness follows from Lemma 19, so it remains to prove existence. We will use an approximation argument. We start by proving the statement for finite Ξ.

Lemma 20 (Finite sets) *For each finite set $\Xi \subset (0, \infty)$, there exists an $F \in \mathcal{F}$ that solves (2.9).*

Proof Let $\Xi = \{t_1, \ldots, t_n\}$ with $0 =: t_0 < t_1 < \cdots < t_n$. We inductively define F so that it is constant on each of the intervals $(-\infty, t_1), [t_1, t_2), \ldots [t_{n-1}, t_n)$, and $[t_n, \infty)$, satisfies $F(0) = 0$, and

$$F(t_k) := F(t_{k-1}) \vee \big(t_k - F(t_{k-1})\big) \qquad (1 \leq k \leq n). \tag{2.11}$$

Then clearly F satisfies (2.9) (ii) and (iii). Moreover, for each $1 \leq k \leq n$, we have either $F(t_k) = F(t_{k-1})$ or $F(t_k) = t_k - F(t_{k-1})$. In either case,

$$t_k\big(F(t_k) - F(t_{k-1})\big) = 2 \cdot \tfrac{1}{2}\big(F(t_k) + F(t_{k-1})\big)\big(F(t_k) - F(t_{k-1})\big), \tag{2.12}$$

which shows that F satisfies (2.9) (i). It is clear that F is right-continuous, nonnegative, and nondecreasing, and by induction (2.11) also implies that $F(t) \leq t$ for all $t \geq 0$, showing that $F \in \mathcal{F}$. ∎

Let d be any metric generating the topology on $[0, \infty]$ and let $\mathcal{K}[0, \infty]$ denote the space of all closed subsets of $[0, \infty]$. For each $A \in \mathcal{K}[0, \infty]$ and $\varepsilon > 0$, we set

$$A_\varepsilon := \big\{t \in [0, \infty] : d(t, A) < \varepsilon\big\} \quad \text{where} \quad d(t, A) := \inf_{s \in A} d(t, s). \tag{2.13}$$

We equip $\mathcal{K}[0,\infty]$ with the *Hausdorff metric*

$$d_{\mathrm{H}}(A,B) := \inf\big\{\varepsilon > 0 : A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\big\}. \tag{2.14}$$

The following lemma lists some elementary properties of the space $\mathcal{K}[0,\infty]$.

Lemma 21 (Properties of the Hausdorff metric) *The space $\mathcal{K}[0,\infty]$ is compact and the set of all finite subsets of $(0,\infty)$ is dense in $\mathcal{K}[0,\infty]$.*

Proof The fact that $\mathcal{K}(E)$ is compact if E is compact is well-known, see, e.g., [SSS14, Lemm B.4]. If $\Xi \subset [0,1]$ is closed, then it is easy to see that the sets $\Xi_n := \{k/n : 1 < k < n, \ d(k/n,\Xi) \leq 1/n\}$ converge to Ξ in the Hausdorff metric. This shows that the set of finite subsets of $(0,1)$ is dense in $\mathcal{K}[0,1]$. Since $[0,\infty]$ is homeomorphic to $[0,1]$, the same is true with $(0,1)$ and $[0,1]$ replaced by $(0,\infty)$ and $[0,\infty]$. ∎

Our next lemma will allow us to construct solutions to (2.9) for general Ξ by approximation with finite Ξ.

Lemma 22 (Limits of solutions) *Let $F, F_n \in \mathcal{F}$ and $\Xi_n, \Xi \in \mathcal{K}[0,\infty]$ satisfy $F_n \to F$ and $\Xi_n \to \Xi$. Assume that F_n solves (2.9) relative to $\Xi_n \cap [0,\infty)$ for each n. Then F solves (2.9) relative to $\Xi \cap [0,\infty)$.*

Proof Recall that $F_n \to F$ means that $\mathrm{d}F_n \to \mathrm{d}F$ vaguely, or equivalently, $F_n(t) \to F(t)$ for each continuity point t of F. Since T is a continuous function, the vague convergence $\mathrm{d}F_n \to \mathrm{d}F$ implies that also $T\mathrm{d}F_n \to T\mathrm{d}F$ vaguely. By Lemma 16, $2\overline{F}\mathrm{d}F = \mathrm{d}F^2$. Now if $F_n(t) \to F(t)$ for each continuity point t of F, then also $F_n^2(t) \to F^2(t)$ for each continuity point t of F^2, so taking the value limit on the left- and right-hand sides of the equation, we see that F solves (2.9) (i). Since $\Xi_n \to \Xi$, we easily obtain that $\mathrm{d}F$ is concentrated on Ξ_ε for each $\varepsilon > 0$, and hence F satisfies (2.9) (ii). To see that F also satisfies (2.9) (iii), fix $t \in \Xi$. Since $\Xi_n \to \Xi$ we can find $t_n \in \Xi_n$ such that $t_n \to t$. Then for each $s > t$, we have $F_n(s) \geq F_n(t_n) \geq \frac{1}{2}t_n$ for all n large enough. Taking the limit, it follows that $F(s) \geq \frac{1}{2}t$ for each $s \geq t$ that is a continuity point of s, and hence $F(t) \geq \frac{1}{2}t$ by right-continuity. ∎

We can now prove existence of solutions to (2.9) for general Ξ.

Lemma 23 (Existence of solutions to the RDE) *For each closed set $\Xi \subset [0,\infty)$, there exists a function $F \in \mathcal{F}$ that solves (2.9).*

Proof By Lemma 21, for each closed $\Xi \subset [0,\infty]$, there exist finite $\Xi_n \subset (0,\infty)$ such that $\Xi_n \to \Xi$. By Lemma 20, for each n there exists an $F_n \in \mathcal{F}$ so that F_n solves (2.9) relative to Ξ_n. Since $\mathcal{F}$ is compact, by going to a subsequence if necessary we can assume that $F_n \to F$ for some $F \in \mathcal{F}$. Then Lemma 22 tells us that F solves (2.9) relative to $\Xi \cap [0,\infty)$. ∎

Before we prove Lemma 5, we recall the general definition of an RTP. Let $\mathbb{T}$ denote the space of all finite words $\mathbf{i} = i_1 \cdots i_n$ ($n \geq 0$) made up from the alphabet $\{1,\ldots,d\}$, where $d \geq 1$ is some fixed integer. All previous notation involving the binary tree generalizes in a straightforward manner to the d-ary tree $\mathbb{T}$. Let I and Ω be Polish spaces, let $\gamma : \Omega \times I^d \to I$ be a measurable function, and let $(\omega_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ be i.i.d. Ω-valued random variables. Let ν be a probability law on I that solves the Recursive Distributional Equation (RDE)

$$X_\varnothing \stackrel{\mathrm{d}}{=} \gamma[\omega_\varnothing](X_1,\ldots,X_d), \tag{2.15}$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution, $X_\varnothing$ has law ν, and $X_1,\ldots,X_d$ are copies of $X_\varnothing$, independent of each other and of $\omega_\varnothing$. A simple argument based on Kolmogorov's extension theorem (see [MSS20, Lemma 8]) tells us that the i.i.d. random variables $(\omega_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ can be coupled to I-valued random variables $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ in such a way that:

(i) For each finite rooted subtree $\mathbb{U} \subset \mathbb{T}$, the r.v.'s $(X_{\mathbf{i}})_{\mathbf{i} \in \partial \mathbb{U}}$ are i.i.d. with common law ν and independent of $(\omega_{\mathbf{i}})_{\mathbf{i} \in \mathbb{U}}$.

(ii) $X_{\mathbf{i}} = \gamma[\omega_{\mathbf{i}}](X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}d})$ $\qquad (\mathbf{i} \in \mathbb{T})$.

Moreover, these conditions uniquely determine the joint law of $(\omega_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$. We call the latter the *Recursive Tree Process* (RTP) corresponding to the maps γ and solution ν of the RDE (2.15).

Proof of Lemma 5 Let $\Xi \subset (0,1]$ be relatively closed and let $\overline{\Xi} := \Xi \cup \{0\}$. By Lemma 23, there exists a solution $F \in \mathcal{F}$ of the differential equation (2.9) relative to $\overline{\Xi}$. Set $\rho_{\Xi}([0,t]) := F(t)$ $(t \in [0,1])$ and $\rho_{\Xi}(\{\infty\}) := 1 - F(1)$ and observe that $\rho_{\Xi}(\{0\}) = 0$. Then ρ_{Ξ} is a probability measure on I that satisfies conditions (ii) and (iii) of Lemma 5, and by Lemma 18 also condition (i). Assume, conversely, that ρ_{Ξ} satisfies conditions (i)–(iii) of Lemma 5, and set $F(t) := \rho_{\Xi}([0,t])$ $(t \in [0,1])$, $F(t) := 0$ $(t < 0)$, $F(t) := F(1)$ $(t > 1)$. Then by Lemma 18, F solves the differential equation (2.9) subject to the initial condition $F(t) := 0$ $(t < 0)$. By Lemma 19, these conditions uniquely determine F and hence also ρ_{Ξ}.

Assume that $\mathbb{F}$ solves the frozen percolation equation (1.3) for the set of possible freezing times Ξ and that $\mathbb{F}$ is stationary, adapted, and respects the tree structure. Generalising (1.13), for any $\Delta \subset (0,1]$ that is relatively closed, we set

$$Y_{\mathbf{i}}^{\Delta} := \inf \left\{ t \in \Delta : \mathbf{i} \xrightarrow{\mathbb{T}^t \setminus \mathbb{F}} \infty \right\} \qquad (\mathbf{i} \in \mathbb{T}). \tag{2.16}$$

Then in particular, $Y_{\mathbf{i}}^{\Xi}$ is the burning time $Y_{\mathbf{i}}$ defined in (1.13). As in (1.4), we write $\omega_{\mathbf{i}} = (\tau_{\mathbf{i}}, \kappa_{\mathbf{i}})$ $(t \in \mathbb{T})$. Since $\mathbb{F}$ is stationary, adapted, and respects the tree structure, the random variables $(\omega_{\mathbf{i}}, Y_{\mathbf{i}}^{\Xi})_{\mathbf{i} \in \mathbb{T}}$ form an RTP corresponding to the map χ in (1.16) and some solution ρ_{Ξ} to the RDE (1.17). To complete the proof, we need to show that ρ_{Ξ} also satisfies conditions (ii) and (iii) of Lemma 5. Since ρ_{Ξ} is the law of $Y_{\mathbf{i}}^{\Xi}$ $(\mathbf{i} \in \mathbb{T})$, it clearly satisfies condition (ii) of Lemma 5. To also prove (iii), we use that by Lemma 3, we have $\mathbb{F} = \{\mathbf{i} \in \mathbb{I} : Y_{\mathbf{i}1}^{\Xi} \leq \tau_{\mathbf{i}}\}$, which allows us to apply [RST19, Prop. 39], which tells us that

$$\mathbb{P}\left[Y_{\mathbf{i}}^{(0,1]} \leq t\right] = F(t) \vee \left(t - F(t)\right) \qquad \left(\mathbf{i} \in \mathbb{T},\ t \in [0,1]\right), \tag{2.17}$$

where $F(t) := \rho_{\Xi}\left([0,t]\right)$ $\left(t \in [0,1]\right)$. Since

$$Y_{\mathbf{i}}^{\Xi} = \inf \left\{ t \in \Xi : t \geq Y_{\mathbf{i}}^{(0,1]} \right\} \qquad (\mathbf{i} \in \mathbb{T}), \tag{2.18}$$

it follows that

$$F(t) = \rho_{\Xi}\left([0,t]\right) = \mathbb{P}\left[Y_{\mathbf{i}}^{\Xi} \leq t\right] = \mathbb{P}\left[Y_{\mathbf{i}}^{(0,1]} \leq t\right] = F(t) \vee \left(t - F(t)\right) \qquad (t \in \Xi), \tag{2.19}$$

which proves that ρ_{Ξ} satisfies condition (iii) of Lemma 5. ∎

Proof of Lemma 6 If ρ solves the RDE (1.17), then by Lemma 18, the function $F \in \mathrm{BV}$ defined in (2.8) is right-continuous and nondecreasing with $F(0) = 0$ and satisfies (2.7). Let $\Xi := \mathrm{supp}(\mathrm{d}F) \cap (0,1]$. Then (2.7) implies that $\overline{F}(t) = \frac{1}{2}t$ for a.e. t w.r.t. $\mathrm{d}F$. Since F is right-continuous with left limits, this implies that $\overline{F}(t) = \frac{1}{2}t$ for all $t \in \Xi$, and hence $F(t) \geq \frac{1}{2}t$ for all $t \in \Xi$. It follows that ρ satisfies conditions (i)–(iii) of Lemma 5 and hence $\rho = \rho_{\Xi}$. ∎

The following lemma settles the existence part of Theorem 1.

Lemma 24 (Frozen points) *Let $\Xi \subset (0,1]$ be closed w.r.t. the relative topology of $(0,1]$, let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ be the RTP corresponding to the solution ρ_{Ξ} to the RDE (1.17) defined in Lemma 5, and let $\mathbb{F}$ be defined by (1.14). Then $\mathbb{F}$ solves the frozen percolation equation (1.3) for the set of possible freezing times Ξ and $\mathbb{F}$ is stationary, adapted, and respects the tree structure. Moreover, the $Y_{\mathbf{i}}$ are given by (1.13).*

Proof It follows from the properties of an RTP that $\mathbb{F}$, defined by (1.14), is stationary, adapted, and respects the tree structure. The inductive relation (1.15) implies that if $Y_{\mathbf{i}} < \infty$, then there exist $(j_k)_{k \geq 1}$ such that $\mathbf{i}j_1 \cdots, j_n$ is a legal descendant of $\mathbf{i}j_1 \cdots, j_{n-1}$ and $Y_{\mathbf{i}} = Y_{\mathbf{i}j_1\cdots,j_n}$ for all $n \geq 1$. For all $n \geq 0$ such that $\kappa_{\mathbf{i}j_1\cdots,j_n} = 1$, the fact that $Y_{\mathbf{i}j_1\cdots,j_n} < \infty$ and (1.15) moreover imply that $Y_{\mathbf{i}j_1\cdots,j_n 1} > \tau_{\mathbf{i}}$. Therefore, we have that

$$\mathbf{i} \xrightarrow{\mathbb{T}^t \setminus \mathbb{F}} \infty \quad \text{if } t = Y_{\mathbf{i}} < \infty. \tag{2.20}$$

Since $Y_{\mathbf{i}}$ takes values in $\Xi \cup \{\infty\}$, it follows that

$$Y_{\mathbf{i}} \geq \inf \left\{ t \in \Xi : \mathbf{i} \xrightarrow{\mathbb{T}^t \setminus \mathbb{F}} \infty \right\} \qquad (\mathbf{i} \in \mathbb{T}). \tag{2.21}$$

To prove that this is actually an equality, let $Y_{\mathbf{i}}'$ denote the right-hand side of (2.21). Since $\mathbb{F}$ is stationary, adapted, and respects the tree structure, the random variables $(\omega_{\mathbf{i}}, Y_{\mathbf{i}}')_{\mathbf{i} \in \mathbb{T}}$ form an RTP corresponding to the map χ in (1.16) and some solution ρ to the RDE (1.17). By Lemma 5, $\rho = \rho_\Xi$, so $Y_{\mathbf{i}}'$ and $Y_{\mathbf{i}}$ are equal in law, which by (2.21) implies that they are a.s. equal. This proves that the $Y_{\mathbf{i}}$ are given by (1.13). Inserting (1.13) into (1.14), we see that $\mathbb{F}$ solves the frozen percolation equation (1.3). ∎

Proof of Theorem 1 Lemma 24 proves existence of a solution $\mathbb{F}$ of the frozen percolation equation (1.3) for the set of possible freezing times Ξ that is stationary, adapted, and respects the tree structure. It remains to prove uniqueness in law. Set $\omega_{\mathbf{i}} := (\tau_{\mathbf{i}}, \kappa_{\mathbf{i}})$ $(\mathbf{i} \in \mathbb{T})$ and let $\Omega = (\omega_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$. Let $Y_{\mathbf{i}}$ $(\mathbf{i} \in \mathbb{T})$ be the burning times defined in (1.13). Since $\mathbb{F}$ is stationary, adapted, and respects the tree structure, the random variables $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ form an RTP corresponding to the map χ in (1.16) and some solution ρ to the RDE (1.17). By Lemma 5, $\rho = \rho_\Xi$, and hence the law of $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ is uniquely determined. By Lemma 3, this implies that the joint law of $(\Omega, \mathbb{F})$ is also uniquely determined. ∎

2.2 Scale invariance

In this subsection, we prove Lemmas 11, 12, and 13 about invariance of solutions of the (bivariate) RDE under the scaling maps $\Gamma_t^{(1)}$ and $\Gamma_t^{(2)}$. We will generalise a bit and define scaling maps $\Gamma_t^{(n)}$ for any $1 \leq n \leq \infty$, where the case $n = \infty$ will play an important role in the proof of Lemma 13.

Recall the definition of the scaling maps $\psi_t : I \to I$ $(t > 0)$ in (1.25). For $t > 0$, we define a cut-off map $c_t : I \to I$ by

$$c_t(y) := \begin{cases} y & \text{if } y \leq t, \\ \infty & \text{otherwise.} \end{cases} \tag{2.22}$$

Note that c_t is the identity map when $t \geq 1$. It is easy to check that

$$\psi_{1/t} \circ \psi_t = c_t \qquad (t > 0). \tag{2.23}$$

For $1 \leq n < \infty$, we write $[n] := \{1, \ldots, n\}$ and we set $[\infty] := \mathbb{N}_+$. We denote a generic element of I^n by $\vec{y} = (y^k)_{k \in [n]}$ and we define $\psi_t^{(n)} : I^n \to I^n$ and $c_t^{(n)} : I^n \to I^n$ in a coordinatewise way by $\psi_t^{(n)}(\vec{y}) := (\psi_t(y^k))_{k \in [n]}$ and $c_t^{(n)}(\vec{y}) := (c_t(y^k))_{k \in [n]}$.

We say that a probability measure on I^n is *symmetric* if it is invariant under a permutation of the coordinates. Generalising the definitions of $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ in Subsection 1.5, for any $0 < t \leq 1$ and $1 \leq n \leq \infty$, we let $\mathcal{M}^{(n)}$ denote the space of symmetric probability measures $\rho^{(n)}$ on I^n such that

$$\rho^{(n)}(J^n[t]) \leq t \qquad (0 < t \leq 1) \quad \text{with} \quad J^n[t] := \{\vec{y} \in I^n : \exists k \in [n] \text{ s.t. } y^k \leq t\}. \tag{2.24}$$

Note that $J^n[1] = I^n \setminus \{\vec{\infty}\}$, where $\vec{\infty}$ denotes the element $\vec{y} \in I^n$ with $y^k := \infty$ for all $k \in [n]$. Generalising the definitions of $\Gamma_t^{(1)}$ and $\Gamma_t^{(2)}$ in Subsection 1.5, for each $1 \leq n \leq \infty$ and $t > 0$, we define

$$\Gamma_t^{(n)} \rho^{(n)} := t^{-1} \rho^{(n)} \circ (\psi_t^{(n)})^{-1} + (1 - t^{-1}) \delta_{\vec{\infty}} \qquad (\rho^{(n)} \in \mathcal{M}^{(n)}). \qquad (2.25)$$

We also define cut-off maps $C_t^{(n)}$ by

$$C_t^{(n)} \rho^{(n)} := \rho^{(n)} \circ (c_t^{(n)})^{-1} \qquad (\rho^{(n)} \in \mathcal{M}^{(n)}) \qquad (2.26)$$

and in particular set $C_t := C_t^{(1)}$. Finally, for all $1 \leq n \leq \infty$, we define a map $T^{(n)}$ acting on probability measures on I^n by

$$T^{(n)} \rho^{(n)} := \text{ the law of } \big(\chi[\omega](Y_1^k, Y_2^k) \big)_{k \in [n]}, \qquad (2.27)$$

where $(Y_1^k)_{k \in [n]}$ and $(Y_2^k)_{k \in [n]}$ are independent random variables with law $\rho^{(n)}$, and ω is an independent random variable that is uniformly distributed on $[0, 1] \times \{1, 2\}$. We call the equation

$$T^{(n)} \rho^{(n)} = \rho^{(n)} \qquad (2.28)$$

the *n-variate RDE*. In particular, for $n = 1$ this is the RDE (1.17) and for $n = 2$ this is bivariate RDE (1.23). The following lemma, which will be proved below, shows that all these maps are well-defined on the space $\mathcal{M}^{(n)}$.

Lemma 25 (Maps are well-defined) *For each $1 \leq n \leq \infty$ and $t > 0$, the maps $\Gamma_t^{(n)}$, $C_t^{(n)}$, and $T^{(n)}$ map the space $\mathcal{M}^{(n)}$ into itself.*

The following lemma says that as long as we are interested in symmetric solutions of the *n*-variate RDE (2.28), it suffices to look for solutions in the space $\mathcal{M}^{(n)}$.

Lemma 26 (Solutions to the RDE are scalable) *If a symmetric probability measure $\rho^{(n)}$ on I^n solves the n-variate RDE (2.28), then $\rho^{(n)} \in \mathcal{M}^{(n)}$.*

The following lemma is the central result of this subsection.

Lemma 27 (Commutation relation) *For each $1 \leq n \leq \infty$, one has*

$$\Gamma_t^{(n)} T^{(n)} \rho^{(n)} = t T^{(n)} \Gamma_t^{(n)} \rho^{(n)} + (1 - t) \Gamma_t^{(n)} \rho^{(n)} \qquad (t > 0, \ \rho^{(n)} \in \mathcal{M}^{(n)}). \qquad (2.29)$$

We first show how Lemmas 25–27 imply Lemmas 11 and 12, and then prove Lemmas 25–27. We start by proving a more general statement.

Lemma 28 (Scale invariance of *n*-variate RDE) *Let $1 \leq n \leq \infty$ and let $\rho^{(n)}$ be a solution to the n-variate RDE (2.28). Then $\rho^{(n)} \in \mathcal{M}^{(n)}$, and for each $t > 0$, the measure $\Gamma_t^{(n)} \rho^{(n)}$ is also a solution to (2.28).*

Proof Let $1 \leq n \leq \infty$ and let $\rho^{(n)}$ be a solution to the *n*-variate RDE (2.28). Then $\rho^{(n)} \in \mathcal{M}^{(n)}$ by Lemma 26. Moreover, for each $t > 0$, Lemma 27 and the fact that $T^{(n)} \rho^{(n)} = \rho^{(n)}$ imply that

$$\Gamma_t^{(n)} \rho^{(n)} = t T^{(n)} \Gamma_t^{(n)} \rho^{(n)} + (1 - t) \Gamma_t^{(n)} \rho^{(n)} \qquad (2.30)$$

which shows that $T^{(n)} \Gamma_t^{(n)} \rho^{(n)} = \Gamma_t^{(n)} \rho^{(n)}$, i.e., the measure $\Gamma_t^{(n)} \rho^{(n)}$ solves the *n*-variate RDE (2.28). $\blacksquare$

Proof of Lemmas 11 and 12 Most of the statements of Lemmas 11 and 12 follow by specialising Lemma 28 to the cases $n = 1$ and $n = 2$, respectively. Apart from this, we only

need to prove (1.27). Let $\Xi \subset (0,1]$ be relatively closed and let Ξ' be as in (1.27). Then $\Gamma_t \rho_\Xi$ solves the RDE (1.17) by Lemma 28, $\Gamma_t \rho_\Xi$ is concentrated on $\Xi' \cup \{\infty\}$, and $\Gamma_t \rho_\Xi([0,t]) \geq \frac{1}{2} t$ for all $t \in \Xi'$, so Lemma 5 allows us to identify $\Gamma_t \rho_\Xi$ as $\rho_{\Xi'}$. ∎

We now provide the proofs of Lemmas 25–27.

Proof of Lemma 25 (partially) Let $\rho^{(n)} \in \mathcal{M}^{(n)}$. It is clear that the right-hand side of (2.25) defines a signed measure that is symmetric with respect to a permutation of the coordinates and that satisfies $\Gamma_t^{(n)} \rho^{(n)}(I^n) = 1$. We observe that by (1.25),

$$(\psi_t^{(n)})^{-1}(J^n[s]) = \{\vec{y} \in I^n : \exists k \in [n] \text{ s.t. } \psi_t(y^k) \leq s\}$$
$$= \{\vec{y} \in I^n : \exists k \in [n] \text{ s.t. } y^k \leq t \text{ and } t^{-1} y^k \leq s\} = \{\vec{y} \in I^n : \exists k \in [n] \text{ s.t. } y^k \leq st \wedge t\}, \tag{2.31}$$

and hence

$$\Gamma_t^{(n)} \rho^{(n)}(J^n[s]) = t^{-1} \rho^{(n)} \circ (\psi_t^{(n)})^{-1}(J^n[s]) = t^{-1} \rho^{(n)}(J^n[st \wedge t]) \leq t^{-1}(st \wedge t) \leq s. \tag{2.32}$$

for each $t > 0$ and $0 < s \leq 1$. Applying this with $s = 1$ and using the fact that $I^n = J^n[1] \cup \{\vec{\infty}\}$ we see that $\Gamma_t^{(n)} \rho^{(n)}$ is a probability measure. More generally, (2.32) shows that $\Gamma_t^{(n)}$ maps the space $\mathcal{M}^{(n)}$ into itself.

It is clear that $C_t^{(n)} \rho^{(n)}$, defined in (2.26), is a (symmetric) probability measure on I^n whenever $\rho^{(n)}$ is. If moreover $\rho^{(n)} \in \mathcal{M}^{(n)}$, then

$$C_t^{(n)} \rho^{(n)}(J^n[s]) = \rho^{(n)} \circ (c_t^{(n)})^{-1}(J^n[s]) = \rho^{(n)}(J^n[s \wedge t]) \leq s \tag{2.33}$$

for each $t > 0$ and $0 < s \leq 1$, which shows that $C_t^{(n)}$ maps the space $\mathcal{M}^{(n)}$ into itself.

This proves the claims for $\Gamma_t^{(n)}$ and $C_t^{(n)}$. We postpone the proof of claim for $T^{(n)}$ until the proof of Lemma 27, where it will follow as a side result of the main argument. ∎

The proof of Lemma 26 uses the following simple lemma, which we cite from [RST19, Lemma 8].

Lemma 29 (Percolation probability) *One has* $\mathbb{P}[\varnothing \xrightarrow{\mathbb{T}^t} \infty] = t$ $(0 \leq t \leq 1)$.

Proof of Lemma 26 We will prove the following, somewhat stronger statement. Let $1 \leq n \leq \infty$ and let $\rho^{(n)}$ be a solution to the n-variate RDE (2.28). Then we will show that $\rho^{(n)}$ satisfies (2.24). In particular, if $\rho^{(n)}$ is symmetric, this implies that $\rho^{(n)} \in \mathcal{M}^{(n)}$.

Let $\rho^{(n)}$ be a solution to the n-variate RDE (2.28) and for $k \in [n]$, let ρ_k denote the n-th marginal of $\rho^{(n)}$. It is clear from (2.28) that ρ_k solves the RDE (1.17), so by Lemma 6, for each $k \in [n]$, there exists a relatively closed set $\Xi_k \subset (0,1]$ such that $\rho_k = \rho_{\Xi_k}$. Since $\rho^{(n)}$ solves the n-variate RDE (2.28), we can construct an n-variate RTP

$$\left(\omega_{\mathbf{i}}, \vec{Y}_{\mathbf{i}}\right)_{\mathbf{i} \in \mathbb{T}} \tag{2.34}$$

where $\vec{Y}_{\mathbf{i}} = (Y_{\mathbf{i}}^k)_{k \in [n]}$ are I^n-valued random variables such that $\vec{Y}_{\mathbf{i}}$ is inductively given in terms of $\vec{Y}_{\mathbf{i}1}$ and $\vec{Y}_{\mathbf{i}2}$ as in (2.28). In particular, for each $k \in [n]$, $(\omega_{\mathbf{i}}, Y_{\mathbf{i}}^k)_{\mathbf{i} \in \mathbb{T}}$ is an RTP corresponding to $\rho_k = \rho_{\Xi_k}$. Let

$$\mathbb{F}^k = \{\mathbf{i} \in \mathbb{I} : Y_{\mathbf{i}1}^k \leq \tau_{\mathbf{i}}\} \qquad (k \in [n]). \tag{2.35}$$

Then Lemma 24 tells us that

$$Y_{\mathbf{i}}^k := \inf \{t \in \Xi : \mathbf{i} \xrightarrow{\mathbb{T}^t \setminus \mathbb{F}^k} \infty\} \qquad (\mathbf{i} \in \mathbb{T}, \ k \in [n]), \tag{2.36}$$

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with the convention that $\inf \emptyset := \infty$. Using Lemma 29, we can now estimate

$$\mathbb{P}\big[\inf_{k \in [n]} Y_\emptyset^k \le t\big] \le \mathbb{P}\big[\emptyset \xrightarrow{\mathbb{T}^t} \infty\big] = t \qquad (0 < t \le 1), \tag{2.37}$$

which proves that $\rho^{(n)}$ satisfies (2.24). ∎

Proof of Lemma 27 We will first prove (2.29) for $0 < t \le 1$, and on the way also establish that $T^{(n)}$ maps $\mathcal{M}^{(n)}$ into itself, which is the missing part of Lemma 25 that still remains to be proved. Fix $1 \le n \le \infty$, $0 < t \le 1$, and $\rho^{(n)} \in \mathcal{M}^{(n)}$. Let $\vec{Y} = (Y^k)_{k \in [n]}$ be a random variable with law $\rho^{(n)}$. It follows from (2.24) that we can couple $\vec{Y}$ to a Bernoulli random variable B such that $\mathbb{P}[B = 1] = t$ and $\mathbb{P}[B = 1 \,|\, \inf_{k \in [n]} Y^k \le t] = 1$. More formally, there exists a probability measure μ on $I^n \times \{0, 1\}$ whose first marginal is $\rho^{(n)}$, whose second marginal is the Bernoulli distribution with parameter t, and that is concentrated on $\{(\vec{y}, b) : b = 1 \text{ if } \inf_{k \in [n]} y_k \le t\}$. Such a measure μ is not unique, but we fix one from now on. Let $(\vec{Y}_1, B_1)$ and $(\vec{Y}_2, B_2)$ be independent random variables with law μ, and let $\omega = (\tau, \kappa)$ be an independent random variable that is uniformly distributed on $[0, 1] \times \{1, 2\}$. We define

$$\begin{aligned}
&\text{(i)} \qquad \vec{Y}_\emptyset := \big(\chi[\omega](Y_1^k, Y_2^k)\big)_{k \in [n]}, \\
&\text{(ii)} \qquad B_\emptyset := 1_{\{\kappa=1\}} 1_{\{\tau \le t\}} B_1 + 1_{\{\kappa=2\}} (B_1 \vee B_2).
\end{aligned} \tag{2.38}$$

We claim that

$$\text{(i)} \quad \mathbb{P}\big[B_\emptyset = 1 \,\big|\, \inf_{k \in [n]} Y_\emptyset^k \le t\big] = 1, \qquad \text{(ii)} \quad \mathbb{P}\big[B_\emptyset = 1\big] = t. \tag{2.39}$$

Indeed, if $Y_\emptyset^k \le t$ for some k, then by the definition of χ in (1.16), a.s. either $\kappa = 1$ and $\tau < Y_1^k \le t$, or $\kappa = 2$ and $Y_1^k \wedge Y_2^k \le t$. In either case, it follows that $B_\emptyset = 1$, proving part (i) of (2.39). Part (ii) follows by writing

$$\mathbb{P}[B_\emptyset = 1] = \tfrac{1}{2} t \mathbb{P}[B_1 = 1] + \tfrac{1}{2} \mathbb{P}[B_1 \vee B_2 = 1] = \tfrac{1}{2} t^2 + \tfrac{1}{2}[1 - (1 - t)^2] = t. \tag{2.40}$$

We next claim that for each measurable subset $A \subset I^n \backslash \{\vec{\infty}\}$,

$$\begin{aligned}
&\text{(i)} \qquad\qquad T^{(n)} \rho^{(n)}(A) = \mathbb{P}\big[\vec{Y}_\emptyset \in A\big], \\
&\text{(ii)} \qquad\qquad \Gamma_t^{(n)} \rho^{(n)}(A) = \mathbb{P}\big[t^{-1} \vec{Y}_i \in A \,\big|\, B_i = 1\big] \qquad (i = 1, 2), \\
&\text{(iii)} \qquad \Gamma_t^{(n)} T^{(n)} \rho^{(n)}(A) = \mathbb{P}\big[t^{-1} \vec{Y}_\emptyset \in A \,\big|\, B_\emptyset = 1\big].
\end{aligned} \tag{2.41}$$

Part (i) of (2.41) is immediate from (2.38) while part (ii) follows immediately from the definition of $\Gamma_t^{(n)}$ in (2.25). Formulas (2.41) (i) and (2.39) (ii) imply that

$$T^{(n)} \rho^{(n)}(J^n[t]) = \mathbb{P}\big[\inf_{k \in [n]} Y_\emptyset^k \le t\big] \le \mathbb{P}\big[B_\emptyset = 1\big] = t. \tag{2.42}$$

Since this holds for general $0 < t \le 1$, and since $T^{(n)}$ also clearly preserves the symmetry of $\rho^{(n)}$, we conclude that $T^{(n)}$ maps the space $\mathcal{M}^{(n)}$ into itself. In particular, this shows that $\Gamma_t^{(n)} T^{(n)} \rho^{(n)}$ is well-defined. Part (iii) of (2.41) now follows by applying part (ii) to the measure $T^{(n)} \rho^{(n)}$, using part (i) and (2.39).

We next prove (2.29) for $0 < t \le 1$. We set

$$B_\circ := 1_{\{\kappa=1\}} 1_{\{\tau \le t\}} B_1 + 1_{\{\kappa=2\}} (B_1 \wedge B_2). \tag{2.43}$$

We claim that for each measurable subset $A \subset I^n \backslash \{\vec{\infty}\}$,

$$\text{(i)} \quad \mathbb{P}\big[B_\circ = 1\big] = t^2, \qquad \text{(ii)} \quad T^{(n)} \Gamma_t^{(n)} \rho^{(n)}(A) = \mathbb{P}\big[t^{-1} \vec{Y}_\emptyset \in A \,\big|\, B_\circ = 1\big]. \tag{2.44}$$

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Part (i) is a consequence of the independence of τ, κ, B_1 and B_2 which yields $\mathbb{P}[B_\circ = 1] = \frac{1}{2} \cdot t \cdot t + \frac{1}{2} \cdot t \cdot t$. To prove part (ii), we introduce the function

$$\Phi[s](y) := \begin{cases} y & \text{if } s < y, \\ \infty & \text{if } y \le s, \end{cases} \qquad (s \in [0,1], \ y \in I). \tag{2.45}$$

and we define $\Phi[s](\vec{y})$ and $\vec{y}_1 \wedge \vec{y}_2$ in a componentwise way, i.e., $\Phi[s](\vec{y}) := (\Phi[s](y^k))_{k \in [n]}$ and $\vec{y}_1 \wedge \vec{y}_2 := (y_1^k \wedge y_2^k)_{k \in [n]}$. With this notation, we can write

$$\begin{aligned}
&\mathbb{P}[t^{-1}\vec{Y}_\varnothing \in A \,|\, B_\circ = 1] \\
&\quad = \mathbb{P}[\kappa = 1, \ t^{-1}\Phi[\tau](\vec{Y}_1) \in A \,|\, B_\circ = 1] + \mathbb{P}[\kappa = 2, \ t^{-1}(\vec{Y}_1 \wedge \vec{Y}_2) \in A \,|\, B_\circ = 1] \\
&\quad = \tfrac{1}{2}\mathbb{P}[\Phi[t^{-1}\tau](t^{-1}\vec{Y}_1) \in A \,|\, \tau \le t, \ B_1 = 1] + \tfrac{1}{2}\mathbb{P}[(t^{-1}\vec{Y}_1 \wedge t^{-1}\vec{Y}_2) \in A \,|\, B_1 = 1 = B_2].
\end{aligned} \tag{2.46}$$

Using (2.41) (ii) and the fact that conditional on $\tau \le t$, the random variable $t^{-1}\tau$ is uniformly distributed on $[0,1]$, we arrive at (2.44) (ii).

Formulas (2.39) (ii), (2.41) (iii), and (2.44) give, for any measurable subset $A \subset I^n \backslash \{\vec{\infty}\}$,

$$\begin{aligned}
t\Gamma_t^{(n)}T^{(n)}\rho^{(n)}(A) &= \mathbb{P}[t^{-1}\vec{Y}_\varnothing \in A, \ B_\varnothing = 1], \\
t^2 T^{(n)}\Gamma_t^{(n)}\rho^{(n)}(A) &= \mathbb{P}[t^{-1}\vec{Y}_\varnothing \in A, \ B_\circ = 1].
\end{aligned} \tag{2.47}$$

We observe that the event $\{B_\circ = 1\}$ is contained in the event $\{B_\varnothing = 1\}$ and the difference of these events is the event that $\kappa = 2$ and precisely one of the random variables B_1 and B_2 is one. In view of this

$$\begin{aligned}
&t\Gamma_t^{(n)}T^{(n)}\rho^{(n)}(A) - t^2 T^{(n)}\Gamma_t^{(n)}\rho^{(n)}(A) \\
&\quad = \mathbb{P}[t^{-1}\vec{Y}_1 \in A, \ \kappa = 2, \ B_1 = 1, \ B_2 = 0] + \mathbb{P}[t^{-1}\vec{Y}_2 \in A, \ \kappa = 2, \ B_1 = 0, \ B_2 = 1] \\
&\quad = \tfrac{1}{2}t\mathbb{P}[t^{-1}\vec{Y}_1 \in A \,|\, B_1 = 1] + \tfrac{1}{2}t\mathbb{P}[t^{-1}\vec{Y}_2 \in A \,|\, B_2 = 1] = t\Gamma_t^{(n)}\rho^{(n)}(A),
\end{aligned} \tag{2.48}$$

where in the last step we have used (2.41) (ii). Dividing by t, we see that (2.29) holds for $0 < t \le 1$.

To derive (2.29) also for $t \ge 1$, replacing t by $1/t$, we may equivalently show that

$$\Gamma_{1/t}^{(n)}T^{(n)}\rho^{(n)} = t^{-1}T^{(n)}\Gamma_{1/t}^{(n)}\rho^{(n)} + (1 - t^{-1})\Gamma_{1/t}^{(n)}\rho^{(n)} \qquad (0 < t \le 1, \ \rho^{(n)} \in \mathcal{M}^{(n)}). \tag{2.49}$$

For $0 < t \le 1$, we set

$$\mathcal{M}^{(n)}[t] := \{\rho^{(n)} \in \mathcal{M}^{(n)} : \rho^{(n)} \text{ is concentrated on } I_t^n\} \quad \text{with} \quad I_t := [0,t] \cup \{\infty\}. \tag{2.50}$$

We observe that $\psi_t : I_t \to I$ is a bijection and $\psi_{1/t}$ is its inverse. As a result, $\Gamma_t^{(n)} : \mathcal{M}^{(n)}[t] \to \mathcal{M}^{(n)}$ is a bijection and $\Gamma_{1/t}^{(n)}$ is its inverse. Using this and applying what we already know to the measure $\Gamma_{1/t}^{(n)}\rho^{(n)}$, we conclude that

$$\begin{aligned}
\Gamma_t^{(n)}T^{(n)}\Gamma_{1/t}^{(n)}\rho^{(n)} &= tT^{(n)}\Gamma_t^{(n)}\Gamma_{1/t}^{(n)}\rho^{(n)} + (1-t)\Gamma_t^{(n)}\Gamma_{1/t}^{(n)}\rho^{(n)} \\
&= tT^{(n)}\rho^{(n)} + (1-t)\rho^{(n)}.
\end{aligned} \tag{2.51}$$

By our earlier remarks, we have $\Gamma_{1/t}^{(n)}\rho^{(n)} \in \mathcal{M}^{(n)}[t]$, which is easily seen to imply that also $T^{(n)}\Gamma_{1/t}^{(n)}\rho^{(n)} \in \mathcal{M}^{(n)}[t]$. Using this, we can multiply (2.51) from the left by $t^{-1}\Gamma_{1/t}^{(n)}$ to obtain

$$t^{-1}T^{(n)}\Gamma_{1/t}^{(n)}\rho^{(n)} = \Gamma_{1/t}^{(n)}T^{(n)}\rho^{(n)} + (t^{-1} - 1)\Gamma_{1/t}^{(n)}\rho^{(n)}, \tag{2.52}$$

which proves (2.49). ∎

The rest of this subsection is devoted to the proof of Lemma 13. The maps $\Gamma_t^{(\infty)}$, $C_t^{(\infty)}$, and $T^{(\infty)}$ will play an important role in the proof. Symmetric probability measures on I^∞ are also known as exchangeable probability measures. We will use De Finetti's theorem to associate the space $\mathcal{M}^{(\infty)}$ with a subspace $\mathcal{M}^*$ of the space of all probability measures on the space of probability measures on I. The space $\mathcal{M}^*$ is naturally equipped with a special kind of stochastic order, called the convex order, and we will use a characterisation, proved in [MSS18], of the measures $\underline{\rho}_{\sqsubseteq}^{(2)}$ and $\overline{\rho}_{\sqsubseteq}^{(2)}$ from (1.24) in terms of the convex order.

We now give the precise definitions. We let $\mathcal{P}(I^n)$ denote the space of all probability measures on I^n and denote the subspace of symmetric probability measures by $\mathcal{P}_{\mathrm{sym}}(I^n)$. We equip $\mathcal{P}(I)$ with the topology of weak convergence and the associated Borel-σ-field and let $\mathcal{P}(\mathcal{P}(I))$ denote the space of all probability measures on $\mathcal{P}(I)$. Each $\nu \in \mathcal{P}(\mathcal{P}(I))$ is the law of a $\mathcal{P}(I)$-valued random variable, i.e., we can construct a random probability measure ξ on I such that $\nu = \mathbb{P}[\xi \in \cdot\,]$ is the law of ξ. By definition, for $1 \le n \le \infty$,

$$\nu^{(n)} := \mathbb{E}\big[\underbrace{\xi \otimes \cdots \otimes \xi}_{n \text{ times}}\big] \tag{2.53}$$

is called the *n-th moment measure* of ν. Here $\xi \otimes \cdots \otimes \xi$ denotes the product measure of n identical copies of ξ and the expectation of a random measure μ on a Polish space Ω is the deterministic measure $E[\mu]$ defined by $\int_\Omega \phi \mathrm{d}E[\mu] := E\big[\int_\Omega \phi\, \mathrm{d}\mu\big]$ for all bounded measurable $\phi : \Omega \to \mathbb{R}$. Let ξ be a $\mathcal{P}(I)$-valued random variable with law ν, and conditional on ξ, let $(Y^k)_{k \in [n]}$ be i.i.d. with law ξ. Then it is easy to see (compare [MSS18, formula (4.1)]) that the unconditional law of $(Y^k)_{k \in [n]}$ is given by $\nu^{(n)}$, i.e.,

$$\nu^{(n)} = \mathbb{P}\big[(Y^k)_{k \in [n]} \in \cdot\,\big] \quad \text{where} \quad \mathbb{P}\big[(Y^k)_{k \in [n]} \in \cdot\,\big|\,\xi\big] = \underbrace{\xi \otimes \cdots \otimes \xi}_{n \text{ times}}. \tag{2.54}$$

We observe that $\nu^{(n)} \in \mathcal{P}_{\mathrm{sym}}(I^n)$ for all $\nu \in \mathcal{P}(\mathcal{P}(I))$ and $1 \le n \le \infty$. In fact, by De Finetti's theorem, the map $\nu \mapsto \nu^{(\infty)}$ is a bijection from $\mathcal{P}(\mathcal{P}(I))$ to $\mathcal{P}_{\mathrm{sym}}(I^\infty)$. This allows us to identify the space $\mathcal{M}^{(\infty)}$ with a subspace of $\mathcal{P}(\mathcal{P}(I))$. We set (compare (2.24))

$$\mathcal{M}^* := \big\{\nu \in \mathcal{P}(\mathcal{P}(I)) : \nu(J^*[t]) \le t\ \forall 0 < t \le 1\big\}$$
$$\text{with} \quad J^*[t] := \big\{\xi \in \mathcal{P}(I) : \xi([0,t]) > 0\big\} \qquad (0 < t \le 1). \tag{2.55}$$

Note that $J^*[1] = \mathcal{P}(I) \backslash \{\delta_\infty\}$, where δ_∞ denotes the delta-measure at ∞. The following lemma identifies $\mathcal{M}^{(\infty)}$ with $\mathcal{M}^*$.

Lemma 30 (Probability measures on probability measures) *The map $\nu \mapsto \nu^{(\infty)}$ is a bijection from $\mathcal{M}^*$ to $\mathcal{M}^{(\infty)}$.*

In order to expose the main line of the argument, we postpone the proof of this and some of the following lemmas till later. It follows immediately from Lemmas 25 and 30 that there exist unique maps Γ_t^*, C_t^*, and T^*, mapping the space $\mathcal{M}^*$ into itself, such that

$$(\Gamma_t^*\nu)^{(\infty)} = \Gamma_t^{(\infty)}\nu^{(\infty)}, \quad (C_t^*\nu)^{(\infty)} = C_t^{(\infty)}\nu^{(\infty)}, \quad \text{and} \quad (T^*\nu)^{(\infty)} = T^{(\infty)}\nu^{(\infty)} \tag{2.56}$$

for any $t > 0$ and $\nu \in \mathcal{M}^*$. The equation $T^*\nu = \nu$ has been called the *higher-level RDE* in [MSS18] and we will refer to Γ_t^*, C_t^*, and T^* as *higher-level maps*. The following lemma gives a more explicit description of Γ_t^* and C_t^*.

Lemma 31 (Higher-level scaling and cut-off maps) *Let $\nu \in \mathcal{M}^*$ and let ξ be a $\mathcal{P}(I)$-valued random variable with law ν. Then for each $t > 0$, the maps Γ_t^* and C_t^* defined in (2.56) are given by*

$$
\begin{aligned}
&\text{(i)} \quad \Gamma_t^* \nu = t^{-1} \mathbb{P}\big[\xi \circ \psi_t^{-1} \in \cdot\big] + (1 - t^{-1})\delta_{\delta_\infty}, \\
&\text{(ii)} \quad C_t^* \nu = \mathbb{P}\big[\xi \circ c_t^{-1} \in \cdot\big],
\end{aligned}
\tag{2.57}
$$

where $\delta_{\delta_\infty} \in \mathcal{P}(\mathcal{P}(I))$ denotes the delta measure at the point $\delta_\infty \in \mathcal{P}(I)$.

The following lemma, which we cite from [MSS18, Lemma 2], identifies the map T^* more explicitly. Recall the definition of the map $\chi[\omega] : I^2 \to I$ in (1.16). In line with earlier notation, in (2.58) below, $\xi_1 \otimes \xi_2 \circ \chi[\omega]^{-1}$ denotes the image of the random measure $\xi_1 \otimes \xi_2$ under the random map $\chi[\omega]$.

Lemma 32 (Higher-level RDE map) *Let $\nu \in \mathcal{M}^*$, let ξ_1, ξ_2 be independent $\mathcal{P}(I)$-valued random variables with law ν, and let ω be an independent random variable that is uniformly distributed on $[0, 1] \times \{1, 2\}$. Then the map T^* defined in (2.56) is given by*

$$
T^*(\nu) = \mathbb{P}\big[\xi_1 \otimes \xi_2 \circ \chi[\omega]^{-1} \in \cdot\big].
\tag{2.58}
$$

We equip the space $\mathcal{P}(\mathcal{P}(I))$ with the *convex order*, which we denote by $\leq_{\mathrm{cv}}$. Two measures $\nu_1, \nu_2 \in \mathcal{P}(\mathcal{P}(I))$ satisfy $\nu_1 \leq_{\mathrm{cv}} \nu_2$ if and only if the following two equivalent conditions are satisfied, see [MSS18, Thm 13]:

(i) $\int \phi \, d\nu_1 \leq \int \phi \, d\nu_2$ for all convex continuous functions $\phi : \mathcal{P}(I) \to \mathbb{R}$.

(ii) There exists an I-valued random variable Y defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and σ-fields $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$ such that $\nu_i = \mathbb{P}\big[\mathbb{P}[Y \in \cdot | \mathcal{F}_i] \in \cdot\big]$ $(i = 1, 2)$.

The convex order is a partial order, in particular, $\nu_1 \leq_{\mathrm{cv}} \nu_2 \leq_{\mathrm{cv}} \nu_1$ implies $\nu_1 = \nu_2$, see [MSS18, Lemma 15]. The following lemma says that the scaling maps Γ_t^* preserve the convex order.

Lemma 33 (Monotonicity with respect to the convex order) *Let $t > 0$ and let Γ_t^* be defined in (2.56). Then $\nu_1, \nu_2 \in \mathcal{M}^*$ and $\nu_1 \leq_{\mathrm{cv}} \nu_2$ imply $\Gamma_t^* \nu_1 \leq_{\mathrm{cv}} \Gamma_t^* \nu_2$.*

Let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ be an RTP corresponding to a solution ρ to the RDE (1.17) and let $\Omega := (\omega_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$. We define $\underline{\rho}, \overline{\rho} \in \mathcal{P}(\mathcal{P}(I))$ by

$$
\underline{\rho} := \mathbb{P}\big[\mathbb{P}[Y_\varnothing \in \cdot \,| \Omega] \in \cdot\big] \quad \text{and} \quad \overline{\rho} := \mathbb{P}\big[\delta_{Y_\varnothing} \in \cdot\big].
\tag{2.59}
$$

We observe that the second moment measures of $\underline{\rho}$ and $\overline{\rho}$ are given by

$$
\underline{\rho}^{(2)} = \mathbb{P}\big[(Y_\varnothing, Y_\varnothing') \in \cdot\big] \quad \text{and} \quad \overline{\rho}^{(2)} = \mathbb{P}\big[(Y_\varnothing, Y_\varnothing) \in \cdot\big],
\tag{2.60}
$$

where $(Y_{\mathbf{i}}')_{\mathbf{i} \in \mathbb{T}}$ is conditionally independent of $(Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ given Ω and conditionally equally distributed with $(Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$. In particular, if $\rho = \rho_\Xi$, then $\underline{\rho}^{(2)}$ and $\overline{\rho}^{(2)}$ are the measures defined in (1.24). The following proposition, which we cite from [MSS18, Props 3 and 4], says that $\underline{\rho}$ and $\overline{\rho}$ are the minimal and maximal solutions, with respect to the convex order, of the higher-level RDE $T^*(\nu) = \nu$.

Proposition 34 (Minimal and maximal solutions) *Let ρ be a solution to the RDE (1.17). Then the set*

$$
\mathcal{S}_\rho := \big\{\nu \in \mathcal{P}(\mathcal{P}(I)) : T^*(\nu) = \nu, \ \nu^{(1)} = \rho\big\}
\tag{2.61}
$$

has a unique minimal element $\underline{\rho}$ and maximal element $\overline{\rho}$ with respect to the convex order, and these are the measures defined in (2.59).

We will derive Lemma 13 from the following, stronger statement. We will first give the proofs of Lemmas 35 and 13 and then provide the proofs of the remaining lemmas.

Lemma 35 (Scaling of minimal and maximal solutions) *Let ρ be a solution to the RDE (1.17) and let $t > 0$. Then*

$$\text{(i) } C_t^* \underline{\rho} = \underline{C_t \rho}, \quad \text{(ii) } C_t^* \overline{\rho} = \overline{C_t \rho}, \quad \text{(iii) } \Gamma_t^* \underline{\rho} = \underline{\Gamma_t \rho}, \quad \text{(iv) } \Gamma_t^* \overline{\rho} = \overline{\Gamma_t \rho}. \tag{2.62}$$

Proof We first prove (2.62) (i) and (ii). Recall from (2.26) that $C_t \rho := \rho \circ c_t^{-1}$ where c_t is the cut-off map defined in (2.22). Let $(\omega_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ be the RTP corresponding to ρ. Then it is easy to see that $(\omega_{\mathbf{i}}, c_t(Y_{\mathbf{i}}))_{\mathbf{i} \in \mathbb{T}}$ is the RTP corresponding to $C_t \rho$. Applying the definition in (2.59), it follows that

$$\underline{C_t \rho} = \mathbb{P}\big[\mathbb{P}[c_t(Y_\varnothing) \in \cdot \,|\, \Omega] \in \cdot\big] = \mathbb{P}\big[\mathbb{P}[Y_\varnothing \in \cdot \,|\, \Omega] \circ c_t^{-1} \in \cdot\big] = C_t^* \underline{\rho}, \tag{2.63}$$

where in the last equlity we have used Lemma 31. This proves (2.62) (i). The proof of (2.62) (ii) is similar, using the fact that $\delta_{c_t(Y_\varnothing)} = \delta_{Y_\varnothing} \circ c_t^{-1}$.

We next prove (2.62) (iii) and (iv). We start by observing that as a consequence of (2.23) and (2.57),

$$\Gamma_{1/t}^* \circ \Gamma_t^* \nu = C_t^* \nu \qquad (\nu \in \mathcal{M}^*, \ t > 0). \tag{2.64}$$

Moreover, (2.56) implies that

$$(\Gamma_t^* \nu)^{(1)} = \Gamma_t \nu^{(1)} \qquad (\nu \in \mathcal{M}^*, \ t > 0). \tag{2.65}$$

Let ρ be a solution to the RDE (1.17), let $0 < t \leq 1$, and let $\rho' := \Gamma_{1/t}\rho$. Then, by Lemma 11, ρ' also solves the RDE (1.17). Moreover, ρ' is concentrated on $[0, t] \cup \{\infty\}$ and hence in view of (2.23) $\Gamma_t \rho' = \rho$. Let us write

$$\mathcal{M}_\rho^* := \big\{\nu \in \mathcal{M}^* : \nu^{(1)} = \rho\big\} \tag{2.66}$$

and let $\mathcal{M}_{\rho'}^*$ be defined similarly with ρ replaced by ρ'. Since $t \leq 1$, the cut-off map $c_{1/t}$ and hence also $C_{1/t}^*$ are the indenity maps and hence it follows from (2.64) with $1/t$ instead of t and from (2.65) that $\Gamma_{1/t}^*$ is a bijection from $\mathcal{M}_\rho^*$ to $\mathcal{M}_{\rho'}^*$ and that Γ_t^* is its inverse.

It follows from Lemma 26 and our identification of $\mathcal{M}^{(\infty)}$ with $\mathcal{M}^*$ in Lemma 30 and (2.56) that the sets $\mathcal{S}_\rho$ and $\mathcal{S}_{\rho'}$ defined in (2.61) are subsets of $\mathcal{M}_\rho^*$ and $\mathcal{M}_{\rho'}^*$, respectively. Using moreover Lemma 28, we see that $\Gamma_{1/t}^*$ maps $\mathcal{S}_\rho$ into $\mathcal{S}_{\rho'}$ and that Γ_t^* maps $\mathcal{S}_{\rho'}$ into $\mathcal{S}_\rho$. Since $\Gamma_{1/t}^*$ is a bijection from $\mathcal{M}_\rho^*$ to $\mathcal{M}_{\rho'}^*$ and that Γ_t^* is its inverse, we conclude that $\Gamma_{1/t}^*$ is a bijection from $\mathcal{S}_\rho$ to $\mathcal{S}_{\rho'}$. By Lemma 33, the map $\Gamma_{1/t}^*$ is monotone with respect to the convex order. By Proposition 34, the set $\mathcal{S}_\rho$ has unique minimal and maximal elements with respect to the convex order, which are $\underline{\rho}$ and $\overline{\rho}$. Likewise, $\underline{\rho}'$ and $\overline{\rho}'$ are the unique minimal and maximal elements of $\mathcal{S}_{\rho'}$. Since $\Gamma_{1/t}^*$ is a monotone bijection from $\mathcal{S}_\rho$ to $\mathcal{S}_{\rho'}$, it must map $\underline{\rho}$ and $\overline{\rho}$ to $\underline{\rho}'$ and $\overline{\rho}'$, respectively. Recalling that $\rho' = \Gamma_{1/t}\rho$, this shows that

$$\Gamma_{1/t}^* \underline{\rho} = \underline{\Gamma_{1/t}\rho}, \quad \Gamma_{1/t}^* \overline{\rho} = \overline{\Gamma_{1/t}\rho}, \tag{2.67}$$

which proves (2.62) (iii) and (iv) in the special case that $t \geq 1$.

To prove (2.62) (iii) for $0 < t \leq 1$, let ρ'' be a solution to the RDE (1.17), let $\rho := \Gamma_t \rho''$, and as before let $\rho' = \Gamma_{1/t}\rho$. Then, by (2.23), $\rho' = \Gamma_{1/t} \circ \Gamma_t \rho'' = C_t \rho''$. Our previous arguments show that $\Gamma_{1/t}^*$ maps $\underline{\rho}$ into $\underline{\rho}'$ and hence the inverse map Γ_t^* maps $\underline{\rho}'$ into $\underline{\rho}$, i.e.,

$$\Gamma_t^* \underline{\rho}' = \underline{\rho}. \tag{2.68}$$

Formulas (2.64) and (2.62) (i) tell us that

$$\Gamma_{1/t}^* \circ \Gamma_t^* \underline{\rho}'' = C_t^* \underline{\rho}'' = \underline{C_t \rho''} = \underline{\rho}'. \tag{2.69}$$

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Applying Γ_t^* from the left, using (2.64) and (2.68), we obtain that

$$\Gamma_t^* \underline{\rho}'' = \Gamma_t^* \underline{\rho}' = \underline{\rho}. \tag{2.70}$$

Since $\rho = \Gamma_t \rho''$, this proves (2.62) (iii) for $0 < t \le 1$. The proof of (2.62) (iv) for $0 < t \le 1$ goes exactly in the same way. ∎

Proof of Lemma 13 It follows from Lemma 35 and (2.56) that $\Gamma_t^{(2)} \underline{\rho_\Xi^{(2)}} = \underline{\Gamma_t \rho_\Xi}^{(2)}$ and $\Gamma_t^{(2)} \overline{\rho_\Xi^{(2)}} = \overline{\Gamma_t \rho_\Xi}^{(2)}$, where $\Gamma_t \rho_\Xi = \rho_{\Xi'}$ by Lemma 11. ∎

We cited Lemma 32 and Proposition 34 from [MSS18], so to complete the proofs of this subsection, it only remains to provide the proofs of Lemmas 30, 31, and 33.

Proof of Lemma 30 By De Finetti's theorem, the map $\nu \mapsto \nu^{(\infty)}$ is a bijection from $\mathcal{P}(\mathcal{P}(I))$ to $\mathcal{P}_{\mathrm{sym}}(I^\infty)$, so it suffices to show that for $\nu \in \mathcal{P}(\mathcal{P}(I))$, one has $\nu \in \mathcal{M}^*$ if and only if $\nu^{(\infty)} \in \mathcal{M}^{(\infty)}$. Let ξ be a $\mathcal{P}(I)$-valued random variable with law ν and conditional on ξ, let $(Y^k)_{k \in \mathbb{N}_+}$ be i.i.d. with law ξ. Then the unconditional law of $(Y^k)_{k \in \mathbb{N}_+}$ is $\nu^{(\infty)}$. By the definition in (2.55), $\nu \in \mathcal{M}^*$ if and only if $\mathbb{P}[\xi([0,t]) > 0] \le t$ for all $0 < t \le 1$. The event $\{\xi([0,t]) > 0\}$ is a.s. equal to the event $\{\exists k \in \mathbb{N}_+ \text{ s.t. } Y^k \le t\}$, so comparing with the definition in (2.24) we see that $\nu \in \mathcal{M}^*$ if and only if $\nu^{(\infty)} \in \mathcal{M}^{(\infty)}$. ∎

Proof of Lemma 31 We need to show that Γ_t^* and C_t^* defined as in (2.57) satisfy (2.56). Conditional on ξ, let $(Y^k)_{k \in \mathbb{N}_+}$ be i.i.d. with law ξ. Then the unconditional law of $(Y^k)_{k \in \mathbb{N}_+}$ is $\nu^{(\infty)}$ and by (2.26) $C_t^{(\infty)} \nu^{(\infty)}$ is the (unconditional) law of $(c_t(Y^k))_{k \in \mathbb{N}_+}$, which is the same as $(C_t^* \nu)^{(\infty)}$. The claim for Γ_t^* follows in the same way, using (2.25) and the fact that $\delta_{\delta_\infty}^{(\infty)} = \delta_{\vec{\infty}}$. ∎

Proof of Lemma 33 Assume that $\nu_1, \nu_2 \in \mathcal{M}^*$ satisfy $\nu_1 \le_{\mathrm{cv}} \nu_2$. By characterisation (ii) of the convex order, we can find a random variable Y and σ-fields $\mathcal{F}_1 \subset \mathcal{F}_2$ such that $\nu_i = \mathbb{P}\big[\mathbb{P}[Y \in \cdot | \mathcal{F}_i] \in \cdot\big]$ $(i = 1, 2)$. Then

$$\Gamma_t^* \nu_i = t^{-1} \tilde\nu_i + (1 - t^{-1}) \delta_{\delta_\infty} \quad \text{with} \quad \tilde\nu_i := \mathbb{P}\big[\mathbb{P}[\psi_t(Y) \in \cdot | \mathcal{F}_i] \in \cdot\big] \qquad (i = 1, 2). \tag{2.71}$$

Since $\mathcal{F}_1 \subset \mathcal{F}_2$, by characterisation (ii) of the convex order, we see that $\tilde\nu_1 \le_{\mathrm{cv}} \tilde\nu_2$. Using characterisation (i) of the convex order, it follows that $\Gamma_t^* \nu_1 \le_{\mathrm{cv}} \Gamma_t^* \nu_2$. ∎

3 Scale invariant solutions to the bivariate RDE

The goal of this section is to prove Theorem 14.

Given some $\theta \in (0,1)$, In Section 3 we will use the shorthand ρ to denote the probability measure ρ_{Ξ_θ} defined in (1.18). For every $k \in \mathbb{N}$ let

$$x_k := \theta^k \quad \text{and} \quad c_k := \frac{1-\theta}{1+\theta} \cdot \theta^k, \tag{3.1}$$

thus we have

$$\rho = \sum_{k=0}^\infty c_k \delta_{x_k} + \frac{\theta}{1+\theta} \delta_\infty. \tag{3.2}$$

For simplification we also introduce the notation

$$x_{-1} := \infty \quad \text{and} \quad c_{-1} := \rho(\{\infty\}) = \frac{\theta}{1+\theta}. \tag{3.3}$$

Using this notation we have $\rho(\{x_k\}) = c_k$ for every $k \ge -1$.

Definition 36 *Let $\mathcal{P}_\theta^{(2)}$ denote the space of symmetric probability measures on $I \times I$ such that its marginal distributions are ρ.*

3.1 Main lemmas

In Section 3.1 we state the key lemmas of Section 3 and prove Theorem 14 using them.

Definition 37 (The signature of a scale invariant measure) *Let $\theta \in (0, 1)$. The signature of a scale invariant measure $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ is the function $f_{\rho^{(2)}} : \mathbb{N} \to \mathbb{R}$ defined by*

$$f_{\rho^{(2)}}(n) := \rho^{(2)}\big(\{[0, x_n] \times I\} \cup \{I \times [0, 1]\}\big), \qquad n \in \mathbb{N}. \tag{3.4}$$

The signature of the diagonal measure $\overline{\rho}^{(2)}$ (c.f. (1.24)) is

$$f_{\overline{\rho}^{(2)}}(n) = \mathbb{P}\big[Y_\varnothing \leq x_n \text{ or } Y_\varnothing \leq 1\big] = \sum_{k=0}^{\infty} c_k = \frac{1}{1+\theta}, \quad n \in \mathbb{N}. \tag{3.5}$$

Lemma 38 (Conditions for f to be a signature) *If $\theta \in (0, 1)$ and $f : \mathbb{N} \to \mathbb{R}$ then there exists a (unique) probability measure $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ such that f is its signature if and only if*

(i) $f(0) \leq 1$,

(ii) $\lim\limits_{n \to \infty} f(n) = \frac{1}{1+\theta}$,

(iii) $f(n)$ *is non-increasing,*

(iv) $(1 + \theta) \cdot f(0) \leq 2f(1)$,

(v) $(1 + \theta) \cdot f(n) \leq \theta \cdot f(n - 1) + f(n + 1)$ *for every $n \geq 1$.*

We will prove this lemma in Section 3.2. Next we define a function $f_{\theta,c}(n), n \in \mathbb{N}$ that will help us identify the signature of a scale invariant solution of the bivariate RDE.

Lemma 39 (Implicit equation for $f_{\theta,c}(n)$) *Let $\theta \in (0, 1)$ and $c \geq 0$ be arbitrary. The system of equations*

$$f_{\theta,c}(0)^2 - \frac{1}{1+\theta} f_{\theta,c}(0) = 2c, \tag{3.6}$$

$$f_{\theta,c}(n-1)^2 - f_{\theta,c}(n)^2 = \theta^{n-1}\left(f_{\theta,c}(n-1) - f_{\theta,c}(n)\right) + c \cdot \theta^{2n-2}(1 - \theta^2), \quad n \geq 1, \tag{3.7}$$

$$f_{\theta,c}(0) > 0, \quad f_{\theta,c}(n) > \frac{\theta^{n-1}}{2}, \quad n \geq 1 \tag{3.8}$$

has a unique solution $f_{\theta,c}(n), n \in \mathbb{N}$.

Lemma 40 (Existence and continuity of $f_{\theta,c}(\infty)$) *If $\theta \in (0, 1), c \geq 0$, then the limit*

$$f_{\theta,c}(\infty) := \lim_{n \to \infty} f_{\theta,c}(n) \tag{3.9}$$

exists and the function $c \mapsto f_{\theta,c}(\infty)$ is continuous.

We will prove Lemmas 39 and 40 in Section 3.3. The $c = 0$ case is simple:

$$f_{\theta,0}(n) = \frac{1}{1+\theta}, \qquad n \in \mathbb{N}. \tag{3.10}$$

Remark 41 *Note that if we rearrange (3.6) and (3.7), we get*

$$f_{\theta,c}(0)^2 - \frac{f_{\theta,c}(0)}{1+\theta} = 2c, \qquad \frac{f_{\theta,c}(n-1) - f_{\theta,c}(n)}{\theta^{n-1} - \theta^n} = \frac{c \cdot \theta^{n-1} \cdot \frac{1+\theta}{2}}{\frac{f_{\theta,c}(n-1)+f_{\theta,c}(n)}{2} - \frac{\theta^{n-1}}{2}}. \tag{3.11}$$

Now if we non-rigorously define the function f by $f(\theta^n) := f_{\theta,c}(n)$ when θ is very close to 1, moreover we denote $r := \theta^n$, then in the $\theta \to 1$ limit we get

$$f(1)^2 - \frac{1}{2}f(1) = 2c, \qquad \frac{\partial}{\partial r}f(r) = \frac{c \cdot r}{f(r) - r/2} \tag{3.12}$$

i.e., conditions (iii) and (i) of equation (2.2) of [RST19]. We also note that condition (ii) of equation (2.2) of [RST19],i.e., $f(0) = \frac{1}{2}$, corresponds to the condition $f_{\theta,c}(\infty) = \frac{1}{1+\theta}$ in our current discrete setting. We will see that the key question is whether there exists $c > 0$ for which $f_{\theta,c}(\infty) = \frac{1}{1+\theta}$ holds.

Lemma 42 (Signature of scale invariant solution of the bivariate RDE) *Let $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ and let $f_{\rho^{(2)}}$ denote its signature.*

(i) *$\rho^{(2)}$ is a solution of the bivariate RDE (1.23) if and only if there exists $c \geq 0$ such that $f_{\rho^{(2)}}(n) = f_{\theta,c}(n)$ holds for every $n \in \mathbb{N}$.*

(ii) *If $\rho^{(2)}$ is a solution of the bivariate RDE and c is the parameter for which $f_{\rho^{(2)}}(n) = f_{\theta,c}(n)$ holds for every $n \in \mathbb{N}$, then*

$$c \leq \max\left(0, \frac{\theta \cdot (2\theta - 1)}{(1+\theta)^2}\right). \tag{3.13}$$

We will prove this lemma in Section 3.4. Note that that the diagonal measure $\overline{\rho}^{(2)}$ defined in (1.24) is a solution of the bivariate RDE (1.23), and indeed $f_{\overline{\rho}^{(2)}}(n) = \frac{1}{1+\theta} = f_{\theta,0}(n)$ for every $n \in \mathbb{N}$ by (3.5) and (3.10), in accordance with Lemma 42.

Definition 43 (Perturbation of the diagonal signature) *Let $\theta \in (0,1)$ and $c \geq 0$ be arbitrary and $f_{\theta,c}$. Let us define*

$$\tilde{f}_\theta(n) := \left(\frac{\partial}{\partial c} f_{\theta,c}(n)\right)\Big|_{c=0_+} \quad and \quad \tilde{f}_\theta(\infty) := \lim_{n\to\infty} \tilde{f}_\theta(n). \tag{3.14}$$

We will prove in Section 3.3 that the limit in (3.14) exists. Recall the notion of $\theta^* \in (\frac{1}{2}, 1)$ from Lemma 8.

Lemma 44 (Critical value) *We have $\tilde{f}_\theta(\infty) > 0$ for every $\theta \in (0, \theta^*)$, $\tilde{f}_\theta(\infty) < 0$ for every $\theta \in (\theta^*, 1)$ and $\tilde{f}_{\theta^*}(\infty) = 0$.*

We will prove this lemma in Section 3.5.

Lemma 45 (Solution of the recursion if $\theta \leq \theta^*$) *If $\theta \in \left(\frac{1}{2}, \theta^*\right]$, then there does not exist $c \in \left(0, \frac{\theta \cdot (2\theta-1)}{(1+\theta)^2}\right]$ for which $f_{\theta,c}(\infty) = \frac{1}{1+\theta}$.*

We will prove this lemma in Section 3.6.

Lemma 46 (Solution of the recursion if $\theta > \theta^*$) *For any $\theta \in (\theta^*, 1)$ there exists a $\hat{c} > 0$ for which $f_{\theta,\hat{c}}(\infty) = \frac{1}{1+\theta}$, moreover $f_{\theta,\hat{c}}$ also satisfies all of the conditions of Lemma 38.*

We will prove this lemma in Section 3.7.

Remark 47 *Figure 1 shows the values of the parameter c for which we have $f_{\theta,c}(\infty) = \frac{1}{1+\theta}$ for different values of $\theta \in (0.6, 1)$. It shows that if $\theta \leq \theta^*$ then the only such value is $c = 0$, but if $\theta > \theta^*$ then there also exists a positive value $\hat{c}$. We also note that if $\theta \to 1$ then the numerical simulations suggest that $\hat{c} \to 0.01770838$, which coincides with parameter value of c which gives the non-diagonal solution in the case $\Xi_1 = (0,1]$, see [RST19, Section 1.6]. In other words, $c = 0.01770838$ is the unique positive value of c for which the differential equation (3.12) together with the boundary condition $f(0) = \frac{1}{2}$ has a solution.*
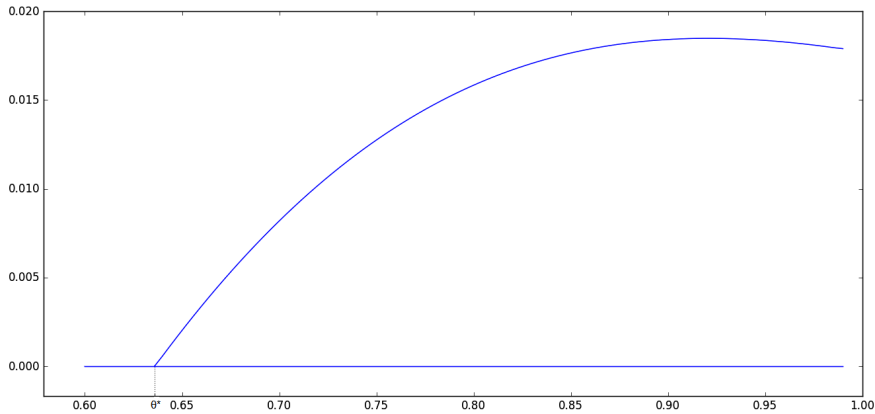
Figure 1: The values of c for which $f_{\theta,c}(\infty) = \frac{1}{1+\theta}$

Proof of Theorem 14 The diagonal measure $\bar{\rho}^{(2)}$ defined in (1.24) is indeed a solution of the bivariate RDE (1.23) for every $\theta \in (0,1)$.

If $\theta \leq \frac{1}{2}$ and $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ is a solution of the bivariate RDE, then by Lemma 42(ii) we must have $c = 0$, where c is the parameter for which $f_{\rho^{(2)}} \equiv f_{\theta,c}$ (such c exists by Lemma 42(i)). Hence in this case there is no scale invariant solution of the bivariate RDE other than the diagonal one.

If $\theta \in \left(\frac{1}{2}, \theta^*\right]$ and we assume that $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ is a solution of the bivariate RDE, then by Lemma 42 we have $c \in \left[0, \frac{\theta \cdot (2\theta-1)}{(1+\theta)^2}\right]$ for the parameter c which gives $f_{\rho^{(2)}} \equiv f_{\theta,c}$. But by Lemma 45 we know that there is no $c \in \left(0, \frac{\theta \cdot (2\theta-1)}{(1+\theta)^2}\right]$ such that $\lim_{n \to \infty} f_{\theta,c}(n) = \frac{1}{1+\theta}$ holds. Therefore by condition (ii) of Lemma 38 we see that again only $c = 0$ produces a signature of a scale invariant solution of the bivariate RDE.

If $\theta > \theta^*$ then by Lemmas 38 and 46 there exists a non-diagonal measure $\hat{\rho}^{(2)} \in \mathcal{M}_\theta^{(2)}$ for which $f_{\theta,\hat{c}}(n) = f_{\hat{\rho}^{(2)}}(n)$ for every $n \in \mathbb{N}$. Moreover, $\hat{\rho}^{(2)}$ is a solution of the bivariate RDE (1.23) by Lemma 42(i). ∎

3.2 Conditions for f to be a signature

In this section we show the necessary and sufficient conditions for a function $f : \mathbb{N} \to \mathbb{R}$ to be the signature of some $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$, i.e., we prove Lemma 38. To do this, first we define the bivariate signature $F_{\rho^{(2)}}$ in Definition 48 for each $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$ (c.f. Definition 36). In Lemma 49 we prove that this function $F_{\rho^{(2)}}$ characterizes the distribution $\rho^{(2)}$ and in Lemma 50 we prove necessary and sufficient conditions for bivariate functions to be the bivariate signature of some $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$. After analysing the relation between scale invariant measures and scale invariant

bivariate signatures in Lemma 52 as well as the relation between $F_{\rho^{(2)}}$ and the univariate signature in Lemma 53, we can easily conclude the proof of Lemma 38.

Definition 48 (Bivariate signature) *Let us assume given $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$. We define the function $F_{\rho^{(2)}} : \{x_k\}_{k=0}^\infty \times \{x_k\}_{k=0}^\infty \to [0,1]$ by*

$$F_{\rho^{(2)}}(x_k, x_j) := \rho^{(2)}\big(\{[0, x_k] \times I\} \cup \{I \times [0, x_j]\}\big), \quad j, k \in \mathbb{N}. \tag{3.15}$$

Recall the notation from the beginning of Section 3.

Lemma 49 (Bivariate signature characterizes the measure) *The measure $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$ is uniquely characterized by $F_{\rho^{(2)}}$, in particular*

$$\rho^{(2)}\big(\{\infty\} \times \{\infty\}\big) = 1 - F_{\rho^{(2)}}(x_0, x_0), \tag{3.16}$$

$$\rho^{(2)}\big([0, x_k] \times \{\infty\}\big) = F_{\rho^{(2)}}(x_k, x_0) - \frac{1}{1 + \theta}, \tag{3.17}$$

$$\rho^{(2)}\big(\{\infty\} \times [0, x_j]\big) = F_{\rho^{(2)}}(x_j, x_0) - \frac{1}{1 + \theta}, \tag{3.18}$$

$$\rho^{(2)}\big([0, x_k] \times [0, x_j]\big) = \frac{x_k}{1 + \theta} + \frac{x_j}{1 + \theta} - F_{\rho^{(2)}}(x_k, x_j). \tag{3.19}$$

Proof The proof of (3.16) follows from Definition 48 using $x_0 = 1$ and $\rho^{(2)}(([0,1] \cup \{\infty\})^2) = 1$. Since the marginal distribution of $\rho^{(2)}$ is ρ, for every $j \in \mathbb{N}$ we have $\rho^{(2)}\big(I \times [0, x_j]\big) = \rho([0, x_j]) = \sum_{i=j}^\infty c_i = \frac{x_j}{1+\theta}$. The equalities (3.17), (3.18) and (3.19) readily follow. The $\rho^{(2)}$ measure of every atom of $\rho^{(2)}$ can be determined using (3.16)-(3.19) and inclusion-exclusion. ∎

Lemma 50 (Necessary and sufficient conditions on F) *Let $\theta \in (0,1)$. Let us assume that we are given a function $F : \{x_k\}_{k=0}^\infty \times \{x_k\}_{k=0}^\infty \to [0, \infty)$. There exists a unique probability measure $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$ such that $F \equiv F_{\rho^{(2)}}$ holds (where $F_{\rho^{(2)}}$ is defined in Definition 48) if and only if the following conditions are fulfilled:*

1. $F(x_0, x_0) \leq 1$,

2. $\lim_{k \to \infty} F(x_k, x_j) = \frac{x_j}{1+\theta} \quad \forall j \in \mathbb{N}$,

3. $F(x_k, x_0)$ is non-increasing in k,

4. $F(x_k, x_j) = F(x_j, x_k) \quad \forall j, k \in \mathbb{N}$,

5. $-F(x_k, x_j) + F(x_{k+1}, x_j) + F(x_k, x_{j+1}) - F(x_{k+1}, x_{j+1}) \geq 0 \quad \forall j, k \in \mathbb{N}$.

Proof If $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$ and we define $F_{\rho^{(2)}}$ as in (3.15), then conditions 1., 2. and 3. trivially hold for $F_{\rho^{(2)}}$. Condition 4. follows from the symmetry of $\rho^{(2)}$ and finally condition 5. also holds, since $-F_{\rho^{(2)}}(x_k, x_j) + F_{\rho^{(2)}}(x_{k+1}, x_j) + F_{\rho^{(2)}}(x_k, x_{j+1}) - F_{\rho^{(2)}}(x_{k+1}, x_{j+1}) = \rho^{(2)}(x_k, x_j)$.

In the other direction, the uniqueness statement follows from Lemma 49.

If F is a function such that all of the conditions of the lemma hold, then we will define $\rho^{(2)}$ pointwise on $\{(x_k, x_j)\}_{k,j=-1}^\infty$ (where $x_{-1} = \infty$, c.f. (3.3)) and prove that $\rho^{(2)}$ is a probability measure, it is in $\mathcal{P}_\theta^{(2)}$ and $F \equiv F_{\rho^{(2)}}$ holds. For every $j, k \in \mathbb{N}$ let

$$\rho^{(2)}(x_k, x_j) := -F(x_k, x_j) + F(x_{k+1}, x_j) + F(x_k, x_{j+1}) - F(x_{k+1}, x_{j+1}), \tag{3.20}$$

$$\rho^{(2)}(\infty, x_k) := F(x_k, x_0) - F(x_{k+1}, x_0), \tag{3.21}$$

$$\rho^{(2)}(x_k, \infty) := F(x_k, x_0) - F(x_{k+1}, x_0), \tag{3.22}$$

$$\rho^{(2)}(\infty, \infty) := 1 - F(x_0, x_0). \tag{3.23}$$

The non-negativity of $\rho^{(2)}$ follows from conditions 1., 3. and 5., moreover $\rho^{(2)}$ is trivially symmetric, since F is also symmetric by 4.

The marginals of $\rho^{(2)}$:

$$\bullet \sum_{i=-1}^{\infty} \rho^{(2)}(x_k, x_i) = F(x_k, x_0) - F(x_{k+1}, x_0) +$$

$$+ \sum_{i=0}^{\infty} (-F(x_k, x_i) + F(x_{k+1}, x_i) + F(x_k, x_{i+1}) - F(x_{k+1}, x_{i+1})) =$$

$$= F(x_k, x_0) - F(x_{k+1}, x_0) - F(x_k, x_0) + \lim_{i \to \infty} F(x_k, x_i) + F(x_{k+1}, x_0) -$$

$$- \lim_{i \to \infty} F(x_{k+1}, x_i) \overset{2.}{=} \frac{(1-\theta) \cdot x_k}{1 + \theta} \overset{(3.1)}{=} c_k, \qquad k \in \mathbb{N} \tag{3.24}$$

$$\bullet \sum_{i=-1}^{\infty} \rho^{(2)}(\infty, x_i) = 1 - F(x_0, x_0) + \sum_{k=0}^{\infty} (F(x_k, x_0) - F(x_{k+1}, x_0)) =$$

$$= 1 - F(x_0, x_0) + F(x_0, x_0) - \lim_{k \to \infty} F(x_k, x_0) \overset{2.}{=} 1 - \frac{x_0}{1 + \theta} = c_{-1} \tag{3.25}$$

So the measure $\rho^{(2)}$ has marginal distributions ρ defined as in (3.2). In particular, $\rho^{(2)}$ is a probability measure on I^2. We still have to check that $F \equiv F_{\rho^{(2)}}$ holds:

$$F_{\rho^{(2)}}(x_k, x_j) \overset{(3.15)}{=} \rho^{(2)}\big(\{[0, x_k] \times I\} \cup \{I \times [0, x_j]\}\big) = \sum_{i=k}^{\infty} c_i + \sum_{l=j}^{\infty} \sum_{i=-1}^{k-1} \rho^{(2)}(x_i, x_l) =$$

$$\frac{x_k}{1 + \theta} + \sum_{l=j}^{\infty} (-F(x_0, x_l) + F(x_k, x_l) + F(x_0, x_{l+1}) - F(x_k, x_{l+1})) + F(x_0, x_j) - \lim_{j \to \infty} F(x_0, x_j) =$$

$$\frac{x_k}{1 + \theta} - F(x_0, x_j) + \lim_{l \to \infty} F(x_0, x_l) + F(x_k, x_j) - \lim_{l \to \infty} F(x_k, x_l) + F(x_0, x_j) - \lim_{j \to \infty} F(x_0, x_j) \overset{2.}{=}$$

$$F(x_k, x_j), \qquad j, k \in \mathbb{N}. \tag{3.26}$$

∎

Definition 51 (Scale invariant bivariate function) $F : \{x_k\}_{k=0}^{\infty} \times \{x_k\}_{k=0}^{\infty} \to [0, \infty)$ *is a scale invariant bivariate function if*

$$F(x_{k+l}, x_{j+l}) = \theta^l F(x_k, x_j), \quad j, k, l \in \mathbb{N}. \tag{3.27}$$

If F is scale invariant then for every $0 \le j \le k$ we have

$$F(x_k, x_j) = \theta^j F(x_{k-j}, x_0). \tag{3.28}$$

Recall the notation of $\mathcal{M}_\theta^{(2)}$ from below (1.31) as well as that of $\mathcal{P}_\theta^{(2)}$ from Definition 36.

Lemma 52 (Scale invariant measures and functions) *Let $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$. $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ holds if and only if $F_{\rho^{(2)}}$ defined in Definition 48 is a scale invariant function.*

Proof First note that if $F_{\rho^{(2)}}$ is a scale invariant function then

$$\rho^{(2)}\big([0, \theta^n] \times I \cup I \times [0, \theta^n]\big) \overset{(3.15)}{=} F_{\rho^{(2)}}(x_n, x_n) \overset{(3.27)}{=} \theta^n F_{\rho^{(2)}}(x_0, x_0) \le \theta^n, \qquad n \ge 0. \tag{3.29}$$

Together with our assumption that both of the marginals of $\rho^{(2)}$ are ρ, this implies that we have $\rho^{(2)}\big([0, t] \times I \cup I \times [0, t]\big) \le t$ for all $0 \le t \le 1$, thus by (1.28) we have $\rho^{(2)} \in \mathcal{M}^{(2)}$.

So it remains to show that if $\rho^{(2)} \in \mathcal{M}^{(2)} \cap \mathcal{P}_\theta^{(2)}$ then $F_{\rho^{(2)}}$ is scale invariant if and only if $\Gamma_\theta \rho^{(2)} = \rho^{(2)}$. Let us denote $\hat{\rho}^{(2)} := \Gamma_\theta \rho^{(2)}$. By the scale invariance of the marginal distribution (c.f. (1.31)) we have $\hat{\rho}^{(2)} \in \mathcal{P}_\theta^{(2)}$. Thus by Lemma 49 we only need to prove that $F_{\rho^{(2)}} \equiv F_{\hat{\rho}^{(2)}}$ if and only if $F_{\rho^{(2)}}$ satisfies (3.27). This equivalence follows as soon as we observe that by the definition (1.29) of $\Gamma_\theta \rho^{(2)}$ and the definition (3.15) of $F_{\hat{\rho}^{(2)}}$ we have $F_{\hat{\rho}^{(2)}}(x_k, x_j) = \theta^{-1} F_{\rho^{(2)}}(x_{k+1}, x_{j+1})$ for any $k, j \in \mathbb{N}$. ∎

Lemma 53 (Relationship between $F_{\rho^{(2)}}$ and the signature) *If $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$, $F_{\rho^{(2)}}$ is the function defined in Definition 48 and $f_{\rho^{(2)}}$ is the signature of $\rho^{(2)}$ (c.f. Definition 37), then*

$$f_{\rho^{(2)}}(n) = F_{\rho^{(2)}}(x_n, x_0) = \frac{1}{x_{k \wedge j}} F_{\rho^{(2)}}(x_k, x_j) \qquad (3.30)$$

holds for every $j, k \in \mathbb{N}$ for which $n = |k - j|$.

Proof The first equality is trivial from the definition of $f_{\rho^{(2)}}$ and $F_{\rho^{(2)}}$. The second equality follows from the fact that $F_{\rho^{(2)}}$ is a scale invariant function (by Lemma 52), (3.28) and symmetry of $F_{\rho^{(2)}}$. ∎

Proof of Lemma 38 First let $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ and $f_{\rho^{(2)}}$ be its signature. Let us also define $F_{\rho^{(2)}}$ as in (3.15), which means $f_{\rho^{(2)}}(n) = F_{\rho^{(2)}}(x_n, x_0)$ by Lemma 53.

Let us now check that $f_{\rho^{(2)}}$ satisfies the properties (i)-(v) of Lemma 38.

(i) $f_{\rho^{(2)}}(0) = F_{\rho^{(2)}}(x_0, x_0) \leq 1$ using condition 1. of Lemma 50.

(ii) $\lim_{n \to \infty} f_{\rho^{(2)}}(n) = \frac{1}{1+\theta}$ by (3.30) and condition 2. of Lemma 50.

(iii) We have $f_{\rho^{(2)}}(n) = F_{\rho^{(2)}}(x_n, x_0) \geq F_{\rho^{(2)}}(x_{n+1}, x_0) = f_{\rho^{(2)}}(n+1)$ for any $n \in \mathbb{N}$ by condition 3. of Lemma 50.

(iv) From condition 5. of Lemma 50 we know

$$- F_{\rho^{(2)}}(x_k, x_j) + F_{\rho^{(2)}}(x_{k+1}, x_j) + F_{\rho^{(2)}}(x_k, x_{j+1}) - F_{\rho^{(2)}}(x_{k+1}, x_{j+1}) \geq 0. \qquad (3.31)$$

Using $F_{\rho^{(2)}}(x_k, x_j) = x_{k \wedge j} \cdot f_{\rho^{(2)}}(|k - j|)$ (c.f. (3.30)), we get

$$- x_k \cdot f_{\rho^{(2)}}(0) + x_k \cdot f_{\rho^{(2)}}(1) + x_k \cdot f_{\rho^{(2)}}(1) - x_{k+1} \cdot f_{\rho^{(2)}}(0) \geq 0. \qquad (3.32)$$

Dividing by x_k, after rearranging we get condition (iv).

(v) If we use $F_{\rho^{(2)}}(x_k, x_j) = x_{k \wedge j} \cdot f_{\rho^{(2)}}(|k - j|)$ in (3.31) if $k > j$, we get

$$-x_j \cdot f_{\rho^{(2)}}(k-j) + x_j \cdot f_{\rho^{(2)}}(k+1-j) + x_{j+1} \cdot f_{\rho^{(2)}}(k-j-1) - x_{j+1} \cdot f_{\rho^{(2)}}(k-j) \geq 0. \qquad (3.33)$$

Let $n := k - j$. If we divide (3.33) by x_j, after rearranging we get the required inequality.

In the other direction, assume that $f : \mathbb{N} \to \mathbb{R}$ satisfies conditions (i)-(v) of Lemma 38. In this case, by Lemma 50 and Lemma 52, it is enough to define a scale invariant function F such that the conditions of Lemma 50 hold and we have $f \equiv f_{\rho^{(2)}}$ for the measure $\rho^{(2)}$ determined by F as in Lemma 50. To do this, for every $j, k \in \mathbb{N}$ let

$$F(x_k, x_j) := x_{k \wedge j} \cdot f(|k - j|). \qquad (3.34)$$

The scale invariance (c.f. (3.27)) and symmetry of F are straightforward from the above definition. Conditions (i), (ii), (iii) of Lemma 38 on f imply respectively conditions 1., 2. and 3. of Lemma 50. Condition 5. of Lemma 50 follows from condition (iv) of Lemma 38 (in the $k = j$ case) and condition (v) of Lemma 38 (in the $k > j$ case, and, by symmetry, in the $j > k$ case). Finally $f \equiv f_{\rho^{(2)}}$ follows from Lemma 50. ∎

3.3 Basic properties of $f_{\theta,c}(n)$

The main goal of Section 3.3 is to prove Lemmas 39 and 40, but we will also collect some other useful properties of $f_{\theta,c}(n)$ in Corollary 56. We will first define an auxiliary function $g_{\theta,c}(n), n \in \mathbb{N}$ and later we will identify $f_{\theta,c}(n)$ as $f_{\theta,c}(n) = \theta^n g_{\theta,c}(n)$. In order to construct $g_{\theta,c}(n)$, we need the following definition.

Definition 54 (Recursion map $\psi_{\theta,c}$) *Given some $\theta \in (0,1)$ and $c \geq 0$, let us define the function $\psi_{\theta,c} : \mathcal{D}_{\theta,c} \to \mathbb{R}$ by*

$$\psi_{\theta,c}(x) = \frac{1 + \sqrt{(2x-1)^2 - 4c \cdot (1-\theta^2)}}{2\theta}, \quad \mathcal{D}_{\theta,c} = (\sqrt{(1-\theta^2)c} + 1/2, +\infty). \tag{3.35}$$

Note that $x \in \mathcal{D}_{\theta,c}$ if and only if $2x - 1 \geq 0$ and $(2x-1)^2 - 4c \cdot (1-\theta^2) > 0$.

Lemma 55 (Recursive definition of $g_{\theta,c}(n)$) *For any $\theta \in (0,1)$ and $c \geq 0$, the recursion*

$$g_{\theta,c}(0) = \frac{1 + \sqrt{1 + 8c(1+\theta)^2}}{2(1+\theta)}, \quad g_{\theta,c}(n) = \psi_{\theta,c}\big(g_{\theta,c}(n-1)\big), \quad n \geq 1 \tag{3.36}$$

has a solution (i.e., $g_{\theta,c}(n) \in \mathcal{D}_{\theta,c}$ holds for all $n \geq 0$). Moreover, the solution satisfies

$$g_{\theta,c}(n) \geq g_{\theta,c}(n-1), \qquad n \geq 1. \tag{3.37}$$

Proof. We first check that $g_{\theta,c}(0) \in \mathcal{D}_{\theta,c}$ holds. In order to do so, let us denote $\gamma = 4c(1+\theta)^2$. After some rearrangements, we only need to check that $\sqrt{1 + 2\gamma} > \sqrt{(1-\theta^2)\gamma} + \theta$ holds. Taking the square of both sides and rearranging, we want to show $(1-\theta^2) + (1+\theta^2)\gamma > 2\theta \sqrt{(1-\theta^2)\gamma}$. Taking the square of both sides again and rearranging, we need

$$(1-\theta^2)^2 + 2(1-\theta^2)^2\gamma + (1+\theta^2)^2\gamma^2 > 0, \tag{3.38}$$

and this inequality indeed holds, since all of the terms are non-negative for any choice of $\theta \in (0,1)$ and $c \geq 0$. We have thus checked $g_{\theta,c}(0) \in \mathcal{D}_{\theta,c}$.

Next we observe that $x \mapsto \psi_{\theta,c}(x)$ is an increasing and concave function of $x \in \mathcal{D}_{\theta,c}$, moreover $\frac{\mathrm{d}}{\mathrm{d}x}\psi_{\theta,c}(x) > 1/\theta > 1$ holds for all $x \in \mathcal{D}_{\theta,c}$. This implies that the equation $\psi_{\theta,c}(x) = x$ has at most one solution in $\mathcal{D}_{\theta,c}$. Let $y_0 := \sqrt{(1-\theta^2)c} + 1/2$ denote the left endpoint of $\mathcal{D}_{\theta,c}$. One easily checks that $\psi_{\theta,c}(y_0) \geq y_0$ holds if and only if $c \leq \frac{1-\theta}{4\theta^2(1+\theta)}$ holds. We will prove Lemma 55 by treating the cases $\psi_{\theta,c}(y_0) \geq y_0$ and $\psi_{\theta,c}(y_0) < y_0$ separately.

If $\psi_{\theta,c}(y_0) \geq y_0$ then $\psi_{\theta,c}(x) > x$ for every $x \in \mathcal{D}_{\theta,c}$ follows from the above listed properties of $\psi_{\theta,c}$. Now it follows from (3.36) by induction on n that $g_{\theta,c}(n) \in \mathcal{D}_{\theta,c}$ and (3.37) hold for all $n \geq 0$.

If $\psi_{\theta,c}(y_0) < y_0$ then the above listed properties of $\psi_{\theta,c}$ imply that there exists a unique $x_0^* \in \mathcal{D}_{\theta,c}$ for which $\psi_{\theta,c}(x_0^*) = x_0^*$, moreover $x \geq x_0^*$ implies $\psi_{\theta,c}(x) \geq x$. One checks that $x_0^* := \frac{1 + \sqrt{1 + 4c(1+\theta)^2}}{2(1+\theta)}$, thus $g_{\theta,c}(0) \geq x_0^*$ holds. It is enough to prove that $g_{\theta,c}(n) \geq x_0^*$ for all $n \geq 0$ to conclude that $g_{\theta,c}(n) \in \mathcal{D}_{\theta,c}$ for all $n \geq 0$. Now both $g_{\theta,c}(n) \geq x_0^*$ and (3.37) follow by induction on n using the recursive definition (3.36) of $g_{\theta,c}(n)$. $\qquad\square$

Now we are ready to prove the existence and uniqueness of $f_{\theta,c}(n), n \in \mathbb{N}$.

Proof of Lemma 39. We will show by induction on n that

$$f_{\theta,c}(n) = g_{\theta,c}(n)\theta^n, \qquad n \in \mathbb{N} \tag{3.39}$$

is the unique solution of the system of equations (3.6)-(3.8). The induction hypothesis holds for $n = 0$, since (3.6) is a quadratic equation for $f_{\theta,c}(0)$, which has two solutions, one of them is

equal to $g_{\theta,c}(0)\theta^0$, while the other solution is less then or equal to zero, therefore only $g_{\theta,c}(0)\theta^0$ satisfies (3.8) for $n = 0$.

Now assume that $n \geq 1$ and (3.39) holds for $n-1$, i.e., we have $f_{\theta,c}(n-1) = g_{\theta,c}(n-1)\theta^{n-1}$. We can view (3.7) as a quadratic equation for $f_{\theta,c}(n)$ which has two solutions:

$$\widetilde{x}_{1,2} = \frac{\theta^{n-1} \pm \sqrt{(2f_{\theta,c}(n-1) - \theta^{n-1})^2 - 4c \cdot \theta^{2n-2} \cdot (1-\theta^2)}}{2}. \tag{3.40}$$

Now $\widetilde{x}_1 = \theta^n \psi_{\theta,c}(g_{\theta,c}(n-1)) = g_{\theta,c}(n)\theta^n$ follows from $f_{\theta,c}(n-1) = g_{\theta,c}(n-1)\theta^{n-1}$, (3.35) and (3.36), moreover $\widetilde{x}_1 > \theta^{n-1}/2$, while $\widetilde{x}_2 < \theta^{n-1}/2$, thus only $\widetilde{x}_1$ satisfies (3.8) and therefore (3.39) holds. $\qquad\square$

Corollary 56 (Recursion for $f_{\theta,c}$) *For any $\theta \in (0,1]$ and $c \geq 0$ we have*

$$f_{\theta,c}(0) = \frac{1 + \sqrt{1 + 8c(1+\theta)^2}}{2(1+\theta)}, \tag{3.41}$$

$$f_{\theta,c}(n) = \frac{\theta^{n-1} + \sqrt{(2f_{\theta,c}(n-1) - \theta^{n-1})^2 - 4c \cdot \theta^{2n-2} \cdot (1-\theta^2)}}{2}, \quad n \geq 1. \tag{3.42}$$

Moreover, the function $f_{\theta,c}(n)$ decreases in n:

$$f_{\theta,c}(n) \leq f_{\theta,c}(n-1), \quad n \geq 1. \tag{3.43}$$

Proof The identities (3.41) and (3.42) follow from (3.36) and (3.39).

In order to prove (3.43), we need to show that $g_{\theta,c}(n) \leq g_{\theta,c}(n-1)/\theta$ holds for any $n \geq 1$: this inequality follows from the fact that $\psi_{\theta,c}(x) \leq \psi_{\theta,0}(x) = x/\theta$ holds for any $x \in \mathcal{D}_{\theta,c}$. $\qquad\blacksquare$

Proof of Lemma 40 The limit $f_{\theta,c}(\infty) = \lim_{n\to\infty} f_{\theta,c}(n)$ exists since $f_{\theta,c}(n)$ decreases as n increases (c.f. (3.43)) and $f_{\theta,c}(n) \geq 0$. It follows from (3.41) and (3.42) by induction on n that for each n the function $c \mapsto f_{\theta,c}(n)$ is continuous. Thus, in order to prove that $c \mapsto f_{\theta,c}(\infty)$ is continuous, we only need to check that the functions $c \mapsto f_{\theta,c}(n)$ converge uniformly as $n \to \infty$ on $[0, c_0]$ for any $0 \leq c_0 < +\infty$. Using (3.42), it is easy to check that

$$f_{\theta,c}(n-1) - f_{\theta,c}(n) \leq \frac{1}{2}\sqrt{4c\theta^{2n-2}(1-\theta^2)}, \tag{3.44}$$

thus for any $c_0 > 0$ the functions $c \mapsto f_{\theta,c}(n)$ form a Cauchy sequence with respect to the sup-norm on $[0, c_0]$. From this the desired uniform convergence readily follows. $\qquad\blacksquare$

3.4 Signature of a scale invariant solution of the bivariate RDE

Our next goal is to prove Lemma 42. First we show a formula in Lemma 58, which characterizes the distribution of the right-hand side of the bivariate RDE (1.23) in terms of the bivariate signature $F_{\rho^{(2)}}$ (c.f. Definition 48). Using this we get an equation for the univariate signature $f_{\rho^{(2)}}$ of a scale invariant measure $\rho^{(2)}$ in Lemma 59, which holds if and only if $\rho^{(2)}$ is a solution of the bivariate RDE. In Lemma 60 we show that $f_{\theta,c}$ (defined in Lemma 39) satisfies a very similar equation. Finally we conclude the proof of Lemma 42.

Recall the notion of $\mathcal{P}_\theta^{(2)}$ from Definition 36.

Definition 57 (Definition of $\tilde{\rho}^{(2)}$) *Let $\theta \in (0,1)$, $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$. Denote by $\tilde{\rho}^{(2)}$ the law of*

$$(\chi[\tau, \kappa](Y_1, Y_2), \chi[\tau, \kappa](Y_1^*, Y_2^*)), \tag{3.45}$$

where the function χ is defined in (1.16) and the other notation are defined in (1.23), so $(Y_1, Y_1^*) \sim \rho^{(2)}$, $(Y_2, Y_2^*) \sim \rho^{(2)}$, $\tau \sim UNI[0,1]$, κ is a random variable such that $\mathbb{P}(\kappa = 1) = \mathbb{P}(\kappa = 2) = \frac{1}{2}$ and $(Y_1, Y_1^*), (Y_2, Y_2^*), \tau$ and κ are mutually independent.

Note that $\rho^{(2)} \in \mathcal{P}_\theta^{(2)}$ is a solution of the bivariate RDE (1.23) if and only if $\rho^{(2)} \equiv \tilde{\rho}^{(2)}$.

Lemma 58 (Expressing $F_{\tilde{\rho}^{(2)}}$ in terms of $F_{\rho^{(2)}}$) *If $\theta \in (0,1), \rho^{(2)} \in \mathcal{P}_\theta^{(2)}$, then for every $j, k \in \mathbb{N}$ we have*

$$F_{\tilde{\rho}^{(2)}}(x_k, x_j) = F_{\rho^{(2)}}(x_k, x_j) - \frac{1}{2} F_{\rho^{(2)}}(x_k, x_j)^2 + \frac{x_k^2}{2(1+\theta)^2} + \frac{x_j^2}{2(1+\theta)^2} +$$

$$+ \frac{1-\theta}{2\theta} \sum_{t=k\vee j+1}^{\infty} \left[\theta^t \Big(F_{\rho^{(2)}}(x_k, x_j) - F_{\rho^{(2)}}(x_t, x_j) - F_{\rho^{(2)}}(x_k, x_t) + F_{\rho^{(2)}}(x_t, x_t) \Big) \right], \quad (3.46)$$

where $F_{\rho^{(2)}}, F_{\tilde{\rho}^{(2)}}$ are the bivariate signatures of $\rho^{(2)}$ and $\tilde{\rho}^{(2)}$ respectively (c.f. Definition 48).

Proof Let us use the notation

$$(\tilde{Y}, \tilde{Y}^*) := (\chi[\tau, \kappa](Y_1, Y_2), \chi[\tau, \kappa](Y_1^*, Y_2^*)), \text{ so that } (\tilde{Y}, \tilde{Y}^*) \sim \tilde{\rho}^{(2)}. \quad (3.47)$$

Let us also use the shorthand $F = F_{\rho^{(2)}}, \widetilde{F} = F_{\tilde{\rho}^{(2)}}$ in this proof. Thus we have

$$\widetilde{F}(x_k, x_j) = \mathbb{P}(\kappa = 1, \tilde{Y} \le x_k \text{ or } \tilde{Y}^* \le x_j) + \mathbb{P}(\kappa = 2, \tilde{Y} \le x_k \text{ or } \tilde{Y}^* \le x_j). \quad (3.48)$$

Here

$$\mathbb{P}(\kappa = 1, \tilde{Y} \le x_k \text{ or } \tilde{Y}^* \le x_j) =$$
$$\frac{1}{2} \Big[\mathbb{P}(\tilde{Y} \le x_k \,|\, \kappa = 1) + \mathbb{P}(\tilde{Y}^* \le x_j \,|\, \kappa = 1) - \mathbb{P}(\tilde{Y} \le x_k, \tilde{Y}^* \le x_j \,|\, \kappa = 1) \Big]. \quad (3.49)$$

By the definition of χ in (1.16) we will calculate all the three terms on the r.h.s. of (3.49).

$$\mathbb{P}(\tilde{Y} \le x_k \,|\, \kappa = 1) = \mathbb{P}(Y_1 \le x_k, Y_1 > \tau) = \sum_{l=k}^{\infty} \mathbb{P}(Y_1 = x_l, x_l > \tau)$$

$$= \sum_{l=k}^{\infty} x_l \cdot c_l = \sum_{l=k}^{\infty} \frac{1-\theta}{1+\theta} \cdot \theta^{2l} = x_k^2 \cdot \frac{1-\theta}{1+\theta} \cdot \sum_{l=0}^{\infty} \theta^{2l} = \frac{x_k^2}{(1+\theta)^2}. \quad (3.50)$$

Similarly, we have

$$\mathbb{P}(\tilde{Y}^* \le x_j \,|\, \kappa = 1) = \frac{x_j^2}{(1+\theta)^2}. \quad (3.51)$$

$$\mathbb{P}(\tilde{Y} \le x_k, \tilde{Y}^* \le x_j \,|\, \kappa = 1) = \mathbb{P}(Y_1 \le x_k, Y_1^* \le x_j, \tau < Y_1 \wedge Y_1^*) =$$

$$\sum_{t=k\vee j+1}^{\infty} \mathbb{P}(x_t < Y_1 \le x_k, x_t < Y_1^* \le x_j, \tau \in [x_t, x_{t-1})) =$$

$$\sum_{t=k\vee j+1}^{\infty} ((F(x_t, x_j) + F(x_k, x_t) - F(x_k, x_j) - F(x_t, x_t))(x_{t-1} - x_t) =$$

$$\frac{\theta - 1}{\theta} \sum_{t=k\vee j+1}^{\infty} \left[\theta^t \Big(F(x_k, x_j) - F(x_t, x_j) - F(x_k, x_t) + F(x_t, x_t) \Big) \right]. \quad (3.52)$$

Now we calculate the other term of (3.48):

$$\mathbb{P}(\kappa = 2, \tilde{Y} \le x_k \text{ or } \tilde{Y}^* \le x_j) = \frac{1}{2} \cdot \Big(1 - \mathbb{P}(Y_1 \wedge Y_2 > x_k, Y_1^* \wedge Y_2^* > x_j) \Big)$$

$$= \frac{1}{2} \cdot \Big(1 - (1 - F(x_k, x_j))^2 \Big) = F(x_k, x_j) - \frac{1}{2} \cdot F(x_k, x_j)^2. \quad (3.53)$$

Now (3.46) follows if we substitute (3.49)-(3.53) into (3.48). ∎

Recall the notation of $\mathcal{M}_\theta^{(2)}$ from below (1.31).

Lemma 59 (Equation for the signature) $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ *is a solution of the bivariate RDE* (1.23) *if and only if its signature* $f_{\rho^{(2)}}$ *satisfies*

$$f_{\rho^{(2)}}(n)^2 = \frac{1}{(1+\theta)^2} + \theta^n \cdot f_{\rho^{(2)}}(n) - (1-\theta) \cdot \sum_{t=n}^{\infty} \theta^t f_{\rho^{(2)}}(t+1) +$$
$$+ \left(\frac{1}{(1+\theta)^2} + \frac{\theta \cdot f_{\rho^{(2)}}(0)}{1+\theta} - (1-\theta) \cdot \sum_{t=0}^{\infty} \theta^t f_{\rho^{(2)}}(t+1) \right) \cdot \theta^{2n}. \quad (3.54)$$

Proof By Lemmas 49 and 58, $\rho^{(2)}$ is a solution of the bivariate RDE (1.23) if and only if $F_{\tilde{\rho}^{(2)}} = F_{\rho^{(2)}}$ in (3.46). We will prove that if $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ then (3.54) holds if and only if $F_{\tilde{\rho}^{(2)}} = F_{\rho^{(2)}}$.

We have $F_{\tilde{\rho}^{(2)}} = F_{\rho^{(2)}}$ in (3.46) if and only if

$$F_{\rho^{(2)}}(x_k, x_j)^2 = \frac{x_k^2}{(1+\theta)^2} + \frac{x_j^2}{(1+\theta)^2} +$$
$$\frac{1-\theta}{\theta} \sum_{t=k\vee j+1}^{\infty} \left[\theta^t \Big(F_{\rho^{(2)}}(x_k, x_j) - F_{\rho^{(2)}}(x_t, x_j) - F_{\rho^{(2)}}(x_k, x_t) + F_{\rho^{(2)}}(x_t, x_t) \Big) \right] \quad (3.55)$$

holds for every $j, k \in \mathbb{N}$.

By symmetry of $F_{\rho^{(2)}}$ we can assume that $0 \le j \le k$ (so $x_k \le x_j$) and let $n := k - j$. We have $F_{\rho^{(2)}}(x_l, x_i) = x_{l\wedge i} f(|l-i|), l, i \in \mathbb{N}$ (c.f. (3.30)). Hence

$$F_{\rho^{(2)}}(x_k, x_j) = x_j \cdot f_{\rho^{(2)}}(n), \quad (3.56)$$

$$\sum_{t=k\vee j+1}^{\infty} \theta^t F_{\rho^{(2)}}(x_k, x_j) = \sum_{t=k+1}^{\infty} \theta^t \cdot x_j \cdot f_{\rho^{(2)}}(n) \overset{(3.1)}{=} \frac{x_{k+j+1} \cdot f_{\rho^{(2)}}(n)}{1-\theta}, \quad (3.57)$$

$$\sum_{t=k\vee j+1}^{\infty} \theta^t F_{\rho^{(2)}}(x_t, x_j) = \sum_{t=k+1}^{\infty} \theta^t \cdot x_j \cdot f_{\rho^{(2)}}(t-j) =$$
$$= x_{k+j+1} \cdot \sum_{t=0}^{\infty} \theta^t f_{\rho^{(2)}}(t+n+1) = x_{k+j+1} \cdot \sum_{t=n}^{\infty} \theta^{t-n} f_{\rho^{(2)}}(t+1), \quad (3.58)$$

$$\sum_{t=k\vee j+1}^{\infty} \theta^t F_{\rho^{(2)}}(x_k, x_t) = \sum_{t=k+1}^{\infty} \theta^t x_k f_{\rho^{(2)}}(t-k) = x_{2k+1} \sum_{t=0}^{\infty} \theta^t f_{\rho^{(2)}}(t+1), \quad (3.59)$$

$$\sum_{t=k\vee j+1}^{\infty} \theta^t F_{\rho^{(2)}}(x_t, x_t) = \sum_{t=k+1}^{\infty} \theta^t \cdot x_t \cdot f_{\rho^{(2)}}(0) \overset{(3.1)}{=} \frac{x_{2(k+1)} \cdot f_{\rho^{(2)}}(0)}{1-\theta^2}. \quad (3.60)$$

If we substitute all of the above into (3.55) and divide by $x_j^2 = \theta^{2j}$, we get:

$$f_{\rho^{(2)}}(n)^2 = \frac{\theta^{2n}}{(1+\theta)^2} + \frac{1}{(1+\theta)^2} + \theta^n f_{\rho^{(2)}}(n) - (1-\theta)\theta^n \cdot$$
$$\cdot \sum_{t=n}^{\infty} \theta^{t-n} f_{\rho^{(2)}}(t+1) - (1-\theta) \cdot \theta^{2n} \cdot \sum_{t=0}^{\infty} \theta^t f_{\rho^{(2)}}(t+1) + \frac{\theta^{2n+1} f_{\rho^{(2)}}(0)}{1+\theta}. \quad (3.61)$$

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We get (3.54) after rearranging above formula, so $F_{\tilde{\rho}^{(2)}} = F_{\rho^{(2)}}$ in (3.46) if and only if $f_{\rho^{(2)}}$ satisfies (3.54), therefore we proved the lemma. ∎

Next we consider the sequence $f_{\theta,c}(n), n \in \mathbb{N}$ (c.f. Lemma 39) and derive some formulas analogous to (3.54).

Lemma 60 (Properties of $f_{\theta,c}$) *Given some $\theta \in (0,1]$ and $c \geq 0$, let us assume that $\lim_{n\to\infty} f_{\theta,c}(n) = \frac{1}{1+\theta}$ holds. Under these conditions we have*

$$f_{\theta,c}(n)^2 = \frac{1}{(1+\theta)^2} + \theta^n \cdot f_{\theta,c}(n) - (1-\theta) \cdot \sum_{t=n}^{\infty} \theta^t f_{\theta,c}(t+1) + c \cdot \theta^{2n}, \quad n \in \mathbb{N}, \qquad (3.62)$$

$$c = \frac{1}{(1+\theta)^2} + \frac{\theta \cdot f_{\theta,c}(0)}{1+\theta} - (1-\theta) \cdot \sum_{t=0}^{\infty} \theta^t f_{\theta,c}(t+1), \qquad (3.63)$$

$$\frac{1}{1+\theta} \wedge \frac{2\theta}{1+\theta} \leq f_{\theta,c}(0) \leq \frac{1}{1+\theta} \vee \frac{2\theta}{1+\theta}, \qquad (3.64)$$

$$c \leq 0 \vee \frac{\theta \cdot (2\theta - 1)}{(1+\theta)^2}. \qquad (3.65)$$

Proof To prove (3.62) let us denote by β_n the difference of the r.h.s. and the l.h.s. of (3.62). Our goal is to show $\beta_n \equiv 0$. For every $n \geq 1$ we have

$$\beta_n - \beta_{n-1} = f_{\theta,c}(n-1)^2 - f_{\theta,c}(n)^2 - \theta^{n-1} f_{\theta,c}(n-1) + \theta^n f_{\theta,c}(n) +$$
$$+ (1-\theta)\theta^{n-1} f_{\theta,c}(n) - c \cdot \theta^{2n-2}(1-\theta^2). \qquad (3.66)$$

The right-hand side of (3.66) is 0 by (3.7). Therefore the sequence β_n is constant, but we also have $\lim_{n\to\infty} \beta_n = 0$ by the definition of β_n and our assumption $\lim_{n\to\infty} f_{\theta,c}(n) = \frac{1}{1+\theta}$. So $\beta_n \equiv 0$, thus we get (3.62).

Next we show (3.63). If we take (3.62) at $n = 0$, we obtain

$$f_{\theta,c}(0)^2 = \frac{1}{(1+\theta)^2} + f_{\theta,c}(0) - (1-\theta) \cdot \sum_{t=0}^{\infty} \theta^t f_{\theta,c}(t+1) + c. \qquad (3.67)$$

If we take the difference of (3.6) and (3.67) and rearrange, we get (3.63).

Next we prove (3.64). From our assumption $\lim_{n\to\infty} f_{\theta,c}(n) = \frac{1}{1+\theta}$ and (3.43) we obtain that $f_{\theta,c}(n) \geq \frac{1}{1+\theta}$ for every $n \in \mathbb{N}$, hence

$$c \overset{(3.63)}{\leq} \frac{1}{(1+\theta)^2} + \frac{\theta f_{\theta,c}(0)}{1+\theta} - (1-\theta)\sum_{t=0}^{\infty} \frac{\theta^t}{1+\theta} = \frac{\theta((1+\theta)f_{\theta,c}(0) - 1)}{(1+\theta)^2}. \qquad (3.68)$$

Putting together (3.6) and (3.68), we obtain the inequality

$$f_{\theta,c}(0)^2 - \frac{1}{1+\theta} f_{\theta,c}(0) \leq 2\frac{\theta((1+\theta)f_{\theta,c}(0) - 1)}{(1+\theta)^2}, \qquad (3.69)$$

which implies (3.64), since the roots of the polynomial $x^2 - \frac{x}{1+\theta} - 2\frac{\theta((1+\theta)x-1)}{(1+\theta)^2}$ are $\frac{1}{1+\theta}$ and $\frac{2\theta}{1+\theta}$. Finally, (3.65) follows by plugging the upper bound of (3.64) into (3.68). ∎

Proof of Lemma 42 First assume that $\rho^{(2)} \in \mathcal{M}_{\theta}^{(2)}$ is a solution of the bivariate RDE (1.23) and let us define

$$c := \frac{1}{(1+\theta)^2} + \frac{\theta \cdot f_{\rho^{(2)}}(0)}{1+\theta} - (1-\theta) \cdot \sum_{t=0}^{\infty} \theta^t f_{\rho^{(2)}}(t+1). \qquad (3.70)$$

We will prove that $f_{\rho^{(2)}}(n) = f_{\theta,c}(n)$ holds for this c for every $n \in \mathbb{N}$. By Lemma 39, it is enough to show that $f_{\rho^{(2)}}(n)$ satisfies (3.6)-(3.8).

By Lemma 59 the equation (3.54) holds. If we plug the definition (3.70) of c into equation (3.54), we get

$$f_{\rho^{(2)}}(n)^2 = \frac{1}{(1+\theta)^2} + \theta^n \cdot f_{\rho^{(2)}}(n) - (1-\theta) \cdot \sum_{t=n}^{\infty} \theta^t f_{\rho^{(2)}}(t+1) + c \cdot \theta^{2n}. \tag{3.71}$$

If we take (3.71) at $n = 0$, subtract $\frac{1}{1+\theta} \cdot f_{\rho^{(2)}}(0)$ from both sides and again use the definition (3.70) of c, we get $f_{\rho^{(2)}}(0)^2 - \frac{1}{1+\theta} f_{\rho^{(2)}}(0) = 2c$, i.e., that (3.6) holds. Now let $n \geq 1$. If we take the difference of (3.71) at $n - 1$ and at n, we obtain

$$f_{\rho^{(2)}}(n-1)^2 - f_{\rho^{(2)}}(n)^2 = \tag{3.72}$$
$$= \theta^{n-1} f_{\rho^{(2)}}(n-1) - \theta^n f_{\rho^{(2)}}(n) - (1-\theta)\theta^{n-1} f_{\rho^{(2)}}(n) + c \cdot \theta^{2n-2}(1-\theta^2),$$

therefore (3.7) holds. Both inequalities required by (3.8) follow from $f_{\rho^{(2)}}(n) \geq \frac{1}{1+\theta}$ (which holds by Lemma 38).

We also need that $c \geq 0$: this follows from $f_{\rho^{(2)}}(0)^2 - \frac{1}{1+\theta} f_{\rho^{(2)}}(0) = 2c$ and $f_{\rho^{(2)}}(n) \geq \frac{1}{1+\theta}$.

In the other direction, we assume that for some $\rho^{(2)} \in \mathcal{M}_\theta^{(2)}$ we have $f_{\rho^{(2)}}(n) = f_{\theta,c}(n)$ for every $n \in \mathbb{N}$ for some $c \geq 0$, and we have to show that $\rho^{(2)}$ is a solution of the bivariate RDE. By Lemma 59 it is enough to show that (3.54) holds for every $n \in \mathbb{N}$. In order to do so, we use Lemma 60 (the conditions of which do hold, since $\lim_{n \to \infty} f_{\rho^{(2)}}(n) = \frac{1}{1+\theta}$ by Lemma 38): the identity (3.54) follows by putting (3.62) and (3.63) together. This completes the proof of statement (i) of Lemma 42. Also, (3.13) follows from (3.65), i.e., statement (ii) of Lemma 42 also holds. The proof of Lemma 42 is complete. ∎

3.5   Definition of θ^*

In this section our goal is to prove Lemmas 8 and 44. We will give an explicit formula for $\tilde{f}_\theta(\infty) = \lim_{n \to \infty} \left(\frac{\partial}{\partial c} f_{\theta,c}(n) \right) \big|_{c=0_+}$ in Lemma 61 and prove that it is strictly decreasing in θ. As we will see, this fact implies Lemmas 8 and 44. This is a key point of the proof of Theorem 14: will see in Sections 3.6, 3.7 that the sign of $\tilde{f}_\theta(\infty)$ determines whether or not we have a non-diagonal scale invariant solution of the RDE.

Recall from Corollary 56 that $f_{\theta,c}$ satisfies (3.41) and (3.42). If we differentiate these equations with respect to c, we obtain

$$\frac{\partial}{\partial c} f_{\theta,c}(0) = \frac{2(1+\theta)}{\sqrt{1 + 8c(1+\theta)^2}}, \tag{3.73}$$

$$\frac{\partial}{\partial c} f_{\theta,c}(n) = \frac{\frac{\partial}{\partial c} f_{\theta,c}(n-1) \cdot (2f_{\theta,c}(n-1) - \theta^{n-1}) - \theta^{2n-2} \cdot (1-\theta^2)}{\sqrt{(2f_{\theta,c}(n-1) - \theta^{n-1})^2 - 4c \cdot \theta^{2n-2} \cdot (1-\theta^2)}}, \quad n \geq 1. \tag{3.74}$$

For any $\theta \in (0,1)$, let us define

$$\gamma_n(\theta) := \frac{\theta^{2n-2} \cdot (1-\theta^2)}{\frac{2}{1+\theta} - \theta^{n-1}} = \frac{(1+\theta)^2 \theta^{n-1}}{1 + 2\sum_{k=1}^{n-1} \theta^{-k}}, \quad n \geq 1. \tag{3.75}$$

Note that the equality of the two formulas in (3.75) holds for all $\theta \in (0,1)$, but the second formula for $\gamma_n(\theta)$ extends continuously to $\theta = 1$ as well.

Recall the notations $\tilde{f}_\theta(n)$ and $\tilde{f}_\theta(\infty)$ of Definition 43.

Lemma 61 (Formulas for $\tilde{f}_\theta$) *We have*

$$\tilde{f}_\theta(0) = 2(1 + \theta), \tag{3.76}$$

$$\tilde{f}_\theta(n) = \tilde{f}_\theta(n - 1) - \gamma_n(\theta) = 2(1 + \theta) - \sum_{k=1}^{n} \gamma_k(\theta), \quad n \geq 1, \tag{3.77}$$

$$\tilde{f}_\theta(\infty) = 2(1 + \theta) - \sum_{k=1}^{\infty} \gamma_k(\theta). \tag{3.78}$$

Proof Substituting $c = 0$ into (3.73), we get (3.76). Similarly, if we substitute $c = 0$ into (3.74) using that $f_{\theta,0}(n) = \frac{1}{1+\theta}$ for every $n \in \mathbb{N}$ (see (3.10)), we get (3.77). From (3.77) we get (3.78) by the definition of $\tilde{f}_\theta(\infty)$ (c.f. (3.14)). ∎

Proof of Lemmas 8 and 44 Note that the function $\theta \mapsto \tilde{f}_\theta(\infty)$ defined in (3.78) coincides with the function $\theta \mapsto g(\theta)$ defined in Lemma 8.

First we show that $\tilde{f}_\theta(\infty)$ is a decreasing function of $\theta \in [0, 1)$. By (3.75) and (3.77) we obtain $\tilde{f}_\theta(1) = 1 - \theta^2$, so $\tilde{f}_\theta(1)$ is a decreasing function of $\theta \in [0, 1)$. Next we observe that $\gamma_n(\theta)$ is an increasing function of $\theta \in [0, 1)$ by the second formula for $\gamma_n(\theta)$ in (3.75). Thus by (3.77) and (3.78) it follows that $\tilde{f}_\theta(\infty)$ is a decreasing function of $\theta \in [0, 1)$. The function $\theta \mapsto \tilde{f}_\theta(\infty)$ is also continuous on any compact sub-interval of $[0, 1)$, since it is the uniform limit of continuous functions.

In order to complete the proof of Lemmas 8 and 44, we just have to show that $\tilde{f}_{1/2}(\infty) > 0$ and $\tilde{f}_{1-\varepsilon}(\infty) < 0$ for some $\varepsilon > 0$. Indeed, $\tilde{f}_{1/2}(1) = \frac{3}{4}$ and

$$\tilde{f}_{1/2}(\infty) = \frac{3}{4} - \sum_{k=2}^{\infty} \frac{2^{2-2k} \cdot (1 - \frac{1}{4})}{\frac{2}{1+1/2} - 2^{1-k}} \geq \frac{3}{4} - \sum_{k=2}^{\infty} \frac{2^{2-2k} \cdot (1 - \frac{1}{4})}{\frac{2}{1+1/2} - 1/2} = \frac{9}{20} > 0. \tag{3.79}$$

On the other hand, $\tilde{f}_1(2) = -4/3$ by (3.75) and (3.77), moreover $\theta \mapsto \tilde{f}_\theta(2)$ is a continuous function on $[0, 1]$, therefore $\tilde{f}_{1-\varepsilon}(2) < 0$ for some $\varepsilon > 0$, from which $\tilde{f}_{1-\varepsilon}(\infty) < 0$ follows, since $\tilde{f}_\theta(\infty) \leq \tilde{f}_\theta(2)$ by (3.77) and (3.78). The proofs of Lemmas 8 and 44 are complete. ∎

3.6 The $\theta \leq \theta^*$ case

The goal of this section is to prove Lemma 45.

Lemma 62 (Lower bound on $f_{\theta,c}(n)$) *If $\theta \in \left(\frac{1}{2}, \theta^*\right]$ and $c \in \left(0, \frac{\theta \cdot (2\theta-1)}{(1+\theta)^2}\right]$, then*

$$f_{\theta,c}(1) - \tilde{f}_\theta(1)c > \frac{1}{1 + \theta}, \tag{3.80}$$

$$f_{\theta,c}(n) - \tilde{f}_\theta(n)c \geq f_{\theta,c}(1) - \tilde{f}_\theta(1)c, \qquad n \geq 1. \tag{3.81}$$

Before we prove Lemma 62, let us deduce Lemma 45 from it.

Proof of Lemma 45 By Lemma 44 we have $\lim_{n\to\infty} \tilde{f}_\theta(n) = \tilde{f}_\theta(\infty) \geq 0$ for any $\theta \leq \theta^*$, where $\tilde{f}_\theta(\infty)$ is defined in Definition 43. Thus

$$\lim_{n\to\infty} f_{\theta,c}(n) \geq \lim_{n\to\infty} \left(f_{\theta,c}(n) - \tilde{f}_\theta(n)c \right) \overset{(3.81)}{\geq} f_{\theta,c}(1) - \tilde{f}_\theta(1)c \overset{(3.80)}{>} \frac{1}{1 + \theta} \tag{3.82}$$

holds for any $\theta \in \left(\frac{1}{2}, \theta^*\right]$ and $c \in \left(0, \frac{\theta \cdot (2\theta-1)}{(1+\theta)^2}\right]$. The proof of Lemma 45 is complete. ∎

Remark 63 *We will prove* (3.81) *by induction on n. We have to start the induction from $n = 1$, since it can be easily seen that the analogue of* (3.80) *does not hold in the $n = 0$ case, i.e., we have $f_{\theta,c}(0) - \tilde{f}_\theta(0)c < \frac{1}{1+\theta}$.*

Proof of (3.80)**.** By (3.75) and (3.77) we have $\tilde{f}_\theta(1) = 1 - \theta^2$, so by (3.42) we need to show $\frac{1}{2}\left(1 + \sqrt{(2f_{\theta,c}(0) - 1)^2 - 4c(1 - \theta^2)}\right) - (1 - \theta^2)c > \frac{1}{1+\theta}$. Applying a series of equivalent transformations, we see that we need

$$f_{\theta,c}(0) > \frac{1}{2}\left(1 + \sqrt{\frac{(1-\theta)^2}{(1+\theta)^2} + 8(1-\theta)c + 4(1-\theta^2)^2c^2}\right). \tag{3.83}$$

Substituting the formula (3.41) for $f_{\theta,c}(0)$ into this, we obtain after some rearrangements that we need to show

$$\sqrt{1 + 8(1+\theta)^2c} - \theta > \sqrt{(1-\theta)^2 + 8(1-\theta)(1+\theta)^2c + 4(1-\theta^2)^2(1+\theta)^2c^2}.$$

Taking the square of both sides of this inequality, introducing the notation $\alpha = (1+\theta)^2c$ and rearranging a bit, we obtain that we need to show that $8\theta\alpha > 2\theta(\sqrt{1 + 8\alpha} - 1) + 4(1-\theta)^2\alpha^2$ holds. Introducing the notation $\beta = \sqrt{1 + 8\alpha} - 1$, we may equivalently rewrite this and obtain that we need to show $\beta < \frac{4\sqrt{\theta}}{1-\theta} - 2$. Using the definition of α and β, our assumption $c \leq \frac{\theta \cdot (2\theta - 1)}{(1+\theta)^2}$ becomes $\beta \leq \sqrt{(1 - 4\theta)^2} - 1$. Using that $\theta > \frac{1}{2}$ we see that we have $\beta \leq 4\theta - 2$, so it is enough to show $4\theta < \frac{4\sqrt{\theta}}{1-\theta}$ to conclude the desired inequality $\beta < \frac{4\sqrt{\theta}}{1-\theta} - 2$. Now $\theta < \frac{\sqrt{\theta}}{1-\theta}$ does hold for all $\theta \in (0, 1)$ (therefore it holds for $\theta \in \left(\frac{1}{2}, \theta^*\right]$), completing the proof of (3.80). ∎

Proof of (3.81)**.** We prove (3.81) by induction on n. The $n = 1$ case trivially holds. Let $n \geq 2$. Let us denote $q = \tilde{f}_\theta(n-1)c + f_{\theta,c}(1) - \tilde{f}_\theta(1)c$. By our induction hypothesis we know that $f_{\theta,c}(n-1) \geq q$ holds, and we want to show that (3.81) also holds, or, equivalently, we want $f_{\theta,c}(n) \geq q - \gamma_n(\theta)c$ to hold (c.f. (3.75), (3.77)). Let us note that we have

$$q - \gamma_n(\theta)c \overset{(3.77),(3.80)}{\geq} \tilde{f}_\theta(n)c + \frac{1}{1+\theta} \overset{(*)}{\geq} \frac{1}{1+\theta}, \tag{3.84}$$

where $(*)$ holds since $\theta \leq \theta^*$ implies $\tilde{f}_\theta(n) \geq 0$ (c.f. Lemmas 44 and 61). Using our induction hypothesis and (3.42), we see that it is enough to prove

$$\frac{1}{2}\left(\theta^{n-1} + \sqrt{(2q - \theta^{n-1})^2 - 4c\gamma_n(\theta)\left(\frac{2}{1+\theta} - \theta^{n-1}\right)}\right) \geq q - \gamma_n(\theta)c \tag{3.85}$$

in order to arrive at the desired $f_{\theta,c}(n) \geq q - \gamma_n(\theta)c$. Noting that the expression under the square root is non-negative by (3.84), we can rearrange (3.85) and see that it is equivalent to $2\left(q - \frac{1}{1+\theta}\right) \geq \gamma_n(\theta)c$, and this inequality indeed holds by (3.84). The proof of the induction step is complete. ∎

The proof of Lemma 62 is complete.

Remark 64 *Our assumption $c \in \left(0, \frac{\theta \cdot (2\theta - 1)}{(1+\theta)^2}\right]$ that appears in the statement of Lemma 62 (or something similar to it) seems indispensable, because numerical simulations suggest that the conclusions of Lemma 62 do not hold for big values of c.*

3.7 The $\theta > \theta^*$ case

In this section we prove Lemma 46. First we show Lemma 65, which implies that $f_{\theta,c}(\infty)$ is large if c is large. We will also argue that $f_{\theta,c}(\infty) < \frac{1}{1+\theta}$ if $\theta > \theta^*$ and c is small. We then combine these facts to show that there exists a $\hat{c} > 0$ for which $f_{\theta,\hat{c}}(\infty) = \frac{1}{1+\theta}$. After that we will see in Lemma 66 and 67 that this $f_{\theta,\hat{c}}$ satisfies the conditions of Lemma 38 (and therefore it is the signature of a non-diagonal solution $\hat{\rho}^{(2)} \in \mathcal{M}_\theta^{(2)}$ of the bivariate RDE (1.23)).

Lemma 65 (Lower bound on $f_{\theta,c}$) *If $\theta \in (0,1)$ and $c \geq 4$, then*

$$f_{\theta,c}(n) \geq \frac{\theta^n}{2} + \sqrt{\left(\frac{1}{2} + \theta^{2n}\right) \cdot c}, \qquad n \in \mathbb{N}. \tag{3.86}$$

Proof We prove (3.86) by induction on n. The $n = 0$ case holds, since

$$f_{\theta,c}(0) \overset{(3.41)}{\geq} \frac{1 + \sqrt{8c(1+\theta)^2}}{2(1+\theta)} \geq \frac{1}{4} + \sqrt{2c} \overset{(*)}{\geq} \frac{1}{2} + \sqrt{\frac{3}{2}c} = \frac{\theta^0}{2} + \sqrt{\left(\frac{1}{2} + \theta^{2\cdot 0}\right) \cdot c}, \tag{3.87}$$

where $(*)$ holds if $c \geq 4$. Now assume that $n \geq 1$ and (3.86) holds for $n - 1$, and we want to deduce that (3.86) also holds with n as well:

$$f_{\theta,c}(n) \overset{(3.42)}{=} \frac{\theta^{n-1} + \sqrt{(2f_{\theta,c}(n-1) - \theta^{n-1})^2 - 4c \cdot \theta^{2n-2} \cdot (1 - \theta^2)}}{2} \overset{(**)}{\geq}$$

$$\frac{\theta^n + \sqrt{4(\frac{1}{2} + \theta^{2(n-1)})c - 4c \cdot \theta^{2n-2} \cdot (1 - \theta^2)}}{2} = \frac{\theta^n}{2} + \sqrt{\left(\frac{1}{2} + \theta^{2n}\right) \cdot c},$$

where in $(**)$ we used the induction hypothesis and also that $\theta^{n-1} \geq \theta^n$. ∎

Lemma 66 ($f_{\theta,c}$ satisfies necessary conditions) *If $\theta \in (0,1)$ and $c \geq 0$ are arbitrary, then $f_{\theta,c}$ satisfies conditions (iii), (iv) and (v) of Lemma 38, i.e.*

1. *$f_{\theta,c}(n)$ is non-increasing in n,*

2. *$(1 + \theta) \cdot f_{\theta,c}(0) \leq 2f_{\theta,c}(1)$,*

3. *$(1 + \theta) \cdot f_{\theta,c}(n) \leq \theta \cdot f_{\theta,c}(n-1) + f_{\theta,c}(n+1)$ for every $n \geq 1$.*

Proof

1. We have already seen this in Corollary 56.

2. Recalling the notation introduced in Section 3.3 as well as (3.39), we want to show

$$(1 + \theta)g_{\theta,c}(0) \leq 2\theta\psi_{\theta,c}\big(g_{\theta,c}(0)\big). \tag{3.88}$$

By (3.35) the function $x \mapsto 2\theta\psi_{\theta,c}(x)$ is increasing and concave on $\mathcal{D}_{\theta,c}$, moreover we have $\frac{\mathrm{d}}{\mathrm{d}x}2\theta\psi_{\theta,c}(x) > 2x$ for every $x \in \mathcal{D}_{\theta,c}$, so the equation $(1 + \theta)x = 2\theta\psi_{\theta,c}(x)$ has at most one solution in $\mathcal{D}_{\theta,c}$. Let $y_0 := \sqrt{(1 - \theta^2)c} + 1/2$ denote the left endpoint of $\mathcal{D}_{\theta,c}$. One easily checks that $2\theta\psi_{\theta,c}(y_0) \geq (1 + \theta)y_0$ holds if and only if $c \leq \frac{1-\theta}{4(1+\theta)^3}$ holds. We will prove (3.88) by treating the cases $2\theta\psi_{\theta,c}(y_0) \geq (1 + \theta)y_0$ and $2\theta\psi_{\theta,c}(y_0) < (1 + \theta)y_0$ separately.

If $2\theta\psi_{\theta,c}(y_0) \geq (1+\theta)y_0$ then $(1+\theta)x \leq 2\theta\psi_{\theta,c}(x)$ for every $x \in \mathcal{D}_{\theta,c}$ follows from the above listed properties of $2\theta\psi_{\theta,c}$, and in particular (3.88) holds.

If $2\theta\psi_{\theta,c}(y_0) < (1+\theta)y_0$ then by the above listed properties of $2\theta\psi_{\theta,c}$ there exists a unique $\tilde{x} \in \mathcal{D}_{\theta,c}$ such that $(1+\theta)\tilde{x} = 2\theta\psi_{\theta,c}(\tilde{x})$, moreover $\tilde{x} \leq x$ implies $(1+\theta)x \leq 2\theta\psi_{\theta,c}(x)$. One easily finds that $\tilde{x} = \frac{1+\sqrt{4(\theta+3)(\theta+1)c+1}}{\theta+3}$, thus we only need to check $\tilde{x} \leq g_{\theta,c}(0)$, i.e., we need to check that $\alpha_\theta(c) \leq \beta_\theta(c)$ holds for all $c \geq 0$, where $\alpha_\theta(c) := \frac{1+\sqrt{4(\theta+3)(\theta+1)c+1}}{\theta+3}$ and $\beta_\theta(c) := \frac{1+\sqrt{1+8(1+\theta)^2 c}}{2(\theta+1)}$. The inverse functions of both $\alpha_\theta(\cdot)$ and $\beta_\theta(\cdot)$ are quadratic polynomials: $\alpha_\theta^{-1}(y) = \frac{((\theta+3)y-1)^2-1}{4(\theta+3)(\theta+1)}$ and $\beta_\theta^{-1}(y) = \frac{(2(\theta+1)y-1)^2-1}{8(\theta+1)^2}$. It is enough to check that $\alpha_\theta^{-1}(y) \geq \beta_\theta^{-1}(y)$ holds for all $y \in \mathbb{R}$, and indeed we have $\alpha_\theta^{-1}(y) - \beta_\theta^{-1}(y) = \frac{(1-\theta)y^2}{4(\theta+1)}$, which is nonnegative for all $\theta \in (0,1], y \in \mathbb{R}$.

3. We have to show $f_{\theta,c}(n) - f_{\theta,c}(n+1) \leq \theta \cdot (f_{\theta,c}(n-1) - f_{\theta,c}(n))$ for every $n \geq 1$. Rewriting this using the notation introduced in Section 3.3 as well as (3.39), we need to show that the inequality $g_{\theta,c}(n) - \theta g_{\theta,c}(n+1) \leq g_{\theta,c}(n-1) - \theta g_{\theta,c}(n)$ holds. Since $g_{\theta,c}(n+1) = \psi_{\theta,c}(g_{\theta,c}(n))$ and $g_{\theta,c}(n) = \psi_{\theta,c}(g_{\theta,c}(n-1))$, moreover we know $g_{\theta,c}(n) \geq g_{\theta,c}(n-1)$ (c.f. (3.37)), it is enough to show that $\varphi_{\theta,c}(x)$ is a decreasing function of x, where $\varphi_{\theta,c}(x) := x - \theta\psi_{\theta,c}(x)$. This is indeed the case, since we have $\varphi'_{\theta,c}(x) = 1 - \frac{2x-1}{\sqrt{(2x-1)^2-4c(1-\theta^2)}} < 0$ for every x in the domain $\mathcal{D}_{\theta,c}$ of $\varphi_{\theta,c}(\cdot)$.

∎

Lemma 67 (Upper bound on $f_{\theta,\hat{c}}(0)$) *If $\theta \in (0,1)$ and $f_{\theta,\hat{c}}(\infty) = \frac{1}{1+\theta}$, then $f_{\theta,\hat{c}}(0) \leq 1$.*

Proof The conditions of Lemma 60 are fulfilled for $f_{\theta,\hat{c}}$, thus we may use (3.64) to conclude $f_{\theta,\hat{c}}(0) \leq \frac{1}{1+\theta} \vee \frac{2\theta}{1+\theta} \leq 1$. ∎

Proof of Lemma 46 We will show that the function $c \mapsto f_{\theta,c}(\infty) - \frac{1}{1+\theta}$ takes both positive and negative values. This is enough to conclude the proof of the first statement of Lemma 46, since this function is continuous by Lemma 40.

We know from (3.10) that $f_{\theta,0}(n) = \frac{1}{1+\theta}$ for all $n \in \mathbb{N}$. By the $\theta > \theta^*$ case of Lemma 44 we have $\tilde{f}_\theta(\infty) = \lim_{n\to\infty} \tilde{f}_\theta(n) < 0$, therefore we can fix an $n \in \mathbb{N}$ such that $\tilde{f}_\theta(n) < 0$. Recall from Definition 43 that $\tilde{f}_\theta(n)$ denotes $\frac{\partial}{\partial c}f_{\theta,c}(n)\big|_{c=0_+}$. We can thus fix a small but positive value of c such that $f_{\theta,c}(n) < f_{\theta,0}(n) = \frac{1}{1+\theta}$. Now $f_{\theta,c}(\infty) < \frac{1}{1+\theta}$ follows from the fact that $f_{\theta,c}(n)$ decreases as n increases (c.f. (3.43)).

Next we show that there exists a $c > 0$ for which $f_{\theta,c}(\infty) > \frac{1}{1+\theta}$. This follows from Lemma 65, since for $c \geq 4$ we have

$$\lim_{n\to\infty} f_{\theta,c}(n) \overset{(3.86)}{\geq} \lim_{n\to\infty}\left(\frac{\theta^n}{2} + \sqrt{\left(\frac{1}{2}+\theta^{2n}\right)\cdot c}\right) = \sqrt{\frac{1}{2}c} > \frac{1}{1+\theta}. \qquad (3.89)$$

Therefore there exists $\hat{c} > 0$ such that $f_{\theta,\hat{c}}(\infty) = \frac{1}{1+\theta}$.

Now we prove the second statement of Lemma 46. Since $f_{\theta,\hat{c}}(\infty) = \frac{1}{1+\theta}$, condition (ii) of Lemma 38 holds and condition (i) also holds by Lemma 67. By Lemma 66 we also know that conditions (iii), (iv) and (v) of Lemma 38 are true. So we can conclude that $f_{\theta,\hat{c}}$ satisfies all of the conditions of Lemma 38. ∎

Remark 68 *In Figure 2 we can see $f_{0.85,c}(\infty)$ as a function of c, where c is an element of the interval $c \in \left[0, \frac{0.85\cdot(2\cdot0.85-1)}{(1+0.85)^2}\right]$. The horizontal red line is the constant $\frac{1}{1+\theta} \overset{\theta=0.85}{=} \frac{20}{37}$. We see that first it is decreasing, then it is increasing and goes to infinity, thus there exists $\hat{c} > 0$ for which $f_{0.85,\hat{c}}(\infty) = \frac{20}{37}$. We get a similar picture for every $\theta \in (\theta^*, 1)$.*

We also note that Figure 2 suggests that Conjecture 15 holds, since this conjecture is equivalent with the fact that there exists exactly one $\hat{c} > 0$ for which $f_{\theta,\hat{c}}(\infty) = \frac{1}{1+\theta}$.
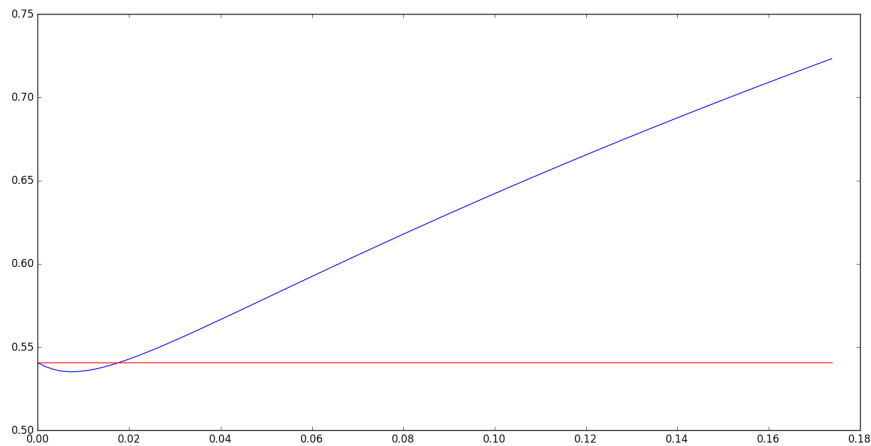
Figure 2: $f_{0.85,c}(\infty)$

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