Monotone and additive Markov process duality

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Abstract

This paper develops a systematic treatment of monotonicity-based dualities for Markov processes taking values in partially ordered sets. We show that every Markov process that takes values in a finite partially ordered set and whose generator can be represented in monotone maps has a pathwise dual, which in the special setting of attractive spin systems has been discovered earlier by Gray. This dual simplifies a lot in the special case that the space is a lattice and all monotone maps satisfy an additivity condition. This leads to a unified treatment of several well-known dualities, including Siegmunds dual for processes with a totally ordered state space, duality of additive spin systems, and a duality due to Krone for the two-stage contact process. It is well-known that additive spin systems can be constructed using a graphical representation involving open paths. We show that more generally, every additive Markov process can be formulated in terms of open paths on a suitably chosen underlying space. However, in order for the process and its dual to be representable on the same underlying space, one needs to assume that the state space is a distributive lattice. In the final section, we show how our results can be generalized from finite state spaces to interacting particle systems with finite local state spaces.

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Contents

1 Introduction

The problem of systematically finding Markov processes that are dual to each other has interested a large number of researchers. In [SL95, SL97, Sud00], a systematic treatment is given of dualities for nearest-neighbor interacting particle systems where the duality function is of a special "local" form (more precisely, it is the product over all sites of the underlying space of a function depending on one coordinate only). In [GKRV09, CGGR15], dual Markov processes are linked to different representations of the same Lie algebra. The paper [JK14] investigates different notions of duality, distinguishing in particular the usual modern definition of Markov process duality from stronger, "pathwise" concepts.

In the present paper, we systematically investigate the link between Markov process duality and order theory. This unified treatment includes some of the oldest known forms of Markov process duality. Building on earlier work, Siegmund [Sie76] proved that (almost) every monotone Markov process taking values in a totally ordered set has a dual, which is also a monotone Markov process, taking values in (almost) the same set. Examples of such dualities (for example between Brownian motion with reflection and absorption at the origin) were known at least 28 years earlier; see [JK14] and also [Lig85, Section II.3] for a (short) historical overview.

Moving away from totally ordered spaces, spin systems are Markov processes taking values in the set of all subsets of a countable underlying space. Such a set of subsets, equipped with the order of set inclusion, is a partially ordered set. It was known since the early 1970ies that certain spin systems such as the contact and voter models have duals. Harris [Har78] showed that the essential feature of these models is that they are additive and that there is a percolation picture behind each additive system. The idea was further formalized in Griffeath's monologue [Gri79] where the term "percolation substructure" was coined. Some years later, Gray [Gra86] introduced a more general, but also more complicated duality for spin systems that are monotone but not necessarily additive, which includes the dualities of additive systems as a special case.

The main contribution of the present paper is that we replace the specific examples of partially ordered sets mentioned above (totally ordered sets respectively the set of all subsets of another set) by a general partially ordered set. For technical simplicity, we will mostly concentrate on finite spaces, but we also show how the theory can be generalized to infinite products of finite partially ordered sets, which allows our results to be applied to interacting particle systems. Widening the view to general partially ordered sets leads to a number of new insights. In particular, we will see that:

I Siegmund's duality for monotone processes taking values in totally ordered spaces and Harris' duality for additive spin systems are based on the same principle. Indeed, both are special cases of a general duality for additive processes taking values in a lattice (in the order-theoretic meaning of the word).

II Additive processes taking values in a general lattice (in our new, more general formulation) can always be constructed in terms of a percolation substructure. If the lattice is moreover distributive, then the process and its dual can be represented in the same percolation substructure and the duality has a graphical interpretation.

III Gray's duality for monotone spin systems can be generalized to monotonically representable Markov processes taking values in general partially ordered spaces. Here, "monotonically representable" is a somewhat stronger concept than monotonicity, as discovered by [FM01] and independently by D.A. Ross (unpublished).

The rest of the paper is organized as follows. In Section 2, we state our main results. In Subsections 2.1–2.3, we develop a general strategy for finding pathwise duals, based on invariant subspaces, and determine the invariant subspaces associated with additive and monotone maps, respectively, that form the basis for our dualities. In Subsections 2.4 and 2.5 we then state our main results, which are a general duality for additively representable Markov processes (Theorem 7, corresponding to Point I above), and general duality for monotonically representable Markov processes (Theorem 9, corresponding to Point III above). In Subsection 2.6, we present the results from Point II above and in Subsection 2.7 we mention some open problems.

Section 3 is dedicated to examples. In Subsections 3.1 and 3.2, we show that Siegmund's duality for monotone processes taking values in totally ordered spaces and Harris' duality for additive spin systems, respectively, are both special cases of the duality for additively representable systems introduced in Subsection 2.4. In Subsection 3.3, we show that a nice duality for the two-stage contact process discovered by Krone [Kro99] also fits into this general scheme. In Subsection 3.4, we show that Gray's [Gra86] duality for monotone spin systems is a special case of the general duality for monotonically representable Markov processes from Subsection 2.5. In Subsection 3.5, finally, we apply the duality of Subsection 2.5 to derive a new duality, announced in [SS14], for particle systems with cooperative branching and coalescence.

Section 4 is dedicated to proofs. Up to this point, all results are for processes with finite state spaces only. In Section 5, we generalize our results to infinite product spaces, which are the typical setting of interacting particle systems. We conclude the paper with a short appendix recalling some basic facts from order theory.

2 Main results

2.1 Duality and pathwise duality

We recall that a (time-homogeneous) Markov process with measurable state space S is a stochastic process $X = (X_t)_{t\geq 0}$ taking values in S, defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with expectation denoted by $\mathbb{E}$, such that

$$\mathbb{E}\big[f(X_u)\,\big|\,(X_s)_{0\leq s\leq t}\big] = P_{u-t}f(X_t) \quad \text{a.s.} \quad (0 \leq t \leq u), \tag{2.1}$$

for each bounded measurable $f : S \to \mathbb{R}$, where $(P_t)_{t\geq 0}$ is a collection of probability kernels called the transition probabilities of X, and $P_t f(x) := \int P_t(x, \mathrm{d}y)f(y)$.

Let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be Markov processes with state spaces S and T, respectively, and let $\psi : S \times T \to \mathbb{R}$ be a measurable function. Then one says that X and Y are *dual* to each other with respect to the *duality function* ψ, if

$$\mathbb{E}\big[\psi(X_t, Y_0)\big] = \mathbb{E}\big[\psi(X_0, Y_t)\big] \qquad (t \geq 0) \tag{2.2}$$

for arbitrary deterministic initial states X_0 and Y_0. If (2.2) holds for deterministic initial states, then it also holds for random initial states provided X_t is independent of Y_0, X_0 is independent of Y_t, and the integrals are well-defined. Note that since we require (2.2) to hold for arbitrary initial conditions, Markov process duality is in fact a property of the transition kernels of two Markov processes, rather than of two concrete processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$. Possibly the first place where duality of Markov processes is defined in this general way is [Lig85, Def. II.3.1], although the term duality was used much earlier for specific duality functions.

It is often possible to construct Markov processes by means of a stochastic flow. Let S be a metrizable space and let $(\mathbf{X}_{s,t})_{s\leq t}$ be a collection of random maps $\mathbf{X}_{s,t} : S \to S$, such that almost surely, for each $x \in S$, the value $\mathbf{X}_{s,t}(x)$ as a function of both s and t is cadlag, i.e., right-continuous with left limits (denoted by $\mathbf{X}_{s-,t}$, $\mathbf{X}_{s,t-}$, etc.). [1] We call $(\mathbf{X}_{s,t})_{s\leq t}$ a *stochastic flow* if

[1]More formally, we require that there exist four functions $\mathbf{X}_{s,t}$, $\mathbf{X}_{s-,t}$, $\mathbf{X}_{s,t-}$, and $\mathbf{X}_{s-,t-}$ such that $\mathbf{X}_{s-,t} = \lim_{u\uparrow s} \mathbf{X}_{u,t}$ and $\mathbf{X}_{s,t} = \lim_{u\downarrow s} \mathbf{X}_{u-,t}$, and likewise with t replaced by $t-$, and with the roles of the first and second variable interchanged.

(i) $\mathbf{X}_{s,t} = \mathbf{X}_{s-,t} = \mathbf{X}_{s,t-} = \mathbf{X}_{s-,t-}$ a.s. for deterministic $s \le t$.

(ii) $\mathbf{X}_{s,s}$ is the identity map and $\mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u}$ $(s \le t \le u)$.

We say that $(\mathbf{X}_{s,t})_{s \le t}$ has *independent increments* if

(iii) $\mathbf{X}_{t_0,t_1}, \ldots, \mathbf{X}_{t_{n-1},t_n}$ are independent for any $t_0 < \cdots < t_n$.

If $(\mathbf{X}_{s,t})_{s \le t}$ is a stochastic flow with independent increments and X_0 is an S-valued random variable, independent of $(\mathbf{X}_{s,t})_{s \le t}$, then for any $s \in \mathbb{R}$, setting

$$X_t := \mathbf{X}_{s,s+t}(X_0) \qquad (t \ge 0) \tag{2.3}$$

defines a Markov process $(X_t)_{t \ge 0}$ with cadlag sample paths. Many well-known Markov processes can be constructed in this way, based on a stochastic flow with independent increments. Examples are interacting particle systems that can be constructed from Poisson processes that form their graphical representation, or diffusion processes that can be constructed as strong solutions to a stochastic differential equation, relative to a (multidimensional) Brownian motion.

Let $(\mathbf{X}_{s,t})_{s \le t}$ and $(\mathbf{Y}_{s,t})_{s \le t}$ be stochastic flows with independent increments, acting on metrizable spaces S and T, respectively, and let $\psi : S \times T \to \mathbb{R}$ be a function. Then we will say that $(\mathbf{X}_{s,t})_{s \le t}$ and $(\mathbf{Y}_{s,t})_{s \le t}$ are *dual* to each other with respect to the *duality function* ψ, if

(i) $(\mathbf{X}_{t_0,t_1}, \mathbf{Y}_{-t_1,-t_0}), \ldots, (\mathbf{X}_{t_{n-1},t_n}, \mathbf{Y}_{-t_n,-t_{n-1}})$ are independent for any $t_0 < \cdots < t_n$.

(ii) For each $x \in S$, $y \in T$, and $s \le u$, the function $[s,u] \ni t \mapsto \psi\big(\mathbf{X}_{s,t-}(x), \mathbf{Y}_{-u,-t}(y)\big)$ is a.s. constant.

Condition (i) implies in particular that $(\mathbf{X}_{s,t})_{s \le t}$ and $(\mathbf{Y}_{s,t})_{s \le t}$ have independent increments but also says that $\mathbf{X}_{s,t}$ and $\mathbf{Y}_{-t,-s}$ use, in a sense, the same randomness, i.e., the direction of time for the second flow is reversed with respect to the first flow. Setting $t = s, u$ in Condition (ii) shows that

$$\psi\big(x, \mathbf{Y}_{-u,-s}(y)\big) = \psi\big(\mathbf{X}_{s,u}(x), y\big) \quad \text{a.s.}, \tag{2.4}$$

which in view of (2.3) implies that the Markov processes X and Y associated with the stochastic flows $(\mathbf{X}_{s,t})_{s \le t}$ and $(\mathbf{Y}_{s,t})_{s \le t}$ are dual. Note that in Condition (ii), we take $\mathbf{X}_{s,t-}(x)$ to be left-continuous in t, which is in general necessary to get a sensible definition.

If two Markov processes X and Y can be constructed from stochastic flows that are dual in the sense defined above, then, loosely following terminology introduced in [JK14], we say that X and Y are *pathwise dual* to each other. Although this seems a priori a much stronger concept, many (though not all) well-known Markov process dualities can be realized as pathwise dualities. In particular, this will be true for all dualities discussed in the present paper.

Sometimes (such as in [AS05]) it may be useful to consider the case where the equality in (2.2) is replaced by a $\le$ (respectively $\ge$) sign. If this is the case, then we say that X is a *subdual* (respectively *superdual*) of Y (with duality function ψ). Likewise, if the function $[s,u] \ni t \mapsto \psi\big(\mathbf{X}_{s,t-}(x), \mathbf{Y}_{-u,-t}(y)\big)$ is a.s. nonincreasing (respectively nondecreasing), then we speak of *pathwise subduality* (respectively *superduality*).

2.2 Random mapping representations

Starting here, throughout the remainder of Section 2 as well as Sections 3 and 4, we will only be concerned with (continuous-time) Markov processes with *finite* state spaces. In the present subsection, we show how stochastic flows for such processes can be constructed from Poisson noise and how this leads to a standard way to construct pathwise duals.

For any two sets S, T, we let $\mathcal{F}(S,T)$ denote the space of all functions $f : S \to T$. If S, T are finite sets, then any linear operator $A : \mathcal{F}(T,\mathbb{R}) \to \mathcal{F}(S,\mathbb{R})$ is uniquely characterized by its matrix $(A(x,y))_{x\in S,\ y\in T}$ through the formula

$$Af(x) = \sum_{y\in T} A(x,y)f(y) \qquad \big(x \in S,\ f \in \mathcal{F}(T,\mathbb{R})\big). \tag{2.5}$$

A linear operator $K : \mathcal{F}(T,\mathbb{R}) \to \mathcal{F}(S,\mathbb{R})$ is a *probability kernel from S to T* if and only if

$$K(x,y) \geq 0 \quad \text{and} \quad \sum_{z\in T} K(x,z) = 1 \qquad (x \in S,\ y \in T). \tag{2.6}$$

If S is a finite set, then a *Markov generator* (sometimes called Q-matrix) of a Markov process with state space S is a matrix $(G(x,y))_{x,y\in S}$ such that

$$G(x,y) \geq 0 \quad (x \neq y) \quad \text{and} \quad \sum_{y\in S} G(x,y) = 0 \quad (x \in S). \tag{2.7}$$

It is well-known that each such Markov generator defines probability kernels $(P_t)_{t\geq 0}$ on S through the formula

$$P_t := e^{tG} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n G^n \qquad (t \geq 0). \tag{2.8}$$

The $(P_t)_{t\geq 0}$ are the *transition kernels* of a Markov process $X = (X_t)_{t\geq 0}$ with cadlag sample paths, and the Markov property (2.1) can be reformulated in matrix notation as

$$\mathbb{P}[X_u = y \,|\, (X_s)_{0\leq s\leq t}] = P_{u-t}(X_t, y) \quad \text{a.s.} \qquad (0 \leq t \leq u,\ y \in S). \tag{2.9}$$

We wish to construct the Markov process X from a stochastic flow. To this aim, we write the generator in the form

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m\big(f(m(x)) - f(x)\big) \qquad (x \in S, f \in \mathcal{F}(S,\mathbb{R})), \tag{2.10}$$

where $\mathcal{G} \subset \mathcal{F}(S,S)$ is a set whose elements are maps $m : S \to S$ and $(r_m)_{m\in\mathcal{G}}$ are nonnegative constants. Loosely following terminology from [LPW09], we call (2.10) a *random mapping representation* for G. It is not hard to see that each Markov generator (on a finite set S) can be written as in (2.10). Random mapping representations are far from unique, although in practical situations, there are usually only a handful which one would deem "natural".

Given a random mapping representation for its generator, there is a standard way to construct a stochastic flow for a Markov process. Let Δ be a Poisson point subset of $\mathcal{G} \times \mathbb{R} = \{(m,t) : m \in \mathcal{G},\ t \in \mathbb{R}\}$ with local intensity $r_m \mathrm{d}t$, where $\mathrm{d}t$ denotes Lebesgue measure. For $s \leq u$, set $\Delta_{s,u} := \Delta \cap (\mathcal{G} \times (s,u])$. Then the $\Delta_{s,u}$ are a.s. finite sets and, because Lebesgue measure is nonatomic, two distinct points $(m,t), (m',t') \in \Delta_{s,u}$ have a.s. different time coordinates $t \neq t'$. Using this, we can unambiguously define random maps $\mathbf{X}_{s,t} : S \to S$ $(s \leq t)$ by

$$\begin{aligned} &\mathbf{X}_{s,t} := m_n \circ \cdots \circ m_1 \\ &\quad \text{if} \quad \Delta_{s,t} = \{(m_1,t_1), \ldots, (m_n,t_n)\}, \quad t_1 < \cdots < t_n, \end{aligned} \tag{2.11}$$

with the convention that $\mathbf{X}_{s,t}(x) = x$ if $\Delta_{s,t} = \emptyset$. We also define $\Delta_{s,u-} := \Delta \cap (\mathcal{G} \times (s,u))$, and define $\mathbf{X}_{s,t-}$ correspondingly, and similarly for $\mathbf{X}_{s-,t}$, $\mathbf{X}_{s-,t-}$. The following lemma is well-known, but for completeness we will sketch a proof in Section 4.2.

Lemma 1 (Stochastic flow) *The random maps $(\mathbf{X}_{s,t})_{s\leq t}$ form a stochastic flow with independent increments, as defined in Subsection 2.1, and the Markov process with generator G can be constructed in terms of $(\mathbf{X}_{s,t})_{s\leq t}$ as in (2.3).*

Let S and T be sets and let $\psi : S \times T \to \mathbb{R}$ be a function. Then we say that two maps $m : S \to S$ and $\hat{m} : T \to T$ are *dual* with respect to the *duality function* ψ if

$$\psi\big(m(x), y\big) = \psi\big(x, \hat{m}(y)\big) \qquad (x \in S,\ y \in T). \tag{2.12}$$

If in (2.12) the equality is replaced by $\leq$ then we also say that $\hat{m}$ is *subdual* to m with respect to the duality function ψ. Superdual maps are defined in the obvious way. Now imagine that S and T are finite sets, that G is the generator of a Markov process in S, and that for a given random mapping representation as in (2.10), all maps $m \in \mathcal{G}$ have a dual $\hat{m}$ with respect to ψ. (In general, such a dual map need not be unique, but we choose one and denote it by $\hat{m}$.) Then we claim that the Markov process Y with state space T and generator

$$Hf(y) := \sum_{m\in\mathcal{G}} r_m\big(f(\hat{m}(y)) - f(y)\big) \qquad (y \in T,\ f \in \mathcal{F}(T, \mathbb{R})) \tag{2.13}$$

is pathwise dual to X. To see this, we define

$$\hat{\Delta} := \{(\hat{m}, -t) : (m, t) \in \Delta\}, \tag{2.14}$$

which is a Poisson point set on $\hat{\mathcal{G}} \times \mathbb{R}$ with $\hat{\mathcal{G}} := \{\hat{m} : m \in \mathcal{G}\}$. We use this Poisson point set to define random maps $(\mathbf{Y}_{s,t})_{s\leq t}$ in the same way as in (2.11). The proof of the following proposition, which can just as well be formulated in terms of sub- or superduality, is entirely straightforward; for completeness, we give the main steps in Section 4.2.

Proposition 2 (Pathwise duality) *Let X and Y be Markov processes with generators G and H of the form (2.10) and (2.13), respectively, where for each $m \in \mathcal{G}$, the map $\hat{m}$ is a dual of m in the sense of (2.12). Construct stochastic flows $(\mathbf{X}_{s,t})_{s\leq t}$ and $(\mathbf{Y}_{s,t})_{s\leq t}$ for these Markov processes as above. Then, almost surely, the map $\mathbf{X}_{s-,t-}$ is dual to $\mathbf{Y}_{-t,-s}$ for each $s \leq t$. Moreover, the stochastic flow $(\mathbf{Y}_{s,t})_{s\leq t}$ is dual to $(\mathbf{X}_{s,t})_{s\leq t}$ with respect to the duality function ψ, in the sense defined in Subsection 2.1, and the Markov processes X and Y are pathwise dual w.r.t. ψ.*

By grace of Proposition 2, in order to prove that a given Markov process X with generator G has a pathwise dual Y with respect to a certain duality function ψ, and in order to explicitly construct such a dual, it suffices to show that G has a random mapping representation (2.10) such that each map $m \in \mathcal{G}$ has a dual w.r.t. ψ. In view of this, much of our paper will be devoted to showing that certain maps have duals with respect to particular duality functions. Once we have shown this for a suitable class of maps, it is clear how to construct pathwise duals for Markov processes whose generators are representable in such maps.

In relation to this, we adopt the following definition. Let S be a finite set and let $\mathcal{G} \subset \mathcal{F}(S, S)$ be a set whose elements are maps $m : S \to S$. We say that a Markov generator G is *representable in* $\mathcal{G}$ if G can be written in the form (2.10) for nonnegative constants $(r_m)_{m\in\mathcal{G}}$.

Similar definitions apply to probability kernels. Let S and T be finite sets. We let $\mathcal{K}(S, T)$ denote the space of all probability kernels from S to T. If M is a random variable taking values in the space $\mathcal{F}(S, T)$, then

$$K(x, y) := \mathbb{P}[M(x) = y] \qquad (x \in S,\ y \in T) \tag{2.15}$$

defines a probability kernel $K \in \mathcal{K}(S, T)$. If a probability kernel K is written in the form (2.15), then, borrowing terminology from [LPW09], we call (2.15) a *random mapping representation* for K. For any set $\mathcal{G} \subset \mathcal{F}(S,T)$ of maps $m : S \to T$, we say that a probability kernel $K \in \mathcal{K}(S,T)$ is *representable in* $\mathcal{G}$ if there exists a $\mathcal{G}$-valued random variable M such that (2.15) holds. It is easy to see that each probability kernel from S to T is representable in $\mathcal{F}(S,T)$ but the representation is again not unique.

We observe that if $(\mathbf{X}_{s,t})_{s \le t}$ is a stochastic flow with independent increments, then the transition kernels of the associated Markov process (in the sense of (2.3)) are given by

$$P_t(x,y) = \mathbb{P}[\mathbf{X}_{0,t}(x) = y] \qquad (t \ge 0,\ x, y \in S), \tag{2.16}$$

and this formula gives a random mapping representation for P_t. In view of this and (2.11), the following lemma, which will be proved in Section 4.2, should not come as a surprise.

Lemma 3 (Representability of generators) *Let S be a finite set, let $\mathcal{G}$ be a set whose elements are maps $m : S \to S$, and let X be a Markov process in S with generator G and transition kernels $(P_t)_{t \ge 0}$. Assume that $\mathcal{G}$ is closed under composition and contains the identity map. Then the following statements are equivalent:*

(i) *G can be represented in $\mathcal{G}$.*

(ii) *P_t can be represented in $\mathcal{G}$ for all $t \ge 0$.*

2.3 Invariant subspaces

As explained previously we will due to Proposition 2 be interested in Markov processes whose generators can be represented in maps that have a dual map with respect to a suitable duality function. In the present subsection, we describe a general strategy for finding such maps, and for choosing the duality function. The following simple observation shows that in fact, each map is dual to its inverse image map. For any set S, we let

$$\mathcal{P}(S) := \{A : A \subset S\} \tag{2.17}$$

denote the set of all subsets of S.

Lemma 4 (Inverse image map) *Let S be a set, let $m : S \to S$ be a map, and let $m^{-1} : \mathcal{P}(S) \to \mathcal{P}(S)$ be the inverse image map defined as $m^{-1}(A) := \{x \in S : m(x) \in A\}$. Then m^{-1} is dual to m with respect to the duality function $\psi(x, A) := 1_{\{x \in A\}}$.*

Combining this with Proposition 2, we see that if the generator G of a Markov process X with state space S has a random mapping representation of the form (2.10), then the Markov process Y with state space $\mathcal{P}(S)$ and generator H given by

$$Hf(A) = \sum_{m \in \mathcal{G}} r_m \big(f(m^{-1}(A)) - f(A)\big) \qquad \big(A \in \mathcal{P}(S)\big) \tag{2.18}$$

is a pathwise dual of X with respect to the duality function ψ from Lemma 4. In practise, this dual process is not very useful since the state space $\mathcal{P}(S)$ is very large compared to S. The situation is better, however, if $\mathcal{P}(S)$ contains a smaller subspace that is left invariant by all maps $m \in \mathcal{G}$. In the present paper, we will be interested in the subspace of all decreasing subsets of S (where S is a partially ordered set) and the subspace of all principal ideals of S (where S is a lattice), which leads to a duality for Markov processes that are representable in monotone maps and additive maps, respectively.

So let S now be a partially ordered set and let us briefly introduce some definitions and recall some basic facts for partially ordered sets. For any set $A \subset S$ we define $A^{\uparrow} := \{x \in S : x \ge y \text{ for some } y \in A\} \supset A$. We say that A is *increasing* if $A^{\uparrow} \subset A$. We define $A^{\downarrow}$ and *decreasing* sets in the same way for the reversed order. A nonempty increasing set A such that for every $x, y \in A$ there exists a $z \in A$ with $z \leq x, y$ is called a *filter* and a nonempty decreasing set A such that for every $x, y \in A$ there exists a $z \in A$ with $x, y \leq z$ is called an *ideal.* A *principal filter* is a filter that contains a minimal element and a *principal ideal* is an ideal that contains a maximal element. Equivalently, principal filters are sets of the form $A = \{z\}^{\uparrow}$ and principal ideals are sets of the form $A = \{z\}^{\downarrow}$, for some $z \in S$. A finite filter or ideal is always principal. We will use the notation

$$\begin{aligned} \mathcal{P}_{\text{inc}}(S) &:= \{A \subset S : A \text{ is increasing}\}, \\ \mathcal{P}_{!\text{inc}}(S) &:= \{A \subset S : A \text{ is a principal filter}\}, \\ \mathcal{P}_{\text{dec}}(S) &:= \{A \subset S : A \text{ is decreasing}\}, \\ \mathcal{P}_{!\text{dec}}(S) &:= \{A \subset S : A \text{ is a principal ideal}\}. \end{aligned} \tag{2.19}$$

A *lattice* is a partially ordered set for which the sets of principal filters and principal ideals are closed under finite intersections. Equivalently, this says that for every $x, y \in S$ there exist (necessarily unique) elements $x \vee y \in S$ and $x \wedge y \in S$ called the *supremum* or *join* and *infimum* or *meet* of x and y, respectively, such that

$$\{x\}^{\uparrow} \cap \{y\}^{\uparrow} =: \{x \vee y\}^{\uparrow} \quad \text{and} \quad \{x\}^{\downarrow} \cap \{y\}^{\downarrow} =: \{x \wedge y\}^{\downarrow}. \tag{2.20}$$

A *join-semilattice* (respectively *meet-semilattice*) is a partially ordered set in which $x \vee y$ (respectively $x \wedge y$) are well-defined. A partially ordered set S is *bounded from below* if it contains an (obviously unique) element, usually denoted by 0, such that $0 \leq x$ for all $x \in S$. Boundedness from above is defined analogously and the (obviously unique) upper bound is often denoted by 1. Finite lattices are always bounded from below and above.

Let S and T be partially ordered sets. By definition, a map $m : S \to T$ is *monotone* if it is a $\leq$-homomorphism, i.e.,

$$x \leq y \quad \text{implies} \quad m(x) \leq m(y) \quad (x, y \in S). \tag{2.21}$$

We denote the set of all monotone maps $m : S \to T$ by $\mathcal{F}_{\text{mon}}(S, T)$. If S and T are join-semilattices that are bounded from below, then, generalizing terminology from the theory of interacting particle systems, we will call a map $m : S \to T$ *additive* if it is a $(0, \vee)$-homomorphism, i.e.,

$$m(0) = 0 \quad \text{and} \quad m(x \vee y) = m(x) \vee m(y) \qquad (x, y \in S). \tag{2.22}$$

The basis of all dualities discussed in the present paper is the following simple lemma, which says that a map m is monotone, respectively additive, if and only if its inverse image map leaves the subspaces of all decreasing subsets, respectively ideals, invariant. Note that since in a finite lattice, all ideals are principal, part (ii) says in particular that if S and T are finite join-semilattices that are bounded from below, then $m : S \to T$ is additive if and only if $m^{-1}(A) \in \mathcal{P}_{!\text{dec}}(S)$ for all $A \in \mathcal{P}_{!\text{dec}}(T)$. The proof of Lemma 5 will be given in Section 4.3.

Lemma 5 (Monotone and additive maps)
(i) *Let S and T be partially ordered sets and let $m : S \to T$ be a map. Then m is monotone if and only if*

$m^{-1}(A) \in \mathcal{P}_{\text{dec}}(S)$ *for all* $A \in \mathcal{P}_{\text{dec}}(T)$.

(ii) *If S and T are join-semilattices that are bounded from below, then m is additive if and only if*

$m^{-1}(A)$ *is an ideal whenever* $A \subset T$ *is an ideal.*

2.4 Additive systems duality

In the present section, we show how the fact that the inverse image of an additive map leaves the subspace of principal ideals invariant, leads in a natural way to a duality for Markov processes that are representable in additive maps.

We start with an abstract definition of a dual for any partially ordered set S. Namely, a *dual* of S is a partially ordered set S' together with a bijection $S \ni x \mapsto x' \in S'$ such that

$$x \leq y \quad \text{if and only if} \quad x' \geq y'. \tag{2.23}$$

Two canonical ways to construct such a dual are as follows. *Example 1:* For any partially ordered set S, we may take $S' := S$ but equipped with the reversed order, and $x \mapsto x'$ the identity map. *Example 2:* If Λ is a set and $S \subset \mathcal{P}(\Lambda)$ is a set of subsets of Λ, equipped with the partial order of inclusion, then we may take for $x' := \Lambda \backslash x$ the complement of x and $S' := \{x' : x \in S\}$.

Returning to the abstract definition, it is easy to see that all duals of a partially ordered set are naturally isomorphic and that the original partially ordered set is in a natural way the dual of its dual, which motivates us to write $x'' = x$. If S is bouded from below, then S' is bounded from above and $0' = 1$. We define a function $S \times S' \ni (x, y) \mapsto \langle x, y\rangle \in \{0, 1\}$ by

$$\langle x, y\rangle := 1_{\{x \leq y'\}} = 1_{\{y \leq x'\}} \qquad (x \in S,\ y \in S'). \tag{2.24}$$

Note that $x, y' \in S$ while $y, x' \in S'$, so the two signs $\leq$ in this formula refer to the partial orders on S and S', respectively. Since S is the dual of S', the formal definition of $\langle\,\cdot\,,\,\cdot\,\rangle$ is symmetric in the sense that $\langle y, x\rangle = \langle x, y\rangle$ for all $y \in S'$ and $x \in S'' = S$. In the concrete examples of a dual space given above, $\langle\,\cdot\,,\,\cdot\,\rangle$ has the following meanings. In Example 1, $\langle x, y\rangle = 1_{\{x \leq y\}}$, while in Example 2, $\langle x, y\rangle = 1_{\{x \cap y = \emptyset\}}$.

Since our aim is to show how additive systems duality arises naturally from Lemma 5, we will now let S be a lattice and prove the following lemma here on the spot.

Lemma 6 (Duals of additive maps) *Let S be a finite lattice and let S' be a dual of S in the sense defined above. Then a map $m : S \to S$ is additive if and only if there exists a (necessarily unique) map $m' : S' \to S'$ that is dual to m with respect to the duality function $\psi(x, y) := \langle x, y\rangle$ from (2.24). This dual map m', if it exists, is also additive.*

Proof By Lemma 5, m is additive if and only if $m^{-1}(A) \in \mathcal{P}_{!\mathrm{dec}}(S)$ for all $A \in \mathcal{P}_{!\mathrm{dec}}(S)$. Since each element $A \in \mathcal{P}_{!\mathrm{dec}}(S)$ can be written in the form $A = \{x \in S : x \leq y'\}$ for some unique $y \in S'$, m is additive if and only if for each $y \in S'$ there exists a (necessarily unique) element $m'(y) \in S'$ such that

$$m^{-1}\big(\{z \in S : z \leq y'\}\big) = \{x \in S : x \leq (m'(y))'\}, \tag{2.25}$$

i.e., there exists a (necessarily unique) map $m' : S' \to S'$ such that

$$m(x) \leq y' \quad \text{if and only if} \quad x \leq (m'(y))' \qquad (x \in S,\ y \in S'), \tag{2.26}$$

which says that $\langle m(x), y\rangle = \langle x, m'(y)\rangle$, i.e., m' is dual to m with respect to the duality function from (2.24).

Since conversely m is dual to m' with respect to the duality function from (2.24), by what we have just proved, m' (having a dual with respect to this duality function) must be additive. ∎

Combining Proposition 2 and Lemma 6, we can write down our first nontrivial result.

Theorem 7 (Additive systems duality) *Let S be a finite lattice and let X be a Markov process in S whose generator has a random mapping representation of the form*

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m\big(f(m(x)) - f(x)\big) \qquad (x \in S), \tag{2.27}$$

where all maps $m \in \mathcal{G}$ are additive. Then the Markov process Y in S' with generator

$$Hf(y) := \sum_{m\in\mathcal{G}} r_m\big(f(m'(y)) - f(y)\big) \qquad (y \in S') \tag{2.28}$$

is pathwise dual to X with respect to the duality function $\psi(x,y) := \langle x, y\rangle$ from (2.24).

In Subsections 3.1 and 3.2 below, we will see that Theorem 7 contains both Siegmund's duality and the well-known duality of additive interacting particle systems as special cases. Moreover, in Section 3.3, we will see that a duality for the two-stage contact process discovered by Krone [Kro99] also fits into this general scheme.

2.5 Monotone systems duality

In the present section, we show how the fact that the inverse image of a monotone map leaves the subspace of decreasing subsets invariant can be used to construct duals of general monotone Markov processes. These dual processes are more complicated than in the case of additively representable processes, but we will stay as close as possible to the formalism of the previous subsection, so that Theorem 7 will be a special case of a more general theorem to be formulated here.

Theorem 7 is based on the fact that if m is an additive map, then its inverse image maps the space of principal ideals into itself, and each principal ideal $A \in \mathcal{P}_{!\mathrm{dec}}(S)$ can be encoded in terms of the unique element $y \in S'$ such that $A = \{y'\}^{\downarrow}$. In our present setting, we will encode a general decreasing set $A \in \mathcal{P}_{\mathrm{dec}}(S)$ in terms of a set $B \subset S'$ such that $A = \{y' : y \in B\}^{\downarrow}$. This means that we will use the duality function

$$\phi(x,B) := 1_{\{x \leq y' \text{ for some } y \in B\}} \qquad (x \in S,\ B \subset S'). \tag{2.29}$$

For a given $A \in \mathcal{P}_{\mathrm{dec}}(S)$, there are usually more ways to choose a set $B \subset S'$ such that $A = \{y' : y \in B\}^{\downarrow}$ and as a result we see that for a given monotone map m there are at least two natural ways to define a dual map with respect to the duality function ϕ from (2.29).

As before, we assume that S is a finite partially ordered set and we let S' (together with the map $x \mapsto x'$) denote a dual of S in the sense defined in (2.23). Contrary to the latter part of the previous subsection, we no longer assume that S is a lattice. For any set $A \subset S$ we write $A' := \{x' : x \in A\}$ and we let

$$A_{\max} := \{x \in A : x \text{ is a maximal element of } A\} = \{x \in A : \nexists y \in A, y \neq x \text{ s.t. } x \leq y\} \tag{2.30}$$

denote the set of maximal elements of A. Similarly, we let $A_{\min}$ denote the set of minimal elements of a set A. For any monotone $m : S \to S$, we can uniquely define maps $m^{\dagger} : \mathcal{P}(S') \to \mathcal{P}(S')$ and $m^* : \mathcal{P}(S') \to \mathcal{P}(S')$ by

$$m^{\dagger}(B)' := (m^{-1}(B'^{\downarrow}))_{\max} \quad \text{and} \quad m^*(B)' := \bigcup_{x\in B} (m^{-1}(\{x'\}^{\downarrow}))_{\max} \tag{2.31}$$

($B \in \mathcal{P}(S')$). The next lemma shows that both $m^{\dagger}$ and m^* are dual to m with respect to the duality function ϕ from (2.29). In addition, both $m^{\dagger}$ and m^* each have a special property justifying their definitions.

Lemma 8 (Duals of monotone maps) *Let S be a finite partially ordered set and let $m \in \mathcal{F}_{\rm mon}(S,S)$. Then $m^\dagger$ and m^* are dual to m with respect to the duality function ϕ from (2.29). Moreover,*

$$m^\dagger(B) = m^\dagger(B)_{\rm min} = m^*(B)_{\rm min} \qquad \big(B \in \mathcal{P}(S')\big), \tag{2.32}$$

and

$$m^*(B \cup C) = m^*(B) \cup m^*(C) \qquad \big(B, C \in \mathcal{P}(S')\big). \tag{2.33}$$

The proof of Lemma 8 is straightforward but a bit tedious, and for this reason we postpone it till Section 4.4. Combining Proposition 2 and Lemma 8, we can write down our second nontrivial result.

Theorem 9 (Monotone systems duality) *Let S be a finite partially ordered set and let X be a Markov process in S whose generator has a random mapping representation of the form*

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x)\big) \qquad (x \in S), \tag{2.34}$$

where all maps $m \in \mathcal{G}$ are monotone. Then the $\mathcal{P}(S')$-valued Markov processes $Y^\dagger$ and Y^* with generators

$$\begin{aligned} H_\dagger f(B) &:= \sum_{m \in \mathcal{G}} r_m \big(f(m^\dagger(B)) - f(B)\big), \\ H_* f(B) &:= \sum_{m \in \mathcal{G}} r_m \big(f(m^*(B)) - f(B)\big), \end{aligned} \qquad \big(B \in \mathcal{P}(S')\big) \tag{2.35}$$

are pathwise dual to X with respect to the duality function ϕ from (2.29).

We note that if a map $m : S \to S$ is monotone, then it is also monotone with respect to the reversed order on S. As a result, the inverse image map m^{-1} also maps increasing subsets into increasing subsets. This naturally leads to the duality function (compare (2.29))

$$\tilde{\phi}(x, B) := 1_{\{x \geq y' \text{ for some } y \in B\}} \qquad (x \in S,\ B \subset S'). \tag{2.36}$$

In analogy with (2.31), we may define maps $m^\circ : \mathcal{P}(S') \to \mathcal{P}(S')$ and $m^\bullet : \mathcal{P}(S') \to \mathcal{P}(S')$ by

$$m^\circ(B)' := (m^{-1}(B'^{\uparrow}))_{\rm min} \quad \text{and} \quad m^\bullet(B)' := \bigcup_{x \in B} (m^{-1}(\{x'\}^{\uparrow}))_{\rm min}. \tag{2.37}$$

As a direct consequence of Lemma 8, applied to the reversed order on S, we obtain the following lemma.

Lemma 10 (Alternative duals of monotone maps) *Let S be a finite partially ordered set and let $m \in \mathcal{F}_{\rm mon}(S,S)$. Then m° and $m^\bullet$ are dual to m with respect to the duality function $\tilde{\phi}$ from (2.36). Moreover,*

$$m^\circ(B) = m^\circ(B)_{\rm max} = m^\bullet(B)_{\rm max} \qquad \big(B \in \mathcal{P}(S')\big), \tag{2.38}$$

and

$$m^\bullet(B \cup C) = m^\bullet(B) \cup m^\bullet(C) \qquad \big(B, C \in \mathcal{P}(S')\big). \tag{2.39}$$

We will mostly focus on the maps $m^\dagger$ and m^* from (2.31) since these are most closely related to the additive systems duality from the previous subsection. The next lemma, which will be proved in Section 4.4, shows that the dual processes of Theorem 9 reduce to the dual of Theorem 7 if all maps occurring in (2.34) are additive and the dual process $Y = Y^\dagger$ or $= Y^*$ is started in a singleton, i.e., a state of the form $Y_0 = \{y\}$ for some $y \in S'$.

Lemma 11 (Relation between additive and monotone duals) *Let S be a finite lattice and let $m : S \to S$ be additive. Let m' be the dual map from Lemma 6 and let m^* and $m^\dagger$ be as in (2.31). Then*

$$m^*(B) = \{m'(y) : y \in B\} \quad \textit{and} \quad m^\dagger(B) = m^*(B)_{\min} \qquad \big(B \in \mathcal{P}(S')\big). \tag{2.40}$$

In Subsection 3.4, we will show that if X is a monotone spin system, then the dual process $Y^\bullet$ coincides with Gray's [Gra86] dual process. Here $Y^\bullet$ denotes the process with generator $H_\bullet$ which is defined analogously as $H_\dagger, H_*$ in (2.35), but with the dual map $m^\bullet$ from (2.37) instead of $m^\dagger, m^*$. As a more concrete application of this sort of dualities, in Subsection 3.5, we use Theorem 9 to derive a new duality, announced in [SS14], for particle systems with cooperative branching.

If the generator of a Markov process can be written in the form (2.34) where all maps $m \in \mathcal{G}$ are monotone, then we say that G is *monotonically representable*. By the remarks at the end of Subsection 2.2, if the generator of a Markov process is monotonically representable, then the same is true for its transition kernels $(P_t)_{t\geq 0}$. It turns out that being monotonically representable is a stronger concept than being monotone in the traditional meaning of that term, which we now explain.

For S and T finite partially ordered sets a probability kernel K from S to T is called *monotone* if

$$f \in \mathcal{F}_{\rm mon}(S, \mathbb{R}) \quad \text{implies} \quad Kf \in \mathcal{F}_{\rm mon}(S, \mathbb{R}), \tag{2.41}$$

where as before $Kf(x) := \sum_y K(x,y)f(y)$. It is easy to see that each monotonically representable probability kernel is also monotone, but it is known that there exist kernels that are monotone yet not monotonically representable. See [FM01, Example 1.1] for an example where $S = T = \mathcal{P}(\Lambda)$ with Λ a set containing just two elements. (Note that what we call monotonically representable is called *realizably monotone* in [FM01].) On the positive side, we cite the following result from [KKO77, FM01].

Proposition 12 (Sufficient conditions for monotone representability) *Let S, T be finite partially ordered sets and assume that at least one of the following conditions is satisfied:*

(i) *S is totally ordered.*

(ii) *T is totally ordered.*

Then any monotone probability kernel from S to T is monotonically representable.

Proof The sufficiency of (i) was proved in [KKO77] and also follows as a special case of [FM01, Thm 4.1], where it is shown that the statement holds more generally if S has a tree-like structure. The sufficiency of (ii) is proved in [FM01, Example 1.2]. ∎

The question how to determine whether a given probability kernel (or Markov generator) can be represented in the set of additive maps was already mentioned in [Har78], but we do not know about any results in this direction. Cursory contemplation suggests that this problem is at least as difficult as monotone representability.

2.6 Percolation substructures

If Λ is a finite set, then the set $S := \mathcal{P}(\Lambda)$ of all subsets of Λ, equipped with the order of set inclusion, is a finite lattice. For lattices of this form, there exists a simple way to characterize additive maps $m : S \to S$ and this naturally leads to a representation of additive Markov processes in S, and their duals, in terms of a form of oriented percolation on $\Lambda \times \mathbb{R}$, where the second coordinate plays the role of time [Har78, Gri79]. In the present subsection, we investigate whether this picture remains intact if S is replaced by a more general lattice. For distributive lattices, the answer is largely positive. In Subsection 3.3 below, we apply the results of the present subsection to give a percolation representation for the two-stage contact process and its dual, which are interacting particle systems with a state space of the form $\{0,1,2\}^\Lambda$ [Kro99].

By definition, a lattice S is *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad (x,y,z \in S). \tag{2.42}$$

If Λ is a finite set and $S \subset \mathcal{P}(\Lambda)$ is closed under intersections and unions and contains $\emptyset$ and Λ as elements, then S is a distributive lattice with $\emptyset$ and Λ as lower and upper bounds. In particular, if Λ is a partially ordered set, then this applies to $S := \mathcal{P}_{\rm dec}(\Lambda)$. Birkhoff's representation theorem ([Bir37]; see, e.g., [DP02, Thm 5.12] for a modern reference) says that conversely, each distributive lattice is of this form. Note that in particular, if Λ is equipped with the *trivial order* $i \not\leq j$ for all $i \neq j$, then $\mathcal{P}_{\rm dec}(\Lambda) = \mathcal{P}(\Lambda)$.

In view of this, let Λ be a finite partially ordered set, and let $S := \mathcal{P}_{\rm dec}(\Lambda)$ be the lattice of decreasing subsets of Λ. Let Λ' denote the set Λ equipped with the reversed order. Then $S' := \mathcal{P}_{\rm dec}(\Lambda') = \mathcal{P}_{\rm inc}(\Lambda)$, together with the complement map $x \mapsto x' := x^{\rm c}$, is a dual of the lattice S in the sense defined in (2.23). The following lemma will be proved in Section 4.5.

Lemma 13 (Characterization of additive maps) *There is a one-to-one correspondence between, on the one hand, additive maps $m : S \to S$ and, on the other hand, sets $M \subset \Lambda \times \Lambda$ such that for all $i, j, \tilde\imath, \tilde\jmath \in \Lambda$*

$$\begin{aligned} &\text{(i)} \quad (i,j) \in M \text{ and } i \leq \tilde\imath \text{ implies } (\tilde\imath, j) \in M, \\ &\text{(ii)} \quad (i,j) \in M \text{ and } j \geq \tilde\jmath \text{ implies } (i, \tilde\jmath) \in M. \end{aligned} \tag{2.43}$$

This one-to-one correspondence comes about by identifying M with the additive map m defined by

$$m(x) := \{j \in \Lambda : (i,j) \in M \text{ for some } i \in x\} \qquad (x \in S). \tag{2.44}$$

If $m' : S' \to S'$ is the dual of m (in the sense of Lemma 6) and M' is the corresponding subset of $\Lambda' \times \Lambda'$, then M and M' are related by

$$M' = \big\{(j,i) : (i,j) \in M\big\}. \tag{2.45}$$

Using Lemma 13, we can represent the stochastic flow of an additive Markov process with values in a general distributive lattice in terms of open paths in a "percolation substructure" in the sense of [Gri79]. Let X be a Markov process whose generator G has a random mapping representation of the form (2.10) where all maps $m \in \mathcal{G}$ are additive maps $m : S \to S$, with $S = \mathcal{P}_{\rm dec}(\Lambda)$ as before. As in Section 2.2, we construct a Poisson point set Δ on $\mathcal{G} \times \mathbb{R}$ and use this to define a stochastic flow $(\mathbf{X}_{s,t})_{s \leq t}$ as in (2.11). We also set $\Delta' := \{(m', -t) : (m,t) \in \Delta\}$ (compare (2.14)) and use this Poisson set to define the dual stochastic flow $(\mathbf{Y}_{s,t})_{s \leq t}$ as in Proposition 2.

Plotting space-time $\Lambda \times \mathbb{R}$ with time upwards, for each $(m,t) \in \Delta$, let M be the set corresponding to m in the sense of Lemma 13, draw an arrow from (i,t) to (j,t) for each $i \neq j$ such that $(i,j) \in M$, and place a "blocking symbol" ▬ at (i,t) whenever $(i,i) \notin M$. Examples of such graphical representations are given in Figures 1 and 2 below. By definition, an *open path* in such a graphical representation is a cadlag function $\gamma : [s,u] \to \Lambda$ such that:

(i) If $\gamma_{t-} \neq \gamma_t$ for some $t \in (s,u]$, then there is an arrow from (γ_{t-}, t) to (γ_t, t).

(ii) If there is a blocking symbol at (γ_t, t) for some $t \in (s,u]$, then $\gamma_{t-} \neq \gamma_t$.

In words, an open path may jump using arrows and cannot stay at the same site if there is a blocking symbol at such a site. For $i, j \in \Lambda$ and $s \leq u$, we write $(i,s) \rightsquigarrow (j,u)$ if there is an open path γ that leads from (i,s) to (j,u), i.e., $\gamma_s = i$ and $\gamma_u = j$. The following lemma, whose formal proof will be given in Section 4.5, gives the promised percolation representation of additive Markov processes in $S = \mathcal{P}_{\rm dec}(\Lambda)$, and their duals. Moreover, formula (2.48) gives a graphical interpretation of the pathwise duality of Theorem 7. Note that in the present setting, the duality function takes the form $\psi(x,y) = 1_{\{x \subset y^{\rm c}\}} = 1_{\{x \cap y = \emptyset\}}$.

Lemma 14 (Percolation representation) *Almost surely, for all $s \leq t$ and $x \in S$,*

$$\mathbf{X}_{s,u}(x) = \{j \in \Lambda : (i,s) \rightsquigarrow (j,u) \text{ for some } i \in x\}, \tag{2.46}$$

and the left-continuous version of the dual stochastic flow is a.s. given by

$$\mathbf{Y}_{s-,u-}(y) = \{j \in \Lambda : (j,-u) \rightsquigarrow (i,-s) \text{ for some } i \in y\} \qquad (s \leq u,\ y \in S'). \tag{2.47}$$

Moreover, for deterministic times $s \leq u$, a.s.

$$1_{\{\mathbf{X}_{s,t-}(x) \cap \mathbf{Y}_{-u,-t}(y) = \emptyset\}} = 1_{\{(i,s) \not\rightsquigarrow (j,u)\ \forall\ i \in x,\ j \in y\}} \qquad (s \leq t \leq u). \tag{2.48}$$

By Birkhoff's representation theorem, Lemmas 13 and14 apply to additive processes taking values in general distributive lattices. For nondistributive lattices, the picture is not as nice, but we can still show that each additive process has a percolation representation. The next lemma will be proved in Section 4.5.

Lemma 15 (Additive maps on general lattices) *Let S be a finite lattice. Then S is $(0,\vee)$-isomorphic to a join-semilattice of sets, i.e., there exists a finite set Λ and a set $T \subset \mathcal{P}(\Lambda)$ such that $\emptyset \in T$ and T is closed under unions, and $S \cong T$. Moreover, if T is such a join-semilattice of sets, then each additive map $m : T \to T$ can be extended to an additive map $\overline{m} : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda)$.*

By Lemma 15, if G is the generator of a Markov process X in T, and X is additive in the sense that G has a random mapping representation of the form (2.10) where all maps $m \in \mathcal{G}$ are additive, then we can extend X to a Markov process in $\mathcal{P}(\Lambda)$ with generator of the form

$$\overline{G}f(x) = \sum_{m \in \mathcal{G}} r_m \big(f(\overline{m}(x)) - f(x)\big) \qquad \big(x \in \mathcal{P}(\Lambda)\big). \tag{2.49}$$

This extended process leaves T invariant, and by Lemma 15, it has a percolation representation of the form (2.46). If S is nondistributive, however, then T can never be chosen such that it is also closed under intersections, which means that $\{x^{\rm c} : x \in T\}$ is not $(0,\vee)$-isomorphic to S', and the parts of Lemma 14 referring to the dual process break down. Of course, the extended process (2.49) has a dual that can be interpreted in terms of open paths, but it is unclear if the extended process can ever be chosen in such a way that it leaves a subspace invariant that is $(0,\vee)$-isomorphic to S'.

2.7 Some open problems

The problem how to decide whether a given Markov generator can be represented in monotone or additive maps has already been mentioned in the text. These problems have been open for a long time and appear to be hard.

We have also only partially resolved the question whether additive systems taking values in a nondistributive lattice have a "nice" percolation representation together with their dual. If S is a nondistributive lattice, then by Lemma 15, S is $(0,\vee)$-isomorphic to a join-semilattice of sets, but by Birkhoff's theorem such a join-semilattice of sets can never be closed under intersections. This also means that we cannot simultaneously represent the dual lattice S' as a join-semilattice of sets on the same space, so that the duality map $x \mapsto x'$ is the complement map. However, to get a useful percolation representation of S and S' together, it would suffice to represent S and S' as join-semilattice of sets on the same space in such a way that the duality map $\langle x, y \rangle$ takes the form $1_{\{x \cap y = \emptyset\}}$. It is not clear if this can be done.

A more urgent question is perhaps if all this abstract theory "is actually good for anything". Additive systems duality is clearly a very useful tool, and as the two-stage contact process shows (see Section 3.3 below), one sometimes needs more general lattices than the Boolean algebra $\{0,1\}^\Lambda$. Our general approach to constructing percolation representations for such lattices seems to be new. The more general but also more complicated duality for monotone systems, that are not necessarily additive, has so far found few applications, although Gray [Gra86] did use it to prove nontrivial statements and we used it in our previous work [SS14] to derive a useful subduality for systems with cooperative branching.

For many additive systems including the contact process, using duality, it is fairly easy to prove that starting from an arbitrary translation invariant initial law, the law of the system converges to a convex combination of the upper invariant law and the delta measure on the zero configuration; see [Lig85, Thm III.5.18]. As far as we know, it is an open problem to generalize these techniques to monotone systems that are not additive, such as the systems with cooperative branching discussed in Section 3.5 below. We hope that our present systematic treatment of monotone systems duality can contribute to such an undertaking.

In this context, we mention one more open problem. In Section 5.4 below, we generalize Theorem 9 about monotone systems duality to infinite underlying spaces. We only show, however, that the dual process is well-defined started from finite initial states. It is an open problem to construct the dual process with infinite initial states, as would be needed, e.g., to study invariant laws of the dual process.

3 Examples

3.1 Siegmund's duality

In this subsection, we show that for finite state spaces, Siegmund's duality [Sie76] is a special case of Theorem 7 which gives a pathwise dual for additive Markov processes taking values in a finite lattice.

Let $S = \{0, \ldots, n\}$ $(n \geq 2)$ be a finite totally ordered set with at least two elements, and let S' denote the same space, equipped with the reversed order, which is a dual of S in the sense defined in Section 2.4. Since $x \vee y$ equals either x or y for each $x, y \in S$, it is easy to see that a map $m : S \to S$ is additive if and only if m is monotone and $m(0) = 0$. By Lemma 6, for each such map there exists a unique map $m' : S \to S$ that is dual to m with respect to the duality function $\psi(x, y) = 1_{\{x \leq y\}}$. This map m' is additive viewed as a map on S', i.e., m' is monotone and $m'(n) = n$. Theorem 7 now gives us the following result. We note that Siegmund's result [Sie76] applies more generally to processes taking values in a real interval, which includes, e.g., a duality between Brownian motions on $[0, \infty)$ with absorption at 0 and those with reflection at 0.

Theorem 16 (Pathwise Siegmund's duality) *Let $S = \{0, \ldots, n\}$ $(n \geq 2)$ be a finite totally ordered set and let X be a Markov process in S whose generator has a random mapping representation of the form*

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x) \big) \qquad (x \in S), \tag{3.1}$$

where all maps $m \in \mathcal{G}$ are monotone and satisfy $m(0) = 0$. Then the Markov process Y in S' with generator

$$Hf(y) := \sum_{m \in \mathcal{G}} r_m \big(f(m'(y)) - f(y)\big) \qquad (y \in S') \tag{3.2}$$

is pathwise dual to X with respect to the duality function $\psi(x, y) := 1_{\{x \leq y\}}$.

A Markov process X taking values in a partially ordered set S is called *monotone* if its transition kernels P_t are monotone for each $t \geq 0$. If $S = \{0, \ldots, n\}$ is totally ordered, then by Proposition 12, it follows that P_t is monotonically representable for each $t \geq 0$, and by Lemma 3 the same is true for the generator G. If 0 is a trap (i.e., the rate of jumps away from 0 is zero), then each random mapping representation for G involves only maps satisfying $m(0) = 0$. Since each pathwise dual is also a dual in the classical sense (2.2), Theorem 16 allows us to conclude:

Proposition 17 (Siegmund's duality) *Let $S = \{0, \ldots, n\}$ ($n \geq 2$) be a finite totally ordered set and let X be a monotone Markov process in S for which 0 is a trap. Then there exists a monotone Markov process Y in S for which n is a trap, such that*

$$\mathbb{P}[X_t \leq Y_0] = \mathbb{P}[X_0 \leq Y_t] \qquad (t \geq 0) \tag{3.3}$$

for arbitrary deterministic initial states X_0 and Y_0.

It is easy to see that (3.3) determines the transition probabilities of Y uniquely, even though there usually is more than one way to represent the generators of X and Y in terms of monotone maps and hence to construct a coupling that realizes the pathwise duality of Theorem 16.

3.2 Additive interacting particle systems

Additive particle systems in the classical sense of [Har78, Gri79] are additively representable Markov processes X with state space of the form $S = \mathcal{P}(\Lambda)$ where Λ is a countable set. Concentrating on finite Λ for the moment, this is a special case of the set-up from Section 2.6, where the set Λ from Lemmas 13 and 14 is equipped with the trivial order, so that $\mathcal{P}_{\rm dec}(\Lambda) = \mathcal{P}(\Lambda)$, and the conditions (2.43) are void. In this case, (2.44) simply defines a one-to-one correspondence between additive maps $m : S \to S$ and sets $M \subset \Lambda \times \Lambda$, and Lemma 14 shows that X and its dual Y can be represented in terms of the same percolation substructure and are pathwise dual in the sense of (2.48).

To illustrate this on a concrete example, let us look at *voter models* which have a generator of the form

$$G_{\rm voter} f(x) := \sum_{i,j \in \Lambda} r_{ij} \big(f\big(\mathrm{vot}_{ij}(x)\big) - f(x)\big) \qquad \big(x \in \mathcal{P}(\Lambda)\big), \tag{3.4}$$

where we define *voter model maps* $\mathrm{vot}_{ij} : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda)$ by

$$\mathrm{vot}_{ij}(x) := \begin{cases} x \cup \{j\} & \text{if } i \in x, \\ x \backslash \{j\} & \text{if } i \notin x, \end{cases} \qquad \big(x \in \mathcal{P}(\Lambda),\ i, j \in \Lambda\big). \tag{3.5}$$

The map vot_{ij} corresponds (in the sense of (2.44)) to the set $M_{ij} \subset \Lambda \times \Lambda$ given by $M_{ij} = \{(k, k) : k \in \Lambda,\ k \neq j\} \cup \{(i, j)\}$, which is represented by an arrow from i to j and simultaneously a blocking symbol at j.

By Lemma 13, the dual map vot'_{ij} corresponds to the set $M' = \{(k, k) : k \in \Lambda,\ k \neq j\} \cup \{(j, i)\}$, which through (2.44), M' defines the dual map

$$\mathrm{rw}_{ji}(y) := \begin{cases} \big(y \backslash \{j\}\big) \cup \{i\} & \text{if } j \in y, \\ y & \text{if } j \notin y, \end{cases} \qquad \big(y \in \mathcal{P}(\Lambda),\ i, j \in \Lambda\big), \tag{3.6}$$

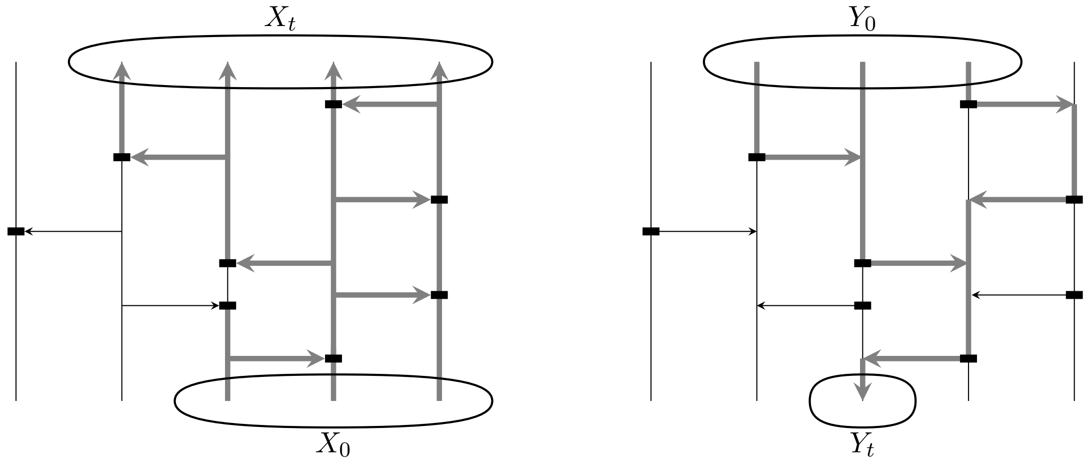


Figure 1: Graphical representation of a voter model and its dual.

which is represented by an arrow from j to i and simultaneously a blocking symbol at j. We interpret rw_{ji} as a *coalescing random walk map*, i.e., when rw_{ji} is applied, if there is a particle at j, then this particle jumps to i, coalescing with any particle that may already be present.

By Theorem 7, the voter model X is pathwise dual to the system of coalescing random walks Y with generator

$$G_{\mathrm{rw}} f(y) := \sum_{i,j\in\Lambda} r_{ij}\big(f\big(\mathrm{rw}_{ji}(y)\big) - f\big(y\big)\big) \qquad \big(y \in \mathcal{P}(\Lambda)\big), \tag{3.7}$$

and by Lemma 14, the stochastic flows associated with X and Y can be represented in terms of open paths in a percolation substructure. In Figure 1, we have drawn an example of such a percolation substructure, together with a voter model (in the upward time direction) and system of coalescing random walks (with time running downwards). In the picture for the dual process Y, we have reversed the direction of all Poisson arrows, in line with formula (2.45) for the dual of an additive map.

3.3 Krone's duality

Steve Krone [Kro99] has studied a two-stage contact process, which is a Markov process with state space of the form $S = \{0,1,2\}^\Lambda$. The main interest is in the case $\Lambda = \mathbb{Z}^d$ but the construction works in the same way if Λ is finite. If $x(i) = 0, 1$, or 2, then he interprets this as the site i being occupied by no individual, a young individual, or an adult individual, respectively. For each $i, j \in \Lambda$, consider the maps $a_i, b_{ij}, c_i, d_i, e_i$ defined by

$$\begin{array}{llll}
\text{grow up} & a_i(x)(k) := 2 & \text{if } k = i,\ x(i) = 1, & := x(k) \text{ otherwise,} \\
\text{give birth} & b_{ij}(x)(k) := 1 & \text{if } k = j,\ x(i) = 2,\ x(j) = 0, & := x(k) \text{ otherwise,} \\
\text{young dies} & c_i(x)(k) := 0 & \text{if } k = i,\ x(i) = 1, & := x(k) \text{ otherwise,} \\
\text{death} & d_i(x)(k) := 0 & \text{if } k = i, & := x(k) \text{ otherwise,} \\
\text{grow younger} & e_i(x)(k) := 1 & \text{if } k = i,\ x(i) = 2, & := x(k) \text{ otherwise.}
\end{array} \tag{3.8}$$

Except for the last one, all these maps have natural biological interpretations. For example, the map b_{ij} has the effect that if i is occupied by an adult individual and j is empty, then the adult individual at i gives birth to a young individual at j. Applying the map c_i with an appropriate rate models the commonly observed fact that young individuals die at a higher rate than adults.

We set $S' := S$ and define a map $S \ni x \mapsto x' \in S'$ by

$$x'(i) := 2 - x(i) \qquad (i \in \Lambda). \tag{3.9}$$

Then S', together with the map $x \mapsto x'$, is a dual of S in the sense of (2.23). The set S, being the product of totally ordered sets, is a distributive lattice and so we can apply Theorem 7 to find pathwise duals of additive Markov processes in S. Our choice of the dual space S' means that the duality function $\psi(x, y) = \langle x, y\rangle$ from (2.24) takes the form

$$\psi(x, y) = 1_{\{x \le y'\}} = 1_{\{x(i) + y(i) \le 2 \ \forall \ i \in \Lambda\}}. \tag{3.10}$$

The good news is that the maps in (3.8) are all additive.

Lemma 18 (Additive maps) *The maps in (3.8) are all additive and their dual maps with respect to the duality function in (3.10) are given by*

$$a'_i = a_i, \quad b'_{ij} = b_{ji}, \quad c'_i = e_i, \quad d'_i = d_i, \quad e'_i = c_i. \tag{3.11}$$

Lemma 18 can be checked by straightforward, but somewhat lengthy considerations. Things become easier if we construct a percolation representation for two-stage contact processes. We equip Λ with the trivial order, we view $\{0, 1\}$ as a totally ordered set with two elements, and we equip $\Lambda \times \{0, 1\}$ with the product order. Then

$$S \cong \mathcal{P}_{\rm dec}\big(\Lambda \times \{0, 1\}\big) \quad \text{and} \quad S' \cong \mathcal{P}_{\rm inc}\big(\Lambda \times \{0, 1\}\big), \tag{3.12}$$

where for the duality map we now take the complement map $x \mapsto x' := x^{\rm c}$ so that in this new representation the duality function takes the usual form $\psi(x, y) = 1_{\{x \cap y = \emptyset\}}$. To compare this with our earlier representation of the process, note that for $x \in \mathcal{P}_{\rm dec}(\Lambda \times \{0, 1\})$ and $i \in \Lambda$,

$$\{\sigma : (i, \sigma) \in x\} = \emptyset, \quad \{0\}, \quad \text{or} \quad \{0, 1\}, \tag{3.13}$$

which we interpret as $x(i) = 0, 1$, or 2 in our previous representation of S. Likewise, for $y \in \mathcal{P}_{\rm inc}(\Lambda \times \{0, 1\})$ and $i \in \Lambda$,

$$\{\sigma : (i, \sigma) \in y\} = \emptyset, \quad \{1\}, \quad \text{or} \quad \{0, 1\}, \tag{3.14}$$

which we interpret as $y(i) = 0, 1$, or 2 in our previous representation of S'.

An example of a percolation representation for a two-stage contact process and its dual are shown in Figure 2. Here, the maps from (3.8) are represented as follows:

a_i arrows between $(i, 0)$ and $(i, 1)$ (in both directions conforming to the rules of Lemma 13),

b_{ij} arrow from $(i, 1)$ to $(j, 0)$,

c_i blocking symbol at $(i, 0)$, arrow from $(i, 1)$ to $(i, 0)$,

d_i blocking symbols at $(i, 0)$ and $(i, 1)$,

e_i blocking symbol at $(i, 1)$, arrow from $(i, 1)$ to $(i, 0)$.

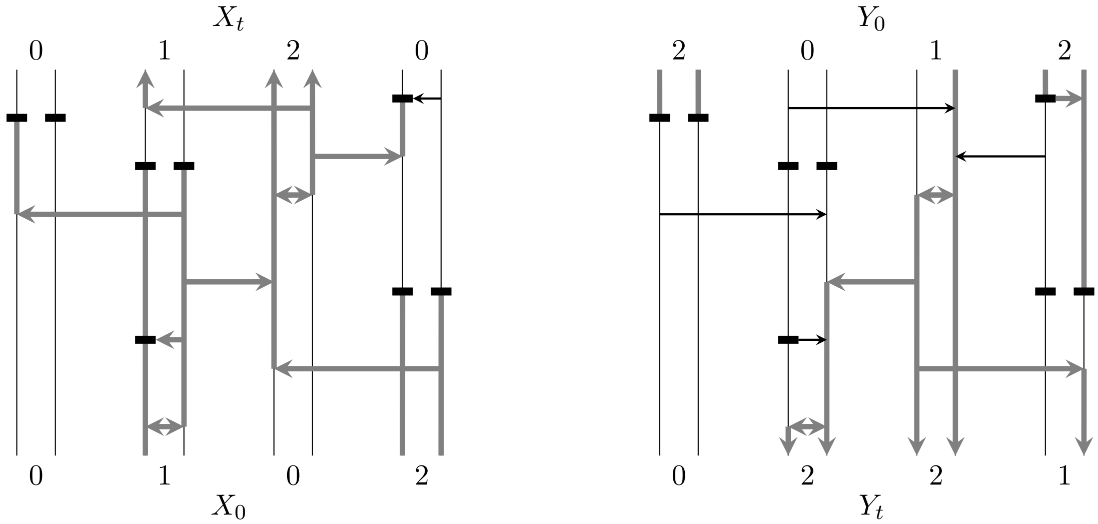


Figure 2: Graphical representation of a two-stage contact process and its dual.

3.4 Gray's duality

Gray [Gra86] proved a general duality for monotone spin systems that need, in general, not be additive. Although Gray's formulation of the duality differs considerably from ours, in the present subsection, we will show that his dual is a special case of the process with generator $H_\bullet$, which is defined as in Theorem 9 but with the dual maps m^* replaced by the analogous maps $m^\bullet$ from Lemma 10.

Gray considers spin systems, which are Markov processes with state space $S := \{0,1\}^\Lambda$ and generator of the form

$$Gf(x) = \sum_{i\in\Lambda} \beta_i(x)\big(f(x \vee \varepsilon_i) - f(x)\big) + \sum_{i\in\Lambda} \delta_i(x)\big(f(x \wedge (1-\varepsilon_i)) - f(x)\big), \tag{3.15}$$

where $\varepsilon_i(j) := 1_{\{i=j\}}$. Gray's emphasis is on the case $\Lambda = \mathbb{Z}$ but the arguments are the same for finite Λ. Gray assumes that his systems are *attractive*, which means that the function $\beta_i : S \to [0,\infty)$ is monotone and the function $\delta_i : S \to [0,\infty)$ is *anti-monotone*, i.e., $-\delta_i$ is monotone. He then proves the following fact.

Lemma 19 (Attractive spin systems are monotonically representable) *Let G be of the form (3.15) and assume that for each $i \in \Lambda$, the function β_i is monotone and δ_i is anti-monotone. Then there exists a set $\mathcal{G}$ of monotone maps $m : S \to S$ and nonnegative constants $(r_m)_{m\in\mathcal{G}}$ such that G has the form*

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m\big(f(m(x)) - f(x)\big) \qquad (x \in S). \tag{3.16}$$

Using the random mapping representation (3.16) of G in terms of monotone maps, Gray then constructs a Poisson point set Δ whose elements are pairs (m,t) with $m \in \mathcal{G}$ and $t \in \mathbb{R}$, and uses this to define a stochastic flow $(\mathbf{X}_{s,t})_{s\le t}$ as in (2.11).[2] Gray then defines an $[s,u]$*-path* from x to y (with $x, y \in S$) to be a cadlag function $\pi : [s,u] \to S$ with $\pi(s) = x$ and $\pi(u) = y$, such that:[3]

[2]See formulas (10)–(12) of [Gra86]. His notation for $\mathbf{X}_{s,t}(x)$ is $\xi(s,t,x)$.

[3]Gray does not state his definition entirely correctly. In his definition of minimality, he also includes, probably inadvertently, the condition that $\tilde{\pi}(s) = x$.

(i) $\mathbf{X}_{t,t'}(\pi(t)) \geq \pi(t')$ for all $s \leq t \leq t' \leq u$.

(ii) π is *minimal* in the sense that if a cadlag function $\tilde{\pi} : [s,u] \to S$ with $\tilde{\pi}(u) = y$ satisfies (i) and $\tilde{\pi}(t) \leq \pi(t)$ for all $t \in [s,u]$, then $\pi = \tilde{\pi}$.

Next, for each $s \leq u$, he defines random maps from S to $\mathcal{P}(S)$ by

$$\zeta_{s,u}(y) := \big\{x \in S : \text{ there exists an } [s,u]\text{-path from } x \text{ to } y\big\} \qquad (y \in S). \tag{3.17}$$

With this notation, Theorem 1 in [Gra86] reads:

Theorem 20 (Gray's duality) *For each $x, y \in S$ and $s \leq u$, one has $y \leq \mathbf{X}_{s,u}(x)$ if and only if $z \leq x$ for some $z \in \zeta_{s,u}(y)$.*

To relate this to our work, for each monotone map $m : S \to S$, let $m^\bullet$ denote the dual map defined in (2.37), where for S' we choose the set S equipped with the reversed order, i.e.,

$$m^\bullet(B) := \bigcup_{x \in B} (m^{-1}(\{x\}^\uparrow))_{\min} \qquad \big(B \in \mathcal{P}(S)\big), \tag{3.18}$$

where on the right-hand side $\{x\}^\uparrow$ and the minimum are taken with respect to the order on S. Set $\Delta^\bullet := \{(m^\bullet, -t) : (m,t) \in \Delta\}$ and use this Poisson set to define a stochastic flow $(\mathbf{Y}^\bullet_{s,t})_{s \leq t}$ that by Proposition 2 and Lemma 10 is dual to $(\mathbf{X}_{s,t})_{s \leq t}$ with respect to the duality function $\tilde{\phi}$ from (2.36). Then Theorem 20 is an immediate consequence of this duality and the following fact, that will be proved in Section 4.7.

Proposition 21 (Reformulation of Gray's dual) *For each $y \in S$ and $s \leq u$, one has $\zeta_{s,u}(y) = \mathbf{Y}^\bullet_{-u,-s}\big(\{y\}\big)$.*

Note that the duality of $(\mathbf{Y}^\bullet_{s,t})_{s \leq t}$ to $(\mathbf{X}_{s,t})_{s \leq t}$ with respect to the duality function from (2.36) implies that for each $x, y \in S$ and $s \leq t$,

$$1_{\{x \geq z \text{ for some } z \in \mathbf{Y}^\bullet_{-t,-s}(\{y\})\}} = 1_{\{\mathbf{X}_{s,t}(x) \geq z \text{ for some } z \in \{y\}\}}, \tag{3.19}$$

which by Proposition 21 implies Gray's result Theorem 20.

3.5 Cooperative branching

In the present subsection, we consider a more concrete example of an interacting particle system that is monotone but not additive and to which Theorem 9 applies. We will give a concrete description of the dual process with generator H_* from (2.35) and make optimal use of the fact that some of the maps involved are additive.

Denoting elements of $\{0,1\}^n$ as finite words made up of the letters 0 and 1, consider the maps a, b, c, d, e acting on $\{0,1\}^2$, $\{0,1\}^3$, $\{0,1\}^2$, $\{0,1\}^1$, and $\{0,1\}^2$, respectively, defined as follows.

$$\begin{array}{lll}
\text{voter move} & a(10) := 00,\ a(01) := 11, & a(x) := x \text{ otherwise,} \\
\text{cooperative branching} & b(110) := 111, & b(x) := x \text{ otherwise,} \\
\text{coalescing RW} & c(11) := 01,\ c(10) := 01, & c(x) := x \text{ otherwise,} \\
\text{death} & d(1) := 0, & d(x) := x \text{ otherwise,} \\
\text{exclusion} & e(10) := 01,\ e(01) := 10, & e(x) := x \text{ otherwise.}
\end{array} \tag{3.20}$$

These maps can be lifted to a larger space of the form $\{0,1\}^\Lambda$ with Λ any finite set as follows. For each ordered triple $(i,j,k) \in \Lambda^3$ such that $i,j,k \in \Lambda$ are all different from each other, let $x(ijk)$ denote the word obtained by writing $x(i), x(j)$, and $x(k)$ after each other, and let $b_{ijk} : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ be the map defined by

$$\big(b_{ijk}x\big)(ijk) := b\big(x(ijk)\big) \quad \text{and} \quad \big(b_{ijk}x\big)(m) := x(m) \quad \text{for} \quad m \notin \{i,j,k\}. \tag{3.21}$$

We define a_{ij}, c_{ij}, d_i, and e_{ij} in a similar way, by lifing the map a, c, d, and e to $\{0,1\}^\Lambda$.

The map a_{ij} corresponds to the voter model map vot_{ji} (note the order of the indices) from (3.5) and the map c_{ij} coincides with the coalescing random walk map rw_{ij} from (3.6). The death map d_i corresponds to deaths of particles at i and e_{ij} corresponds to exclusion model dynamics, where the states of the sites i and j are interchanged. The map b_{ijk} describes the situation where two particles at i, j are both needed to produce a third particle at k; such maps have been used to model, e.g., sexual reproduction. In [Nob92, Neu94], particle systems whose generators can be represented in the maps b_{ijk}, d_i, and e_{ij} are studied, while [SS14] is concerned with systems that involve the maps b_{ijk} and c_{ij}.

We set $S := \{0,1\}^\Lambda$. Here, we will only consider finite Λ. The extension to infinite Λ such as $\Lambda = \mathbb{Z}$, considered in [Nob92, Neu94, SS14], will be treated in Section 5.4. Choose $S' := S$, $x' := 1 - x$, so that S' is a dual of S as in the sense of (2.23). With this choice, the function $\langle \cdot\,, \cdot \rangle$ from (2.24) becomes

$$\langle x, y \rangle = 1_{\{x \leq y'\}} = 1_{\{x \wedge y = 0\}} \qquad (x, y \in S). \tag{3.22}$$

We start by observing that the maps a_{ij}, c_{ij}, d_i, and e_{ij} are additive and hence have duals in the sense of Lemma 6.

Lemma 22 (Additive maps) *For each $i, j \in \Lambda$ such that $i \neq j$, the maps a_{ij}, c_{ij}, d_i, and e_{ij} are additive and their dual maps in the sense of Lemma 6 are given by*

$$a'_{ij} = c_{ij}, \quad c'_{ij} = a_{ij}, \quad d'_i = d_i, \quad \textit{and} \quad e'_{ij} = e_{ij}. \tag{3.23}$$

The maps b_{ijk}, on the other hand, are not additive, but only monotone. Sticking to our choice $S' := S$ and $x' := 1 - x$, we fall back on the duality function ϕ from (2.29), which now reads

$$\phi(x, B) = 1_{\{x \leq y' \text{ for some } y \in B\}} = 1_{\{x \wedge y = 0 \text{ for some } y \in B\}} \tag{3.24}$$

$\big(x \in S,\ B \in \mathcal{P}(S)\big)$. Our next aim is to determine the dual map b^*_{ijk} defined in (2.31). We start with the map b from (3.20). We observe that

$$\big(b^{-1}(\{x\}^\downarrow)\big)_{\max} = \begin{cases} \{100, 010\} & \text{if } x = 110, \\ \{x\} & \text{otherwise.} \end{cases} \tag{3.25}$$

By (2.31), taking into account that $x' := 1 - x$, it follows that

$$b^*\big(\{x\}\big) = \begin{cases} \{011, 101\} & \text{if } x = 001, \\ \{x\} & \text{otherwise.} \end{cases} \tag{3.26}$$

In order to find a more convenient expression for b^*, we define maps $b^{(1)}, b^{(2)} : \{0,1\}^3 \to \{0,1\}^3$ by

$$\begin{aligned} b^{(1)}(001) &:= 011, \qquad b^{(1)}(x) := x \text{ otherwise}, \\ b^{(2)}(001) &:= 101, \qquad b^{(2)}(x) := x \text{ otherwise}. \end{aligned} \tag{3.27}$$

Then, using (2.33), we see that for any $B \subset \{0,1\}^3$,

$$b^*(B) = \{b^{(1)}(x) : x \in B\} \cup \{b^{(2)}(x) : x \in B\} = b^{(1)}(B) \cup b^{(2)}(B), \tag{3.28}$$

where $b^{(1)}(B)$ and $b^{(2)}(B)$ denote the images of B under the maps $b^{(1)}$ and $b^{(2)}$, respectively. Similarly, lifting these maps to the larger space $S = \{0,1\}^\Lambda$ in the same way as before,

$$b^*_{ijk}(B) = b^{(1)}_{ijk}(B) \cup b^{(2)}_{ijk}(B) \qquad \big(B \in \mathcal{P}(S)\big). \tag{3.29}$$

Lemma 11 tells us that $a^*_{ij}(B) = a'_{ij}(B)$, where $a'_{ij}(B)$ denotes the image of B under the map a'_{ij}, and similarly for the additive maps c_{ij}, d_i, and e_{ij}. Using also Lemma 22, we see that

$$a^*_{ij}(B) = c_{ij}(B), \quad c^*_{ij}(B) = a_{ij}(B), \quad d^*_i(B) = d_i(B), \quad \text{and} \quad e^*_{ij}(B) = e_{ij}(B). \tag{3.30}$$

By Lemma 8, the maps $a^*_{ij}, b^*_{ij}, \ldots$ in (3.29) and (3.30) are dual to $a_{ij}, b_{ij}, \ldots$ with respect to the duality function ϕ from (3.24), so Proposition 2 tells us how to construct a pathwise dual for a Markov process whose generator is represented in these maps.

It is instructive to do this in a concrete example. Let us consider a process X with a generator of the form

$$Gf(x) := \sum_{ijk} r_{ijk}\big(f(b_{ijk}(x)) - f(x)\big) + \sum_{ij} s_{ij}\big(f(c_{ij}(x)) - f(x)\big), \tag{3.31}$$

where the first sum runs over all ordered triples $(i,j,k) \in \Lambda^3$ such that $i,j,k \in \Lambda$ are all different from each other, the second sum runs over all $(i,j) \in \Lambda^2$ with $i \neq j$, and r_{ijk} and s_{ij} are nonnegative rates. The process X is a process of particles performing cooperative branching and coalescing random walk dynamics such as studied in [SS14].

By Proposition 2, Lemma 8, (3.29), and (3.30), the process X is pathwise dual with respect to the function ϕ from (3.24) to the $\mathcal{P}(S)$-valued process $Y^* = (Y^*_t)_{t\geq 0}$ with generator

$$H_*f(B) := \sum_{ijk} r_{ijk}\big(f\big(b^{(1)}_{ijk}(B) \cup b^{(2)}_{ijk}(B)\big) - f(B)\big) + \sum_{ij} s_{ij}\big(f(a_{ij}(B)) - f(B)\big). \tag{3.32}$$

We can think of the set-valued process $(Y^*_t)_{t\geq 0}$ as an evolving collection of voter-model configurations. With rate s_{ij}, the voter map a_{ij} is applied to all configurations $y \in Y^*_t$ simultaneously. With rate r_{ijk}, we apply both of the maps $b^{(1)}_{ijk}$ and $b^{(2)}_{ijk}$ to each configuration $y \in Y^*_t$, and collect all different outcomes in a new collection of voter-model configurations.

Since each pathwise dual is also a (normal) dual, we conclude that if X and Y are processes with generators as in (3.31) and (3.32) deterministic initial states X_0 and Y_0, then

$$\mathbb{P}\big[X_t \wedge y = 0 \text{ for some } y \in Y_0\big] = \mathbb{P}\big[X_0 \wedge y = 0 \text{ for some } y \in Y_t\big] \qquad (t \geq 0), \tag{3.33}$$

or equivalently

$$\mathbb{P}\big[X_t \wedge y \neq 0 \ \forall y \in Y_0\big] = \mathbb{P}\big[X_0 \wedge y \neq 0 \ \forall y \in Y_t\big] \qquad (t \geq 0). \tag{3.34}$$

Using (3.30), it is straightforward to extend this duality so that the generator of X also includes application of the maps d_i and e_{ij} with certain rates as in [Nob92, Neu94].

4 Proofs

4.1 Overview

In this section, we provide the proofs for all facts stated so far without proof or reference. Lemma 1, Proposition 2, Lemma 3, and Lemma 4 are proved in Section 4.2. Lemma 5 is proved in Section 4.3 while Lemma 6 has already been proved "on the spot" in Section 2.4. Theorem 7 follows directly from Proposition 2 and Lemma 6. Section 4.4 contains the proofs of Lemma 8 and the later Lemma 11. The preceding Theorem 9 follows directly from Proposition 2 and Lemma 8, and, as already pointed out in the text, Lemma 10 follows by applying Lemma 8 to the reversed order. Proposition 12 has been cited in the text from [KKO77, FM01]. Lemmas 13, 14, and 15 are proved in Section 4.5. As explained in the text preceding it, Theorem 16 follows from Theorem 7. It has also already been explained that Proposition 17 follows from Theorem 7. Lemma 18 is proved in Section 4.6, which also contains the proof of the later Lemma 22, while the preceding Lemma 19 and Proposition 21 are proved in Section 4.7. Theorem 20, finally, is cited from [Gra86] and as explained in the text also follows from Proposition 2, Lemma 10, and Proposition 21.

4.2 Markov process duality

In this section we prove Lemma 1 and Proposition 2 about the Poisson construction of Markov processes with finite state spaces and pathwise duality, as well as Lemma 3 about representability of Markov generators and semigroups and the simple observation Lemma 4 about the inverse image map.

Proof of Lemma 1 Recall the definition of a stochastic flow from Section 2.1. The fact that $(\mathbf{X}_{s,t})_{s\leq t}$ forms a stochastic flow follows immediately from its definition via the Poisson point sets Δ. More precisely, defining $\Delta_{s-,t} := \Delta \cap (\mathcal{G}\times[s,u])$, $\Delta_{s,t-} := \Delta\cap(\mathcal{G}\times(s,u))$, $\Delta_{s-,t-} := \Delta\cap(\mathcal{G}\times[s,u))$ and constructing $\mathbf{X}_{s-,t}$ etc. correspondingly we see that property (i) holds a.s. Since $\Delta_{s,s}=\emptyset$ the map $\mathbf{X}_{s,s}$ is the identity map, and the flow property $\mathbf{X}_{t,u}\circ\mathbf{X}_{s,t}=\mathbf{X}_{s,u}$ follows directly from the composition structure in definition (2.11). The independence of the increments (property (iii)) follows since for any $t_0<\cdots<t_n$ the Poisson sets $\Delta_{t_0,t_1},\Delta_{t_1,t_2},\ldots,\Delta_{t_{n-1},t_n}$ are independent.

Now let X be defined as in (2.3). From the definition of the stochastic flow it is clear that X has cadlag sample paths. Since for each $0\leq t\leq u$, the Poisson set $\Delta_{s+t,s+u}$ is independent of $(X_0,\Delta_{s,s+t})$, using also translation invariance, the Markov property (2.9) follows as long as we show that for each $t\geq 0$, the probability kernel P_t has the random mapping representation

$$P_t(x,y)=\mathbb{P}[\mathbf{X}_{0,t}(x)=y]\qquad(t\geq 0,\ x,y\in S).\tag{4.1}$$

Let $r:=\sum_{m\in\mathcal{G}}r_m$. If $r=0$ the statement is trivial. Otherwise, define a probability kernel K by

$$K(x,y):=r^{-1}\sum_{m\in\mathcal{G}}r_m 1_{\{m(x)=y\}}\qquad(x,y\in S).\tag{4.2}$$

Write $\Delta_{0,t}=\{(m_s,s):s\in\Gamma\}$ where Γ is a Poisson point set on $(0,t]$ with intensity r and conditional on Γ, the maps $(m_s)_{s\in\Gamma}$ are i.i.d. with law $r^{-1}r_m$. Then, using the definition of the Poisson distribution, we see that

$$\mathbb{P}[\mathbf{X}_{0,t}(x)=y]=e^{-rt}\sum_{k=0}^{\infty}\frac{(rt)^k}{k!}K^k(x,y).\tag{4.3}$$

On the other hand, since $G=r(K-1)$ where 1 denotes the identity matrix,

$$\begin{aligned}P_t=e^{tG}&=\sum_{n=0}^{\infty}\frac{(rt)^n}{n!}(K-1)^n=\sum_{n=0}^{\infty}\frac{(rt)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}K^k\\&=\sum_{k=0}^{\infty}\frac{(rt)^k}{k!}K^k\sum_{n=k}^{\infty}\frac{(-rt)^{n-k}}{(n-k)!},\end{aligned}\tag{4.4}$$

which agrees with the right-hand side of (4.3). ∎

Proof of Proposition 2 If

$$\Delta_{t-,u-}=\{(m_1,t_1),\ldots,(m_n,t_n)\},\qquad t_1<\cdots<t_n,\tag{4.5}$$

then

$$\mathbf{X}_{t-,u-} := m_n \circ \cdots \circ m_1 \quad \text{and} \quad \mathbf{Y}_{-u,-t} := \hat{m}_1 \circ \cdots \circ \hat{m}_n, \tag{4.6}$$

so repeated application of (2.12) implies that $\mathbf{X}_{t-,u-}$ is dual to $\mathbf{Y}_{-u,-t}$. Using the semigroup property, it follows that the expression

$$\psi\big(\mathbf{X}_{s,t-}(x), \mathbf{Y}_{-u,-t}(y)\big) = \psi\big(\mathbf{X}_{t-,u-} \circ \mathbf{X}_{s-,t-}(x), y\big) = \psi\big(\mathbf{X}_{s,u-}(x), y\big) \tag{4.7}$$

is constant as a function of $t \in [s,u]$. For any $t_0 < \cdots < t_n$, the pairs

$$(\mathbf{X}_{t_0,t_1}, \mathbf{Y}_{-t_1,-t_0}), \ldots, (\mathbf{X}_{t_{n-1},t_n}, \mathbf{Y}_{-t_n,-t_{n-1}})$$

are functions of the restrictions of the Poisson point process Δ to disjoint sets $\mathcal{G} \times (t_0, t_1]$ etc., and therefore independent, completing the proof that the stochastic flows $(\mathbf{X}_{s,t})_{s\leq t}$ and $(\mathbf{Y}_{s,t})_{s\leq t}$ are dual in the sense defined in Section 2.1. In particular, this proves that the Markov processes X and Y are pathwise dual as defined in Section 2.1. ∎

Proof of Lemma 3 If G can be represented in $\mathcal{G}$, then we can construct a stochastic flow $(\mathbf{X}_{s,t})_{s\leq t}$ based on such a random mapping representation as in (2.11). By the fact that $\mathcal{G}$ is closed under composition and contains the identity map, it follows that $\mathbf{X}_{s,t} \in \mathcal{G}$ for each $s \leq t$, so (2.16) proves that P_t can be represented in $\mathcal{G}$ for all $t \geq 0$.

Assume, conversely, that P_t can be represented in $\mathcal{G}$ for all $t \geq 0$. Then, for each $t \geq 0$, we can find a probability distribution π_t on $\mathcal{G}$ such that

$$P_t(x,y) = \sum_{m\in\mathcal{G}} \pi_t(m) 1_{\{m(x) = y\}} \qquad (x, y \in S). \tag{4.8}$$

Since G is the generator of $(P_t)_{t\geq 0}$,

$$\lim_{t\downarrow 0} t^{-1} \sum_{m\in\mathcal{G}} \pi_t(m) 1_{\{m(x) = y\}} = G(x,y) \qquad (x, y \in S,\ x \neq y). \tag{4.9}$$

Let $m \in \mathcal{G}$ satisfy $m \neq 1$, i.e., m is different from the identity map. Then we can find $x \neq y$ such that $m(x) \neq y$. Now (4.9) shows that $t^{-1}\pi_t(m) \leq G(x,y)$, so using compactness we can find a sequence of times $t_n \downarrow 0$ such that the limit

$$r_m := \lim_{n\to\infty} t_n^{-1} \pi_{t_n}(m) \tag{4.10}$$

exists for every $m \in \mathcal{G}' := \{m \in \mathcal{G} : m \neq 1\}$. Using (4.9), it follows that

$$G(x,y) = \sum_{m\in\mathcal{G}'} r_m 1_{\{m(x) = y\}} \qquad (x, y \in S,\ x \neq y). \tag{4.11}$$

Using also that G is a Markov generator, this is easily seen to imply (2.10), i.e., we have found a random mapping representation of G in terms of maps in $\mathcal{G}$. ∎

Proof of Lemma 4 This is immediate from the observation that

$$\psi\big(m(x), A\big) = 1_{\{m(x) \in A\}} = 1_{\{x \in m^{-1}(A)\}} = \psi\big(x, m^{-1}(A)\big).$$

∎

4.3 Monotone and additive maps

In this section we prove Lemma 5, which characterizes montone and additive maps in terms of their inverse images.

Proof of Lemma 5 (i) If S and T are partially ordered sets, $m : S \to T$ is monotone, and $A \subset T$ is decreasing, then for each $x, y \in S$ with $x \leq y$ and $y \in m^{-1}(A)$, we have $m(x) \leq m(y) \in A$ by the fact that m is monotone and hence $m(x) \in A$ or equivalently $x \in m^{-1}(A)$ by the fact that A is decreasing. This shows that

$$m^{-1}(A) \in \mathcal{P}_{\rm dec}(S) \text{ for all } A \in \mathcal{P}_{\rm dec}(T). \tag{4.12}$$

Conversely, if (4.12) holds, and $x, y \in S$ satisfy $x \leq y$, then $m^{-1}(\{y\}^{\downarrow})$ is a decreasing set containing y, so $x \in m^{-1}(\{y\}^{\downarrow})$ or equivalently $m(x) \leq m(y)$, proving that m is monotone.

(ii) We use the fact that if S is a join-semilattice, then a set $A \subset S$ is an ideal if and only if A is nonempty, decreasing, and $x, y \in A$ imply $x \vee y$ in A (see Lemma 35 in the appendix). Now let S and T be join-semilattices that are bounded from below. Assume that $m : S \to T$ is additive and that $A \in \mathcal{P}_{!\rm dec}(T)$ is an ideal. Since additive functions are monotone, by what we have just proved, $m^{-1}(A)$ is decreasing. Since A is nonempty and decreasing, it contains 0, and since $m(0) = 0$ we have $0 \in m^{-1}(A)$, which shows that $m^{-1}(A)$ is nonempty. Finally, if $x, y \in m^{-1}(A)$, then by additivity and the fact that A is an ideal, $m(x \vee y) = m(x) \vee m(y) \in A$, so $x \vee y \in m^{-1}(A)$, which completes the proof that $m^{-1}(A)$ is an ideal.

Assume, conversely, that $m : S \to T$ has the property that $m^{-1}(A)$ is an ideal whenever $A \subset T$ is an ideal. Then, since $m^{-1}(\{0\})$ is nonempty and decreasing, we have $0 \in m^{-1}(\{0\})$ and hence $m(0) = 0$. For each $y \in S$, the set $m^{-1}(\{m(y)\}^{\downarrow})$ is a decreasing set containing y, so $x \leq y$ implies $x \in m^{-1}(\{m(y)\}^{\downarrow})$ or equivalently $m(x) \leq m(y)$, showing that m is monotone. Since $x \leq x \vee y$ and $y \leq x \vee y$, this implies that $m(x) \vee m(y) \leq m(x \vee y)$ for each $x, y \in S$. To get the opposite inequality, we observe that $m^{-1}(\{m(x) \vee m(y)\}^{\downarrow})$ is an ideal containing x and y, so using the properties of ideals also $x \vee y \in m^{-1}(\{m(x) \vee m(y)\}^{\downarrow})$ which says that $m(x \vee y) \leq m(x) \vee m(y)$. ∎

4.4 Monotone systems duality

In this section we prove Lemmas 8 and 11 that form the basis of Section 2.5 about monotone systems duality.

To prepare for the proof of Lemma 8, we make the following general observation on sets of maximal and minimal elements. If $A \subset S$ is finite, then there exists to any $x \in A$ a maximal element $y \in A$ with $y \geq x$ (see Lemma 34 in the appendix). Therefore, we have $(A_{\rm max})^{\downarrow} \supset A$ and similarly $(A_{\rm min})^{\uparrow} \supset A$ for each $A \in \mathcal{P}(S)$, so in particular

$$(A_{\rm max})^{\downarrow} = A \quad \big(A \in \mathcal{P}_{\rm dec}(S)\big) \quad \text{and} \quad (A_{\rm min})^{\uparrow} = A \quad \big(A \in \mathcal{P}_{\rm inc}(S)\big). \tag{4.13}$$

(In particular, this also holds if $A = \emptyset$ since $\emptyset_{\rm min} = \emptyset = \emptyset_{\rm max}$ and $\emptyset^{\uparrow} = \emptyset = \emptyset^{\downarrow}$.) In view of (4.13), we can "encode" a set $A \in \mathcal{P}_{\rm dec}(S)$ by describing its set of maximal elements, or more generally by any subset $B \subset S$ such that $A = B^{\downarrow}$.

Proof of Lemma 8 Through the bijection $x \mapsto x'$, the monotone map $m : S \to S$ naturally gives rise to a monotone map $n : S' \to S'$ defined by

$$n(x) := m(x')' \qquad (x \in S'). \tag{4.14}$$

In terms of the map n, the definitions (2.31) take the simpler form

$$m^{\dagger}(B) := n^{-1}(B^{\uparrow})_{\rm min} \quad \text{and} \quad m^{*}(B) := \bigcup_{x \in B} n^{-1}(\{x\}^{\uparrow})_{\rm min} \tag{4.15}$$

$(B \in \mathcal{P}(S'))$. Instead of showing that $m^\dagger$ and m^* are dual to m with respect to the duality function in (2.29), we may equivalently show that $m^\dagger$ and m^* are dual to n with respect to the duality function

$$\tilde{\phi}(x, B) := 1_{\{x' \in B'^{\downarrow}\}} = 1_{\{x \in B^{\uparrow}\}} \qquad \big(x \in S',\ B \in \mathcal{P}(S')\big). \tag{4.16}$$

Letting $\sharp = \dagger$ or $*$, this means that we must show that

$$n(x) \in B^{\uparrow} \quad \text{if and only if} \quad x \in m^{\sharp}(B)^{\uparrow} \qquad \big(x \in S',\ B \in \mathcal{P}(S')\big). \tag{4.17}$$

Since the event in the left-hand side of this equation is $\{x \in n^{-1}(B^{\uparrow})\}$, this is equivalent to

$$m^{\sharp}(B)^{\uparrow} = n^{-1}(B^{\uparrow}) \qquad \big(B \in \mathcal{P}(S')\big). \tag{4.18}$$

Since n is monotone, Lemma 5 tells us that $n^{-1}(A) \in \mathcal{P}_{\rm inc}(S')$ for each $A \in \mathcal{P}_{\rm inc}(S')$. Therefore, by (4.13) applied to the increasing set $A = n^{-1}(B^{\uparrow})$, we see that

$$m^\dagger(B)^{\uparrow} = \big(n^{-1}(B^{\uparrow})_{\rm min}\big)^{\uparrow} = m^{-1}(B^{\uparrow}) \qquad \big(B \in \mathcal{P}(S')\big), \tag{4.19}$$

proving that $m^\dagger$ satisfies (4.18). Similarly, for the map m^*, applying (4.13) to the increasing sets $A = n^{-1}(\{x\}^{\uparrow})$ with $x \in B$, we have for $B \in \mathcal{P}(S')$

$$m^*(B)^{\uparrow} = \big(\bigcup_{x \in B} n^{-1}(\{x\}^{\uparrow})_{\rm min}\big)^{\uparrow} = \bigcup_{x \in B} \big(n^{-1}(\{x\}^{\uparrow})_{\rm min}\big)^{\uparrow} = \bigcup_{x \in B} n^{-1}(\{x\}^{\uparrow}) = n^{-1}(B^{\uparrow}), \tag{4.20}$$

which shows that also m^* satisfies (4.18).

It remains to prove the properties (2.32) and (2.33). It is clear from (4.15) that

$$m^\dagger(B) = m^\dagger(B)_{\rm min} \quad \text{and} \quad m^*(B \cup C) = m^*(B) \cup m^*(C) \qquad \big(B, C \in \mathcal{P}(S')\big). \tag{4.21}$$

Formula (4.18) tells us that

$$m^\dagger(B)^{\uparrow} = n^{-1}(B^{\uparrow}) = m^*(B)^{\uparrow}, \tag{4.22}$$

Taking minima on both sides, using the facts that $m^\dagger(B) = m^\dagger(B)_{\rm min}$ and $(A^{\uparrow})_{\rm min} = A_{\rm min}$ for any $A \subset S$, we see that $m^\dagger(B) = (m^*(B))_{\rm min}$ for $B \in \mathcal{P}(S')$. ∎

Proof of Lemma 11 By Lemma 6, we have

$$\begin{aligned} m^{-1}(\{y'\}^{\downarrow}) &= \{x \in S : m(x) \leq y'\} = \{x \in S : \langle m(x), y\rangle = 1\} \\ &= \{x \in S : \langle x, m'(y)\rangle = 1\} = \{m'(y)'\}^{\downarrow} \qquad (y \in S'). \end{aligned} \tag{4.23}$$

It follows that $(m^{-1}(\{y'\}^{\downarrow}))_{\rm max} = \{m'(y)'\}$, so by (2.31), we see that $m^*(B) = \{m'(y) : y \in B\}$ $(B \in \mathcal{P}(S))$. The fact that $m^\dagger(B) = m^*(B)_{\rm min}$ has already been proved in (2.32). ∎

4.5 Percolation representations

In this section we prove Lemmas 13, 14, and 15, that show that each additive system taking values in a lattice has a percolation representation, and if the lattice is moreover distributive, then the process and its dual can be represented in the same percolation substructure.

Proof of Lemma 13 We start by showing that if $M \subset \Lambda \times \Lambda$ satisfies (2.43), then (2.44) defines an additive map $m : \mathcal{P}_{\rm dec}(\Lambda) \to \mathcal{P}_{\rm dec}(\Lambda)$. To see that m maps $\mathcal{P}_{\rm dec}(\Lambda)$ into itself, we note that $\tilde{\jmath} \le j$ and $j \in m(x)$ implies $(i,j) \in M$ for some $i \in x$ which by (2.43) (ii) implies that $(i,\tilde{\jmath}) \in M$ and hence $\tilde{\jmath} \in m(x)$. We see from (2.44) that

$$m(x) = \bigcup_{i \in x} m\big(\{i\}\big) \qquad \big(x \in \mathcal{P}_{\rm dec}(\Lambda)\big), \tag{4.24}$$

which is easily seen to imply that m is additive.

We next show that if m is defined via M as in (2.44), then we can recover M from m since

$$M = \big\{(i,j) \in \Lambda \times \Lambda : j \in m\big(\{i\}^{\downarrow}\big)\big\}. \tag{4.25}$$

To see this, note that $(i,j) \in M$ implies by (2.44) $j \in m\big(\{i\}^{\downarrow}\big)$ since $i \in \{i\}^{\downarrow}$. This proves the inclusion $\subset$ in (4.25). Conversely, if $j \in m\big(\{i\}^{\downarrow}\big)$, then by (2.44) there must exist some $\tilde{\imath} \le i$ such that $(\tilde{\imath},j) \in M$, which by (2.43) (i) implies $(i,j) \in M$ and shows $\supset$ in (4.25).

To see that every additive map from $\mathcal{P}_{\rm dec}(\Lambda)$ into itself is of the form (2.44), let m be such a map. We will show that m is of the form (2.44) where M is given by (4.25). Indeed, if M is given by (4.25), then M satisfies (2.43) (i) by the monotonicity of m, namely if $(i,j) \in M$ and $i \le \tilde{\imath}$ then $j \in m\big(\{i\}^{\downarrow}\big) \subset m\big(\{\tilde{\imath}\}^{\downarrow}\big)$, which implies $(\tilde{\imath},j) \in M$. Property (2.43) (ii) holds due to the fact that $m\big(\{i\}^{\downarrow}\big) \in \mathcal{P}_{\rm dec}(\Lambda)$ so $j \in m\big(\{i\}^{\downarrow}\big)$ implies $\tilde{\jmath} \in m\big(\{i\}^{\downarrow}\big)$ for any $\tilde{\jmath} \le j$.

To see that m is given by (2.44), first define n by (2.44). We claim that for any $i \in \Lambda$,

$$n\big(\{i\}^{\downarrow}\big) = \big\{j : (\tilde{\imath},j) \in M \text{ for some } \tilde{\imath} \le i\big\} = \big\{j : (i,j) \in M\big\} = m\big(\{i\}^{\downarrow}\big). \tag{4.26}$$

Indeed, in the second equality, we have $\subset$ by (2.43) (i) and $\supset$ by choosing $\tilde{\imath} = i$, while the third equality is immediate from (4.25). It follows that $m = n$ on sets of the form $\{x\}^{\downarrow}$. Since both m and n are additive, it follows that they agree on $\mathcal{P}_{\rm dec}(\Lambda)$.

It remains to prove (2.45). Recall that $S := \mathcal{P}_{\rm dec}(\Lambda)$ and that $S' := \mathcal{P}_{\rm dec}(\Lambda') = \mathcal{P}_{\rm inc}(\Lambda)$, which together with the map $x \mapsto x' := x^{\rm c}$ is a dual to S. Then the duality function takes the form $\langle x,y\rangle = 1_{\{x \cap y = \emptyset\}}$ and two additive maps $m : S \to S$ and $m' : S' \to S'$ are dual if and only if

$$1_{\{m(x) \cap y = \emptyset\}} = 1_{\{x \cap m'(y) = \emptyset\}} \qquad \big(x \in \mathcal{P}_{\rm dec}(\Lambda),\ y \in \mathcal{P}_{\rm inc}(\Lambda)\big). \tag{4.27}$$

We observe that if M satisfies (2.43) and M' is given by (2.45), then M' satisfies (2.43) with respect to the reversed order, i.e., with respect to the order on Λ'. In view of this, it suffices to show that if $m : S \to S$ is defined in terms of M as in (2.44) and similarly $m' : S' \to S'$ is defined in terms of M', then m and m' are dual in the sense of (4.27). Indeed,

$$\begin{aligned} m(x) \cap y = \emptyset \quad &\Leftrightarrow \quad \{j \in \Lambda : (i,j) \in M \text{ for some } i \in x\} \cap y = \emptyset \\ &\Leftrightarrow \quad \{(i,j) \in M : i \in x,\ j \in y\} = \emptyset \\ &\Leftrightarrow \quad x \cap \{i \in \Lambda : (j,i) \in M' \text{ for some } j \in y\} = \emptyset \quad \Leftrightarrow \quad x \cap m'(y) = \emptyset. \end{aligned} \tag{4.28}$$

∎

Proof of Lemma 14 The left- and right-hand side of (2.46) both equal x when $s = u$ and for fixed $s \in \mathbb{R}$, both sides change only at times u when $(m,u) \in \Delta$ for some $m \in \mathcal{G}$. If just before time u, both sides are equal to some $y \in \mathcal{P}_{\rm dec}(\Lambda)$, then at time u, using the definition of open paths, we see that the left- and right-hand side of (2.46) equal

$$m(y) \quad \text{and} \quad \{j \in \Lambda : (i,j) \in M \text{ for some } i \in y\}, \tag{4.29}$$

respectively. In view of this, (2.46) follows by induction from (2.44). The proof of (2.47) is the same: we first observe that equality holds at $u = s$ and then check that the equation remains true when we increase u while keeping s fixed, where

$$m'(y) = \{j \in \Lambda : (i,j) \in M' \text{ for some } i \in y\}, \tag{4.30}$$

with M' as in (2.45).

Let us write $(i,s) \rightsquigarrow (j,u-)$ if there is an open path "from (i,s) to $(j,u-)$", where in the definition of an open path, we replace the time interval $(s,u]$ by (s,u). In the same way, we can give a meaning to $(i,s-) \rightsquigarrow (j,u)$ and $(i,s-) \rightsquigarrow (j,u-)$, where arrows and blocking symbols at the end of the time interval have an effect, or not, depending on whether we want our definition to be left- or right-continuous in s or u. Then formulas (2.46) and (2.47) have obvious analogues where right-continuity in a variable is replaced by left-continuity, or vice versa. Now (2.48) follows from the observation that

$$\begin{aligned} \mathbf{X}_{s,t-}(x) \cap \mathbf{Y}_{-u,-t}(y) = \emptyset \quad &\Leftrightarrow \quad \nexists i,j,k \text{ such that } i \in x,\ j \in y,\ (i,s) \rightsquigarrow (k,t-) \rightsquigarrow (j,u-) \\ &\Leftrightarrow \quad (i,s) \not\rightsquigarrow (j,u)\ \forall\ i \in x,\ j \in y \quad \text{a.s.,} \end{aligned} \tag{4.31}$$

where in the last step we have used a.s. continuity at the deterministic time u. ∎

Proof of Lemma 15 Let S be a finite lattice. Then we claim that the map[4]

$$x \mapsto \big(\{x\}^{\uparrow}\big)^{\mathrm{c}} \tag{4.32}$$

is a $(0,\vee)$-homomorphism from S into the lattice of sets $\mathcal{P}_{\mathrm{dec}}(S)$. Indeed,

$$\big(\{0\}^{\uparrow}\big)^{\mathrm{c}} = \emptyset \quad \text{and} \quad \big(\{x \vee y\}^{\uparrow}\big)^{\mathrm{c}} = \big(\{x\}^{\uparrow} \cap \{y\}^{\uparrow}\big)^{\mathrm{c}} = \big(\{x\}^{\uparrow}\big)^{\mathrm{c}} \cup \big(\{y\}^{\uparrow}\big)^{\mathrm{c}}. \tag{4.33}$$

Since $x \neq y$ implies $\{x\}^{\uparrow} \neq \{y\}^{\uparrow}$, the map in (4.32) is one-to-one and as a result a $(0,\vee)$-isomorphism to its image, proving that each finite lattice is $(0,\vee)$-isomorphic to a join-semilattice of sets.

Now let Λ be a finite set and let $T \subset \mathcal{P}(\Lambda)$ be a join-semilattice of sets, i.e., $\emptyset \in T$ and T is closed under unions. We claim that each $m : T \to T$ can be extended to an additive map $\overline{m} : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda)$. We will actually prove a somewhat stronger statement. Assume moreover that Λ is partially ordered and that $T \subset \mathcal{P}_{\mathrm{dec}}(\Lambda)$. Then we will show that $m : T \to T$ can be extended to an additive map $\overline{m} : \mathcal{P}_{\mathrm{dec}}(\Lambda) \to \mathcal{P}_{\mathrm{dec}}(\Lambda)$. In particular, equipping Λ with the trivial order, this includes the statement in Lemma 15 as a special case. The following argument was suggested to us by László Csirmaz.

Assume that $x \in \mathcal{P}_{\mathrm{dec}}(\Lambda)$ and $x \notin T$. Then $\overline{T} := \{y, x \cup y : y \in T\} \subset \mathcal{P}_{\mathrm{dec}}(\Lambda)$ is $(\emptyset,\cup)$-closed and contains x. By the finiteness of S, the claim will follow by induction if we can prove that m can be extended to an additive map $\overline{m} : \overline{T} \to \mathcal{P}_{\mathrm{dec}}(\Lambda)$.

For $y \in T$, we define

$$\begin{aligned} \overline{m}(y) &:= m(y), \\ \overline{m}(x \cup y) &:= m(y) \cup \bigcap\{m(z) : x \subset z \in T\} \\ &= \bigcap\{m(y) \cup m(z) : x \subset z \in T\} \\ &= \bigcap\{m(y \cup z) : x \subset z \in T\}. \end{aligned} \tag{4.34}$$

Note that since $\mathcal{P}_{\mathrm{dec}}(\Lambda)$ is closed under finite intersections, the last line in (4.34) defines an element of $\mathcal{P}_{\mathrm{dec}}(\Lambda)$. We need to show that (4.34) is a good definition, i.e.,

(i) If $y \in T$ and $x \cup y \in T$, then $\bigcap\{m(y \cup z) : x \subset z \in T\} = m(x \cup y)$.

(ii) If $y, y' \in T$ and $x \cup y = x \cup y'$, then
$\bigcap\{m(y \cup z) : x \subset z \in T\} = \bigcap\{m(y' \cup z) : x \subset z \in T\}$.

[4] Alternatively, one may consider the map $x \mapsto \big(\{x\}^{\uparrow} \cap \Lambda\big)^{\mathrm{c}}$ where Λ is the set of meet-irreducible elements of S. As a result of Birkhoff's representation theorem, one can prove that this map is onto, and as a result sets up an isomorphism between the lattices S and $\mathcal{P}_{\mathrm{dec}}(\Lambda)$, if and only if S is distributive.

Indeed, in (i), the inclusion $\supset$ follows from the monotonicity of m, while the inclusion $\subset$ follows by setting $z = x \cup y$. Property (ii) follows from the observation that

$$x \cup y = x \cup y' \quad \text{and} \quad x \subset z \in T \quad \text{imply} \quad y \cup z = y' \cup z. \tag{4.35}$$

To show that $\overline{m}$ is additive, we must prove that

(i) If $y, y' \in T$, then $\overline{m}(y \cup y') = \overline{m}(y) \cup \overline{m}(y')$.

(ii) If $y, y' \in T$, then $\overline{m}\big((x \cup y) \cup y'\big) = \overline{m}(x \cup y) \cup \overline{m}(y')$.

(iii) If $y, y' \in T$, then $\overline{m}\big((x \cup y) \cup (y' \cup x)\big) = \overline{m}(x \cup y) \cup \overline{m}(y' \cup x)$.

Property (i) follows from the additivity of m. Properties (ii) and (iii) say that for $y, y' \in T$,

$$\overline{m}(y \cup y' \cup x) = \overline{m}(x \cup y) \cup \overline{m}(y') = \overline{m}(x \cup y) \cup \overline{m}(x \cup y'), \tag{4.36}$$

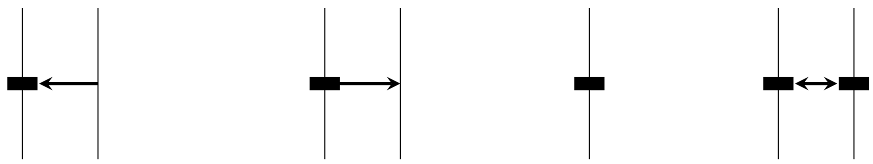
which is easily seen to follow from our definition $\overline{m}(x \cup y) := m(y) \cup \bigcap\{m(z) : x \subset z \in T\}$ as well as from the additivity of m. ∎

4.6 Examples of dual maps

In this section we prove Lemmas 18 and 22 by verifying that the maps there are really dual to each other as claimed.

Proof of Lemma 18 The easiest way to see this is from the graphical representation (see Figure 2). First, one represents the maps a_i, b_{ij}, c_i, d_i, and e_i by arrows and blocking symbols as described at the end of Section 3.3, and checks that this really corresponds to the maps in (3.8) with the interpretation of the two-stage contact process as a set-valued process in the forward time direction (3.13). The dual maps $a'_i, b'_{ij}, c'_i, d'_i$, and e'_i can then be found according to the recipe "reverse the arrows, keep the blocking symbols" (as is clear from Figure 2 and which also corresponds to (2.45)). Using the interpretation of the dual set-valued process (3.14), this yields (3.11). ∎

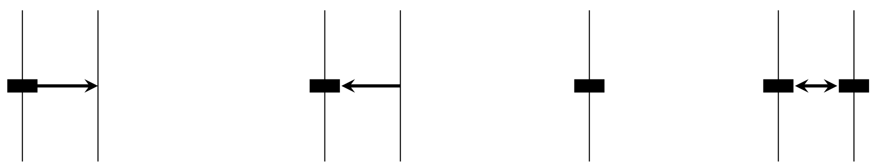
Proof of Lemma 22 As in the proof of Lemma 18, we use the graphical representation. The maps a_{ij}, c_{ij}, d_i, and e_{ij} can be represented in terms of arrows and blocking symbols as:



i j $\qquad$ i j $\qquad$ i $\qquad$ i j

a_{ij} $\qquad$ c_{ij} $\qquad$ d_i $\qquad$ e_{ij}

According to the recipe "reverse the arrows, keep the blocking symbols", the dual maps are given by:



i j $\qquad$ i j $\qquad$ i $\qquad$ i j

a'_{ij} $\qquad$ c'_{ij} $\qquad$ d'_i $\qquad$ e'_{ij}

Comparing the pictures, we see that

$$a'_{ij} = c_{ij}, \quad c'_{ij} = a_{ij}, \quad d'_i = d_i, \quad \text{and} \quad e'_{ij} = e_{ij}.$$

∎

4.7 Gray's duality

In this section, we prove Proposition 21, which shows that Gray's dual for general monotone spin systems is a special case of the dual $Y^\bullet$ from Section 2.5. For completeness, we also prove Lemma 19 which says that attractive spin systems are monotonically representable. We start with a simple preparatory lemma.

Lemma 23 (Monotone real functions) *Let S be a finite partially ordered set. Then $f \in \mathcal{F}_{\rm mon}(S,\mathbb{R})$ if and only if f can be written as*

$$f(x) = \sum_{A \in \mathcal{P}_{\rm inc}(S)} r_A 1_{\{x \in A\}} \qquad (x \in S) \tag{4.37}$$

for real constants $(r_A)_{A \in \mathcal{P}_{\rm inc}(S)}$ satisfying $r_A \geq 0$ for all $A \neq \emptyset, S$.

Proof If $f \in \mathcal{F}(S,\mathbb{R})$ is monotone, then[5] $f^{-1}([r,\infty)) \in \mathcal{P}_{\rm inc}(S)$ for all $r \in \mathbb{R}$. Let $f(S) := \{f(x) : x \in S\}$ be the image of S under f. Since S is finite, so is $f(S)$ and we may write $f(S) = \{r_1, \ldots, r_n\}$ with $r_1 < \cdots < r_n$. The sets $A_k := \{x \in S : f(x) \geq r_k\}$ are increasing and

$$f(x) = r_1 1_{\{x \in S\}} + \sum_{k=2}^{n} (r_k - r_{k-1}) 1_{\{x \in A_k\}}. \tag{4.38}$$

This proves that every $f \in \mathcal{F}_{\rm mon}(S,\mathbb{R})$ can be written in the form (4.37). Conversely, if f is of the form (4.37), then it is a sum of monotone functions, so $f \in \mathcal{F}_{\rm mon}(S,\mathbb{R})$. ∎

Proof of Lemma 19 It suffices to prove that the first term in (3.15) is monotonically representable. The same arguments applied to the second term and S equipped with the reversed order then prove the general statement.

Since each β_i is monotone and nonnegative, by Lemma 23, for each $i \in \Lambda$ we can find some set $\mathcal{A}_i$ whose elements are increasing, nonempty subsets of S, as well as nonnegative constants $(r_{i,A})_{A \in \mathcal{A}_i}$, such that

$$\beta_i(x) = \sum_{A \in \mathcal{A}_i} r_{i,A} 1_{\{x \in A\}}. \tag{4.39}$$

For each $i \in \Lambda$ and $A \in \mathcal{A}_i$, define a map $m_{i,A}$ by

$$m_{i,A}(x) := \begin{cases} x \vee \varepsilon_i & \text{if } x \in A, \\ x & \text{otherwise.} \end{cases} \tag{4.40}$$

Then the first term in (3.15) can be written as

$$\sum_{i \in \Lambda} \sum_{A \in \mathcal{A}_i} r_{i,A} \big(f(m_{i,A}(x)) - f(x) \big), \tag{4.41}$$

which is the desired representation in monotone maps. ∎

Proof of Proposition 21 For any $s \leq u$ and $B \in \mathcal{P}(S)$, let us write

$$\zeta_{s,u}(B) := \bigcup_{y \in B} \zeta_{s,u}(y). \tag{4.42}$$

We will show that $\mathbf{Y}^\bullet_{-u,-s}(B) = \zeta_{s,u}(B)$. In particular, setting $B = \{y\}$ then yields the statement in Proposition 21. Thus, in light of the definition of Gray's dual in (3.17), we claim that for any $s \leq u$ and $B \in \mathcal{P}(S)$, a.s.

$$\mathbf{Y}^\bullet_{-u,-s}(B) = \big\{ z \in S : \text{there exists an } [s,u]\text{-path } \pi \text{ from } z \text{ to some } y \in B \big\}. \tag{4.43}$$

[5]Let S and T be partially ordered sets. If a map $m : S \to T$ is monotone, then it is also monotone with respect to the reversed orders on S and T. In view of this and Lemma 5 (i), a map $m : S \to T$ is monotone if and only if $m^{-1}(A) \in \mathcal{P}_{\rm inc}(S)$ for all $A \in \mathcal{P}_{\rm inc}(T)$.

As in (2.11), we order the elements of $\Delta_{s,u}$ according to the time they occur:

$$\Delta_{s,u} := \{(m_1, t_1), \ldots, (m_n, t_n)\}, \quad t_1 < \cdots < t_n. \tag{4.44}$$

Then condition (i) in the definition of an $[s, u]$-path in Section 3.4 is equivalent to

(i)' $m_k\big(\pi(t_k-)\big) \geq \pi(t_k)$ for all $k = 1, \ldots, n$,

(i)" $t \mapsto \pi(t)$ is nondecreasing on $[s, t_1), \ldots, [t_{n-1}, t_n), [t_n, u]$.

If π would strictly increase at some time $t_{k-1} < t < t_k$ (with $t_0 := s$ and $t_{n+1} := u$), then we could make it smaller on $[t, t_k)$, violating minimality (see (ii) in the definition of an $[s, u]$-path), so we see that an $[s, u]$-path π must actually satisfy

$$t \mapsto \pi(t) \text{ is constant on } [s, t_1), \ldots, [t_{n-1}, t_n), [t_n, u]. \tag{4.45}$$

Condition (i)' says that $\pi(t_k-) \in m_k^{-1}(\{\pi(t_k)\}^{\uparrow})$, which again by minimality implies that

$$\pi(t_k-) \in \big(m_k^{-1}(\{\pi(t_k)\}^{\uparrow})\big)_{\min} \quad \Leftrightarrow \quad \pi(t_k-) \in m^{\bullet}(\{\pi(t_k)\}) \qquad (k = 1, \ldots, n), \tag{4.46}$$

where we have also used (3.18). By induction, decreasing s wile keeping u fixed, it follows that $\zeta_{s,u}(B) \subset \mathbf{Y}^{\bullet}_{-u,-s}(B)$.

Conversely, one can verify that any cadlag function $\pi : [s, u] \to S$ satisfying (4.45) and (4.46) is an $[s, u]$-path. This means that the function $[s, u] \ni t \mapsto \zeta_{t,u}$ is constant between the times $t_1, \ldots, t_n$ and satisfies

$$\zeta_{t_k-,u}(B) = \bigcup_{z \in \zeta_{t_k,u}(B)} \big(m_k^{-1}(\{z\}^{\uparrow})\big)_{\min}. \tag{4.47}$$

Again using (3.18), it follows that $[s, u] \ni t \mapsto \zeta_{t,u}$ is the right-continuous modification of $[s, u] \ni t \mapsto \mathbf{Y}^{\bullet}_{-u,-t}$, proving (4.43). ∎

5 Infinite product spaces

5.1 Introduction

In Sections 2.4 and 2.5, we have shown that Markov processes with finite state spaces have pathwise duals if their generator can be represented in additive or monotone maps, respectively. In most of the applications, such as those in Sections 3.2–3.5, the state space is of the product form T^{Λ} where T is a finite partially ordered set and Λ is another finite set that we will call the *underlying space*. One is usually interested in the case that the underlying space is large, so it is natural to take the limit and consider infinite Λ; this is the setting of *interacting particle systems* as treated, e.g., in Liggett's classical book [Lig85]. For the study of interacting particle systems, it is important that dual processes are available also in the infinite setting. The aim of the present section is to show how, under suitable technical assumptions, such duals can indeed be constructed.

In Sections 5.2 and 5.3, we show how the Poisson construction of Markov processes of Lemma 1 generalizes to interacting particle systems. In the finite setting, we sometimes used the term *graphical representation* for the Poisson point set Δ of Lemma 1 and we will continue to do so in the infinite setting. For additively representable processes, there is a natural way of drawing Δ in terms of arrows and blocking symbols as explained in Section 2.6. Originally, the word graphical representation was used for such pictures only, which can also be used to construct cancellative particle systems as explained in [Gri79]. Nowadays, more general graphical representations that also may involve other symbols are a standard tool in the study of interacting particle systems that need not be additive or cancellative. This asks for a unified treatment of such constructions and, in particular, a proof that under suitable technical conditions, they yield a well-defined process. It seems hard, however, to find a reference for general results of this sort. We base our treatment on the lecture notes [Swa11].

Once the construction of such processes is settled, in Section 5.4 we turn our attention to interacting particle systems that can be represented in terms of monotone maps. We show that the dual set-valued processes constructed in Section 2.5 are well-defined also when the underlying space is infinite, provided that they are started in a "finite" initial state, i.e., a finite set whose elements are configurations that are nonzero at finitely many lattice points only. For such finite initial states, we show that the dual process a.s. remains finite for all times and is pathwise dual to the monotonically representable interacting particle system.

In Section 5.5, we assume moreover that the local state space in each point of the underlying space is a lattice and that the interacting particle system is additively representable. Under these assumptions, we show that the additive dual of Section 2.4 is well-defined also in the infinite setting and pathwise dual to the additively represented interacting particle system. Under weak additional technical assumptions, this dual process can also be started in infinite initial states. The question whether monotone duals in the sense of Section 2.5 can also be started in infinite initial states is left as an open problem.

5.2 Poisson construction of particle systems

Let T be a finite set, let Λ be countably infinite, and let $S := T^\Lambda$ be the space of all collections $(x(i))_{i\in\Lambda}$ with $x(i) \in T$ for all $i \in \Lambda$. Let $m : T^\Lambda \to T^\Lambda$ be a function. We say that a point $j \in \Lambda$ is *m-relevant* for $i \in \Lambda$ if

$$\exists x, y \in T^\Lambda \text{ s.t. } m(x)(i) \neq m(y)(i) \text{ and } x(k) = y(k)\ \forall k \neq j, \tag{5.1}$$

i.e., changing the value of x in j may change the value of $m(x)$ in i. We will use the notation

$$\mathcal{R}_i(m) := \big\{j \in \Lambda : j \text{ is } m\text{-relevant for } i\big\}. \tag{5.2}$$

We note that below, property (ii) may fail even if (i) holds, if $m(x)(i)$ depends on the tail behavior of $(x(j))_{j\in\Lambda}$. (For example, if $T = \{0, 1\}$, it may happen that $m(x)(i) = 1$ if $x(j) = 1$ for infinitely many j's and $m(x)(i) = 0$ otherwise; in such a case $\mathcal{R}_i(m) = \emptyset$.)

Lemma 24 (Continuous maps) *A map $m : T^\Lambda \to T^\Lambda$ is continuous with respect to the product topology on T^Λ if and only if for each $i \in \Lambda$, the following two conditions are satisfied.*

(i) *The set $\mathcal{R}_i(m)$ is finite.*

(ii) *If $x, y \in T^\Lambda$ satisfy $x(j) = y(j)$ for all $j \in \mathcal{R}_i(m)$, then $m(x)(i) = m(y)(i)$.*

Proof Fix $i \in \Lambda$. We will show that the map $x \mapsto m(x)(i)$ is continuous if and only if (i) and (ii) hold. Let $(\alpha_j)_{j\in\Lambda}$ be strictly positive constants such that $\sum_{j\in\Lambda} \alpha_j < \infty$. Then the metric

$$d(x, y) := \sum_{j\in\Lambda} \alpha_j 1_{\{x(j) \neq y(j)\}} \qquad (x, y \in T^\Lambda) \tag{5.3}$$

generates the product topology on T^Λ. By Tychonoff's theorem, T^Λ is compact, so the function $x \mapsto m(x)(i)$ is uniformly continuous. Since the target space T is finite, this means that there exists an $\varepsilon > 0$ such that $d(x, y) < \varepsilon$ implies $m(x)(i) = m(y)(i)$. Since $\sum_{j\in\Lambda} \alpha_j < \infty$, there exists some finite $\Lambda' \subset \Lambda$ such that $\sum_{j\in\Lambda\backslash\Lambda'} \alpha_j < \varepsilon$. It follows that

(ii)' If $x, y \in T^\Lambda$ satisfy $x(j) = y(j)$ for all $j \in \Lambda'$, then $m(x)(i) = m(y)(i)$.

We conclude from this that $\mathcal{R}_i(m) \subset \Lambda'$, proving (i). If this is a strict inclusion, then we can inductively remove those points from Λ' that are not elements of $\mathcal{R}_i(m)$ while preserving the property (ii)', until in a finite number of steps we see that (ii) holds.

Conversely, if (i) and (ii) hold and $x_k \to x$ pointwise, then by (i) there exists some n such that $x_k = x$ on $\mathcal{R}_i(m)$ and hence by (ii) $m(x_k)(i) = m(x)(i)$ for all $k \geq n$. ∎

For any map $m : T^\Lambda \to T^\Lambda$, let

$$\mathcal{D}(m) := \big\{i \in \Lambda : \exists x \in T^\Lambda \text{ s.t. } m(x)(i) \neq x(i)\big\} \tag{5.4}$$

denote the set of underlying space points whose values can possibly be changed by m. We say that a map $m : T^\Lambda \to T^\Lambda$ is *local* if m is continuous with respect to the product topology and $\mathcal{D}(m)$ is finite. Note that

$$\mathcal{R}_i(m) = \{i\} \qquad \big(i \notin \mathcal{D}(m)\big). \tag{5.5}$$

For $i \in \mathcal{D}(m)$, the set $\mathcal{R}_i(m)$ can be any finite subset of Λ, including the empty set.

We will only be interested in Markov processes whose generators can be represented in terms of local maps. Thus, we assume that $\mathcal{G}$ is a set whose elements are local maps $m : T^\Lambda \to T^\Lambda$, and we consider Markov processes with formal generator of the form

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m\big(f(m(x)) - f(x)\big) \qquad (x \in S), \tag{5.6}$$

where $(r_m)_{m\in\mathcal{G}}$ be nonnegative constants. In general, we will need to impose summability conditions (as stated below) on the rates $(r_m)_{m\in\mathcal{G}}$ for such a Markov process to be well-defined.

As in Section 2.2, we let Δ be a Poisson point subset of $\mathcal{G} \times \mathbb{R} = \{(m,t) : m \in \mathcal{G},\ t \in \mathbb{R}\}$ with local intensity $r_m \mathrm{d}t$, and for $s \leq u$, we set $\Delta_{s,u} := \Delta \cap (\mathcal{G} \times (s,u])$. Unlike in Section 2.2, it will usually be too restrictive to assume that the sets $\Delta_{s,u}$ are a.s. finite, so we can no longer simply concatenate the elements of $\Delta_{s,u}$ as in (2.11). Instead, we assume that

$$K_0 := \sup_i \sum_{\substack{m \in \mathcal{G} \\ \mathcal{D}(m) \ni i}} r_m < \infty, \tag{5.7}$$

which guarantees that for each $i \in \Lambda$ and $s < u$, the set $\Delta_{s,u}$ contains only finitely many maps that have the potential to change the state of the underlying space point i. Next, we define a *path of potential influence* to be a cadlag function $\gamma : [s,u] \to \Lambda$ such that

$$\begin{array}{ll} \text{(i)} & \text{if } \gamma_{t-} \neq \gamma_t \text{ for some } t \in (s,u], \text{ then there exists some } m \in \mathcal{G} \\ & \text{such that } (m,t) \in \Delta,\ \gamma_t \in \mathcal{D}(m) \text{ and } \gamma_{t-} \in \mathcal{R}_{\gamma_t}(m), \\ \text{(ii)} & \text{for each } (m,t) \in \Delta \text{ with } t \in (s,u] \text{ and } \gamma_t \in \mathcal{D}(m), \\ & \text{one has } \gamma_{t-} \in \mathcal{R}_{\gamma_t}(m). \end{array} \tag{5.8}$$

We write $(i,s) \leadsto (j,u)$ to denote the presence of a path of potential influence with $\gamma(s) = i$ and $\gamma(u) = j$. If $T = \{0,1\}$ so that $T^\Lambda \cong \mathcal{P}(\Lambda)$, and all maps in $\mathcal{G}$ are additive, then paths of potential influence are the open paths in the percolation representation of Δ involving arrows and blocking symbols (see Sections 2.6 and 3.2), but the present setting is clearly much more general.

Lemma 25 (Exponential bound) *Assume that the rates $(r_m)_{m\in\mathcal{G}}$ satisfy (5.7) and that*

$$K := \sup_{i \in \Lambda} \sum_{\substack{m \in \mathcal{G} \\ \mathcal{D}(m) \ni i}} r_m\big(|\mathcal{R}_i(m)| - 1\big) < \infty. \tag{5.9}$$

Then, for each $j \in \Lambda$, one has

$$\mathbb{E}\big[\big|\{i \in \Lambda : (i,s) \rightsquigarrow (j,u)\}\big|\big] \leq e^{K(u-s)} \qquad (0 \leq s \leq u). \tag{5.10}$$

Assuming moreover that

$$K_1 := \sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G} \\ \mathcal{D}(m)\ni i}} r_m |\mathcal{R}_i(m)| < \infty, \tag{5.11}$$

the set

$$\big\{(m,t) \in \Delta_{s,u} : (i,t) \rightsquigarrow (j,u) \text{ for some } i \in \mathcal{D}(m)\big\} \tag{5.12}$$

is a.s. finite for all $s < u$ and $j \in \Lambda$.

Proof (sketch) Let $\zeta_t := \{i \in \Lambda : (i, u-t) \rightsquigarrow (j,u)\}$. Choose finite $\Lambda_n \uparrow \Lambda$ with $j \in \Lambda_n$, let $\rightsquigarrow_n$ denote the presence of a path of potential influence that stays in Λ_n, and set $\zeta^n_t := \{i \in \Lambda_n : (i, u-t) \rightsquigarrow_n (j,u)\}$. Then $(\zeta^n_t)_{t\geq 0}$ is a set-valued Markov process that jumps

$$A \mapsto \Lambda_n \cap \big(\bigcup_{i\in A} \mathcal{R}_i(m)\big) \tag{5.13}$$

with rate r_m. Note that by (5.5), the set $\bigcup_{i\in A} \mathcal{R}_i(m)$ equals A when $A \cap \mathcal{D}(m) = \emptyset$. Letting H_n denote the generator of the process $(\zeta^n_t)_{t\geq 0}$ and letting f denote the function $f(A) := |A|$, it is easy to check that $H_n f(A) \leq K f(A)$, where K is the constant from (5.9). It follows that $\frac{\partial}{\partial t} e^{-Kt}\mathbb{E}[|\zeta^n_t|] \leq 0$. Letting $\Lambda_n \uparrow \Lambda$, this proves (5.10).

To get also (5.12), note that in view of (5.13), the process ζ jumps $A \mapsto \bigcup_{i\in A} \mathcal{R}_i(m)$ with rate r_m. Couple ζ_t with a process $\zeta_t \subset \tilde{\zeta}_t$ that instead jumps as

$$A \mapsto A \cup \bigcup_{i\in A} \mathcal{R}_i(m) \tag{5.14}$$

with rate r_m. The condition (5.11) guarantees that the expected size of this process grows at most exponentially with rate K_1. Since $\tilde{\zeta}_t$ (unlike ζ_t) is nondecreasing in t, its finiteness at time u implies that it must be finite for all $0 \leq t \leq u$, which gives us (5.12). ∎

Conditions (5.7), (5.9) and (5.11) can be combined in the condition

$$\sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G} \\ \mathcal{D}(m)\ni i}} r_m\big(|\mathcal{R}_i(m)| + 1\big) < \infty. \tag{5.15}$$

Under this summability condition on the rates, Lemma 25 guarantees that the Poisson set Δ defines an a.s. unique Markov process. More precisely, in analogy with (2.11), let Γ be any finite subset of $\Delta_{s,u}$ such that

$$\Gamma \supset \big\{(m,t) \in \Delta_{s,u} : (i,t) \rightsquigarrow (j,u) \text{ for some } i \in \mathcal{D}(m)\big\}. \tag{5.16}$$

We unambiguously define random maps $\mathbf{X}_{s,u} : T^\Lambda \to T^\Lambda$ $(s \leq u)$ by

$$\begin{aligned} &\mathbf{X}_{s,u}(x)(j) := m_n \circ \cdots \circ m_1(x)(j) \\ &\quad\text{with} \quad \Gamma = \{(m_1,t_1),\ldots,(m_n,t_n)\}, \quad t_1 < \cdots < t_n. \end{aligned} \tag{5.17}$$

(Here, we implicitly use property (ii) of Lemma 24, which says that all information needed to determine $m(x)(i)$ is contained in $(x(j))_{j\in\mathcal{R}_i(m)}$.) By (5.12), the set on the right-hand side of (5.16) is a.s. finite, and by our definition of a path of potential influence, including more points in this set has no effect on the state of the underlying space point j at time u, which shows that $\mathbf{X}_{s,u}$ is well-defined.

As we sometimes already did in the finite setting, we continue to call the Poisson point set Δ used in the construction of the maps $(\mathbf{X}_{s,u})_{s\leq u}$ a *graphical representation.* In pictures, we plot the underlying space Λ horizontally, time upwards, and indicate the presence of a point $(m,t)\in\Delta$ by drawing a symbol (e.g., composed of arrows and blocking symbols) indicating the nature of the map m and the sites affected by m (i.e., those in $\mathcal{D}(m)$ and $\bigcup_{i\in\mathcal{D}(m)}\mathcal{R}_i(m)$).

To make full use of graphical representations, one needs to know that just as in the finite case (Lemma 1), they define Markov processes whose generator is given by (5.6). This will be proved in the next section.

5.3 Generator construction of particle systems

Let E be a compact metrizable space, let $\mathcal{C}(E)$ denote the Banach space of continuous real functions on E, equipped with the supremum norm $\|f\|:=\sup_{x\in E}|f(x)|$, and let $\mathcal{M}_1(E)$ denote the space of probability measures on E, equipped with the topology of weak convergence. By definition, a *continuous transition probability* on E is a collection $(P_t(x,\mathrm{d}y))_{t\geq 0}$ of probability kernels on E such that

(i) $(x,t)\mapsto P_t(x,\,\cdot\,)$ is a continuous map from $E\times[0,\infty)$ into $\mathcal{M}_1(E)$,

(ii) $\displaystyle\int_E P_s(x,\mathrm{d}y)P_t(y,\mathrm{d}z)=P_{s+t}(x,\mathrm{d}z)\quad\text{and}\quad P_0(x,\cdot\,)=\delta_x\quad(x\in E,\ s,t\geq 0),$

where δ_x denotes the delta-measure at x. Each continuous transition probability defines a strongly continuous semigroup $(P_t)_{t\geq 0}$ on the Banach space $\mathcal{C}(E)$ by

$$P_tf(x):=\int_E P_t(x,\mathrm{d}y)f(y)\qquad\big(f\in\mathcal{C}(E)\big).\tag{5.18}$$

Such a semigroup, arising from a continuous transition probability as above, is called a *Feller semigroup*. By definition, the *generator* of a strongly continuous semigroup, defined on a general Banach space, is the operator $\mathbf{G}$ defined by

$$\mathbf{G}f:=\lim_{t\downarrow 0}t^{-1}\big(P_tf-f\big),\tag{5.19}$$

where the domain $\mathcal{D}(\mathbf{G})$ of $\mathbf{G}$ consists of all f for which the limit exists w.r.t. the norm on the Banach space (in our case the supremum norm). By the Hille-Yosida theorem, such a generator is a closed linear operator. A linear operator G is called *closable* if there exists a linear operator $\overline{G}$, called the *closure* of G, such that $\{(f,\overline{G}f):f\in\mathcal{D}(\overline{G})\}$ is the closure of $\{(f,Gf):f\in\mathcal{D}(G)\}$. We cite the following fact from [EK86, Thms 4.2.2 and 4.2.7]. Below, one says that G satisfies the *positive maximum principle* if and only if $Gf(x)\leq 0$ whenever $f\in\mathcal{D}(G)$, $f(x)\geq 0$ and f assumes its maximum over E in x.

Proposition 26 (Generators of Feller semigroups) *A linear operator G on $\mathcal{C}(E)$ is closable and its closure generates a Feller semigroup if and only if:*

(i) $(1,0)$ *lies in the closure of* $\{(f,Gf):f\in\mathcal{D}(G)\}$.

(ii) $\mathcal{D}(G)$ *is dense in* $\mathcal{C}(E)$.

(iii) G *satisfies the positive maximum principle.*

(iv) *The range of* $\lambda-G$ *is dense in* $\mathcal{C}(E)$ *for some* $\lambda>0$.

Returning to the set-up of the previous section, we equip T^Λ with the product topology, making it into a compact metrizable space. For any continuous function $f : T^\Lambda \to \mathbb{R}$ and $i \in \Lambda$, we define

$$\delta f(i) := \sup\big\{|f(x) - f(y)| : x, y \in T^\Lambda,\ x(j) = y(j)\ \forall j \neq i\big\}. \tag{5.20}$$

Note that $\delta f(i)$ measures how much $f(x)$ can change if we change x only in the point i. We call δf the *variation* of f and we define a space of functions of 'summable variation' by

$$\mathcal{C}_{\rm sum} = \mathcal{C}_{\rm sum}(T^\Lambda) := \big\{f \in \mathcal{C}(T^\Lambda) : \sum_i \delta f(i) < \infty\big\}. \tag{5.21}$$

Theorem 27 (Poisson and generator constructions) *Under the summability condition (5.15), the generator G in (5.6) with domain $\mathcal{D}(G) := \mathcal{C}_{\rm sum}$ is well-defined and its closure generates a Feller semigroup $(P_t)_{t\geq 0}$. Moreover,*

$$\sum_{i\in\Lambda} \delta P_t f(i) \leq e^{Kt} \sum_{i\in\Lambda} \delta f(i) \qquad \big(t \geq 0,\ f \in \mathcal{C}_{\rm sum}\big), \tag{5.22}$$

where K is the constant from (5.9). Let Δ be a Poisson point subset of $\mathcal{G} \times \mathbb{R}$ with local intensity $r_m \mathrm{d}t$, and define random maps $(\mathbf{X}_{s,u})_{s\leq u}$ as in (5.17). Then

$$\mathbb{P}[\mathbf{X}_{s,u}(x) \in \cdot\,] = P_{u-s}(x, \cdot\,) \qquad (s \leq u,\ x \in T^\Lambda). \tag{5.23}$$

Proof We only sketch the main steps in the proof; for the details, we refer to [Swa11]. Taking (5.23) as the definition of $(P_t)_{t\geq 0}$, the first step is to verify that $(P_t)_{t\geq 0}$ is a continuous transition probability. The semigroup property (ii) is straightforward. To prove also the continuity property (i), it suffices to show that

$$(x_n, t_n) \underset{n\to\infty}{\longrightarrow} (x, t) \quad \text{implies} \quad \mathbf{X}_{-t_n,0}(x_n)(i) \underset{n\to\infty}{\longrightarrow} \mathbf{X}_{-t,0}(x)(i) \quad \text{a.s.} \quad (i \in \Lambda), \tag{5.24}$$

which is easily seen to follow from the finiteness of the set in (5.12).

Letting $\mathbf{G}$ denote the (full) generator of $(P_t)_{t\geq 0}$, the next step is to verify that $\mathcal{C}_{\rm sum} \subset \mathcal{D}(\mathbf{G})$ and $\mathbf{G}f$ is given by the right-hand side of (5.6) for such f. For functions that depend on finitely many coordinates, one can check directly from the definition that

$$\lim_{t\downarrow 0} t^{-1}\big(P_t f - f\big) = \sum_{m\in\mathcal{G}} r_m\big(f(m(x)) - f(x)\big) \qquad (x \in S). \tag{5.25}$$

For $f \in \mathcal{C}_{\rm sum}$, one can check that

$$\sum_{m\in\mathcal{M}} r_m\big|f(m(x)) - f(x)\big| \leq K_0 \sum_{i\in\Lambda} \delta f(i), \tag{5.26}$$

where K_0 is the constant from (5.7). This shows that the right-hand side of (5.6) is well-defined. A little argument shows that each $f \in \mathcal{C}_{\rm sum}$ can be approximated by f_n depending on finitely many coordinates such that $\|f_n - f\| \to 0$ and $\|Gf_n - Gf\| \to 0$, which shows that $\mathbf{G}$ coincides on $\mathcal{C}_{\rm sum}$ with the operator G defined in (5.6).

To prove (5.22), let $x, y \in T^\Lambda$ differ only at the underlying space point i and let $f \in \mathcal{C}_{\rm sum}$. Then

$$\begin{aligned}\big|P_t f(x) - P_t f(y)\big| &\leq \mathbb{E}\big[\big|f(\mathbf{X}_{0,t}(x)) - f(\mathbf{X}_{0,t}(y))\big|\big] \\ &\leq \sum_{j\in\Lambda} \mathbb{P}\big[\mathbf{X}_{0,t}(x)(j) \neq \mathbf{X}_{0,t}(x)(j)\big]\delta f(j) \leq \sum_{j\in\Lambda} \mathbb{P}\big[(i,0) \rightsquigarrow (j,t)\big]\delta f(j).\end{aligned} \tag{5.27}$$

By (5.10), it follows that

$$\sum_{i\in\Lambda}\delta P_t(i) \le \sum_{i,j\in\Lambda}\mathbb{P}\big[(i,0)\rightsquigarrow(j,t)\big]\delta f(j) \le e^{Kt}\sum_{i\in\Lambda}\delta f(i). \tag{5.28}$$

To show that the closure of G is the full generator of $(P_t)_{t\ge0}$, we check the conditions (i)–(iv) of Proposition 26. The conditions (i)–(iii) are simple and left to the reader. Using formula (5.22), one can check that for each $\lambda > K$ and $f \in \mathcal{C}_{\rm sum}$, the function

$$p := \int_0^\infty e^{-\lambda t} P_t f \,\mathrm{d}t. \tag{5.29}$$

satisfies $p \in \mathcal{C}_{\rm sum}$ and $(\lambda - G)p = f$. This shows that the range of $\lambda - G$ contains $\mathcal{C}_{\rm sum}$, which is dense in $\mathcal{C}(T^\Lambda)$, completing the proof. ∎

Remark 1 If the constant K from (5.9) is negative, then (5.22) can be used to show that the law of the Markov process with semigroup $(P_t)_{t\ge0}$ converges exponentially fast to a unique invariant law.

Remark 2 Liggett's classical result [Lig85, Thm I.3.9] gives sufficient conditions for the closure of an operator G on the Banach space $\mathcal{C}(T^\Lambda)$ to generate a Feller semigroup. Liggett writes his generators in the form

$$Gf(x) = \sum_{C\in\mathcal{K}}\int C(x,\mathrm{d}y)\big(f(y) - f(x)\big), \tag{5.30}$$

where $\mathcal{K}$ is a countable collection of finite measure kernels C on T^Λ, i.e., for fixed $x \in T^\Lambda$, $C(x,\,\cdot\,)$ is a finite measure on T^Λ that measures the intensity of jumps from x. Liggett assumes that all kernels $C \in \mathcal{K}$ are local, in the following sense. For any measure kernel C on T^Λ, write

$$\mathcal{D}(C) := \big\{i \in \Lambda : C\big(x, \{y : y(i) \neq x(i)\}\big) > 0 \text{ for some } x \in T^\Lambda\big\} \tag{5.31}$$

for the set of lattice points whose value can possibly be changed by the application of C. Then we say that C is *local* if $x \mapsto C(x,\,\cdot\,)$ is continuous w.r.t. weak convergence of finite measures and $\mathcal{D}(C)$ is finite.[6] In particular, if each kernel $C \in \mathcal{K}$ is deterministic, i.e., $C(x,\mathrm{d}y) = r_m\delta_{m(x)}(\mathrm{d}y)$ for some local map $m : T^\Lambda \to T^\Lambda$ and constant $r_m \ge 0$, then this includes our set-up. The conditions of [Lig85, Thm I.3.9] are similar to our condition (5.15), although it seems that for deterministic kernels, they do not completely agree with our condition. Also the conditions of Liggett's [Lig85, Thm I.4.1], proving exponential ergodicity for interacting particle systems, are similar, but not quite identical to the condition $K < 0$ mentioned in Remark 1 above.

Remark 3 Although the classical result [Lig85, Thm I.3.9] can be used to show that the closure of a generator of the form (5.6) generates a Feller semigroup, this does not immediately imply that the construction based on the graphical representation as in (5.17) is well-defined, nor, indeed, that it yields (through (5.23)) the *same* Feller semigroup. In some special cases, in particular, for additive particle systems as in Section 3.2, one may alternatively argue using approximation with finite systems, invoking [Lig85, Cor. I.3.14], but in general one needs Lemma 25 and Theorem 27 or equivalent results. Except for [Swa11], we do not know a good reference for these facts.

[6] Liggett labels his kernels by finite subsets of Λ, i.e., he writes $\sum_\Gamma \int C_\Gamma \ldots$ where the sum runs over all finite sets $\Gamma \subset \Lambda$ and C_Γ is a local probability kernel such that $\mathcal{D}(C) \subset \Gamma$. This particular way of labeling is not essential for his result.

Remark 4 Let X_0 be a T^Λ-valued random variable, independent of the graphical representation Δ, let $(\mathbf{X}_{s,u})_{s\leq u}$ be the random maps defined in (5.17), and set

$$X_t := \mathbf{X}_{0,t}(X_0) \qquad (t \geq 0). \tag{5.32}$$

Then, using the fact that the restrictions of a Poisson point set to disjoint parts of space are independent, it is easy to see that

$$\mathbb{P}\big[X_u \in \cdot \,\big|\, (X_s)_{0\leq s\leq t}\big] = P_{u-t}(X_t, \cdot\,) \quad \text{a.s.} \qquad (0 \leq t \leq u), \tag{5.33}$$

i.e., $(X_t)_{t\geq 0}$ is a Markov process with semigroup $(P_t)_{t\geq 0}$.

5.4 Monotone particle system duality

In the present section, we generalize Theorem 9 to monotonically representable interacting particle systems with state space of the form $S = T^\Lambda$, where T is a finite partially ordered set and Λ is countable. For reasons that will become clear below, we assume that T is bounded from above with upper bound denoted by 1. We equip T^Λ with the product order $x \leq y$ iff $x(i) \leq y(i)$ for all $i \in \Lambda$. Then T^Λ is also bounded from above, with upper bound 1 given by $1(i) := 1$ $(i \in \Lambda)$. If T' is a dual of T as defined in Section 2.4, then ${T'}^\Lambda$ is in a natural way a dual of T^Λ, where we define the map $x \mapsto x'$ in a pointwise way as $x'(i) := (x(i))'$. Since T is bounded from above, T' is bounded from below with lower bound $0 := 1'$. We also write 0 to denote the constant function $0(i) := 0$ $(i \in \Lambda)$ that is the lower bound of T'.

For any $x \in {T'}^\Lambda$ and $B \in \mathcal{P}\big({T'}^\Lambda\big)$, we let

$$\begin{aligned} \operatorname{supp}(x) &:= \{i \in \Lambda : x(i) \neq 0\}, \\ \operatorname{supp}(B) &:= \{i \in \Lambda : x(i) \neq 0 \text{ for some } x \in B\} = \bigcup_{x\in B} \operatorname{supp}(x) \end{aligned} \tag{5.34}$$

denote the "support" of x and B, respectively. We let

$${T'}^\Lambda_{\text{loc}} := \big\{x \in {T'}^\Lambda : |\operatorname{supp}(x)| < \infty\big\} \tag{5.35}$$

denote the set of finitely supported $x \in {T'}^\Lambda$, and write $\mathcal{P}_{\text{fin}}\big({T'}^\Lambda_{\text{loc}}\big)$ for the set of finite subsets of ${T'}^\Lambda_{\text{loc}}$. Equivalently,

$$\mathcal{P}_{\text{fin}}\big({T'}^\Lambda_{\text{loc}}\big) = \big\{B \in \mathcal{P}\big({T'}^\Lambda\big) : |\operatorname{supp}(B)| < \infty\big\}. \tag{5.36}$$

As the state spaces for the processes Y^* and $Y^\dagger$ from Theorem 9, we will choose

$$P_* := \mathcal{P}_{\text{fin}}({T'}^\Lambda_{\text{loc}}) \quad \text{and} \quad P_\dagger := \{B \in P_* : B = B_{\min}\}. \tag{5.37}$$

Note that these sets are countable, so the dual processes will be continuous-time Markov chains. For any local map $m : T^\Lambda \to T^\Lambda$ and $B \in P_*$, we define $m^\dagger(B)$ and $m^*(B)$ as in (2.31), i.e.,

$$m^\dagger(B)' := (m^{-1}(B'^{\downarrow}))_{\max} \quad \text{and} \quad m^*(B)' := \bigcup_{x\in B} (m^{-1}(\{x'\}^{\downarrow}))_{\max}, \tag{5.38}$$

where we use the notation $B' := \{y' : y \in B\}$. We define ϕ as in (2.29), i.e.,

$$\phi(x, B) := 1_{\{x \,\in\, B'^{\downarrow}\}} = 1_{\{x \leq y' \text{ for some } y \in B\}} \qquad \big(x \in S,\ B \in P_*\big). \tag{5.39}$$

Lemma 28 (Duals of monotone local maps) *Let T be a finite partially ordered set that is bounded from above, let Λ be countable, and let $m : T^\Lambda \to T^\Lambda$ be a local map that is monotone. Then the maps m^* and $m^\dagger$ defined in (5.38) map the space P_* into itself, and are dual to m with respect to the duality function ϕ from (5.39). Moreover, (2.32) and (2.33) hold for all $B, C \in P_*$. In particular, by (2.32), $m^\dagger$ maps P_* into $P_\dagger$.*

Proof Let us say that a set $A \subset \mathcal{P}(T'^\Lambda)$ is *locally defined* if there exists a finite set $\Gamma \subset \Lambda$ and a set $C \subset T'^\Gamma$ such that

$$A = C \times T'^{\Lambda\setminus\Gamma}. \tag{5.40}$$

For a set of this form, by Lemma 34

$$\big(C \times T'^{\Lambda\setminus\Gamma}\big)_{\min} = C_{\min} \times \{0\} \quad \text{and} \quad \big((C \times T'^{\Lambda\setminus\Gamma}\big)_{\min}\big)^\uparrow = C^\uparrow_{\min} \times \{0\}^\uparrow \supset C \times T'^{\Lambda\setminus\Gamma}, \tag{5.41}$$

where 0 here denotes the minimal element of $T'^{\Lambda\setminus\Gamma}$. In particular, this shows that for any locally defined increasing set A,

$$A_{\min} \in P_* \quad \text{and} \quad (A_{\min})^\uparrow = A. \tag{5.42}$$

Also, obviously, $B \in P_*$ implies that $B^\uparrow$ is locally defined. The proof of Lemma 8 now carries over without a change, where we use that since m is a local map, the map $n : T'^\Lambda \to T'^\Lambda$ defined in (4.14) has the property that

$$A \text{ locally defined implies } n^{-1}(A) \text{ locally defined} \qquad \big(A \in \mathcal{P}(T'^\Lambda)\big). \tag{5.43}$$

■

Let $\mathcal{G}$ be a collection of monotone local maps $m : T^\Lambda \to T^\Lambda$, let $(r_m)_{m\in\mathcal{G}}$ be nonnegative rates satisfying the summability condition (5.15), and let Δ be a graphical representation for the interacting particle system X with generator G as in (5.6), i.e., Δ is a Poisson point subset of $\mathcal{G} \times \mathbb{R} = \{(m, t) : m \in \mathcal{G},\ t \in \mathbb{R}\}$ with local intensity $r_m\mathrm{d}t$. Using Δ, we unambiguously define random maps $(\mathbf{X}_{s,u})_{s\le u}$ as in (5.17), which can be used to construct an interacting particle system $X = (X_t)_{t\ge 0}$ as in (5.32).

To show that X is pathwise dual to the continuous-time Markov chains Y^* and $Y^\dagger$ with countable state spaces P_* and $P_\dagger$,, respectively, and generators H_* and $H_\dagger$ as in (2.35), we need to construct stochastic flows for Y^* and $Y^\dagger$. As in (2.14), we define

$$\Delta^* := \{(m^*, -t) : (m, t) \in \Delta\} \quad \text{and} \quad \Delta^\dagger := \{(m^\dagger, -t) : (m, t) \in \Delta\}. \tag{5.44}$$

For $s \le u$ and $\sharp = *$ or $\dagger$, let $\Delta^\sharp_{s,u} := \{(m, t) \in \Delta^\sharp : s < t \le u\}$. We would like to define $\mathbf{Y}^\sharp_{s,u}$ as the concatenation of all maps in $\Delta^\sharp_{s,u}$, ordered by the time at which they apply, but just as for the forward process we run into the problem that $\Delta^\sharp_{s,u}$ a.s. has infinitely many events and we need to show that only finitely many of those are actually needed.

It turns out that this is indeed all right, and in fact guaranteed by the same condition (5.15) that guarantees that the forward process is well-defined. The proof (that we are about to give) is not difficult, but needs some notation, which is made more complicated by the fact that to obtain a right-continuous dual process, we need left-continuous modifications of certain objects that have already been introduced for the forward process.

Let us write $(j, t-) \rightsquigarrow (i, u-)$ to indicate the presence of a path of potential influence from j to i that starts just before time t and ends just before time u (i.e., events at time t are taken into account but events at time u are not). Next, for any $i \subset \Lambda$ and $s \le u \in \mathbb{R}$, we let (compare (5.12))

$$\Delta_{s,(i,u)} := \big\{(m, t) \in \Delta : s \le t < u,\ (j, t-) \rightsquigarrow (i, u-) \text{ for some } j \in \mathcal{D}(m)\big\} \tag{5.45}$$

denote the set of points in the Poisson set Δ that are relevant for $\mathbf{X}_{s-,u-}(x)(i)$, and we set

$$\Delta^*_{(i,s),u} := \big\{(m^*, -t) : (m,t) \in \Delta_{-u,(i,-s)}\big\}, \tag{5.46}$$

and define $\Delta^\dagger_{(i,s),u}$ similarly. We note that by Lemma 25 and our summability assumption (5.15), the sets $\Delta^\sharp_{(i,s),u}$ with $\sharp = *$ or $\dagger$ are a.s. finite for all $s \leq u$ and $i \in \Lambda$.

Lemma 29 (Well-definedness of the dual processes) *Let $\sharp = *$ or $\dagger$, $B \in P_\sharp$, $s \leq t$, and let Γ be any finite set such that*

$$\bigcup_{i \in \mathrm{supp}(B)} \Delta^\sharp_{(i,s),t} \subset \Gamma \subset \Delta^\sharp_{s,t}. \tag{5.47}$$

Then

$$\begin{aligned} \mathbf{Y}^\sharp_{s,t}(B) &:= m^\sharp_n \circ \cdots \circ m^\sharp_1(B) \\ \textit{with} \quad \Gamma &= \big\{(m_1,t_1),\ldots,(m_n,t_n)\big\}, \quad t_1 < \cdots < t_n. \end{aligned} \tag{5.48}$$

unambiguously defines a set $\mathbf{Y}^\sharp_{s,t}(B) \in P_\sharp$. Moreover,

$$\mathbb{E}\big[\big|\mathrm{supp}\big(\mathbf{Y}^\sharp_{s,t}(B)\big)\big|\big] \leq |\mathrm{supp}(B)|\, e^{K(t-s)}, \tag{5.49}$$

where K is the constant in (5.9).

Proof Fix $B \in P_\sharp$ and $s \in \mathbb{R}$, and for $t \geq s$, set

$$\zeta_t := \big\{j \in \Lambda : (j, (-t)-) \rightsquigarrow (i, (-s)-) \text{ for some } i \in \mathrm{supp}(B)\big\}, \tag{5.50}$$

where as in (5.45), $(j,(-t)-) \rightsquigarrow (i,(-s)-)$ indicates the presence of a path of potential influence from j to i that starts just before time $-t$ and ends just before time $-s$. By Lemma 25,

$$\mathbb{E}\big[|\zeta_t|\big] \leq |\mathrm{supp}(B)|\, e^{K(t-s)} \qquad (t \geq s). \tag{5.51}$$

We have seen in the proof of Lemma 25 (compare (5.13)) that for each $(m,-t) \in \Delta$, the set-valued Markov process ζ jumps from its present state A as

$$A \mapsto \bigcup_{i \in A} \mathcal{R}_i(m). \tag{5.52}$$

We will show that for any $B \in P_\sharp$,

$$\mathrm{supp}\big(m^\sharp(B)\big) \subset \bigcup_{i \in \mathrm{supp}(B)} \mathcal{R}_i(m) \tag{5.53}$$

and

$$\mathrm{supp}(B) \cap \mathcal{D}(m) = \emptyset \quad \text{implies} \quad m^\sharp(B) = B. \tag{5.54}$$

Using (5.53), we see by induction that regardless of how we choose the set Γ in (5.47), it will be true that

$$\mathrm{supp}\big(\mathbf{Y}^\sharp_{s,t}(B)\big) \subset \zeta_t \qquad (s \leq t), \tag{5.55}$$

so (5.49) follows from (5.51). By (5.54), the definition in (5.48) does not depend on the choice of Γ in (5.47).

In view of this, it remains to prove (5.53) and (5.54). We start with the former. By (2.32), which by Lemma 28 holds for all $B \in P_*$, we have $m^\dagger(B) \subset m^*(B)$ $(B \in P_*)$, so it suffices to prove (5.53) for m^* only. By (2.33), which by Lemma 28 holds for all $B \in P_*$, both the left- and right-hand side of (5.53) are additive as a function of B, so it suffices to prove the statement for one-point sets of the form $B = \{x\}$ with $x \in T'^{\Lambda}_{\rm loc}$. In this case,

$$m^*\big(\{x\}\big) = n^{-1}\big(\{x\}^{\uparrow}\big)_{\rm min}, \tag{5.56}$$

where n is defined in terms of m as in (4.14). In other words, this says that

$$m^*\big(\{x\}\big) = \big\{y \in T'^{\Lambda} : m(y')' \geq x\big\}_{\rm min} = \big\{y \in T'^{\Lambda} : m(y') \leq x'\big\}_{\rm min}. \tag{5.57}$$

For each $k \in \mathrm{supp}\big(m^*\big(\{x\}\big)\big)$ there exists a $z \in m^*\big(\{x\}\big)$ such that $z(k) \neq 0$. Define y by $y(k) := 0$ and $y(j) = z(j)$ for all $j \neq k$. Then, by (5.57), $m(z') \leq x'$ but $m(y') \not\leq x'$ by the minimality of z. It follows that $k \in \mathcal{R}_i(m)$ for some i such that $x'(i) \neq 1$, proving that

$$\mathrm{supp}\big(m^*(\{x\})\big) \subset \bigcup_{i \in \mathrm{supp}(\{x\})} \mathcal{R}_i(m). \tag{5.58}$$

To prove also (5.54), we observe that $\mathrm{supp}(B) \cap \mathcal{D}(m) = \emptyset$ implies $m^{-1}(B'^{\downarrow}) = B'^{\downarrow}$ and hence, by (5.38)

$$m^{\dagger}(B)' := (B'^{\downarrow})_{\rm max} \quad \text{which implies} \quad m^{\dagger}(B) = (B^{\uparrow})_{\rm min} = B_{\rm min} = B \qquad (B \in P_{\dagger}) \tag{5.59}$$

and similarly

$$m^*(B) := \bigcup_{x \in B} (\{x\}^{\uparrow})_{\rm min} = \bigcup_{x \in B} \{x\} = B \qquad (B \in P_*). \tag{5.60}$$

∎

Using Lemma 29, it is straightforward to check that $(\mathbf{Y}^{\sharp}_{s,u})_{s \leq u}$ is a stochastic flow with independent increments and that if B_0 is a $P_{\sharp}$-valued random variable, independent of the Poisson set $\Delta^{\sharp}$, then

$$Y^{\sharp}_t := \mathbf{Y}^{\sharp}_{0,t}(B_0) \qquad (t \geq 0) \tag{5.61}$$

defines an (obviously nonexplosive) Markov process $(Y^{\sharp}_t)_{t \geq 0}$ with countable state space $P_{\sharp}$ and generator $H_{\sharp}$ as in (2.35) for $\sharp = *$ or $\dagger$.

Proposition 30 (Pathwise duality for monotone particle systems) *Let T be a finite partially ordered set that is bounded from above, let Λ be countable, and let X be an interacting particle system whose generator has a random mapping representation of the form (5.6), where all local maps $m \in \mathcal{G}$ are monotone and the rates satisfy the summability condition (5.15). Let Δ be a graphical representation for X and define a stochastic flow $(\mathbf{X}_{s,u})_{s \leq u}$ as in (5.17). For $\sharp = *$ or $\dagger$, define a Poisson set $\Delta^{\sharp}$ as in (5.44) and use this to define random maps $(\mathbf{Y}^{\sharp}_{s,u})_{s \leq u}$ on the space $P_{\sharp}$ from (5.37) as in (5.48). Then $(\mathbf{Y}^{\sharp}_{s,u})_{s \leq u}$ is a stochastic flow that is dual to $(\mathbf{X}_{s,u})_{s \leq u}$ with respect to the duality function ϕ in (5.39), in the sense defined in Section 2.1. In particular, for each $s < u$, $x \in T^{\Lambda}$, and $B \in P_{\sharp}$, the function*

$$[s,u] \ni t \mapsto 1_{\{\mathbf{X}_{s,t-}(x) \leq z \text{ for some } z \in \mathbf{Y}^{\sharp}_{-u,-t}(B)\}} \tag{5.62}$$

is a.s. constant.

Proof This follows just as in the proof of Proposition 2, using Lemma 28, which says that $m^{\sharp}$ is dual to m with respect to the duality function ϕ in (5.39), and Lemmas 25 and 29, which show that only finitely many points of the Poisson set $\Delta_{s,u}$ are needed to define $\mathbf{X}_{s,t-}(x)$ and $\mathbf{Y}^{\sharp}_{-u,-t}(B)$ for all $t \in [s,u]$. ∎

5.5 Additive particle systems

In the present subsection, specializing from the set-up of the previous section, we look at interacting particle systems that are defined by additive local maps and whose state space is of the form T^Λ with T a finite lattice. As before, we equip T^Λ with the product order; then T^Λ is also a lattice, where $(x \vee y)(i) = x(i) \vee y(i)$ and $(x \wedge y)(i) = x(i) \wedge y(i)$. We let T' denote a dual of T so that T'^Λ is in a natural way a dual of T^Λ. We recall from (2.24) that

$$\langle x, y\rangle := 1_{\{x \leq y'\}} = 1_{\{y \leq x'\}} \qquad (x \in T^\Lambda,\ y \in T'^\Lambda). \tag{5.63}$$

The following lemma generalizes Lemmas 6 and 11 to infinite product spaces.

Lemma 31 (Additive local maps) *Let T be a finite lattice, let T' be its dual, and let Λ be a countable set. Then, for each additive local map $m : T^\Lambda \to T^\Lambda$ there exists a unique additive local map $m' : T'^\Lambda \to T'^\Lambda$ such that*

$$\langle m(x), y\rangle = \langle x, m'(y)\rangle \qquad (x \in T^\Lambda,\ y \in T'^\Lambda), \tag{5.64}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is as defined in (2.24). For $\sharp = *$ or $\dagger$, let $m^\sharp : P_\sharp \to P_\sharp$ be defined as in (5.38). Then

$$m^*(B) = \{m'(y) : y \in B\} \quad \textit{and} \quad m^\dagger(B) = m^*(B)_{\min} \tag{5.65}$$

for all $B \in P_*$ resp. $B \in P_\dagger$.

Proof Let $\tilde\Lambda := \mathcal{D}(m) \cup \bigcup_{i \in \mathcal{D}(m)} \mathcal{R}_i(m)$. Then $\tilde\Lambda$ is a finite set by the definition of a local map and Lemma 24. Moreover, by the same lemma, there exists an additive map $\tilde m : T^{\tilde\Lambda} \to T^{\tilde\Lambda}$ such that

$$m(x)(i) = \begin{cases} \tilde m\big((x(j))_{j \in \tilde\Lambda}\big)(i) & \text{if } i \in \tilde\Lambda, \\ x(i) & \text{otherwise.} \end{cases} \tag{5.66}$$

By Lemma 6, $\tilde m$ has a unique dual map $\tilde m'$, which is also additive. Lifting this map to the larger space yields a local map m' as in (5.64). Since knowing $\langle x, m'(y)\rangle$ for all $x \in T^\Lambda$ determines $m'(y)$ uniquely, such a map is unique.

The proof of formula (5.65) is the same as in the finite case (Lemma 11). ∎

Let $\mathcal{G}$ be a collection of additive local maps $m : T^\Lambda \to T^\Lambda$ and let $(r_m)_{m \in \mathcal{G}}$ be nonnegative rates. For each $m \in \mathcal{G}$, let m' denote the dual map as in Lemma 31. We will be interested in the interacting particle systems X and Y with state space T^Λ and T'^Λ, respectively, and generators

$$\begin{aligned} Gf(x) &= \sum_{m \in \mathcal{G}} r_m\big(f(m(x)) - f(x)\big) \qquad (x \in T^\Lambda, f \in \mathcal{F}(T^\Lambda, \mathbb{R})), \\ Hf(y) &= \sum_{m \in \mathcal{G}} r_m\big(f(m'(y)) - f(y)\big) \qquad (y \in T'^\Lambda, f \in \mathcal{F}(T'^\Lambda, \mathbb{R})). \end{aligned} \tag{5.67}$$

Note that the random mapping representation of the dual generator H satisfies the summability condition (5.15) if and only if

$$\sup_{i \in \Lambda} \sum_{\substack{m \in \mathcal{G} \\ \mathcal{D}(m') \ni i}} r_m\big(|\mathcal{R}_i(m')| + 1\big) < \infty. \tag{5.68}$$

We let Δ be a graphical representation for X, i.e., a Poisson point subset of $\mathcal{G} \times \mathbb{R} = \{(m, t) : m \in \mathcal{G},\ t \in \mathbb{R}\}$ with local intensity $r_m \mathrm{d}t$, and for $s \leq u$, set $\Delta_{s,u} := \Delta \cap (\mathcal{G} \times (s, u])$. Since S is a finite lattice, it is bounded from below. In analogy with (5.34), for any $x \in T^\Lambda$, we write $\mathrm{supp}(x) := \{i \in \Lambda : x(i) \neq 0\}$, and similar to (5.35), we define

$$T^\Lambda_{\mathrm{loc}} := \big\{x \in T^\Lambda : |\mathrm{supp}(x)| < \infty\big\}. \tag{5.69}$$

We make the following observation.

Lemma 32 (Finite and infinite initial states) *Let $s \le u$ and let Γ_k be finite sets such that $\Gamma_k \uparrow \Delta_{s,u}$. If the rates $(r_m)_{m\in\mathcal{G}}$ satisfy the summability condition (5.15), then we can unambiguously define a random map $\mathbf{X}_{s,u} : T^\Lambda \to T^\Lambda$ by requiring that for all k large enough,*

$$\begin{aligned} &\mathbf{X}_{s,u}(x)(j) := m_n \circ \cdots \circ m_1(x)(j) \\ &\quad \text{with} \quad \Gamma_k = \{(m_1,t_1),\ldots,(m_n,t_n)\}, \quad t_1 < \cdots < t_n. \end{aligned} \tag{5.70}$$

Similarly, if the dual summability condition (5.68) is satisfied, then we can unambiguously define a random map $\mathbf{X}_{s,u} : T^\Lambda_{\rm loc} \to T^\Lambda_{\rm loc}$ such that (5.70) holds for all k large enough.

Proof If the rates $(r_m)_{m\in\mathcal{G}}$ satisfy the summability condition (5.15), then the statement follows from Lemma 25. (See (5.17).)

To prove the second statement, we may equivalently show that if the rates $(r_m)_{m\in\mathcal{G}}$ satisfy the summability condition (5.15), then the dual process Y with generator H as in (5.67) is well-defined as a ${T'}^\Lambda_{\rm loc}$-valued process. In analogy with (2.14), set

$$\Delta' := \big\{(m',-t) : (m,t) \in \Delta\big\}. \tag{5.71}$$

By formula (5.65), the map m^* maps the space of all singleton-sets $\{y\}$ with $y \in {T'}^\Lambda_{\rm loc}$ into itself, and

$$m^*\big(\{y\}\big) = \{m'(y)\} \qquad (y \in {T'}^\Lambda_{\rm loc}). \tag{5.72}$$

In view of this, the maps $(\mathbf{Y}^*_{s,u})_{s\le u}$ defined in Lemma 29 also map such singletons into singletons. Defining $\Delta'_{(i,s),u}$ similar to (5.46), by grace of Lemma 29, we can unambiguously define maps $(\mathbf{Y}_{s,u})_{s\le u}$ on the space ${T'}^\Lambda_{\rm loc}$ by

$$\begin{aligned} &\mathbf{Y}_{s,u}(y) := m'_n \circ \cdots \circ m'_1(y) \\ &\quad \text{with} \quad \Gamma = \big\{(m_1,t_1),\ldots,(m_n,t_n)\big\}, \quad t_1 < \cdots < t_n, \end{aligned} \tag{5.73}$$

where Γ is any finite set such that

$$\bigcup_{i\in\mathrm{supp}(y)} \Delta'_{(i,s),u} \subset \Gamma \subset \Delta'_{s,u}. \tag{5.74}$$

In particular, $\Gamma = \Gamma_k$ for k large enough will do. ∎

Lemma 32 tells us that under mild technical assumptions, an additively representable interacting particle system X has the property that $X_0 \in T^\Lambda_{\rm loc}$ implies that a.s. $X_t \in T^\Lambda_{\rm loc}$ for all $t \ge 0$. Indeed, a sufficient condition for X to be well-defined as a $T^\Lambda_{\rm loc}$-valued process is that its rates $(r_m)_{m\in\mathcal{G}}$ satisfy the dual summability condition (5.68), which is the condition under which we have shown that the dual process Y is well-defined as a ${T'}^\Lambda$-valued process. If both the forward and dual summability conditions (5.15) and (5.68) are satisfied, then both processes can be constructed for arbitrary initial states.

Proposition 33 (Duality for additive particle systems) *Let T be a finite lattice and let Λ be a countable set. Let $\mathcal{G}$ be a collection of additive local maps $m : T^\Lambda \to T^\Lambda$ and let $(r_m)_{m\in\mathcal{G}}$ be nonnegative rates satisfying the summability condition (5.15). Let $(\mathbf{X}_{s,t})_{s\le t}$ and $(\mathbf{Y}_{s,t})_{s\le t}$ be the stochastic flows defined in (5.70) and (5.73), acting on the state spaces T^Λ and ${T'}^\Lambda_{\rm loc}$, respectively, which correspond to Markov processes X and Y with generators G and H as in (5.67). Then $(\mathbf{X}_{s,t})_{s\le t}$ and $(\mathbf{Y}_{s,t})_{s\le t}$ are dual with respect to the duality function $\psi(x,y) := \langle x,y\rangle$, in the sense defined in Section 2.1. In particular, for each $x \in T^\Lambda$, $y \in {T'}^\Lambda_{\rm loc}$, and $s \le u$, the function*

$$[s,u] \mapsto \langle \mathbf{X}_{s,t-}(x), \mathbf{Y}_{-u,-t}(y)\rangle \tag{5.75}$$

is a.s. constant. If moreover the dual summability condition (5.68) is satisfied, then the same statements hold for the stochastic flow $(\mathbf{Y}_{s,t})_{s\le t}$ on the larger state space ${T'}^\Lambda$ (instead of ${T'}^\Lambda_{\rm loc}$).

Proof As explained in the proof of Lemma 32, by (5.65), for $y \in {T'}^{\Lambda}_{\rm loc}$, the dual map $\mathbf{Y}_{s,t}$ is a special case of the dual map $\mathbf{Y}^*_{s,t}$ from the previous subsection, so the result follows from Proposition 30. If moreover the dual summability condition (5.68) is satisfied, then by Lemma 32, $\mathbf{Y}_{s,t}$ is also well-defined on ${T'}^{\Lambda}$. For any $y \in {T'}^{\Lambda}$, we can find $y_n \in {T'}^{\Lambda}_{\rm loc}$ that increase to y. It is not hard to see that this implies that $\mathbf{Y}_{s,t}(y_n)$ increases to $\mathbf{Y}_{s,t}(y)$ and hence, for all $t \in [s,u]$,

$$\langle \mathbf{X}_{s,t-}(x), \mathbf{Y}_{-u,-t}(y_n)\rangle = 1_{\{\mathbf{Y}_{-u,-t}(y_n) \leq \mathbf{X}'_{s,t-}(x)\}} \downarrow \langle \mathbf{X}_{s,t-}(x), \mathbf{Y}_{-u,-t}(y)\rangle \tag{5.76}$$

so the limit is a.s. constant as a function of t. ∎

A A bit of lattice theory

In this appendix we collect some elementary properties of partially ordered sets and lattices that are used in the paper.

Let S be a any set. Recall that a relation $\leq$ on S is called a *partial order* if it satisfies the axioms

(i) $x \leq x$ $(x \in S)$.

(ii) $x \leq y$ and $y \leq x$ implies $x = y$ $(x, y \in S)$.

(iii) $x \leq y \leq z$ implies $x \leq z$ $(x, y \in S)$.

A *partially ordered set* (also called *poset*) is a set with a partial order defined on it. A *total order* is a partial order such that moreover

(iv) $x \leq y$ or $y \leq x$ (or both) $(x, y \in S)$.

Increasing and decreasing sets and the notation $A^{\uparrow}$ and $A^{\downarrow}$ have already been defined in Section 2.3.

By definition, a *minimal* element of a set $A \subset S$ is an $x \in A$ such that $\{y \in A : y \leq x\} = \{x\}$. *Maximal* elements are minimal elements for the reversed order. The following simple observation is well-known.

Lemma 34 (Maximal elements) *Let S be a partially ordered set and let $A \subset S$ be finite. Then, for every $x \in A$ there exists a maximal element $y \in A$ such that $y \geq x$.*

Proof Either x is a maximal element or there exists some $x' \in A$, $x' \neq x$, such that $x' \geq x$. Continuing this process, using the finiteness of A, we arrive after a finite number of steps at a maximal element. ∎

In particular, Lemma 34 shows that every nonempty finite $A \subset S$ has a maximal element. Applying this to the reversed order, we see that A also contains a minimal element.

Let S be a partially ordered set and $A \subset S$. An *upper bound* of A is an element $x \in S$ such that $y \leq x$ for all $y \in A$. Note that in general x does not have to be an element of A. If there exists an element $x \in A$ that is an upper bound of A, then such an element is necessarily unique. (Indeed, imagine that $x' \in A$ is also an upper bound for A, then $x' \leq x$ and $x \leq x'$.)

If A is decreasing and there exists an element $x \in A$ that is an upper bound of A, then $A = \{x\}^{\downarrow}$. Indeed, the fact that x is an upper bound means that $A \subset \{x\}^{\downarrow}$ while by the facts that $x \in A$ and A is decreasing, we have $A \supset \{x\}^{\downarrow}$. A *least upper bound* of A is an element $x \in S$ that is an upper bound of A and that satisfies $x \leq x'$ for any (other) upper bound x' of A. Note that $\bigcap_{x \in A}\{x\}^{\uparrow}$ is the set of all upper bounds of A, which is an increasing set. Thus, a least upper bound of A is an element $x \in \bigcap_{x' \in A}\{x'\}^{\uparrow}$ that is a lower bound of

$\bigcap_{x' \in A}\{x'\}^{\uparrow}$. By our earlier remarks, such an element is unique. Also, since $\bigcap_{x' \in A}\{x'\}^{\uparrow}$ is an increasing set, if there exists an element $x \in \bigcap_{x' \in A}\{x'\}^{\uparrow}$ that is a lower bound of $\bigcap_{x' \in A}\{x'\}^{\uparrow}$, then $\bigcap_{x' \in A}\{x'\}^{\uparrow} = \{x\}^{\uparrow}$.

A partially ordered set S is a join-semilattice if and only if for each $y, z \in S$, the set $\{y, z\}$ has a least upper bound. Equivalently, this says that there exists an element $x \in \{y\}^{\uparrow} \cap \{z\}^{\uparrow}$ that is a lower bound of $\{y\}^{\uparrow} \cap \{z\}^{\uparrow}$. By our earlier remarks, such an element is unique and one must in fact have $\{y\}^{\uparrow} \cap \{z\}^{\uparrow} = \{x\}^{\uparrow}$. We denote this unique element by $x =: y \vee z$ and call it the *supremum* or *join* of x and y. Note that this (more traditional) definition of suprema and join-semilattices is equivalent to the definition in (2.20). Infima and meet-semilattices are defined in the same way, but with respect to the reversed order.

Each finite join-semilattice S is clearly bounded from above, since $\bigvee_{x \in S} x$ is an upper bound; similarly finite meet-semilattices are bounded from below.

As already mentioned in Section 2.3, if S is a partially ordered set, then a nonempty increasing set $A \subset S$ such that for every $x, y \in A$ there exists a $z \in A$ with $z \leq x, y$ is called a *filter* and a nonempty decreasing set A such that for every $x, y \in A$ there exists a $z \in A$ with $x, y \leq z$ is called an *ideal*.

The following lemma characterizes ideals in join-semilattices. An analogue statement holds for filters. In a lattice-theoretic setting, this lemma is often taken as the definition of an ideal.

Lemma 35 (Ideals in join-semilattices) *Let S be a join-semilattice and let $A \subset S$. Then A is an ideal if and only if A is nonempty, decreasing, and $x, y \in A$ imply $x \vee y$ in A.*

Proof Let A be nonempty and decreasing. If $x, y \in A$ imply $x \vee y$ in A, then for every $x, y \in A$ there is an element of A, namely $x \vee y$, such that $x \vee y \geq x, y$, showing that A is an ideal. Conversely, if A is an ideal, then for every $x, y \in A$ there is an element $z \in A$ such that $z \geq x, y$ and hence $z \geq x \vee y$, which by the fact that A is decreasing proves that $x \vee y \in A$. ■

As already mentioned in Section 2.3, a *principal filter* is a filter that contains a minimal element and a *principal ideal* is an ideal that contains a maximal element. We state the following facts for ideals only; applying them to the reversed order yields analogue statements for filters.

Lemma 36 (Principal ideals) *Let S be a partially ordered set and let $A \subset S$. Then:*

1. *If A is an ideal, then A contains at most one maximal element.*
2. *A is a principal ideal if and only if there exists some $z \in S$ such that $A = \{z\}^{\downarrow}$.*

If A is moreover finite, then the following conditions are equivalent:

(i) *A is a principal ideal.*

(ii) *A is an ideal.*

(iii) *A is decreasing and contains a unique maximal element.*

Proof 1. If A is an ideal and $x, y \in A$ are maximal, then there exists some $z \in A$ such that $z \geq x, y$ and hence $x = z = y$ by the definition of maximality.

2. Let A be a principal ideal with maximal element z. Then $\{z\}^{\downarrow} \subset A$ by the fact that A is decreasing. The inclusion $\{z\}^{\downarrow} \supset A$ follows by observing that if $x \in A$, then by the definition of an ideal there exists some $y \in A$ with $y \leq x, z$, and hence $y = z$ by the maximality of z, so $x \leq y = z$. Conversely, it is straightforward to check that each set of the form $\{z\}^{\downarrow}$ is a principal ideal.

The implications (i)⇒(ii) and (i)⇒(iii) are trivial. Assume that $A \subset S$ is finite. Then (ii)⇒(i) by Lemma 34. We claim that moreover (iii)⇒(i). Let z be the unique maximal element of A. By 2, it suffices to prove that $A = \{z\}^{\downarrow}$. Since A is decreasing, we have $\{z\}^{\downarrow} \subset A$. To prove the other inclusion, we observe that for any $x \in A$, by Lemma 34, there exists a maximal element $y \in A$ such that $y \geq x$. By assumption, z is the unique maximal element of A, so $y = z$ and hence $x \leq z$, proving that $x \in \{z\}^{\downarrow}$. ∎

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References

[AS05] S.R. Athreya and J.M. Swart. Branching-coalescing particle systems. *Prob. Theory Relat. Fields.* 131(3) (2005), 376–414.

[Bir37] G. Birkhoff. Rings of sets. *Duke Mathematical J.* 3(3) (1937), 443–454.

[CGGR15] G. Carinci, C. Giardinà, C. Giberti, F. Redig. Dualities in population genetics: a fresh look with new dualities, *Stochastic Process. Appl.* 125 (2015), no. 3, 941–969.

[DP02] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order* (2nd ed.) Cambridge University Press, Cambridge, 2002.

[EK86] S.N. Ethier and T.G. Kurtz. *Markov processes: characterization and convergence* Wiley & sons, 1986.

[FM01] J.A. Fill and M. Machida. Stochastic monotonicity and realizable monotonicity *Ann. Probab.* 29(2) (2001), 938–978.

[GKRV09] C. Giardinà, J. Kurchan, F. Redig, and K. Vafayi. Duality and hidden symmetries in interacting particle systems. *J. Stat. Phys.* 135(1) (2009), 25–55.

[Gra86] L. Gray Duality for general attractive spin systems with applications in one dimension. *Ann. Probab.* 14(2) (1986), 371–396.

[Gri79] D. Griffeath. *Additive and Cancellative Interacting Particle Systems.* Lecture Notes in Math. 724, Springer, Berlin, 1979.

[Har78] T.E. Harris. Additive set-valued Markov processes and graphical methods. *Ann. Probab.* 6 (1978), 355–378.

[JK14] S. Jansen and N. Kurt. On the notion(s) of duality for Markov processes. *Prob. Surveys* 11 (2014), 59–120.

[KKO77] T. Kamae, U. Krengel and G.L. O'Brien. *Stochastic inequalities on partially ordered spaces Ann. Probab.* 5(6) (1977), 899–912.

[Kro99] S. Krone. The two-stage contact process. *Ann. Appl. Probab.* 9(2), (1999), 331–351.

[Lig85] T.M. Liggett. *Interacting Particle Systems.* Springer-Verlag, New York, 1985.

[LPW09] D.A. Levin, Y. Peres and E.L. Wilmer. *Markov Chains and Mixing Times.* AMS, Providence, 2009.

[Neu94] C. Neuhauser. A long range sexual reproduction process. *Stochastic Processes Appl.* 53, 193–220, 1994.

[Nob92] C. Noble. Equilibrium behavior of the sexual reproduction process with rapid diffusion. *Ann. Probab.* 20(2), 724–745, 1992.

[Sie76] D. Siegmund. The equivalence of absorbing and reflecting barrier problems for stochastically monotone Markov processes. *Ann. Probab.* 4(6) (1976), 914–924.

[SL95] A. Sudbury and P. Lloyd. Quantum operators in classical probability theory. II: The concept of duality in interacting particle systems. *Ann. Probab.* 23(4) (1995), 1816–1830.

[SL97] A. Sudbury and P. Lloyd. Quantum operators in classical probability theory. IV: Quasi-duality and thinnings of interacting particle systems *Ann. Probab.* 25(1) (1997), 96–114.

[SS14] A. Sturm and J.M. Swart. A particle system with cooperative branching and coalescence. *Ann. Appl. Probab.* 25(3) (2015), 1616–1649.

[Sud00] A. Sudbury. Dual families of interacting particle systems on graphs. *J. Theor. Probab.* 13(3) (2000), 695–716.

[Swa11] J.M. Swart. *Interacting Particle Systems.* Lecture notes for the course NSTP190 at Charles University, Prague, 2011.

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