Pathwise uniqueness for a SDE with non-Lipschitz coefficients

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Abstract

We consider the ordinary stochastic differential equation $dX = -cXdt + \sqrt{2(1-|X|^2)}dB$ on the closed unit ball E in $\mathbb{R}^n$. While it is easy to prove existence and distribution uniqueness for solutions of this SDE for each $c \geq 0$, pathwise uniqueness can be proved by standard methods only in dimension $n = 1$ and in dimensions $n \geq 2$ if $c = 0$ or if $c \geq 2$ and the initial condition is in the interior of E. We sharpen these results by proving pathwise uniqueness for $c \geq 1$. More precisely, we show that for X^1, X^2 solutions relative to the same Brownian motion, the function $t \mapsto |X^1(t) - X^2(t)|^2 + |\sqrt{1-|X^1(t)|^2} - \sqrt{1-|X^2(t)|^2}\,|^2$ is almost surely nonincreasing. Whether or not pathwise uniqueness holds in dimensions $n \geq 2$ for $0 < c < 1$ is still open.

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1 Introduction

1.1 Motivation and introduction of the problem

It is known that pathwise uniqueness holds for the n-dimensional stochastic differential equation (SDE)

$$dX_i(t) = b_i(X(t))dt + \sum_{j=1}^{m} \sigma_{i,j}(X)dB_j(t) \qquad (t \geq 0,\ i = 1, \ldots, n). \tag{1.1}$$

(where B is m-dimensional Brownian motion) if the drift b is Lipschitz continuous and the diffusion coefficient σ is Lipschitz continuous (in dimensions $n \geq 2$) or Hölder-$\frac{1}{2}$-continuous (in dimension $n = 1$, see Yamada and Watanabe (1971a)). These results are sharp, in some sense: for each $\gamma < 1$ and $n \geq 2$ there exists an n-dimensional SDE of the form $dX = \sigma(X)dB$ for which distribution uniqueness does not hold (and

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hence pathwise uniqueness also fails), while σ is Hölder-continuous with exponent γ (see Yamada and Watanabe (1971b) and Swart (2001)).

On the other hand, there exist quite a number of SDE's in dimensions $n \geq 2$ with non-Lipschitz diffusion coefficients, which are known or believed to have a unique solution. If uniqueness is known, then mostly only in distribution. See den Hollander and Swart (1998) and Swart (1999) for some examples of SDE's for which distribution uniqueness is open. Often, such SDE's are defined on a domain with a boundary, and (a component of) the square of the diffusion coefficient (i.e., the matrix $\sigma\sigma^\top$) vanishes at the boundary and has a positive slope there. In the present paper we focus on pathwise uniqueness for one example of such a SDE. Although this SDE has some special features that will facilitate our analysis, the difficulties one encounters in proving pathwise uniqueness are typical for many other SDE's with a boundary.

We consider the SDE

$$dX_i(t) = -cX_i(t)dt + \sqrt{2(1-|X(t)|^2)}dB_i(t) \qquad (t \geq 0,\ i = 1, \ldots, n), \tag{1.2}$$

where $c \geq 0$. A (weak) solution of (1.2) is a process $X = (X_1, \ldots, X_n)$ with sample paths in the space $\mathcal{C}_E[0,\infty)$ of continuous functions from $[0,\infty)$ to the closed unit ball $E := \{x \in \mathbb{R}^n; |x| \leq 1\}$, together with an n-dimensional Brownian motion $B = (B_1, \ldots, B_n)$, such that (1.2) holds in integral form (see, for example, chapter 5 in Ethier & Kurtz (1986)). B and X are adapted processes on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ containing the P-null sets. We write $|x| := \sqrt{\sum_{i=1}^n x_i^2}$ for the Euclidean norm of a vector $x \in \mathbb{R}^n$. One can view equation (1.2) as a multi-dimensional analogue (perhaps not the most natural) of the one-dimensional Wright-Fisher diffusion with migration, which occurs in population biology.

Note that the function $x \mapsto \sqrt{2(1-|x|^2)}$ is Hölder-$\frac{1}{2}$-continuous but not Lipschitz continuous at the boundary of E, so that the results mentioned at the start give pathwise uniqueness for the SDE (1.2) only in dimension $n = 1$.

The organization of the paper is as follows. In Sections 1.2–1.4 we show existence and distribution uniqueness of solutions of (1.2), and investigate what can be proved about their pathwise uniqueness by standard techniques. In Section 2 we present our new results. Section 3 contains proofs.

Some remarks on notation. If $U \subset \mathbb{R}^n$, we write U° for its interior and $\overline{U}$ for its closure. We call $\partial U := \overline{U} \backslash U^\circ$ its boundary. If U is (closed and) the closure of its interior, we denote by $\mathcal{C}^n(U)$ the class of real functions on U that can be extended to a function in $\mathcal{C}^n(\mathbb{R}^n)$, and we use this extension to define partial derivatives of such a function as continuous functions on U. Since U is the closure of its interior, the result does not depend on the choice of the extension.

We write 'a.s. $\forall t$' behind a formula to denote the existence, for all t, of a measurable set Ω^*, depending on t, such that $P[\Omega^*] = 1$ and the formula holds for all $\omega \in \Omega^*$. We write '$\forall t$ a.s.' when Ω^* can be chosen independent of t.

1.2 Existence and distribution uniqueness

For each probability measure μ on E and for each $c \geq 0$, equation (1.2) has a solution with initial condition $\mathcal{L}(X(0)) = \mu$, and this solution is unique in distribution. (In other words: (1.2) has a unique weak solution with initial condition μ.)

Existence of a (weak) solution follows from standard results. Let A be the linear operator

$$Af(x) := -c\sum_{i=1}^{n} x_i \tfrac{\partial}{\partial x_i} f(x) + (1-|x|^2)\sum_{i=1}^{n} \tfrac{\partial^2}{\partial x_i{}^2} f(x) \tag{1.3}$$

with domain $\mathcal{D}(A) := \mathcal{C}^2(E)$. Standard results (see Ethier & Kurtz (1986), Theorem 5.4 and Problem 19 from Chapter 4) show that there exists, for each probability measure μ on E, a solution X to the martingale problem for A with initial condition $\mathcal{L}(X(0)) = \mu$. Here a solution to the martingale problem for A is a process $(X(t))_{t\geq 0}$ with continuous sample paths, such that for every function $f \in \mathcal{D}(A)$ the process $(M(t))_{t\geq 0}$, given by

$$M(t) := f(X(t)) - \int_0^t Af(X(s))ds \tag{1.4}$$

is a martingale with respect to the filtration generated by X. Each solution X to the martingale problem for A has a version (not necessarily defined on the same probability space as X) that is a weak solution of the SDE (1.2) (see Ethier & Kurtz (1986), Theorem 3.3 from Chapter 5).

Weak uniqueness for equation (1.2) can be proved by a moment calculation. For $x \in \mathbb{R}^n$ and $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$, write $x^p := \prod_{i=1}^n x_i^{p_i}$ and set $|p| := \sum_{i=1}^n p_i$. Using Itô's formula (or, alternatively, the fact that X solves the martingale problem for A) one can show that the moments $E[X^p(t)]$ are continuously differentiable functions of t, solving a system of differential equations of the form

$$\tfrac{\partial}{\partial t} E[X^p(t)] = \sum_{q\in\mathbb{N}^n : |q|\leq|p|} \lambda(p,q) E[X^q(t)], \tag{1.5}$$

where the $\lambda(p,q)$ are constants depending only on p and q. For each fixed $n \in \mathbb{N}$, the equations (1.5) with $|p| \leq n$ form a finite-dimensional system of linear differential equations, which has a unique solution for each initial condition. Since E is bounded, the moments of $X(t)$ determine its distribution, and hence two solutions X^1, X^2 of (1.2) with $\mathcal{L}(X^1(0)) = \mathcal{L}(X^2(0))$ satisfy $\mathcal{L}(X^1(t)) = \mathcal{L}(X^2(t))$ for all $t \geq 0$. This implies weak uniqueness for equation (1.2).

1.3 Pathwise uniqueness

In dimension $n = 1$, pathwise uniqueness (also called strong uniqueness) holds for the SDE (1.2) by the result of Yamada and Watanabe (1971a). Their result is not applicable in dimensions $n \geq 2$, but since the function $x \mapsto \sqrt{2(1-|x|^2)}$ is locally Lipschitz on the interior of E, the classical result of Itô gives pathwise uniqueness up to the stopping time

$$\tau := \inf\{t \geq 0 : |X(t)| = 1\}, \tag{1.6}$$

i.e., $X^1(t\wedge\tau) = X^2(t\wedge\tau)$ a.s. for all $t \geq 0$ and for any two solutions X^1, X^2 to (1.2) relative to the same Brownian motion with $X^1(0) = X^2(0)$ a.s.

For $c = 0$, solutions of (1.2) are martingales, and this together with the strict convexity of E implies that $X^\alpha(t) = X^\alpha(\tau)$ on $t \geq \tau$, a.s. ($\alpha = 1, 2$), so that in this case pathwise uniqueness holds for the SDE (1.2).

The previous argument also makes clear that pathwise uniqueness holds for the SDE (1.2) if the stopping time τ is almost surely infinite. The following proposition describes the behavior of solutions to (1.2) near the boundary of E.

Proposition 1 *Let P^x be the law of the solution of the SDE (1.2) starting in $X(0) = x$, and let τ be the stopping time in (1.6). Then*

$$\begin{array}{lll} E^x[\,\tau\,] < \infty & \forall x \in E & \text{if } 0 \leq c < 2 \\ P^x[\tau = \infty] = 1 & \forall x \in E^\circ & \text{if } c \geq 2. \end{array} \tag{1.7}$$

If $0 < c < 2$, then there exists a random function $(t, x) \mapsto L_t(x)$ (local time of the process $|X|$) such that $L : [0, \infty) \times (0, 1) \to [0, \infty)$ is continuous, $t \mapsto L_t(x)$ is nondecreasing for all $x \in (0, 1)$, and

$$\int_0^t f(|X(s)|)ds = \int_0^1 f(x)L_t(x)dx \qquad \forall t \geq 0,\ f \in \mathcal{N}[0, 1] \text{ a.s.}, \tag{1.8}$$

where $\mathcal{N}[0, 1]$ denotes the class of measurable functions $f : [0, 1] \to [0, \infty]$. The function L satisfies

$$L_t(x) \sim (1 - x)^{\frac{1}{2}c-1} l_t \qquad \text{as } x \to 1 \tag{1.9}$$

where $t \mapsto l_t$ is continuous, nondecreasing, and

$$P^x[l_t > 0] = 1 \qquad \forall x \in \partial E,\ t > 0. \tag{1.10}$$

We postpone the (standard) proof to Section 3. Formula (1.7) makes clear that solutions of the SDE (1.2) are pathwise unique if $c \geq 2$ and $X(0) = x \in E^\circ$. We saw already that pathwise uniqueness holds for $c = 0$. On the other hand, for $0 < c < 2$, the process X reaches the boundary of E in a finite time, bounces back, and hits the boundary infinitely often. The process spends enough time near the boundary of E to really 'feel' the non-Lipschitzness of the diffusion coefficient, and one cannot hope to prove pathwise uniqueness by a simple adaptation of Itô's method.

1.4 Rotational symmetry

At this moment, the reader might think that the rotational symmetry of SDE (1.2) could help proving pathwise uniqueness. In particular, one might be tempted to derive an equation for the radial component of X, prove pathwise uniqueness for this equation by one-dimensional methods first, and then treat the transversal components afterwards. It should be stressed that this idea *does not work* for our equation. In fact, a simple application of Itô's formula shows that

$$d(\tfrac{1}{2}|X(t)|^2) = -c|X(t)|^2dt + 2n(1 - |X(t)|^2)dt + \sqrt{2(1 - |X(t)|^2)} \sum_{i=1}^n X_i(t)dB_i(t). \tag{1.11}$$

Assume for the moment that it is possible to define

$$\tilde{B}(t) := \int_0^t \sum_{i=1}^n \frac{X_i(s)}{|X(s)|} dB_i(s) \tag{1.12}$$

(replacing $X(s)/|X(s)|$ by some arbitrary unit vector in $\mathbb{R}^n$ if $X(s) = 0$), and that one can show that $\tilde{B}$ is a Brownian motion and

$$d(\tfrac{1}{2}|X(t)|^2) = -c|X(t)|^2 dt + 2n(1 - |X(t)|^2)dt + \sqrt{2(1 - |X(t)|^2)}|X(t)|d\tilde{B}(t). \tag{1.13}$$

Then, indeed, one finds a one-dimensional SDE for the process $\frac{1}{2}|X|^2$ to which the pathwise uniqueness result of Yamada and Watanabe (1971a) is applicable. However, this does not imply that $|X^1(t)| = |X^2(t)|$ for any two solutions X^1, X^2 to the SDE (1.2) with initial conditions $X^1(0) = X^2(0)$. The reason is that the Brownian motion constructed in (1.12) depends on the process X^α, $\alpha = 1, 2$. Thus, we get uniqueness of $|X^1(t)|$ and $|X^2(t)|$ relative to two a priori different Brownian motions. From this we cannot conclude that $|X^1|$ and $|X^2|$ are equal.[2]

2 Results

2.1 Transformation of the space

The main technique that will allow us to improve the pathwise uniqueness results described in Section 1.3 is a transformation of the state space. Let X be a solution of the SDE (1.2), and consider the process Y given by

$$Y(t) := (\sqrt{1 - |X(t)|^2}, X_1(t), \ldots, X_n(t)) \qquad (t \geq 0). \tag{2.1}$$

Y takes values in the upper-half ball surface $F := \{(y_0, \ldots, y_n) \in \mathbb{R}^{n+1} : |y| = 1,\ y_0 \geq 0\}$. *Formally* applying Itô's formula to the function $x \mapsto \sqrt{1 - |x|^2}$ and inserting $\sqrt{1 - |X|^2} = Y_0$, one finds that Y, considered as a $\mathbb{R}^{n+1}$-valued process, solves the SDE

$$\begin{aligned} dY_0(t) &= -nY_0(t)dt - \sqrt{2}\sum_{i=1}^n Y_i(t)dB_i(t) + (c-1)(Y_0(t))^{-1}\sum_{i=1}^n (Y_i(t))^2 dt \\ dY_i(t) &= -cY_i(t)dt + \sqrt{2}Y_0(t)dB_i(t) \qquad (i = 1, \ldots, n). \end{aligned} \tag{2.2}$$

Since the function $x \mapsto \sqrt{1 - |x|^2}$ is not $\mathcal{C}^2$ at the boundary of E, however, the formal application of Itô's formula is not justified here. In fact, it turns out that (2.2) is correct only for $c > 1$.

[2] In fact, the method sketched above can be applied to prove distribution uniqueness for our and other SDE's exhibiting rotational symmetry. It can also be used to prove pathwise uniqueness for rotationally symmetric SDE's in which separate Brownian motions (1-dimensional and $(n-1)$-dimensional, respectively) drive the radial and transversal components of X. We can find such a SDE whose solutions are equal in distribution to solutions of the SDE (1.2), but the latter is itself not of this type.

To describe the true behavior of Y, define vectorfields $b, \sigma^1, \ldots, \sigma^n$ on $\mathbb{R}^{n+1}$ by

$$\begin{aligned} b(y) &:= (-ny_0, -y_1, \ldots, -y_n) \\ \sigma^k(y) &:= (-\sqrt{2}y_k, 0, \ldots, \sqrt{2}y_0, \ldots, 0) \qquad (k = 1, \ldots, n), \end{aligned} \tag{2.3}$$

where all coordinates of $(\sigma_0^k(y), \ldots, \sigma_n^k(y))$ are zero except $\sigma_0^k(y)$ and $\sigma_k^k(y)$. Moreover, define $\gamma : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n+1}$ by

$$\gamma(y) := ((y_0)^{-1} \sum_{i=1}^{n} (y_i)^2 \ , \ -y_1, \ldots, -y_n). \tag{2.4}$$

Then equation (2.2) can be written as

$$\begin{aligned} dY_i(t) = b_i(Y(t))dt + \sum_{k=1}^{n} \sigma_i^k(Y(t))dB_k(t) + (c-1)\gamma_i(Y(t))dt \\ (t \geq 0, \ i = 0, \ldots, n). \end{aligned} \tag{2.5}$$

The true behavior of the process Y is now described by the following theorem.

Theorem 2 *Assume that $n \geq 1$ and $c > 0$. Let X be a solution of the SDE (1.2) and define Y by (2.1). Then*

$$\begin{aligned} Y_0(t) - Y_0(0) &= \int_0^t b_0(Y(s))ds + \sum_{k=1}^{n} \int_0^t \sigma_0^k(Y(s))dB_k(s) + \Psi(t) \\ Y_i(t) - Y_i(0) &= \int_0^t b_i(Y(s))ds + \sum_{k=1}^{n} \int_0^t \sigma_i^k(Y(s))dB_k(s) + (c-1) \int_0^t \gamma_i(Y(s))ds \\ & \qquad (i = 1, \ldots, n) \qquad \forall t \geq 0, \ \text{a.s.}, \end{aligned} \tag{2.6}$$

where Ψ is a real-valued process with continuous sample paths such that $\Psi(0) = 0$ and

$$\begin{aligned} \Psi(t) &= (c-1) \int_0^t \gamma_0(Y(s))ds \qquad \forall t \geq 0 \quad \text{a.s.} \qquad \textit{if } c > 1 \\ \Psi(t_2) - \Psi(t_1) &= (c-1) \int_{t_1}^{t_2} \gamma_0(Y(s))ds \qquad \forall 0 \leq t_1 < t_2 \textit{ such that} \\ & \qquad Y_0(s) > 0 \quad \forall s \in (t_1, t_2) \quad \text{a.s.} \qquad \textit{if } c \leq 1. \end{aligned} \tag{2.7}$$

If $c \leq 1$, then

$$\lim_{\varepsilon \to 0} \varepsilon^{1-c} \, \Psi_\varepsilon^+(t) = l_t \qquad \forall t \geq 0, \ \text{a.s.}, \tag{2.8}$$

where l is the process in (1.9) and $\Psi_\varepsilon^+(t) := \Psi(t) - \Psi_\varepsilon^-(t)$,

$$\Psi_\varepsilon^-(t) := (c-1) \int_0^t 1_{\{Y_0(s) > \varepsilon\}} \gamma_0(Y(s))ds. \tag{2.9}$$

Note that Proposition 1 implies that the Lebesgue measure of the set $\{s \geq 0 : Y_0(s) = 0\}$ is almost surely zero, so that $\int_0^t \gamma_0(Y(s))ds$ is well-defined even though $\gamma_0(y)$ is not defined when $y_0 = 0$. For $c = 1$, formula (2.8) implies that $\Psi(t) = l_t \; \forall t \geq 0$ a.s. If $c \geq 1$, the process Ψ is almost surely nondecreasing. If $c < 1$ and $X(0) = x \in \partial E$, the process Ψ is almost surely of unbounded variation on every time interval $[0, t]$ with $t > 0$.

For $c > 1$, formula (1.9) implies that $\int_0^t \gamma_0(Y(s))ds$ is almost surely finite for all $t \geq 0$, and the process Y solves the SDE (2.5), written in integral form. We will see (Theorem 3 below) that solutions of this SDE are pathwise unique.

For $c = 1$, formula (2.7) and the fact that Ψ is almost surely nondecreasing imply that

$$\int_0^\infty 1_{\{Y_0(t) > 0\}} d\Psi(t) = 0 \quad \text{a.s.} \tag{2.10}$$

This means that Y solves the SDE (2.5) with reflecting boundary conditions. The terms in (2.5) containing γ vanish, and the remaining coefficients are Lipschitz continuous (in fact, they are even linear). It is well-known that solutions of such a SDE with reflection are pathwise unique (see Tanaka (1979)).

For $c < 1$, we get a type of boundary behavior that we could call super-reflection. Naively, we would like to write $\Psi = \Psi^+ + \Psi^-$, where $\Psi^-(t) = (c-1)\int_0^t \gamma_0(Y_0(s))ds$ and Ψ^+ is a pure reflection term. However, formulas (1.9) and (1.10) show that Ψ^-, so defined, reaches infinity immediately after the first time that X hits the boundary of E, and therefore no such decomposition of Ψ is possible. For $0 < c < 1$ and $n \geq 2$ I do not know if solutions to equations (2.6) and (2.7) are pathwise unique.

2.2 The distance beteen two pathwise solutions

In order to find out if pathwise uniqueness holds for the SDE (1.2), consider two solutions X^1, X^2 of the SDE (1.2), relative to the same Brownian motion, and construct their transformed processes Y^1, Y^2 as in (2.1). If we can prove that $Y^1(0) = Y^2(0)$ implies $Y^1(t) = Y^2(t)$ a.s. for all $t \geq 0$, then pathwise uniqueness holds for (1.2). The following theorem shows that this is OK for $c \geq 1$.

Theorem 3 *Assume that $n \geq 1$ and $c \geq 1$. Let X^1, X^2 be solutions of the SDE (1.2), relative to the same Brownian motion, and define Y^1, Y^2 and Ψ^1, Ψ^2 by (2.1) and (2.6). Then $t \mapsto |Y^1(t) - Y^2(t)|$ is almost surely nonincreasing. For $c > 1$:*

$$\begin{aligned} |Y^1(t) - Y^2(t)| \; &= \; |Y^1(0) - Y^2(0)| \\ &+ (c-1)\int_0^t \sum_{i=0}^n \frac{Y_i^1(s) - Y_i^2(s)}{|Y^1(s) - Y^2(s)|}\Big(\gamma_i(Y^1(s)) - \gamma_i(Y^2(s))\Big)ds \qquad \forall t \geq 0 \;\text{ a.s.} \end{aligned} \tag{2.11}$$

For $c = 1$:

$$|Y^1(t) - Y^2(t)| = |Y^1(0) - Y^2(0)| + \int_0^t \frac{Y_0^1(s) - Y_0^2(s)}{|Y^1(s) - Y^2(s)|}(d\Psi^1(s) - d\Psi^2(s)) \qquad \forall t \geq 0 \;\text{ a.s.} \tag{2.12}$$

Here we define $(Y_i^1(s) - Y_i^2(s))/(|Y^1(s) - Y^2(s)|) := 0$ when $Y^1(s) = Y^2(s)$.

The aim of Theorem 2 was to transform the SDE (1.2) into a SDE with Lipschitz diffusion coefficient σ. It turns out that the diffusion coefficient σ and the drift b do not enter the expressions (2.11) and (2.12) at all. To see why this is so, note that for $c = 1$ the processes Y^1 and Y^2 solve the SDE

$$\begin{aligned} dY_0(t) &= -nY_0(t)dt - \sqrt{2}\sum_{i=1}^n Y_i(t)dB_i(t) \\ dY_i(t) &= -Y_i(t)dt + \sqrt{2}Y_0(t)dB_i(t) \qquad (i = 1, \dots, n) \end{aligned} \tag{2.13}$$

with orthogonally reflecting boundary conditions. It is an easy exercise to show that a solution $\tilde{Y}$ of this equation without reflection (defined on all of $\mathbb{R}^{n+1}$) with initial condition $\tilde{Y}(0)$ stays on the surface of the ball with radius $|\tilde{Y}(0)|$ around the origin. Moreover, since equation (2.13) is linear, the difference of two solutions $\tilde{Y}^1$ and $\tilde{Y}^2$ (relative to the same Brownian motion) is again a solution, and hence

$$|\tilde{Y}^1(t) - \tilde{Y}^2(t)| = |\tilde{Y}^1(0) - \tilde{Y}^2(0)| \qquad \forall t \geq 0 \text{ a.s.} \tag{2.14}$$

We can now understand the behavior of solutions of (2.13) with reflection as follows. As long as the processes Y^1 and Y^2 do not reach the plane $\{y \in \mathbb{R}^{n+1} : y_0 = 0\}$, they behave as solutions to the SDE without reflection, and hence the distance between them remains constant. When one of them, say Y^1, reaches this plane, and according to the SDE (2.13) would make an infinitesimal time step dY^1 which would lead it outside $\{y \in \mathbb{R}^{n+1} : y_0 \geq 0\}$, the increment dY^1 is reflected (i.e. dY_0^1 is changed to $-dY_0^1$). At such a moment, the process Y^2 may come closer to Y^1, and the distance between Y^1 and Y^2 may suddenly decrease. Figure 1 on page 9 shows the result of a computer simulation[3] of the behavior of $|Y^1(t) - Y^2(t)|$ as a function of t during one random run.

It is known that in dimension $n = 1$ and for $c = 1$ the stopping time

$$\tau' := \inf\{t \geq 0 : Y^1(t) = Y^2(t)\} \tag{2.15}$$

is almost surely finite (see Weerasinghe (1985)). On the other hand, in dimensions $n \geq 2$ I conjecture that $|Y^1(t) - Y^2(t)| \to 0$ as $t \to \infty$ almost surely, but that the stopping time τ' above is almost surely infinite if the initial conditions are different. Cranston and Le Jan (1990) prove noncoalescence for two solutions of a SDE describing Brownian motion on a disc with reflecting boundaries; their methods may work here too.[4]

We can now also easily understand the behavior of $|Y^1 - Y^2|$ for $c > 1$. It is not hard to see that the drift γ is attractive:

$$\sum_i (y_i - y_i')(\gamma_i(y) - \gamma_i(y')) \leq 0 \qquad \forall y, y' \in F,\ y_0, y_0' > 0, \tag{2.16}$$

and hence the right-hand side in (2.11) is nonincreasing in t. See Figure 2 on page 10 for a simulation of $|Y^1(t) - Y^2(t)|$ as a function of t for $c = 6/5$.

[3] This computer simulation is just a solution of $X_i^\alpha(t+d) = -cX_i^\alpha(t)d + \sqrt{2(1 - |X^\alpha(t)|) \vee 0}\, W_i(t)$ (on $\mathbb{R}^2$), with d small and the $W_i(t)$ independent with mean zero and variance $\sqrt{d}$. We use $\pm\sqrt{d}$-valued $W_i(t)$.

[4] In this context, we note that it is possible to add random rotations to the processes Y^1 and Y^2 (not affecting the distance beteen them) such that Y^1 and Y^2 are Brownian motions on the upper-half sphere surface $F := \{y \in \mathbb{R}^{n+1} : |y| = 1,\ y_0 \geq 0\}$, with orthogonal reflection at the boundary.

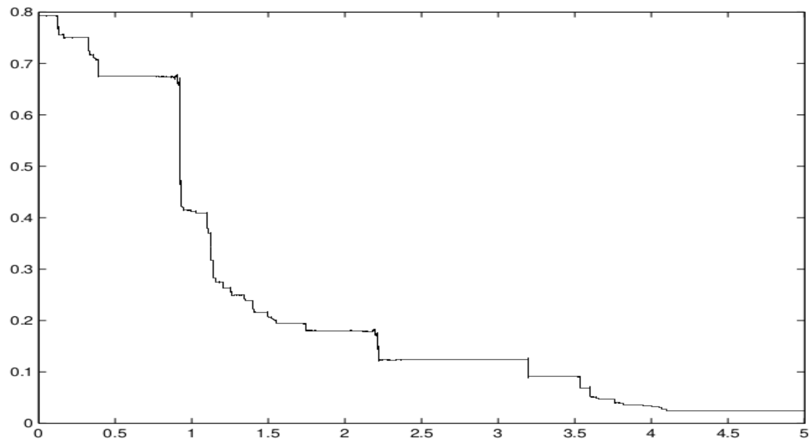

Figure 1: Simulation of two solutions $(X^1(t))_{t\in[0,5]}$ and $(X^2(t))_{t\in[0,5]}$, relative to the same Brownian motion, of the equation $dX = -Xdt + \sqrt{2(1-|X|^2)}dB$ in dimension $n = 2$. The initial conditions are $X^1(0) = (0, 0)$ and $X^2(0) = (-0.7, 0.2)$. We use 312500 time steps. The plot shows the behavior of $|Y^1(t) - Y^2(t)|$ as a function of t in one random run, where $Y^\alpha(t) := (\sqrt{1-|X^\alpha(t)|^2}, X_1^\alpha(t), X_2^\alpha(t))$, $\alpha = 1, 2$.

Figure 2 also shows the results of computer simulations of the behavior of $|Y^1(t) - Y^2(t)|$ as a function of t for two values of c smaller than 1. For $c < 1$, one is tempted to write for $|Y^1(t) - Y^2(t)|$ the heuristic formula

$$\begin{aligned} |Y^1(t) - Y^2(t)| = &|Y^1(0) - Y^2(0)| + \int_0^t \frac{Y_0^1(s) - Y_0^2(s)}{|Y^1(s) - Y^2(s)|}(d\Psi^1(s) - d\Psi^2(s)) \\ &+(c-1)\int_0^t \sum_{i=1}^n \frac{Y_i^1(s) - Y_i^2(s)}{|Y^1(s) - Y^2(s)|}(\gamma_i(Y^1(s)) - \gamma_i(Y^2(s)))ds. \end{aligned} \tag{2.17}$$

Here the distance between Y^1 and Y^2 should increase as long as they do not reach the plane $\{y \in \mathbb{R}^{n+1} : y_0 = 0\}$, and decrease when either Y_0^1 or Y_0^2 becomes zero. It is not obvious how to make mathematical sense of formula (2.17). The processes Ψ^1 and Ψ^2 are no semimartingales (i.e., the sum of a process of bounded variation and a martingale), so that the first line in (2.17) cannot be interpreted as a (stochastic) integral in a traditional sense.

Figure 2 suggests that for $c < 1$, but not too small, $|Y^1(t) - Y^2(t)|$ still tends to zero as $t \to \infty$. On the other hand, it seems that for c sufficiently small this is no longer the case[5] in dimension $n = 2$. We note that[6] in dimension $n = 1$

$$E[|X^1(t) - X^2(t)|] = E[|X^1(0) - X^2(0)|]e^{-ct}, \tag{2.18}$$

[5] I have not made a serious numerical investigation of these phenomena. As long as one does not know whether pathwise uniqueness holds for the SDE (1.2), one should worry about which (if any) solution of (1.2) the simulations approximate.

[6] To prove this, use the method of Yamada and Watanabe (see Yamada and Watanabe (1971a) or Section 5.2 in Karatzas & Shreve (1991)) to show that $|X^1(t) - X^2(t)|e^{ct}$ is a martingale.

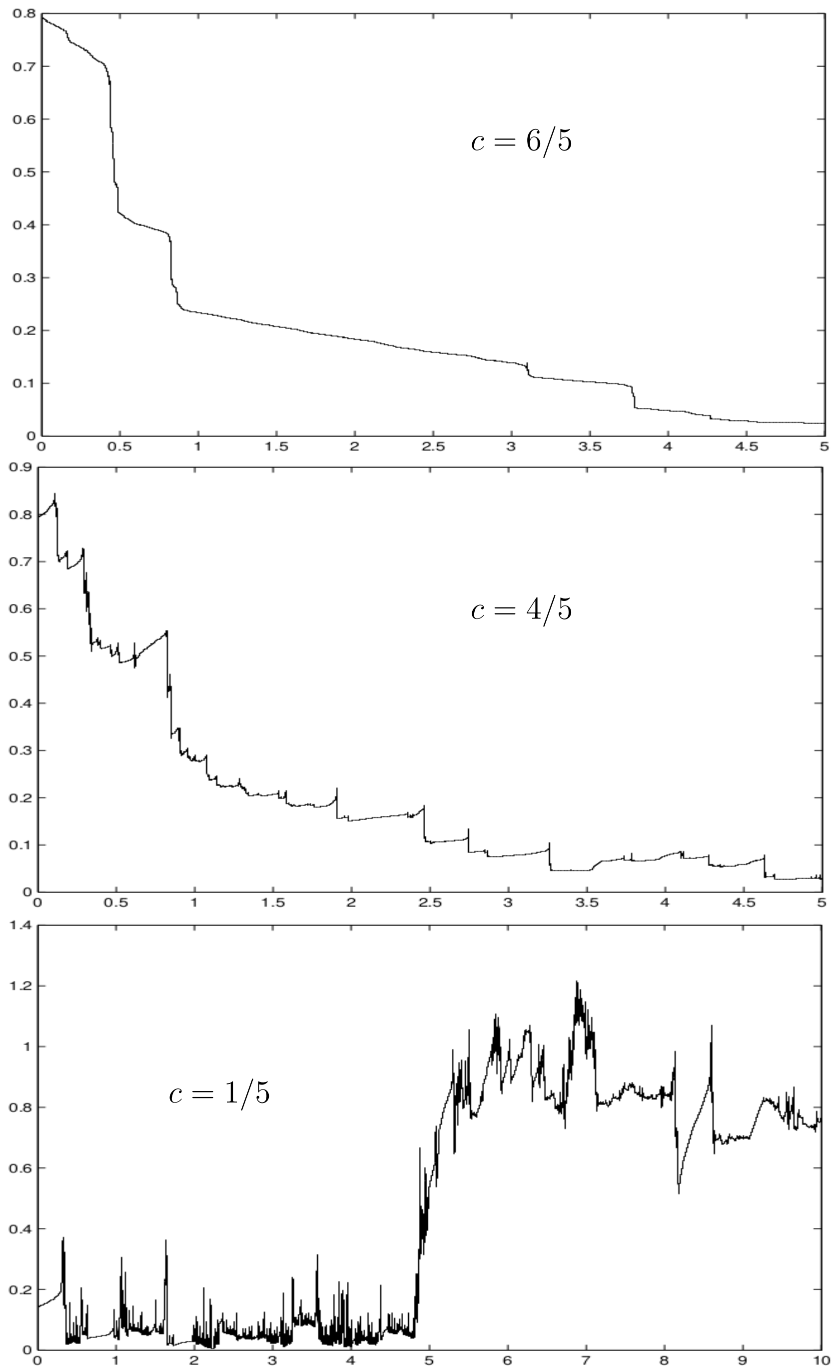

Figure 2: The same as Figure 1, but for the equation $dX = -cXdt + \sqrt{2(1-|X|^2)}dB$ for the values $c = 6/5, 4/5$ and $1/5$. The pictures for $c = 6/5$ and $4/5$ use 112500 time steps for the time interval $[0, 5]$ and use initial conditions $X^1(0) = (0, 0)$, $X^2(0) = (-0.7, 0.2)$. The picture for $c = 1/5$ uses 100000 time steps for the time interval $[0, 10]$ and uses initial conditions $X^1(0) = (0, 0)$, $X^2(0) = (-0.1, 0.1)$.

which implies that $|Y^1(t) - Y^2(t)|$ tends to zero in probability as $t \to \infty$ for all $c > 0$.

In order to prove pathwise uniqueness for the SDE (1.2) for $c < 1$, it is of course not necessary to show that the distance between $Y^1(t)$ and $Y^2(t)$ tends to zero as $t \to \infty$. Rather, it would suffice to show that Y^1 and Y^2, starting in $Y^1(0) = Y^2(0)$, do not lose each other in finite time.

2.3 Open problems

The drift in (1.2) can almost certainly be generalized. For example, it seems likely (as is supported by simulations) that for the SDE $dX = c(\theta - X)dt + \sqrt{2(1 - |X|^2)}dB$ the function $t \mapsto |Y^1(t) - Y^2(t)|$ is almost surely nonincreasing if $c(1 - |\theta|) \geq 1$ (where $\theta \in E$ and $c \geq 0$). This SDE, as opposed to (1.2), does not exhibit rotational symmetry, and this makes it more difficult to prove statements about the local time of its solutions near the boundary of E. Distribution uniqueness for this SDE can be proved by a moment calculation as in Section 1.2.

One may also try to treat other diffusion coefficients that the one in (1.2). This would require an another transformation than the $x \mapsto (\sqrt{1 - |x|^2}, x_1, \ldots, x_n)$ described in Theorem 2. Any proof of pathwise uniqueness for a SDE must show that the distance between two solutions, relative to the same Brownian motion, can not grow too fast, i.e., cannot grow from zero to something nonzero in a finite time. Theorem 2 suggests that it is important to find the 'right' concept of distance for a given diffusion coefficient. We have projected X^1 and X^2 from the (hyper-) plane $\mathbb{R}^n$ onto the surface of the unit ball in $\mathbb{R}^{n+1}$, and measured their distance through the interior of this ball. Of course, we could equivalently have measured their distance along the surface of the ball, i.e., the distance that is produced by the metric associated with the imbedding of this ball surface in $\mathbb{R}^{n+1}$.[7]

At the moment the most challenging problem seems to be:

> Does pathwise uniqueness hold for the SDE (1.2) in dimensions $n \geq 2$ for $0 < c < 1$?

As a possible step towards an answer, one might try to give meaning to the heuristic formula (2.17). An answer, in either way, would be valuable. There are only a few known examples of SDE's for which pathwise uniqueness fails, while distribution uniqueness holds. The only such example with continuous coefficients know to me is due to Barlow (1982); this concerns a one-dimensional SDE which holds no relation to the sort of uniqueness problems occuring in higher dimension discussed in the present paper.

[7]When we consider E as a differentiable manifold, it is natural to write the SDE (1.2) in Stratonovich form. (This can be done at least locally on E°.) It is worth noting that for the SDE (1.2) in Stratonovich form the case $c = 1$ corresponds to vanishing drift.

3 Proofs

3.1 Proof of Proposition 1

We first prove the statements concerning the stopping time τ. For $\alpha \geq 0$, define $f_\alpha : E^\circ \to \mathbb{R}$ by

$$\begin{aligned} f_0(x) &:= -\log(1-|x|^2) \\ f_\alpha(x) &:= \alpha^{-1}\Big(1-(1-|x|^2)^\alpha\Big) \qquad (\alpha > 0). \end{aligned} \tag{3.1}$$

A little calculation shows that for $x \in E^\circ$

$$Af_\alpha(x) = 4\{(1-\tfrac{1}{2}c)-\alpha\}|x|^2(1-|x|^2)^{\alpha-1} + 2n(1-|x|^2)^\alpha. \tag{3.2}$$

Here we change f_α in a neigborhood of ∂E to make it into a $\mathcal{C}^2$-function on E, on which A is properly defined.

Introduce stopping times

$$\tau_r := \inf\{t \geq 0 : |X(t)| \geq r\} \qquad (0 < r < 1), \tag{3.3}$$

and note that by the continuity of sample paths $\tau_r \uparrow \tau$ as $r \uparrow 1$. If $c < 2$, then we can choose $0 < \alpha < 1 - \frac{1}{2}c$, in which case (3.2) shows that there exists an $\varepsilon > 0$ such that $Af_\alpha(x) \geq \varepsilon$ for all $x \in E^\circ$. It follows that for any solution X to the SDE (1.2)

$$\varepsilon E^x[t\wedge\tau_r] \leq E^x\Big[\int_0^{t\wedge\tau_r} Af_\alpha(X_s)ds\Big] = E^x[f_\alpha(X(t\wedge\tau_r))] - f_\alpha(x) \leq \alpha^{-1}, \tag{3.4}$$

where we have used optional stopping and the fact that X solves the martingale problem for A. Letting $t \uparrow \infty$ and $r \uparrow 1$ (in this order) we find that $E[\tau] \leq (\alpha\varepsilon)^{-1}$.

On the other hand, if $c \geq 2$ then we may use that $f_0 \geq 0$ and $Af_0 \leq 2n$ to conclude that for $x \in E^\circ$

$$\begin{aligned} &-\log(1-r^2)P^x[\tau_r \leq t] \leq E^x[f_0(X(t\wedge\tau_r))] \\ &\quad = f_0(x) + E^x\Big[\int_0^{t\wedge\tau_r} Af_0(X_s)ds\Big] \leq f_0(x) + 2nE^x[t\wedge\tau_r] \leq f_0(x) + 2nt. \end{aligned} \tag{3.5}$$

Thus

$$P^x[\tau_r \leq t] \leq \frac{f_0(x)+2nt}{-\log(1-r^2)}. \tag{3.6}$$

Letting $r \uparrow 1$ and $t \uparrow \infty$ (in this order) we find that $P^x[\tau < \infty] = 0$. This proves the statements about the stopping time τ.

We now prove the statements about the local time L. Set $R_t := 1 - |X(t)|^2$. A little calculation shows that R solves the martingale problem for the operator

$$A_R f(r) := \{2c(1-r) - 2nr\}f'(r) + 4r(1-r)f''(r) \qquad (r \in [0,1]), \tag{3.7}$$

with domain $\mathcal{D}(A_R) := \mathcal{C}^2[0,1]$, and thus we can find a version of R solving the SDE

$$dR_t = \{2c(1-R_t) - 2nR_t\}dt + 2\sqrt{2R_t(1-R_t)}dB_t. \tag{3.8}$$

By the result of Yamada and Watanabe (1971a), solutions of this SDE are pathwise unique and hence solutions to the martingale problem for A_R are unique. The process R can be reduced to Brownian motion by a standard technique, consisting of two steps: removal of the drift and a random time transformation. (For this technique and its terminology see Chapter 16 of Breiman (1968).) Define $u : [0,1] \to [0,\infty]$ by

$$u(r) := \int_0^r dp \, \exp\Big(- \int_{1/2}^p dq \, \frac{2c(1-q) - 2nq}{4q(1-q)} \Big). \tag{3.9}$$

Then u is $\mathcal{C}^2$ on $(0,1)$ and $A_R u = 0$ there. Moreover, $u' > 0$ on $(0,1)$ so that u is invertible and one can check that, for some (strictly) positive constant d,

$$u'(r) \sim d\, r^{-c/2} \qquad \text{as } r \to 0. \tag{3.10}$$

For each initial condition the $[0, u(1)]$-valued process $U_t := u(R_t)$ is the unique solution to the martingale problem for the operator

$$A_U f(v) := a(v) f''(v) \qquad \text{with} \qquad a(u(r)) = 4r(1-r)(u'(r))^2 \qquad (r \in (0,1)), \tag{3.11}$$

where the domain of A_U is the class of functions $f \in \mathcal{C}[0, u(1)]$ such that $f \circ u \in \mathcal{C}^2[0,1]$. Such functions satisfy $f' = 0$ in all finite boundary points, which corresponds to reflecting boundaries.

The process U is a time-changed Brownian motion. Let W be Brownian motion on $[0, u(1)]$, reflected at finite boundary points, with initial condition $W_0 = u(R_0)$. Define stopping times $\tau(t)$ by

$$t =: \int_0^{\tau(t)} \frac{ds}{a(W_s)} \qquad (t \geq 0). \tag{3.12}$$

Then the function $\tau : [0,\infty) \to [0,\infty)$ is almost surely continuous and increasing (i.e., $s < t \Rightarrow \tau(s) < \tau(t)$), and a version of the process U is given by (see Breiman, Theorem 16.56 (1968)):

$$U_t = W_{\tau(t)}. \tag{3.13}$$

Let L^W be the local time of W, i.e., $(t, v) \mapsto L^W_t(v)$ is a (random) continuous map from $[0,\infty) \times [0, u(1)]$ to $[0, \infty)$ such that $t \mapsto L_t(v)$ is nondecreasing for all v and such that

$$\int_0^t f(W_s) ds = \int_0^{u(1)} f(v) L^W_t(v) dv \quad \forall f \in \mathcal{N}[0, u(1)],\ t \geq 0 \ \text{ a.s.} \tag{3.14}$$

It follows that

$$\int_0^t f(U_s) ds = \int_0^t f(W_{\tau(s)}) ds = \int_0^{\tau(t)} f(W_\sigma) \frac{d\sigma}{a(W_\sigma)} = \int_0^{u(1)} \frac{f(v)}{a(v)} L^W_{\tau(t)}(v) dv, \tag{3.15}$$

where we have substituted $\tau(s) = \sigma$, $ds = d\sigma / a(W_\sigma)$ and $s = t \Leftrightarrow \sigma = \tau(t)$. Thus

$$\int_0^t f(U_s) ds = \int_0^{u(1)} f(v) L^U_t(v) dv \qquad \text{with } L^U_t(v) := \frac{L^W_{\tau(t)}(v)}{a(v)}. \tag{3.16}$$

Since a is continuous and positive on $(0, u(1))$ and since τ is continuous and nondecreasing, $(t, v) \mapsto L_t^U(v)$ is a continuous function from $[0, \infty) \times (0, u(1))$ to $[0, \infty)$ and $t \mapsto L_t^U(v)$ is nondecreasing for each v. Moreover (since τ is increasing), the limit

$$\lim_{v \to 0} a(v) L_t^U(v) = L_{\tau(t)}^W(0). \tag{3.17}$$

exists and is positive for all $t > 0$ if $W_0 = 0$. A change of coordinates now gives, for all $f \in \mathcal{N}[0, u(1)]$:

$$\int_0^t f(R_s) ds = \int_0^1 f(r) L_t^R(r) dr \qquad \text{with } L_t^R(r) := L_t^U(u(r)) u'(r). \tag{3.18}$$

Here $L^R : [0, \infty) \times (0, 1) \to [0, \infty)$ is continuous, $t \mapsto L_t^R(r)$ is nondecreasing for each $r \in (0, 1)$ and the limit

$$\begin{aligned} &L_{\tau(t)}^W(0) = \lim_{v \to 0} a(v) L_t^U(v) = \lim_{r \to 0} a(u(r)) L_t^U(u(r)) \\ &= \lim_{r \to 0} 4r(1 - r)(u'(r))^2 L_t^R(r)(u'(r))^{-1} = \lim_{r \to 0} 4r(1 - r) u'(r) L_t^R(r) \end{aligned} \tag{3.19}$$

exists and is positive for all $t > 0$ if $R_0 = 0$. Inserting (3.10) we see that also the limit

$$l_t := \lim_{r \to 0} 2 r^{1 - \frac{1}{2} c} L_t^R(r) \tag{3.20}$$

exists with the same properties. By another change of coordinates (similar to the one going from the process U to R, but this time more explicit) we can translate the properties of the local time of R into the statements about the local time of $|X|$ in Proposition 1.∎

3.2 Proof of Theorem 2

First we show that formula (2.6) holds. The formulas for $Y_i(t) - Y_i(0)$ with $i \geq 1$ are trivial. For the case $i = 0$, we proceed as follows.

For $m = 1, 2, \ldots$ choose $\rho_m \in \mathcal{C}[0, \infty)$ such that $\rho_m \geq 0$, $\int_0^\infty \rho_m(x) dx = 1$ for all m and $\int_\varepsilon^\infty \rho_m(x) dx \to 0$ for all $\varepsilon > 0$ as $m \to \infty$. Define $\phi_m \in \mathcal{C}^2[0, \infty)$ by

$$\phi_m(x) := \int_0^x dy \int_0^y dz \, \rho_m(z). \tag{3.21}$$

For each m, the function $x \mapsto \phi_m(\sqrt{1 - x^2})$ is $\mathcal{C}^2$ on E and we may apply Itô's formula to deduce that

$$\begin{aligned} \phi_m(Y_0(t)) - \phi_m(Y_0(0)) = &\int_0^t \phi_m'(Y_0(s)) b_0(Y(s)) ds + \sum_{k=1}^n \int_0^t \phi_m'(Y_0(s)) \sigma_0^k(Y(s)) dB_k(s) \\ &+ (c - 1) \int_0^t \phi_m'(Y_0(s)) \gamma_0(Y(s)) ds \\ &+ \int_0^t \phi_m''(Y_0(s)) \sqrt{1 - (Y_0(s))^2} ds \qquad \forall t \geq 0 \text{ a.s.} \end{aligned} \tag{3.22}$$

Since $0 \le \phi'_m \le 1$ and $\phi'_m(x) \to 1_{\{x>0\}}$ as $m \to \infty$, it is easy to see that

$$\sup_{0\le s\le t}\left|\int_0^s \phi'_m(Y_0(u))b_0(Y(u))du - \int_0^s b_0(Y(u))du\right| \to 0 \qquad \text{as } m\to\infty \qquad \forall t\ge 0 \text{ a.s.} \tag{3.23}$$

(Note that $b_0(y) = 0$ if $y_0 = 0$.) Moreover:

$$E\Big[\Big|\int_0^t \phi'_m(Y_0(s))\sigma_0^k(Y(s))dB_k(s) - \int_0^t \sigma_0^k(Y(s))dB_k(s)\Big|^2\Big]$$
$$= E\Big[\int_0^t \Big|\phi'_m(Y_0(s))\sigma_0^k(Y(s)) - \sigma_0^k(Y(s))\Big|^2 ds\Big] \le 2E\Big[\int_0^t \Big(1-\phi'_m(Y_0(s))\Big)^2 ds\Big], \tag{3.24}$$

with

$$\lim_{m\to\infty} E\Big[\int_0^t \Big(1-\phi'_m(Y_0(s))\Big)^2 ds\Big] = E\Big[\int_0^t 1_{\{Y_0(s)=0\}}ds\Big] = 0, \tag{3.25}$$

where the last equality is a consequence of Proposition 1. Thus

$$\int_0^t \phi'_m(Y_0(s))\sigma_0^k(Y(s))dB_k(s) \longrightarrow \int_0^t \sigma_0^k(Y(s))dB_k(s) \text{ in } L^2\text{-norm as } m\to\infty \quad \forall t\ge 0. \tag{3.26}$$

Setting

$$\Psi(t) := Y_0(t) - Y_0(0) - \int_0^t b_0(Y(s))ds - \sum_{k=1}^n \int_0^t \sigma_0^k(Y(s))dB_k(s), \tag{3.27}$$

which is a process with continuous sample paths, we arrive at formula (2.6). Moreover, with

$$\Psi_m(t) := (c-1)\int_0^t \phi'_m(Y_0(s))\gamma_0(Y_0(s))ds + \int_0^t \phi''_m(Y_0(s))\sqrt{1-(Y_0(s))^2}ds, \tag{3.28}$$

formulas (3.22), (3.23) and (3.26) show that:

$$\Psi_m(t) \to \Psi(t) \text{ in } L^2\text{-norm as } m\to\infty \quad \forall t\ge 0. \tag{3.29}$$

Observe that $Y_0 = \sqrt{R}$, with R the process defined above formula (3.7). The local time of Y_0 is therefore given by $L_t^Y(y) := 2yL_t^R(y^2)$, and the process l in (3.20) is related to L^Y by

$$l_t = \lim_{y\to 0} 2(y^2)^{1-\frac{1}{2}c}L_t^R(y^2) = \lim_{y\to 0} y^{1-c}L_t^Y(y). \tag{3.30}$$

We now treat the cases $c > 1$, $c = 1$ and $c < 1$ separately.
Case $\mathbf{c} > \mathbf{1}$. Since $\int_{0+}\frac{dx}{x} = \infty$, we can choose the $\rho_m = \phi''_m$ in such a way that

$$\phi''_m(0) = 0, \qquad \phi''_m(y) \le \frac{1}{y} \qquad (y\ge 0,\ m = 1,2\ldots). \tag{3.31}$$

Proposition 1 shows that for $c > 1$, the integral $\int_0^t (Y_0(s))^{-1} ds$ is finite $\forall t \geq 0$ a.s., and since we are assuming $\phi''_m(y) \leq \frac{1}{y}$ we may apply dominated convergence to conclude that

$$\int_0^t \phi''_m(Y_0(s))\sqrt{1-(Y_0(s))^2}ds \to 0 \quad \text{as } m \to \infty \quad \forall t \geq 0 \text{ a.s.} \tag{3.32}$$

Moreover, since $0 \leq \phi'_m \leq 1$ and $\phi'_m(y) \to 1_{\{y>0\}}$ as $m \to \infty$, again by dominated convergence:

$$\int_0^t \phi'_m(Y_0(s))Y_0(s)^{-1}\sqrt{1-(Y_0(s))^2}ds \to \int_0^t Y_0(s)^{-1}\sqrt{1-(Y_0(s))^2}ds \\ \text{as } m \to \infty \quad \forall t \geq 0 \text{ a.s.,} \tag{3.33}$$

and we conclude that

$$\Psi_m(t) \to (c-1)\int_0^t \gamma_0(Y(s))ds \quad \text{as } m \to \infty \quad \forall t \geq 0 \text{ a.s.} \tag{3.34}$$

Case $\mathbf{c} = \mathbf{1}$. In this case formula (3.30) implies that

$$\begin{aligned} \lim_{m\to\infty} \Psi_m(t) &= \lim_{m\to\infty} \int_0^t \sqrt{1-(Y_0(s))^2}\phi''_m(Y_0(s))ds \\ &= \lim_{m\to\infty} \int_0^1 \sqrt{1-y^2}\rho_m(y)L_t^Y(y)dy = \lim_{y\to 0} L_t^Y(y) = l_t \quad \forall t \geq 0 \text{ a.s.} \end{aligned} \tag{3.35}$$

This formula shows that $\Psi(t) = l_t$. It also shows that $d\Psi_m$ converges almost surely weakly to the measure dl_t. We therefore see immediately that

$$\int_0^\infty |Y_0(t)|dl_t = \lim_{m\to\infty} \int_0^\infty |Y_0(t)|\, 2d\Psi_m(t) = 0 \qquad \text{a.s.,} \tag{3.36}$$

which implies that both (2.7) and (2.8) are correct for $c = 1$.
Case $\mathbf{c} < \mathbf{1}$. With a view towards (3.28) set, for $\varepsilon > 0$,

$$\Psi^-_{\varepsilon,m}(t) := \int_0^t 1\{Y_0(s) > \varepsilon\}\Big\{\phi''_m(Y_0(s)) + (c-1)\phi'_m(Y_0(s))Y_0(s)^{-1}\Big\}\sqrt{1-(Y_0(s))^2}ds, \tag{3.37}$$

and define $\Psi^+_{\varepsilon,m}(t) := \Psi_m(t) - \Psi^-_{\varepsilon,m}(t)$. Then

$$\sup_{0\leq s\leq t}\left|\Psi^-_{\varepsilon,m}(s) - \Psi^-_\varepsilon(s)\right| \to 0 \quad \text{as } m \to \infty \quad \forall t \geq 0 \text{ a.s.} \tag{3.38}$$

Since $\Psi(t) \to \Psi(t)$ in L^2-norm, it follows that

$$\Psi^+_{\varepsilon,m}(t) \to \Psi^+_\varepsilon(t) \text{ in } L^2\text{-norm as } m \to \infty \quad \forall t \geq 0, \tag{3.39}$$

Note that for each $m = 1, 2, \ldots$

$$\Psi^+_{\varepsilon,m}(t_1) = \Psi^+_{\varepsilon,m}(t_2) \quad \forall 0 \leq t_1 < t_2 \text{ such that } Y_0(s) > \varepsilon \; \forall s \in (t_1, t_2), \tag{3.40}$$

and, since Y_0 has continuous sample paths:

$$\Psi^+_{\varepsilon,m}(t_1) = \Psi^+_{\varepsilon,m}(t_2) \quad \forall 0 \le t_1 < t_2 \text{ such that } Y_0(s) > \varepsilon \ \forall s \in (t_1, t_2) \cap \mathbb{Q}. \tag{3.41}$$

Because of (3.39), we can select a subsequence that converges almost surely for all $t \in \mathbb{Q}$:

$$\Psi^+_{\varepsilon,\tilde{m}}(t) \to \Psi^+_\varepsilon(t) \qquad \forall t \in \mathbb{Q} \quad \text{a.s.} \tag{3.42}$$

Combining this with (3.41) we see that almost surely

$$\Psi^+_\varepsilon(t_1) = \Psi^+_\varepsilon(t_2) \quad \forall t_1, t_2 \in \mathbb{Q},\ 0 \le t_1 < t_2 \text{ such that } Y_0(s) > \varepsilon \ \forall s \in (t_1, t_2) \cap \mathbb{Q}. \tag{3.43}$$

By the continuity of sample paths of Ψ^+_ε we can remove the condition $t_1, t_2 \in \mathbb{Q}$. Letting $\varepsilon \downarrow 0$ and using the continuity of sample paths of Y_0 we arrive (2.7).

To see that (2.8) holds, use (3.30) to write

$$\sqrt{1-y^2} L^Y_t(y) = y^{c-1} l_t(y), \tag{3.44}$$

where $y \mapsto l_t(y)$ is continuous and $l_t(0) = l_t$. Then

$$\varepsilon^{1-c} \Psi^-_\varepsilon(t) = \varepsilon^{1-c}(c-1) \int_\varepsilon^1 \sqrt{1-y^2} y^{-1} L^Y_t(y) dy = -\int_\varepsilon^1 \varepsilon^{1-c}(1-c) y^{c-2} l_t(y) dy. \tag{3.45}$$

The functions $y \mapsto 1_{[\varepsilon,1]}(y) \varepsilon^{1-c}(1-c) y^{c-2}$ approximate the delta-measure in zero, and hence

$$\lim_{\varepsilon \to 0} \varepsilon^{1-c} \Psi^-_\varepsilon(t) = -l_t \qquad \forall t \ge 0 \text{ a.s.} \tag{3.46}$$

It follows that $\lim_{\varepsilon\to 0} \varepsilon^{1-c} \Psi^+_\varepsilon(t) = \lim_{\varepsilon \to 0}\{\varepsilon^{1-c}\Psi(t) - \varepsilon^{1-c}\Psi^-_\varepsilon(t)\} = 0 + l_t \ \forall t \ge 0$ a.s. ∎

3.3 Proof of Theorem 3

Assume that $h \in \mathcal{C}^2[0,\infty)$ satisfies $h'(0) = 0$, and define

$$f(y^1, y^2) := h(|y^1 - y^2|) \qquad (y^1, y^2 \in \mathbb{R}^{n+1}). \tag{3.47}$$

It is not hard to see that $f \in \mathcal{C}^2(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$. Let X^1, X^2 be solutions of the SDE (1.2), relative to the same Brownian motion, and define Y^1, Y^2 and Ψ^1, Ψ^2 by (2.1) and (2.6). For $c \ge 1$, the processes Ψ^1, Ψ^2 are of bounded variation on bounded intervals, and therefore Itô's formula (for example in the formulation of Ethier & Kurtz (1986), Theorem 2.9 from Chapter 5) gives

$$
\begin{aligned}
&f(Y^1(t),Y^2(t)) - f(Y^1(0),Y^2(0)) \\
&\qquad = \sum_{\alpha=1}^{2}\sum_{i=0}^{n}\int_0^t (\tfrac{\partial}{\partial y_i^\alpha} f)(Y^1(s),Y^2(s)) b_i(Y^\alpha(s))ds \\
&\qquad + \sum_{\alpha=1}^{2}\sum_{i=0}^{n}\sum_{k=1}^{n}\int_0^t (\tfrac{\partial}{\partial y_i^\alpha} f)(Y^1(s),Y^2(s)) \sigma_i^k(Y^\alpha(s))dB_k(s) \\
&\qquad + \tfrac{1}{2}\sum_{\alpha,\beta=1}^{2}\sum_{i,j=0}^{n}\int_0^t (\tfrac{\partial^2}{\partial y_i^\alpha \partial y_j^\beta} f)(Y^1(s),Y^2(s)) \Big(\sum_{k=1}^{n} \sigma_i^k(Y^\alpha(s))\sigma_j^k(Y^\beta(s))\Big)ds \\
&\qquad + \sum_{\alpha=1}^{2}\sum_{i=1}^{n}(c-1)\int_0^t (\tfrac{\partial}{\partial y_i^\alpha} f)(Y^1(s),Y^2(s)) \gamma_i(Y^\alpha(s))ds \\
&\qquad + \sum_{\alpha=1}^{2}\int_0^t (\tfrac{\partial}{\partial y_i^\alpha} f)(Y^1(s),Y^2(s)) d\Psi^\alpha(t) \qquad \forall t \geq 0 \text{ a.s.}
\end{aligned}
\tag{3.48}
$$

A small calculation shows that the terms involving b and σ in this formula cancel, as we expect (see formula (2.14)). Now

$$
\tfrac{\partial}{\partial y_i^1} f(y^1,y^2) \;=\; \begin{cases} h'(|y^1-y^2|)\dfrac{y_i^1-y_i^2}{|y^1-y^2|} & \text{if } y^1 \neq y^2 \\ 0 & \text{if } y^1 = y^2. \end{cases} \tag{3.49}
$$

A similar formula holds for $\frac{\partial}{\partial y_i^2} f(y^1,y^2)$ and we find that, for $c > 1$:

$$
\begin{aligned}
&f(Y^1(t),Y^2(t)) - f(Y^1(0),Y^2(0)) \\
&\quad = (c-1)\int_0^t h'(|Y^1(s)-Y^2(s)|)\sum_{i=1}^{n}\frac{Y_i^1(s)-Y_i^2(s)}{|Y^1(s)-Y^2(s)|}\Big(\gamma_i(Y^1(s)) - \gamma_i(Y^1(s))\Big)ds \\
&\qquad\qquad \forall t \geq 0 \text{ a.s.,}
\end{aligned}
\tag{3.50}
$$

and for $c = 1$:

$$
\begin{aligned}
&f(Y^1(t),Y^2(t)) - f(Y^1(0),Y^2(0)) \\
&\quad = \int_0^t h'(|Y^1(s)-Y^2(s)|)\frac{Y_0^1(s)-Y_0^2(s)}{|Y^1(s)-Y^2(s)|}\Big(d\Psi^1(s) - d\Psi^2(s))\Big) \\
&\qquad\qquad \forall t \geq 0 \text{ a.s.,}
\end{aligned}
\tag{3.51}
$$

Write

$$
\Phi_c(t) \;:=\; \begin{cases} (c-1)\displaystyle\int_0^t \sum_{i=1}^{n}\frac{Y_i^1(s)-Y_i^2(s)}{|Y^1(s)-Y^2(s)|}\Big(\gamma_i(Y^1(s)) - \gamma_i(Y^1(s))\Big)ds & \text{if } c > 1 \\ \displaystyle\int_0^t \frac{Y_0^1(s)-Y_0^2(s)}{|Y^1(s)-Y^2(s)|}\Big(d\Psi^1(s) - d\Psi^2(s))\Big) & \text{if } c = 1. \end{cases} \tag{3.52}
$$

Then Φ_c is almost surely continuous and nonincreasing (see formula (2.16) and note that $Y_0^1 - Y_0^2 \le 0$ almost surely with respect to $d\Psi^1$ and vice versa), and we may summarize (3.50) and (3.51) as:

$$h(|Y^1(t) - Y^2(t)|) = h(|Y^1(0) - Y^2(0)|) + \int_0^t h'(|Y^1(s) - Y^2(s)|)d\Phi_c(s) \qquad (c \ge 1) \qquad \forall t \ge 0 \text{ a.s.} \tag{3.53}$$

For $\varepsilon > 0$, set $h_\varepsilon(x) := \sqrt{\varepsilon^2 + x^2}$ $(x \ge 0)$, and note that $h_\varepsilon(x) \downarrow x$ and $h'_\varepsilon(x) \uparrow 1_{\{x>0\}}$ as $\varepsilon \downarrow 0$ for all $x \ge 0$. Taking the limit $\varepsilon \downarrow 0$ in (3.53) we arrive at the statements in Theorem 3. ∎

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