I Spatial Models in Population Biology

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Lecture 1: Interacting Particle Systems

Outline

- Poisson construction of continuous-time Markov chains
- Poisson construction of interacting particle systems
- Unique ergodicity
- Examples

Continuous-time Markov chains

Let S be a countable set. A *probability kernel* on S is a function $K : S^2 \to [0,1]$ such that $\sum_y K(x,y) = 1$. We calculate with kernels as with matrices:

$$KL(x,z) := \sum_y K(x,y)L(y,z) \quad \text{and} \quad Kf(x) := \sum_y K(x,y)f(y).$$

Let $X = (X_t)_{t\geq 0}$ be a stochastic process taking values in a countable set S. We assume that the sample paths of X are *piecewise constant* and *right-continuous*. By definition, X is a (time-homogenous) *Markov process* if

$$\mathbb{P}\big[X_u \in \cdot \,\big|\, (X_s)_{0\leq s\leq t}\big] = P_{u-t}(X_t, \,\cdot\,) \quad \text{a.s.} \qquad (0 \leq t \leq u),$$

where the *transition kernels* $(P_t)_{t\geq 0}$ form a collection of probability kernels on S such

$$P_s P_t = P_{s+t} \quad \text{and} \quad \lim_{t\downarrow 0} P_t = P_0 = 1.$$

Continuous-time Markov chains

We would like to define $(P_t)_{t\geq 0}$ in terms of its *generator*

$$G(x,y) := \lim_{t\downarrow 0} t^{-1}\big(P_t(x,y) - 1(x,y)\big),$$

where $1(x,y) := 1_{\{x=y\}}$ denotes the identity matrix. Such a generator must satisfy

$$G(x,y) \geq 0 \quad (x \neq y) \quad \text{and} \quad \sum_y G(x,y) = 0.$$

We interpret $G(x,y)$ $(x \neq y)$ as the *rate* of transitions $x \mapsto y$. We should have

$$\mathbb{P}^x[X_t = y] = P_t(x,y) = 1(x,y) + tG(x,y) + O(t^2) \quad \text{as } t\downarrow 0,$$

where $\mathbb{P}^x$ denotes the law of the process started in $X_0 = x$.

Generator construction of continuous-time Markov chains

If S is finite, then we can define

$$P_t = e^{tG} := \sum_{n=0}^{\infty} \frac{1}{n!} t^n G^n \qquad (t \geq 0).$$

This works more generally if G is bounded in the sense that

$$\sup_x \sum_{y:\, y \neq x} G(x,y) < \infty.$$

Generator construction of continuous-time Markov chains

Let G be a Markov generator on a countable set S. Then, for each $z \in S$, there exists a unique *minimal* solution to the *backward equations*

$$\frac{\partial}{\partial t} P_t(x,z) = \sum_y G(x,y) P_t(y,z) \qquad (t \geq 0,\ x \in S).$$

Moreover, $(P_t)_{t\geq 0}$ is a semigroup of *subprobability kernels*. For each initial state $X_0 = x \in S$, there exists a process $X = (X_t)_{t\geq 0}$ with values in $S \cup \{\infty\}$ such that

$$\begin{aligned}
&X_t = \infty \quad \forall t \geq \tau := \inf\{t \geq 0 : X_t = \infty\},\\
&\lim_{t\uparrow\tau} X_t = \infty \quad \text{if} \quad \tau < \infty,\\
&\mathbb{P}^x\big[X_u \in y \big| (X_s)_{0\leq s\leq t}\big] = P_{u-t}(X_t, y) \quad \text{a.s.} \quad (y \in S,\ t \leq u).
\end{aligned}$$

If $\mathbb{P}^x[\tau = \infty] = 1$ for all $x \in S$, then X is *nonexplosive*.

Example: a finite contact process

Let (Λ, E) be a finite directed graph with vertex set Λ and set of directed edges $E \subset \Lambda \times \Lambda$. Let $S = \{0,1\}^\Lambda$ be the space of all functions $x : \Lambda \to \{0,1\}$. We interpret $x = (x(i))_{i \in \Lambda}$ as a *particle configuration* where for $i \in \Lambda$,

$$x(i) = 0 \text{ means the site } i \text{ is empty,}$$

$$x(i) = 1 \text{ means there is a particle at } i.$$

The *contact process* with *infection rate* $\lambda \geq 0$ is a continuous-time Markov chain $X = (X_t)_{t \geq 0}$ with state space S. In each jump, the number of occupied sites can increase or decrease by one. Let $x \in S$ and $j \in \Lambda$ such that $x(j) = 0$. Then

$$G(x, x + \delta_j) := \lambda \sum_{(i,j) \in E} x(i) \quad \text{and} \quad G(x + \delta_j, x) := 1.$$

We set $G(x,y) := 0$ if x and y differ in more than one site and choose $G(x,x)$ such that $\sum_y G(x,y) = 0$.

Example: a finite contact process

For each $(i,j) \in E$, define a *branching map* $\mathtt{bra}_{ij} : S \to S$ by

$$\mathtt{bra}_{ij}x(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

For each $i \in \Lambda$, define a *death map* $\mathtt{death}_i : S \to S$ by

$$\mathtt{death}_i x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Fix x. Assume that X_t satisfies for all $(i,j) \in E$ resp. $i \in \Lambda$,

$$X_t = \begin{cases} \mathtt{bra}_{ij}x & \text{with probability } \lambda t + O(t^2), \\ \mathtt{death}_i x & \text{with probability } t + O(t^2), \\ x & \text{with probability } 1 - \lambda t|E| - t|\Lambda| + O(t^2). \end{cases}$$

Then $\mathbb{P}[X_t = y] = 1(x,y) + tG(x,y) + O(t^2)$.

Poisson construction of continuous-time Markov chains

Idea: construct a continuous time Markov chain by applying maps $m : S \to S$ at times of a Poisson process.

Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space with σ-finite, nonatomic measure μ. Recall that a *Poisson point set* with *intensity* μ is a random subset $\omega \subset \Omega$ such that

$$\big|\omega \cap A\big| \quad \text{is Poisson distributed with mean} \quad \mu(A)$$

whenever $A \in \mathcal{F}$, $\mu(A) < \infty$, and

$$\big|\omega \cap A_1\big|, \ldots, \big|\omega \cap A_n\big| \quad \text{are independent}$$

whenever $A_1, \ldots, A_n$ are disjoint. In particular, if $\varepsilon := \mu(A)$ is small, then

$$\mathbb{P}\big[\big|\omega \cap A\big| = 1\big] = \varepsilon + O(\varepsilon^2), \quad \mathbb{P}\big[\big|\omega \cap A\big| \geq 2\big] = O(\varepsilon^2).$$

Poisson construction of continuous-time Markov chains

Each generator G has a *random mapping representation*

$$G(x,y) = \sum_{m \in \mathcal{G}} r_m \big(1_{\{m(x) = y\}} - 1_{\{x = y\}} \big),$$

where $(r_m)_{m \in \mathcal{G}}$ are nonnegative rates and $\mathcal{G}$ is a collection of maps $m : S \to S$. Let ω be a Poisson point set on $\mathcal{G} \times \mathbb{R}$ with intensity

$$\mu\big(\{m\} \times A\big) = r_m \, \ell(A) \qquad \big(A \in \mathcal{B}(\mathbb{R})\big),$$

where $\mathcal{B}(\mathbb{R})$ is the Borel-σ-field on $\mathbb{R}$ and ℓ denotes Lebesgue measure. If $\sum_{m \in \mathcal{G}} r_m < \infty$, then we may order the elements of

$$\omega_{s,t} := \omega \cap \mathcal{G} \times (s,t] = \big\{(m_1, t_1), \ldots, (m_n, t_n)\big\}$$

with $t_1 < \cdots < t_n$.

Poisson construction of continuous-time Markov chains

Define random maps $\mathbf{X}_{s,t} : S \to S$ $(s \leq t)$ by

$$\mathbf{X}_{s,t} := m_n \circ \cdots \circ m_1.$$

(Poisson construction of Markov processes) *Define maps* $(\mathbf{X}_{s,t})_{s \leq t}$ *as above in terms of a Poisson point set* ω. *Let* X_0 *be an* S*-valued random variable, independent of* ω. *Then*

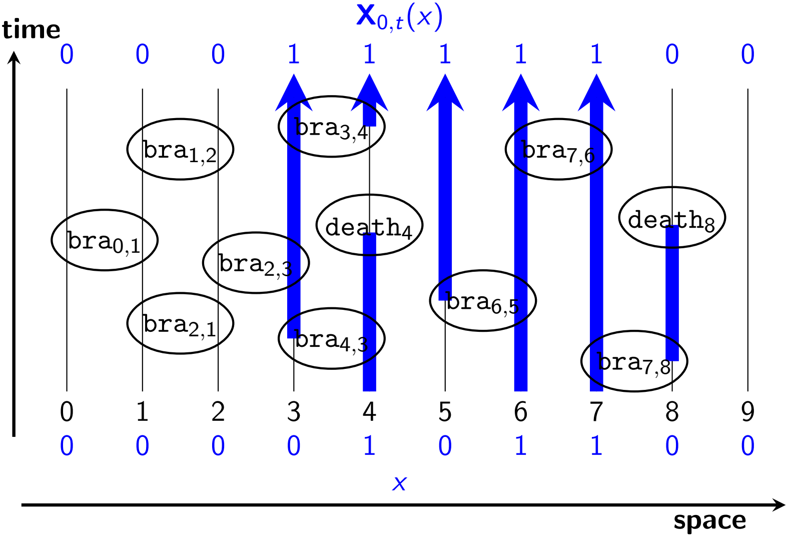
$$X_t := \mathbf{X}_{0,t}(X_0) \qquad (t \geq 0)$$

is a Markov process with generator G.

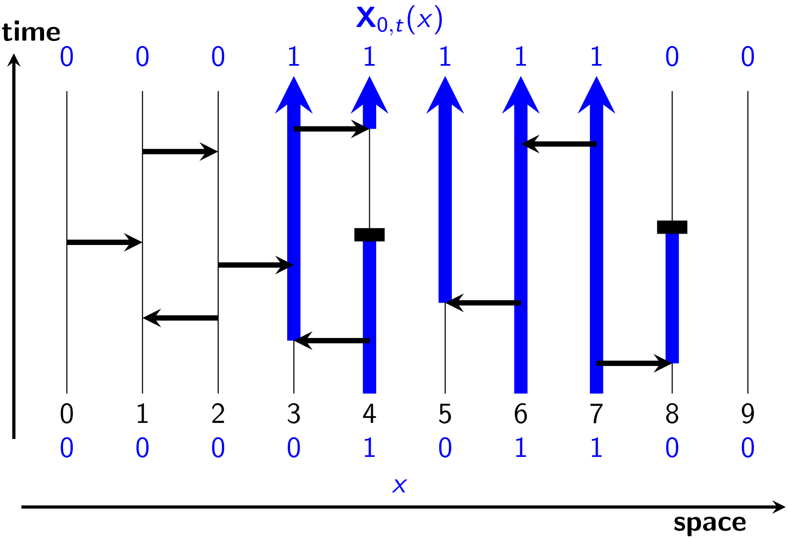
Remark The sample paths of X are right-continuous. We get left-continuous paths by defining

$$\mathbf{X}_{s,t-} \quad \text{in terms of} \quad \omega_{s,t-} := \omega \cap \mathcal{G} \times (s,t).$$

Example: a finite contact process



Example: a finite contact process



Stochastic flows

The random maps $(\mathbf{X}_{s,t})_{s\leq t}$ form a *stochastic flow*

$$\mathbf{X}_{s,s} = 1 \quad \text{and} \quad \mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u}$$

with *independent increments*, in the sense that

$$\mathbf{X}_{t_0,t_1}, \ldots, \mathbf{X}_{t_{n-1},t_n}$$

are independent for each $t_0 < \cdots < t_n$.

Different stochastic flows can define the same Markov process, as there may be many different ways of writing down a random mapping representation

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\},$$

for a given generator G.

Compare: Different SDE's can define the same diffusion process.

Form of the generator

Here we have used that for any function $f : S \to \mathbb{R}$,

$$
\begin{aligned}
Gf(x) &= \sum_y G(x,y) f(y) \\
&= \sum_y \sum_{m \in \mathcal{G}} r_m \big(1_{\{m(x)=y\}} - 1_{\{x=y\}}\big) f(y) \\
&= \sum_{m \in \mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\}.
\end{aligned}
$$

Then, as $t \downarrow 0$,

$$
\begin{aligned}
\mathbb{E}^x\big[f(X_t)\big] &= f(x) + tGf(x) + O(t^2) \\
&= \Big(1 - t \sum_{m \in \mathcal{G}} r_m\Big) f(x) + t \sum_{m \in \mathcal{G}} r_m f\big(m(x)\big) + O(t^2).
\end{aligned}
$$

Example: the contact process

Let (Λ, E) be a finite directed graph. The *contact process* on (Λ, E) with *infection rate* $\lambda \geq 0$ is the continuous-time Markov chain $X = (X_t)_{t\geq 0}$ with state space $\{0,1\}^\Lambda$ and generator

$$
\begin{aligned}
Gf(x) := \lambda \sum_{(i,j)\in E} &\big\{ f\big(\mathtt{bra}_{ij} x\big) - f(x) \big\} \\
&+ \sum_{i\in\Lambda} \big\{ f\big(\mathtt{death}_i x\big) - f(x) \big\}.
\end{aligned}
$$

We would like to define contact processes on infinite lattices, e.g., $\Lambda = \mathbb{Z}^d$ with $E = \{(i,j) : |i-j| = 1\}$.
If Λ is infinite, then we equip $\{0,1\}^\Lambda$ with the *product topology* which corresponds to *pointwise convergence*

$$
x_n \to x \quad \text{iff} \quad x_n(i) \to x(i) \quad \forall i \in \Lambda.
$$

By *Tychonoff's theorem*, $\{0,1\}^\Lambda$ is a compact space.

Feller processes

Let S be a compact metrizable space. Let $\mathcal{C}(S)$ denote the space of all continuous functions $f : S \to \mathbb{R}$, equipped with the supremumnorm $\| \cdot \|_\infty$, and let $\mathcal{M}_1(S)$ denote the space of all probability measures on S, equipped with the topology of weak convergence.

For any probability kernel $K(x, \mathrm{d}y)$ on S and measurable function $f : S \to \mathbb{R}$, we define

$$Kf(x) := \int_S K(x, \mathrm{d}y) f(y) \qquad (x \in S).$$

By definition, a *Feller semigroup* is a collection of probability kernels $(P_t)_{t \geq 0}$ on S such that:

(i) $P_0 = 1$ and $P_s P_t = P_{s+t}$ $(s, t \geq 0)$,

(ii) $S \times [0, \infty) \ni (x, s) \mapsto P_s(x, \cdot\,) \in \mathcal{M}_1(S)$ is continuous.

Feller processes

The *generator* of a Feller semigroup is the operator G defined as

$$Gf := \lim_{t \downarrow 0} t^{-1}(P_t f - f) \qquad\qquad (*),$$

with domain

$$\mathcal{D}(G) := \big\{ f \in \mathcal{C}(S) : \text{ the limit in } (*) \text{ exists in the norm } \| \cdot \|_\infty \big\}.$$

The *Hille-Yosida theorem* gives necessary and sufficient conditions for an operator G to generate a Feller semigroup.

The domain $\mathcal{D}(G)$ is a dense subspace of $\mathcal{C}(S)$. For each $f \in \mathcal{D}(G)$, the function $t \mapsto P_t f$ is given by the unique solution to the *backward equation*

$$\tfrac{\partial}{\partial t} P_t f = G P_t f \qquad (t \geq 0) \quad \text{with} \quad P_0 f = 1.$$

Feller processes

Let S be a compact metrizable space and let $(P_t)_{t\geq 0}$ be a Feller semigroup of transition kernels on S.
Then, for each probability law μ on S, there exists a process $(X_t)_{t\geq 0}$ such that

- The sample paths $t \mapsto X_t$ are a.s. *cadlag*, i.e., right-continuous with left limits.
- $\mathbb{P}[X_0 \in \cdot\,] = \mu$.
- $\mathbb{P}^x\big[X_u \in \cdot \,\big|\, (X_s)_{0\leq s\leq t}\big] = P_{u-t}(X_t, \cdot\,)$ a.s. $(t \leq u)$.

Such a process is called a *Feller process*.

Poisson construction of Interacting Particle Systems

For any map $m : S^\Lambda \to S^\Lambda$, let

$$\mathcal{D}(m) := \big\{ i \in \Lambda : \exists x \in S^\Lambda \text{ s.t. } m(x)(i) \neq x(i) \big\}$$

denote the set of lattice points whose values can possibly be changed by m. By definition, m is a *local map* if and only if

(i) $\mathcal{D}(m)$ is finite,

(ii) m is continuous w.r.t. the product topology.

Say that a site $j \in \Lambda$ is *m-relevant* for some $i \in \Lambda$ if

$$\exists x, y \in S^\Lambda \text{ s.t. } m(x)(i) \neq m(y)(i) \text{ and } x(k) = y(k)\ \forall k \neq j,$$

and define

$$\mathcal{R}_i(m) := \big\{ j \in \Lambda : j \text{ is } m\text{-relevant for } i \big\}.$$

Remark A map $m : S^\Lambda \to S^\Lambda$ is continuous w.r.t. the product topology if and only if for each $i \in \Lambda$:

(i) $\mathcal{R}_i(m)$ is finite,

(ii) $x(j) = y(j)$ for all $j \in \mathcal{R}_i(m)$ implies $m(x)(i) = m(y)(i)$.

Trivially

$$\mathcal{R}_i(m) = \{i\} \quad \text{if } i \notin \mathcal{D}(m).$$

In our example:

$$\begin{aligned} &\mathcal{D}(\mathtt{bra}_{ij}) = \{j\} && \mathcal{R}_j(\mathtt{bra}_{ij}) = \{i, j\}, \\ &\mathcal{D}(\mathtt{death}_i) = \{i\} && \mathcal{R}_i(\mathtt{death}_i) = \emptyset. \end{aligned}$$

Poisson construction of Interacting Particle Systems

Let $\mathcal{G}$ me a collection of local maps $m : S^\Lambda \to S^\Lambda$ and let $(r_m)_{m\in\mathcal{G}}$ be nonnegative constants. As before, we let ω be a Poisson point set on $\mathcal{G}\times\mathbb{R}$ with intensity

$$\mu\big(\{m\}\times A\big) = r_m\,\ell(A) \qquad \big(A\in\mathcal{B}(\mathbb{R})\big).$$

For infinite lattices (e.g., for the contact process on $\mathbb{Z}^d$) we typically have $\sum_{m\in\mathcal{G}} r_m = \infty$ which means

$$\omega_{s,t} := \omega\cap\mathcal{G}\times(s,t]$$

has a.s. infinitely many elements for any $s<t$. As a result, we cannot order the elements of $\omega_{s,t}$ by their time coordinates. Nevertheless, for each finite subset $\tilde\omega\subset\omega_{s,t}$, we can still define

$$\mathbf{X}^{\tilde\omega}_{s,t}(x) := m_n\circ\cdots\circ m_1(x) \qquad (x\in S^\Lambda,\ s\le t),$$

$$\text{where}\quad \tilde\omega = \{(m_1,t_1),\ldots,(m_n,t_n)\}\quad\text{with}\quad t_1<\cdots<t_n.$$

Theorem Assume that

$$\sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G}\\ \mathcal{D}(m)\ni i}} r_m|\mathcal{R}_i(m)| < \infty.$$

Then a.s., for each $x \in S^\Lambda$ the limit

$$\mathbf{X}_{s,t}(x) := \lim_{\tilde\omega_n\uparrow\omega_{s,t}} \mathbf{X}^{\tilde\omega_n}_{s,t}(x)$$

exists, does not depend on the sequence of finite sets $\tilde\omega_n \uparrow \omega_{s,t}$, and defines a stochastic flow $(\mathbf{X}_{s,t})_{s\leq t}$. If X_0 is an S^Λ-valued random variable, independent of ω, then

$$X_t := \mathbf{X}_{0,t}(X_0) \qquad (t \geq 0)$$

defines a Feller process with generator

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m\big\{f\big(m(x)\big) - f\big(x\big)\big\}.$$

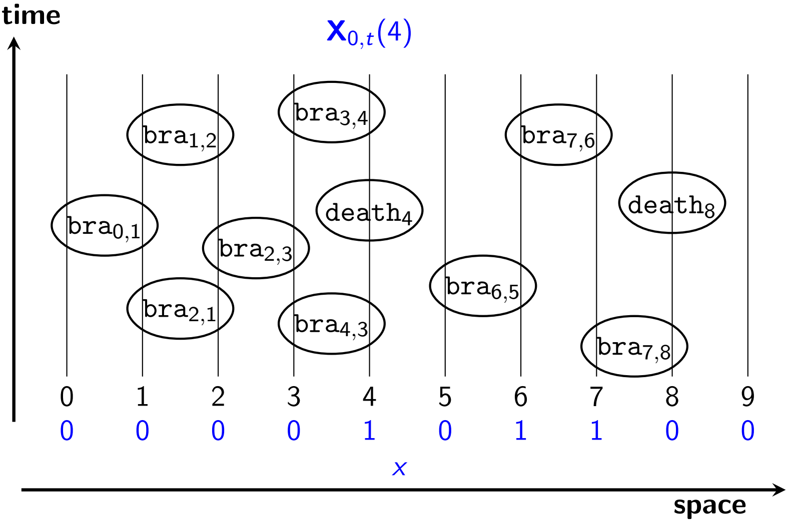
Lemma 1 One has

$$\mathbb{E}\big[\big|\mathcal{R}_i(\mathbf{X}_{s,t})\big|\big] \leq e^{R(t-s)} \qquad (s \leq t),$$

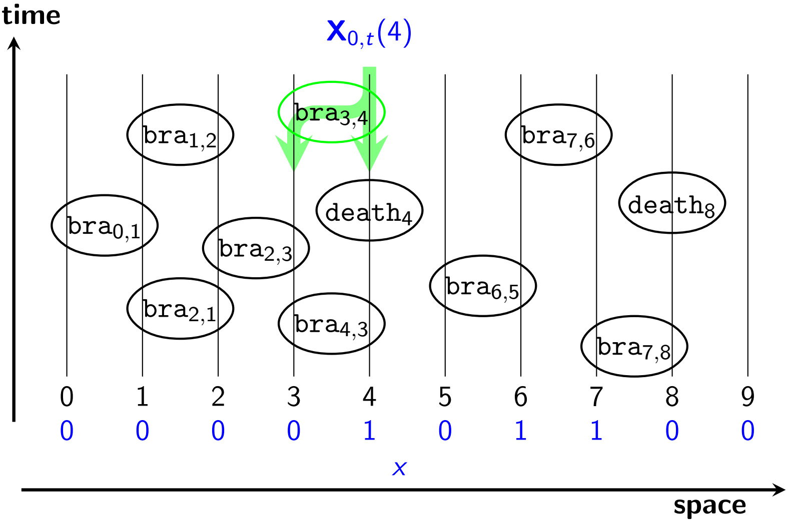
where

$$R := \sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G} \\ \mathcal{D}(m)\ni i}} r_m\big(|\mathcal{R}_i(m)| - 1\big). \tag{1}$$

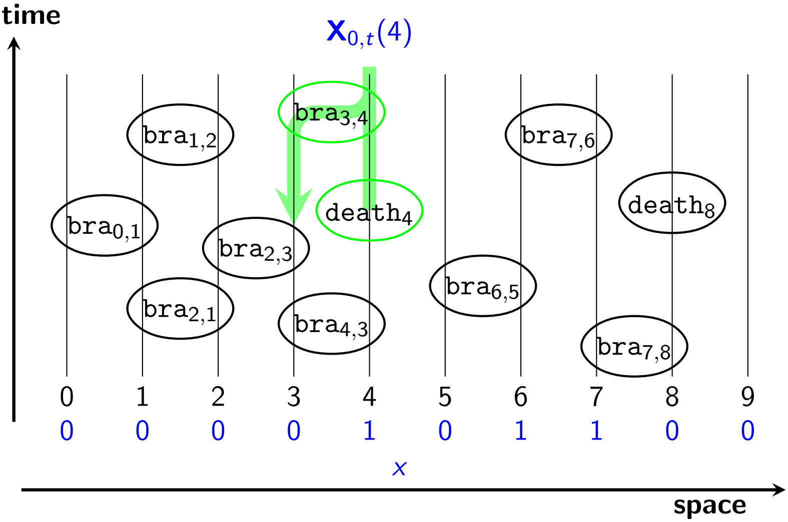
Poisson construction of Interacting Particle Systems



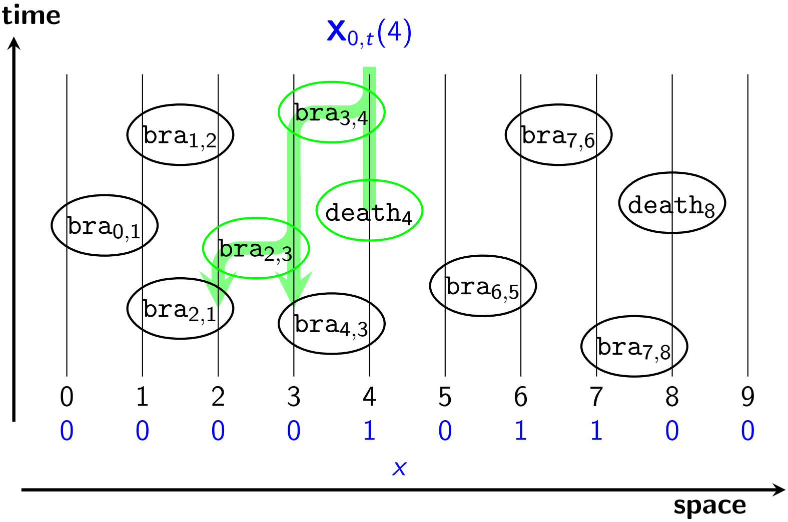
Poisson construction of Interacting Particle Systems



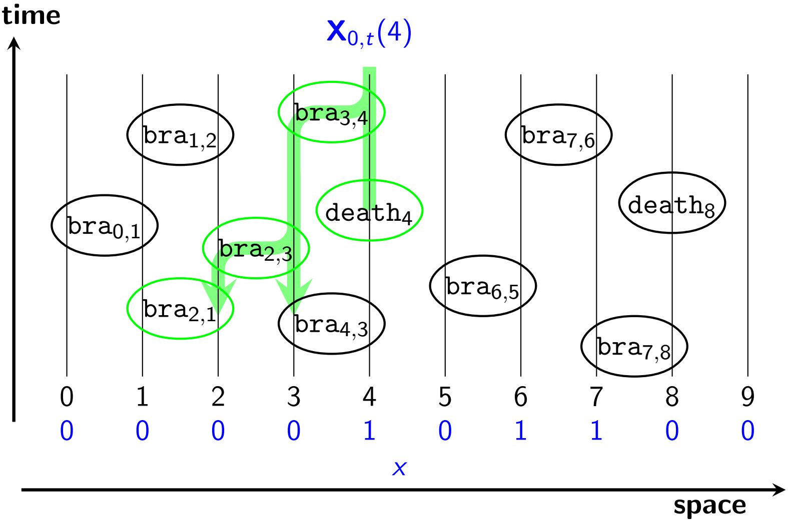
Poisson construction of Interacting Particle Systems



Poisson construction of Interacting Particle Systems



Poisson construction of Interacting Particle Systems



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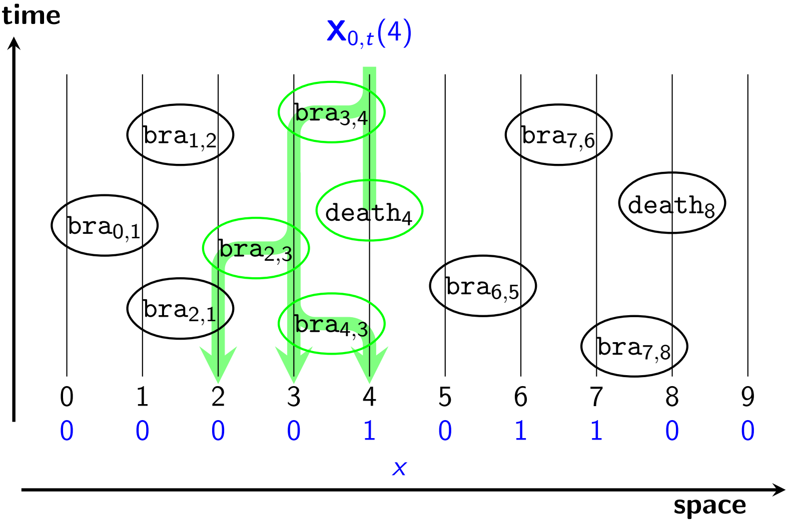
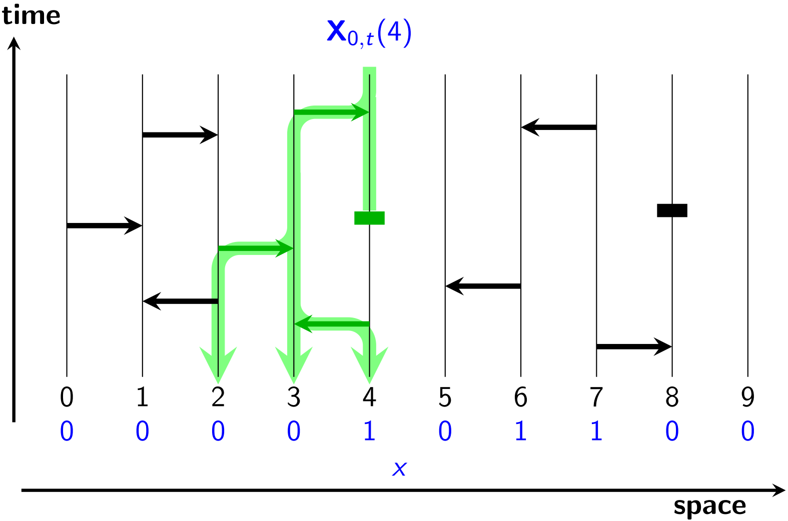


Poisson construction of Interacting Particle Systems



Generator construction of Interacting Particle Systems

For any $f \in \mathcal{C}(S^\Lambda)$ and $i \in \Lambda$, we define

$$\delta f(i) := \sup\big\{|f(x) - f(y)| : x, y \in S^\Lambda,\ x(j) = y(j)\ \forall j \neq i\big\},$$
$$\mathcal{C}_{\rm sum} := \big\{f \in \mathcal{C}(S^\Lambda) : |||f||| := \sum_{i\in\Lambda} \delta f(i) < \infty\big\}$$

We call $f \in \mathcal{C}_{\rm sum}$ a function of *summable variation*. One has

$$\big|f(x) - f(y)\big| \leq \sum_{i:\, x(i)\neq y(i)} \delta f(i) \leq |||f|||.$$

Lemma 2 If $(P_t)_{t\geq 0}$ is the Feller semigroup of an interacting particle system with generator G, then $\mathcal{C}_{\rm summ} \subset \mathcal{D}(G)$ and

$$|||P_t f||| \leq e^{Rt} |||f||| \qquad (t \geq 0),$$

where R is the constant in (1).

Generator construction of Interacting Particle Systems

For any probability kernel κ on S^Λ, set

$$\mathcal{D}(\kappa) := \big\{ i \in \Lambda : \exists x \text{ s.t. } \kappa\big(x, \{y : y(i) \neq x(i)\}\big) > 0 \big\}.$$

By definition, a *local probability kernel* is a probab. kernel such that

(i) $\mathcal{D}(\kappa)$ is finite,

(ii) $x \mapsto \kappa(x, \cdot\,)$ is continuous w.r.t. weak convergence.

Liggett (1985) writes generators in the form

$$Gf(x) = \sum_{\kappa \in \mathcal{K}} r_\kappa \Big\{ \int \kappa(x, \mathrm{d}y)\, f(y) - f(x) \Big\},$$

where $\mathcal{K}$ is a collection of local probability kernels and $(r_\kappa)_{\kappa \in \mathcal{K}}$ are nonnegative rates. Under suitable conditions on the rates, he proves the closure of G generates a Feller semigroup and

$$|||P_t f||| \leq e^{R' t} |||f||| \qquad (t \geq 0)$$

for some constant $R' < \infty$.

Proofs

Proof of Lemma 1 By a cut-off argument, w.l.o.g. we assume that Λ is finite. Then we can apply our previous result about the Poisson construction of continuous-time Markov chains to conclude that

$$\big(\mathcal{R}_i(\mathbf{X}_{-t,0})\big)_{t\geq 0}$$

is a set-valued Markov process that can be bounded from above by a process that jumps as

$$A \mapsto (A\backslash\mathcal{D}(m)) \cup \bigcup_{i\in\mathcal{D}(m)\cap A} \mathcal{R}_i(m) \quad \text{with rate } r_m$$

for each $m \in \mathcal{G}$. Let $(P_t)_{t\geq 0}$ denote the semigroup of the set-valued process and let G be its generator. Let f be the function $f(A) := |A|$. Then

$$Gf(A) = \sum_{m\in\mathcal{G}} r_m \Big\{ f\Big((A\backslash\mathcal{D}(m)) \cup \bigcup_{i\in\mathcal{D}(m)\cap A} \mathcal{R}_i(m)\Big) - f(A) \Big\}$$

Proofs

It follows that

$$\begin{aligned} Gf(A) &\le \sum_{m\in\mathcal{G}} r_m \sum_{i\in\mathcal{D}(m)\cap A} \big(|\mathcal{R}_i(m)|-1\big) \\ &= \sum_{i\in A}\ \sum_{m:\,\mathcal{D}(m)\ni i} r_m\big(|\mathcal{R}_i(m)|-1\big) \le R|A|, \end{aligned}$$

and hence

$$\frac{\partial}{\partial t}\big(e^{-Rt}P_t f\big) = -Re^{-Rt}P_t f + e^{-Rt}P_t G_n f = e^{-Rt}P_t(G_n f - Rf) \le 0$$

and therefore $e^{-Rt}P_t f \le e^{-R0}P_0 f = f$, which means in particular that

$$\mathbb{E}\big[|\mathcal{R}_i(\mathbf{X}_{-t,0})|\big] = P_t f(\{i\}) \le e^{Rt} f(\{i\}) = e^{Rt} \qquad (t \ge 0).$$

$\blacksquare$

Proof of Lemma 2 We want to prove that

$$|||P_t f||| \le e^{Rt} |||f||| \quad \text{with} \quad |||f||| := \sum_{i\in\Lambda} \delta f(i).$$

Fix $i \in \Lambda$ and let $x, y \in S^\Lambda$ with $x(j) = y(j)\ \forall j \neq i$. Then

$$\begin{aligned}
\big|P_t f(x) - P_t f(y)\big| &= \big|\mathbb{E}[f(\mathbf{X}_{0,t}(x))] - \mathbb{E}[f(\mathbf{X}_{0,t}(y))]\big| \\
&\le \mathbb{E}\big[\big|f(\mathbf{X}_{0,t}(x)) - f(\mathbf{X}_{0,t}(y))\big|\big] \\
&\le \sum_j \mathbb{P}\big[\mathbf{X}_{0,t}(x)(j) \neq \mathbf{X}_{0,t}(y)(j)\big]\,\delta f(j) \\
&\le \sum_j \mathbb{P}\big[i \in \mathcal{R}_j(\mathbf{X}_{0,t})\big]\,\delta f(j)
\end{aligned}$$

It follows that

$$\sum_i \delta P_t f(i) \le \sum_{ij} \mathbb{P}\big[i \in \mathcal{R}_j(\mathbf{X}_{0,t})\big]\,\delta f(j) \le e^{Rt} \sum_j \delta f(j).$$

Unique ergodicity

For any probability measure μ on S^Λ, write

$$\mu P_t := \int \mu(\mathrm{d}x) P_t(x,\,\cdot\,) \qquad (t \geq 0).$$

If $X = (X_t)_{t\geq 0}$ has initial law $\mathbb{P}[X_0 \in \,\cdot\,] = \mu$, then,

$$\mathbb{P}[X_t \in A] = \int \mathbb{P}[X_0 \in \mathrm{d}x]\mathbb{P}[X_t \in A \,|\, X_0 \in \mathrm{d}x] = \int \mu(\mathrm{d}x) P_t(x, A),$$

i.e., X_t has law μP_t.
By definition, a probability measure ν on S^Λ is an *invariant law* if

$$\nu P_t = \nu \qquad (t \geq 0).$$

The process $X = (X_t)_{t\geq 0}$ is stationary if and only if $\mathbb{P}[X_0 \in \,\cdot\,]$ is an invariant law.

Unique ergodicity

Proposition If the constant R from (1) satisfies $R < 0$, then the interacting particle system has a unique invariant law ν, and

$$\mu P_t \underset{t\to\infty}{\Longrightarrow} \nu$$

for any initial law μ.

Proof If $R < 0$, then

$$\mathbb{P}\big[\mathcal{R}_i(\mathbf{X}_{0,t}) = \emptyset\big] \underset{t\to\infty}{\longrightarrow} 1.$$

As a consequence, the limit

$$X_t(i) := \lim_{s\to\infty} \mathbf{X}_{t-s,t}(x)(i) \qquad (i \in \Lambda)$$

exists a.s. for each $i \in \Lambda$ and $t \in \mathbb{R}$, and does not depend on the choice of $x \in S^\Lambda$.

Unique ergodicity

One can check that this defines a stationary process $(X_t)_{t\in\mathbb{R}}$. Let Y have law μ and be independent of the Poisson processes used to construct the flow $(\mathbf{X}_{s,t})_{s\leq t}$. Then $\mathbf{X}_{-t,0}(Y)$ has law μP_t and

$$\mathbf{X}_{-t,0}(Y) \underset{t\to\infty}{\longrightarrow} X_0 \qquad \text{a.s.},$$

proving that $\mu P_t \underset{t\to\infty}{\Longrightarrow} \nu$. ∎

Remark 1 This method is called *coupling from the past*.

Remark 2 The proof works more generally if

$$\mathbb{P}\big[\mathcal{R}_i(\mathbf{X}_{0,t}) = \emptyset\big] \underset{t\to\infty}{\longrightarrow} 1 \qquad (i \in \Lambda),$$

which may happen even if $R \geq 0$.

The contact process

Recall that

$$\mathcal{D}(\mathtt{bra}_{ij}) = \{j\} \qquad \mathcal{R}_j(\mathtt{bra}_{ij}) = \{i, j\},$$
$$\mathcal{D}(\mathtt{death}_i) = \{i\} \qquad \mathcal{R}_i(\mathtt{death}_i) = \emptyset.$$

For the nearest neighbor contact process on $\mathbb{Z}^d$, this yields

$$R = 2d\lambda - 1,$$

which is < 0 iff $\lambda < 1/2d$. Since $\mathbf{X}_{s,t}(\underline{0}) = \underline{0}$, we conclude that for $\lambda < 1/2d$, the measure $\delta_{\underline{0}}$ is the unique invariant law.

The contact process

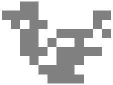
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 0$.

The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 1$.

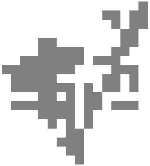
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 2$.

The contact process



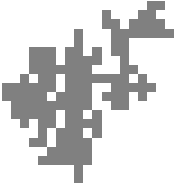
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 3$.

The contact process



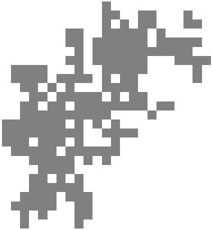
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 4$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 5$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 6$.

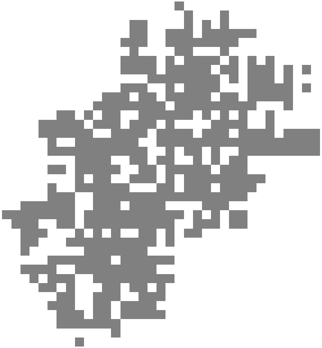
The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 7$.

The contact process

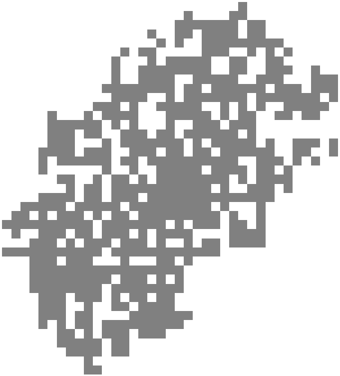
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 8$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 9$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 10$.

The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 11$.

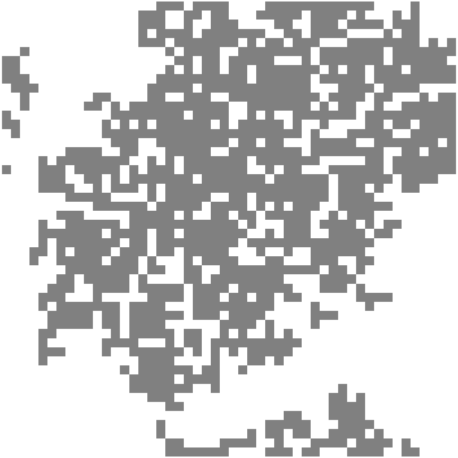
The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 12$.

The contact process

Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 13$.

The contact process

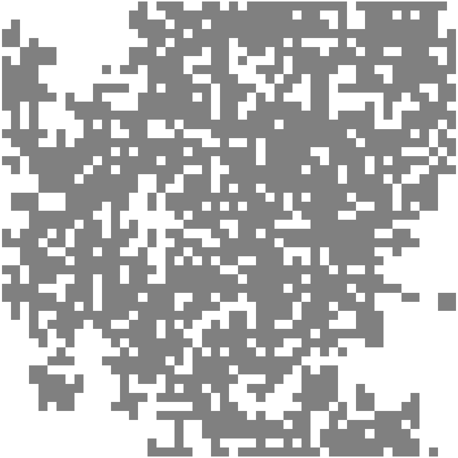


Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 14$.

The contact process

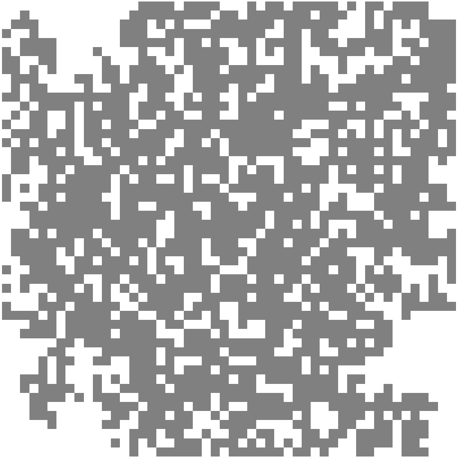
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 15$.

The contact process



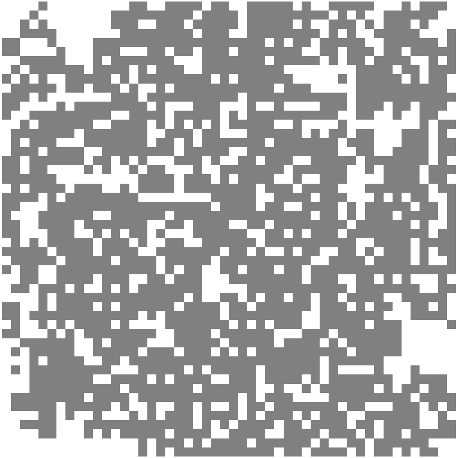
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 16$.

The contact process



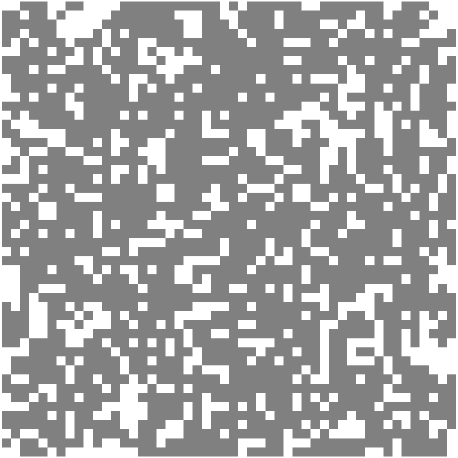
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 17$.

The contact process



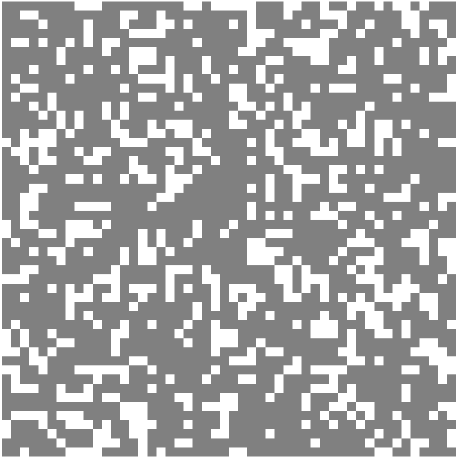
Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 18$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 19$.

The contact process



Contact process with infection rate $\lambda = 2$ and death rate $d = 1$.
Time $t = 20$.

A stochastic Ising model

Let (Λ, E) be an undirected graph and let $S = \{-1, +1\}$. For any $x \in S^\Lambda$, we call

$$M_i(x) := \sum_{j \colon \{i,j\} \in E} x(j)$$

the *local magnetization* around $i \in \Lambda$. Let $\kappa_i(x, \cdot\,)$ denote the law of a random variable X such that

$$\mathbb{P}[X(i) = \pm 1] = \frac{e^{\beta \pm M_i(x)}}{e^{\beta \pm M_i(x)} + e^{\beta \mp M_i(x)}},$$

and $X(j) = x(j)$ a.s. for all $j \neq i$. Then κ_i is a local probability kernel and

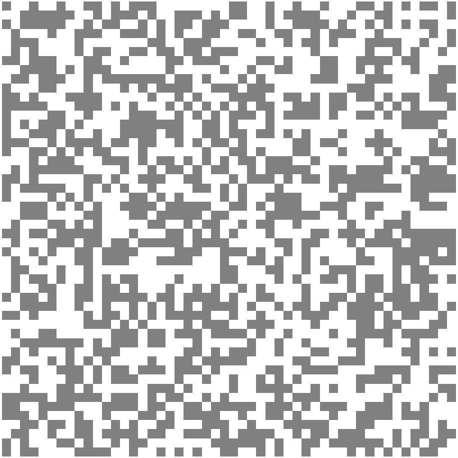
$$Gf(x) = \sum_{i \in \Lambda} \Big(\int \kappa_i(x, \mathrm{d}y) f(y) - f(x) \Big)$$

defines the generator of a *stochastic Ising model* with *Glauber dynamics*.

A stochastic Ising model

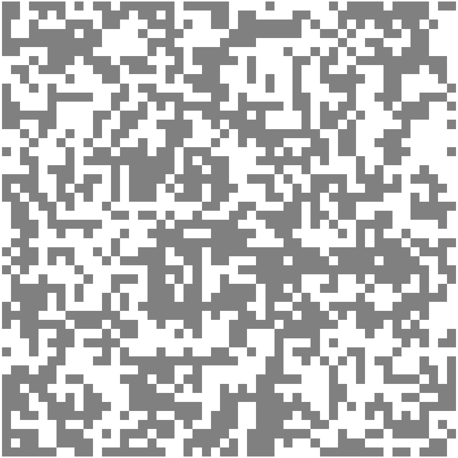
When the parameter β is large, nearby spins like have the same sign.
We start the process in product measure for different values of β and see what happens.

The Ising model



$\beta = 0.3$, time $t = 0$.

The Ising model

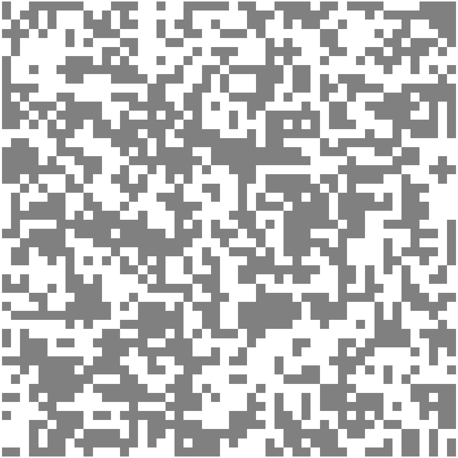


$\beta = 0.3$, time $t = 1$.

The Ising model

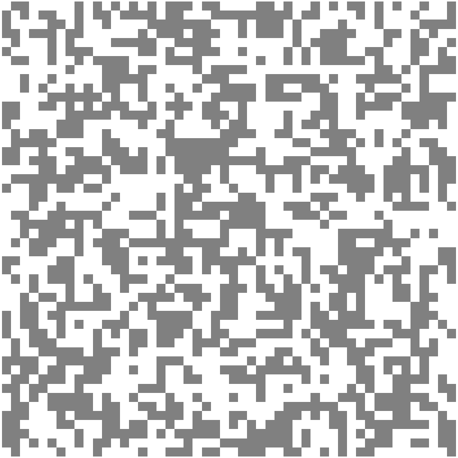
$\beta = 0.3$, time $t = 2$.

The Ising model



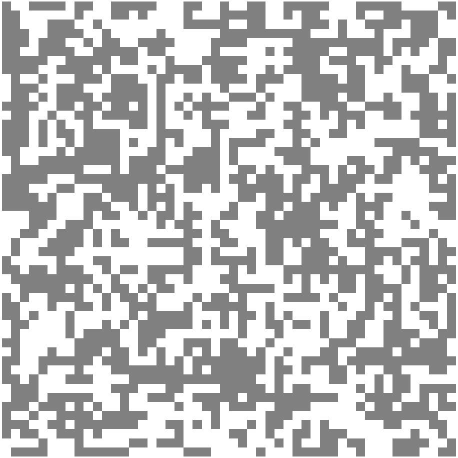
$\beta = 0.3$, time $t = 4$.

The Ising model



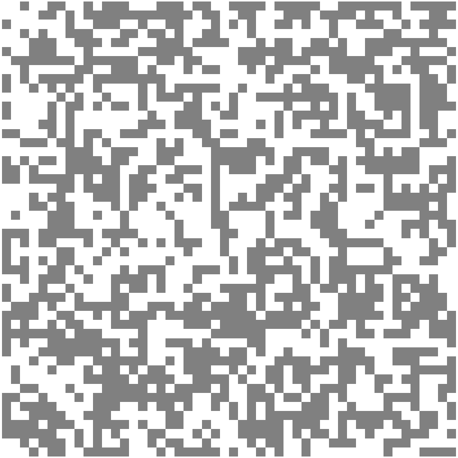
$\beta = 0.3$, time $t = 8$.

The Ising model



$\beta = 0.3$, time $t = 16$.

The Ising model

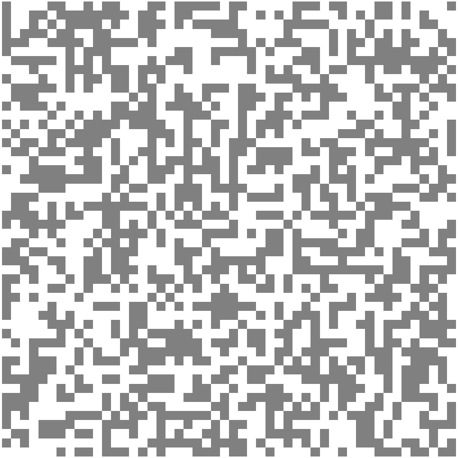


$\beta = 0.3$, time $t = 32$.

The Ising model

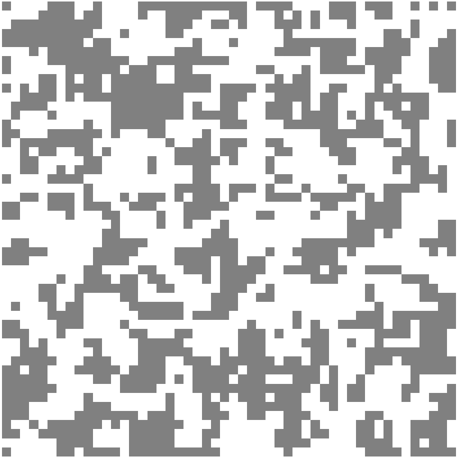
$\beta = 0.3$, time $t = 64$.

The Ising model



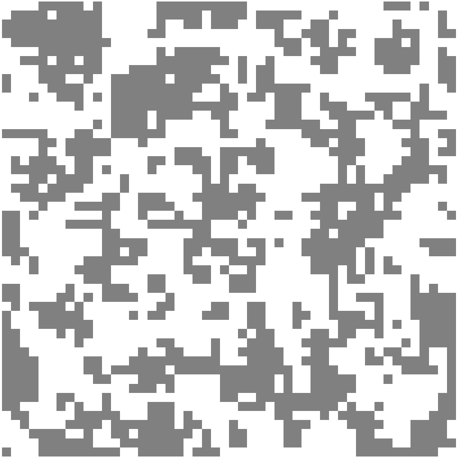
$\beta = 0.7$, time $t = 0$.

The Ising model



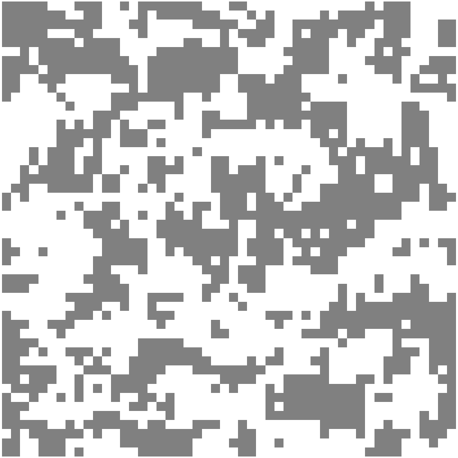
$\beta = 0.7$, time $t = 1$.

The Ising model



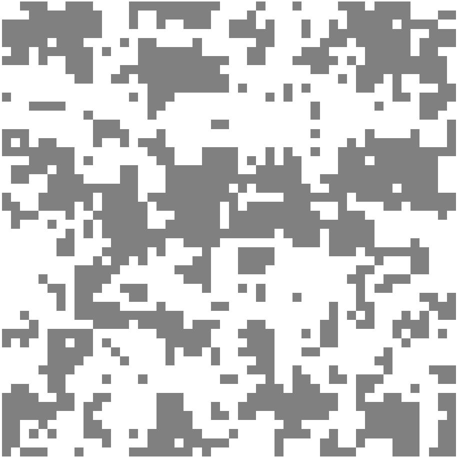
$\beta = 0.7$, time $t = 2$.

The Ising model



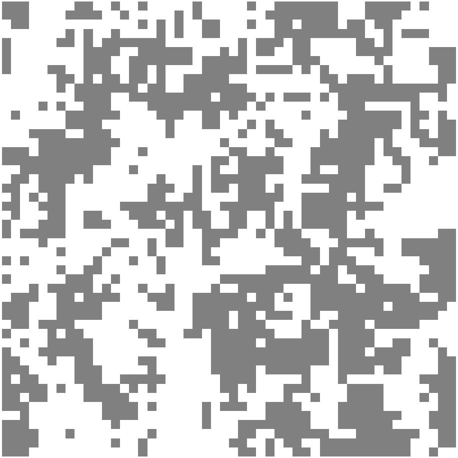
$\beta = 0.7$, time $t = 4$.

The Ising model



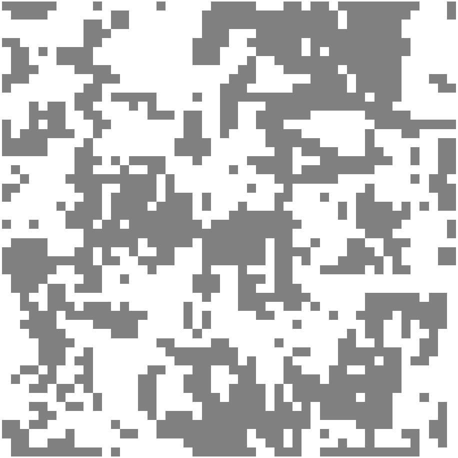
$\beta = 0.7$, time $t = 8$.

The Ising model



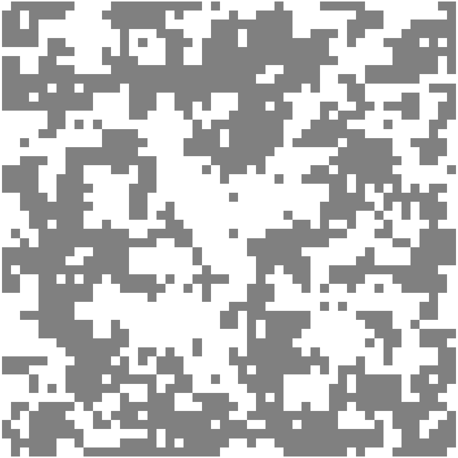
$\beta = 0.7$, time $t = 16$.

The Ising model



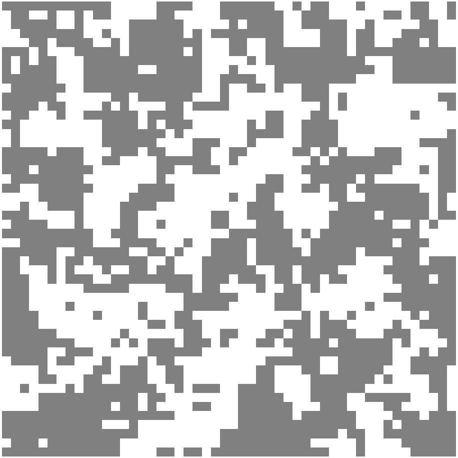
$\beta = 0.7$, time $t = 32$.

The Ising model



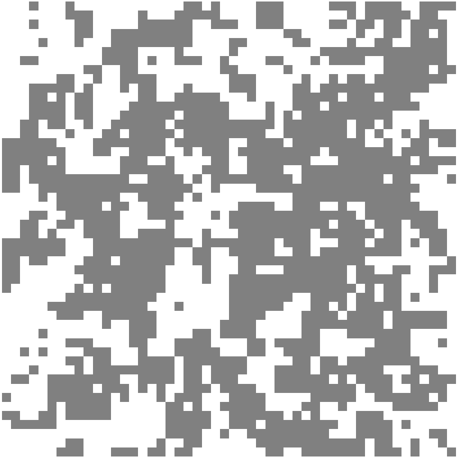
$\beta = 0.7$, time $t = 64$.

The Ising model



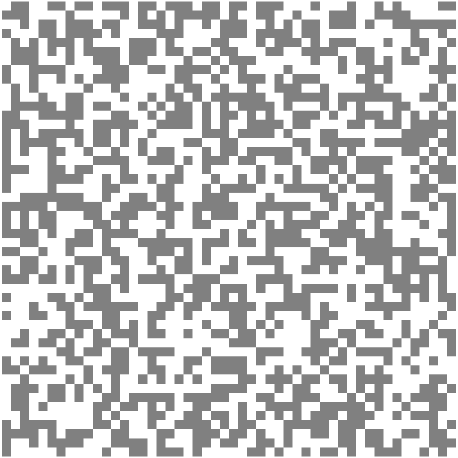
$\beta = 0.7$, time $t = 125$.

The Ising model



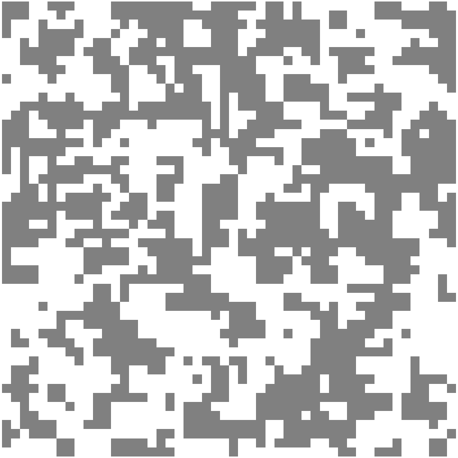
$\beta = 0.7$, time $t = 250$.

The Ising model



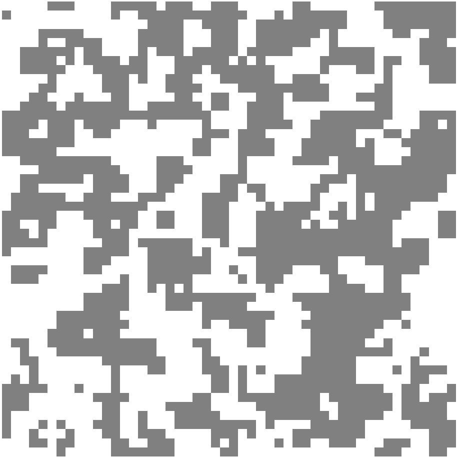
$\beta = 1$, time $t = 0$.

The Ising model



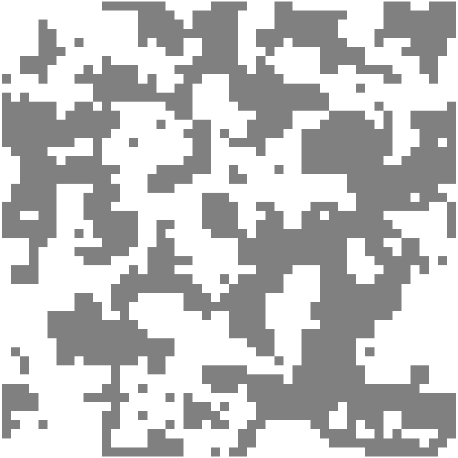
$\beta = 1$, time $t = 1$.

The Ising model



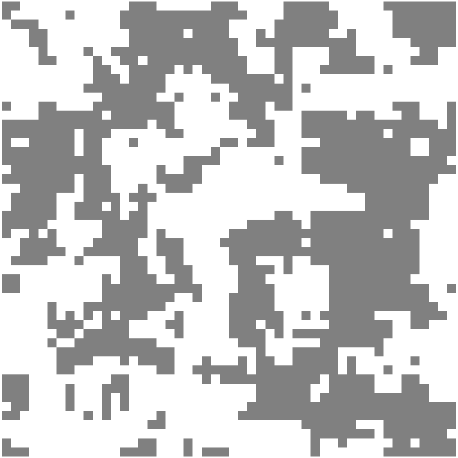
$\beta = 1$, time $t = 2$.

The Ising model



$\beta = 1$, time $t = 4$.

The Ising model

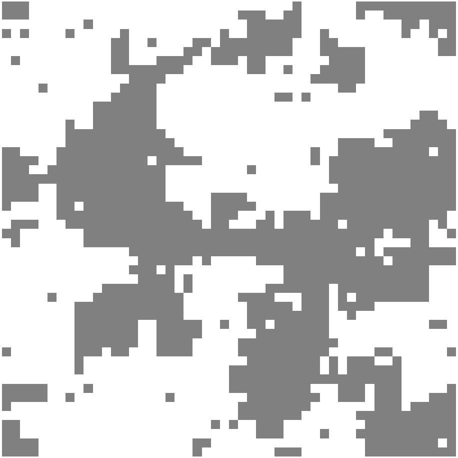


$\beta = 1$, time $t = 8$.

The Ising model

$\beta = 1$, time $t = 16$.

The Ising model

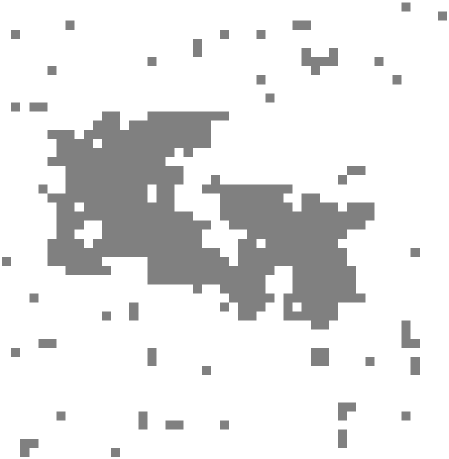


$\beta = 1$, time $t = 32$.

The Ising model

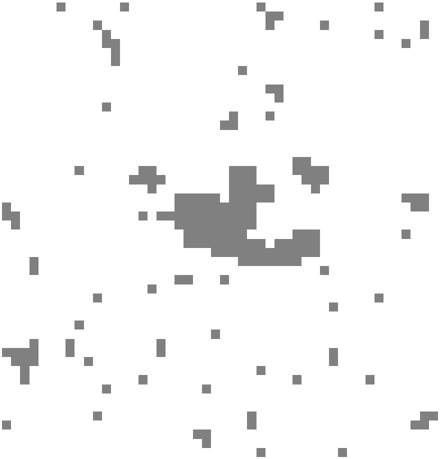
$\beta = 1$, time $t = 64$.

The Ising model



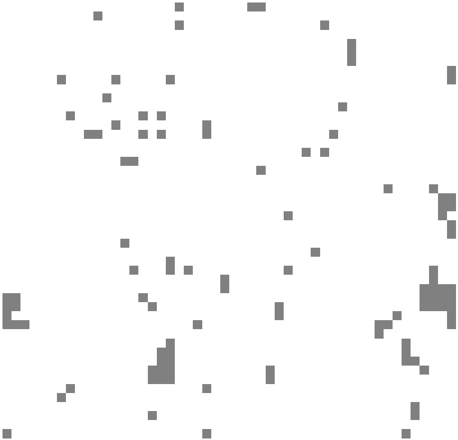
$\beta = 1$, time $t = 125$.

The Ising model



$\beta = 1$, time $t = 250$.

The Ising model



$\beta = 1$, time $t = 500$.

A stochastic Potts model

Instead of allowing only two states $-1, +1$, we can more generally allow $q \geq 2$ states $1, \ldots, q$.

Each person i chooses a new state at times of a Poisson process with rate 1.

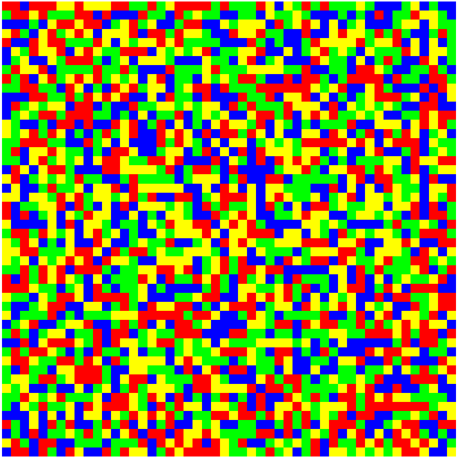
The probability that the newly chosen state is $k \in \{1, \ldots, q\}$ equals

$$\frac{e^{\beta M_i(k)}}{\sum_{m=1}^{q} e^{\beta M_i(m)}},$$

where $M_i(k)$ denotes the number of neighbors of i that are in the state k.

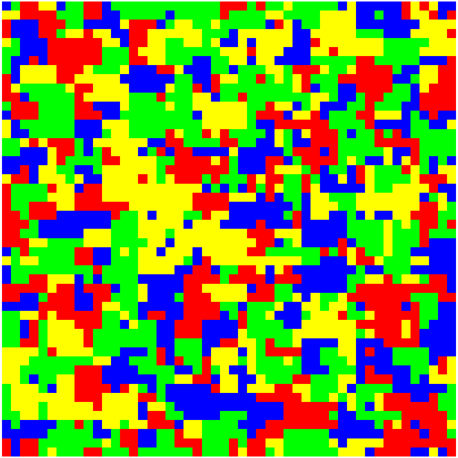
Setting $q = 2$ and replacing β by 2β yields the Ising model.

The Potts model



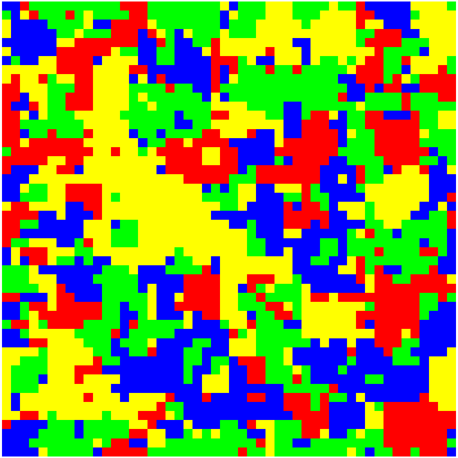
$\beta = 1.2$, time $t = 0$.

The Potts model



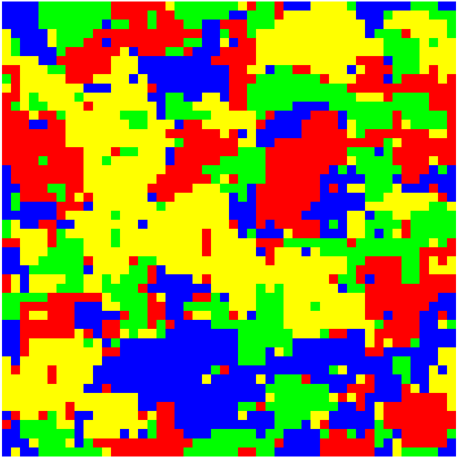
$\beta = 1.2$, time $t = 1$.

The Potts model



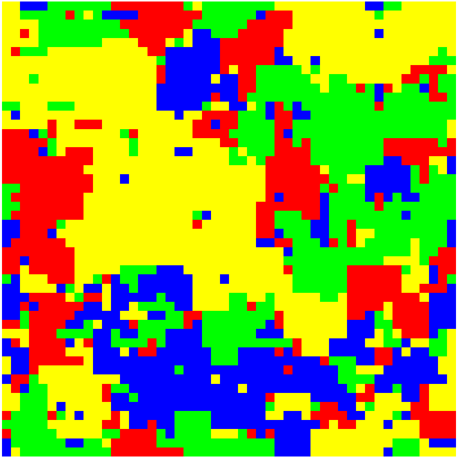
$\beta = 1.2$, time $t = 2$.

The Potts model



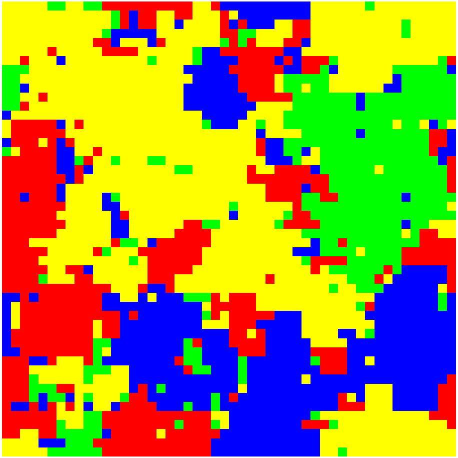
$\beta = 1.2$, time $t = 4$.

The Potts model



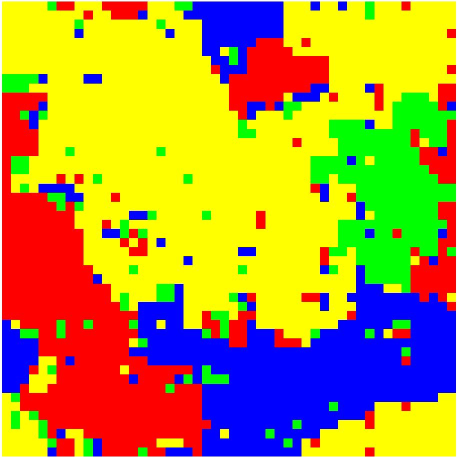
$\beta = 1.2$, time $t = 8$.

The Potts model



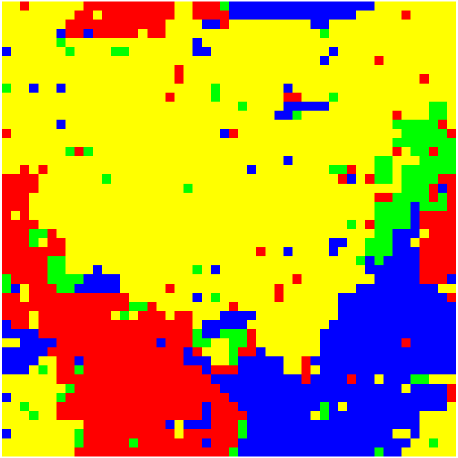
$\beta = 1.2$, time $t = 16$.

The Potts model



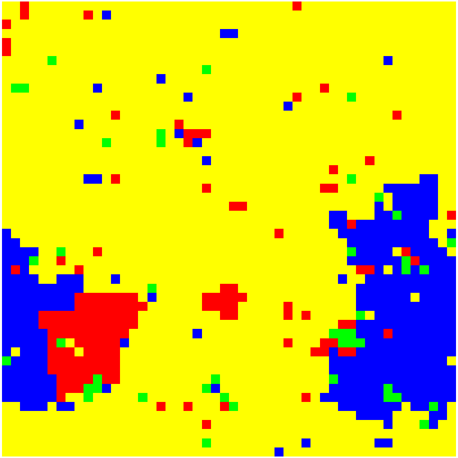
$\beta = 1.2$, time $t = 32$.

The Potts model



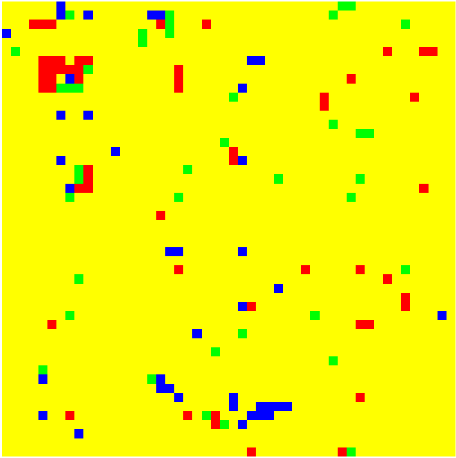
$\beta = 1.2$, time $t = 64$.

The Potts model



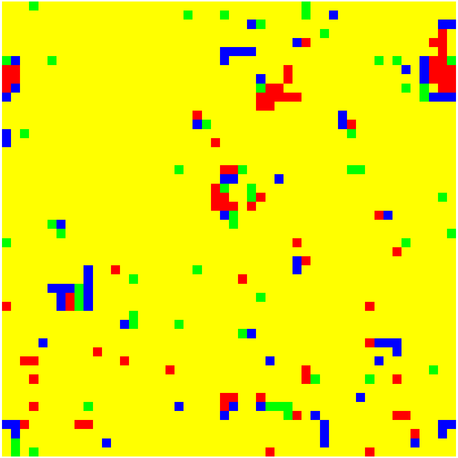
$\beta = 1.2$, time $t = 125$.

The Potts model



$\beta = 1.2$, time $t = 250$.

The Potts model



$\beta = 1.2$, time $t = 500$.

The voter model

In the *voter model*, each site $i \in \Lambda$ is occupied by an organism with genetic type $x(i) \in S$.

The organism at site i dies at times of a Poisson process with rate 1 and is replaced by the offspring of a randomly chosen neighbor.

Using the *voter map*

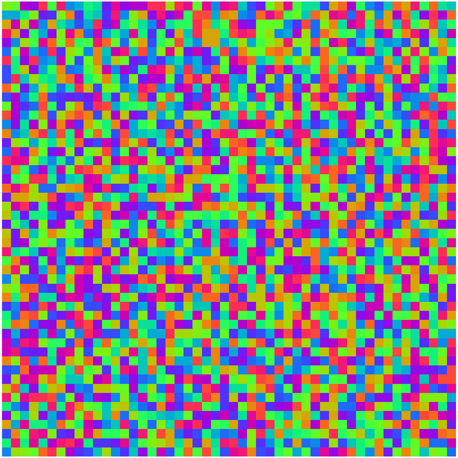
$$\mathtt{vot}_{ji}(x)(k) := \begin{cases} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise,} \end{cases}$$

we can give the following random mapping representation of the generator:

$$Gf(x) = \sum_{(i,j) \in E} \frac{1}{N_i} \big\{ f\big(\mathtt{vot}_{ji} x\big) - f\big(x\big) \big\},$$

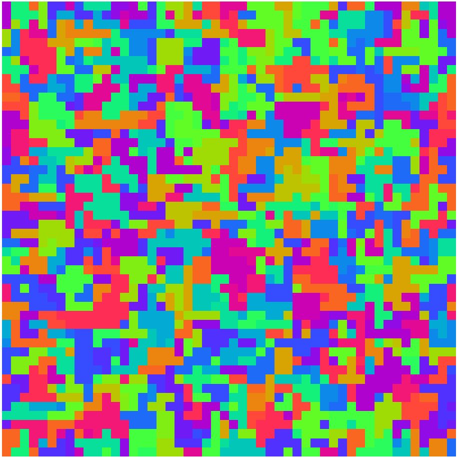
where $N_i := \big|\{j : (j,i) \in E\}\big|$ is the number of neigbors of i.

The voter model



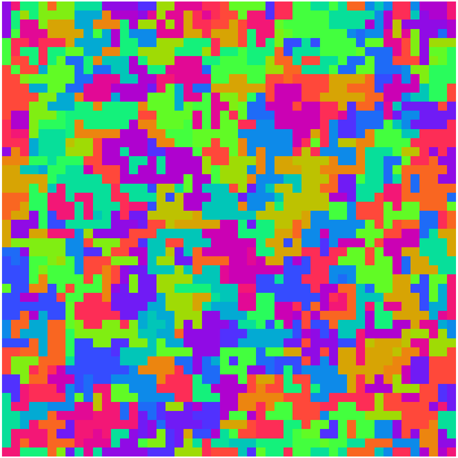
Time $t = 0$.

The voter model



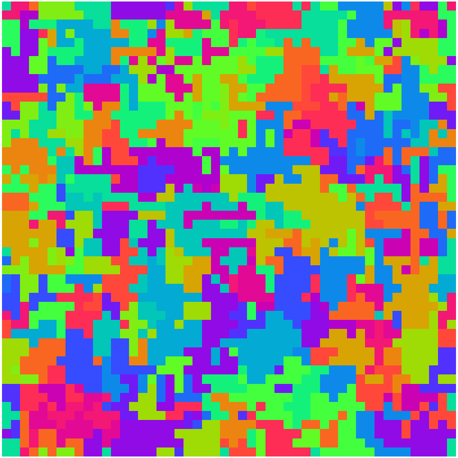
Time $t = 0.25$.

The voter model



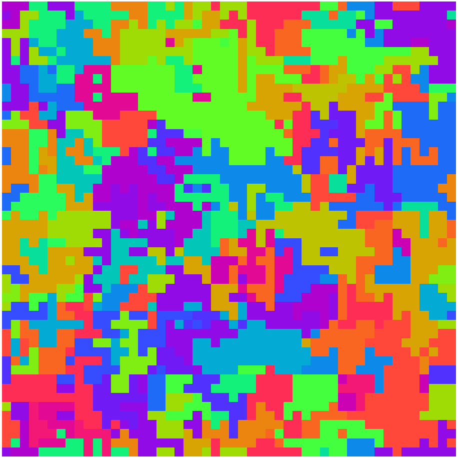
Time $t = 0.5$.

The voter model



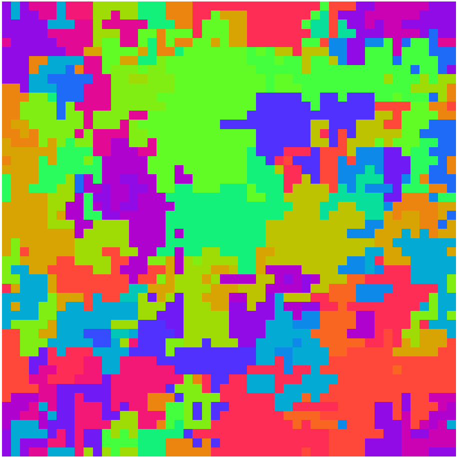
Time $t = 1$.

The voter model



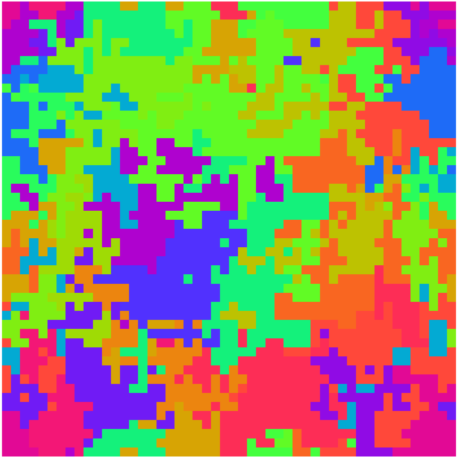
Time $t = 2$.

The voter model



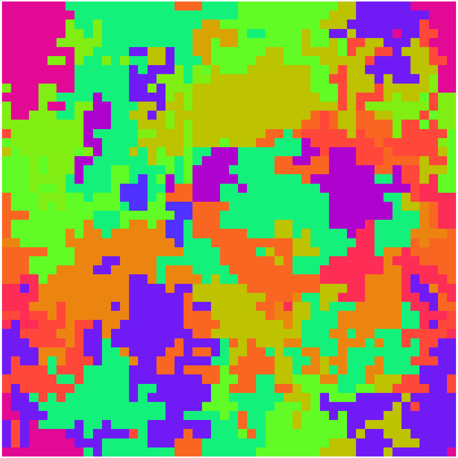
Time $t = 4$.

The voter model



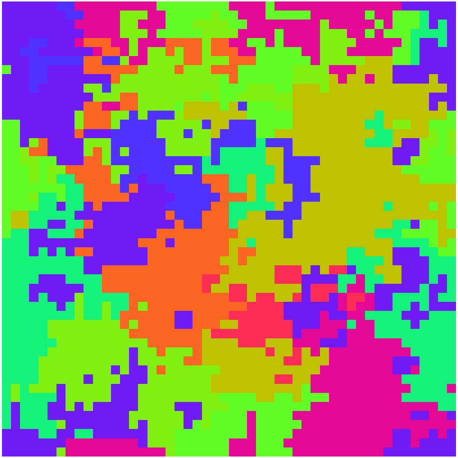
Time $t = 8$.

The voter model



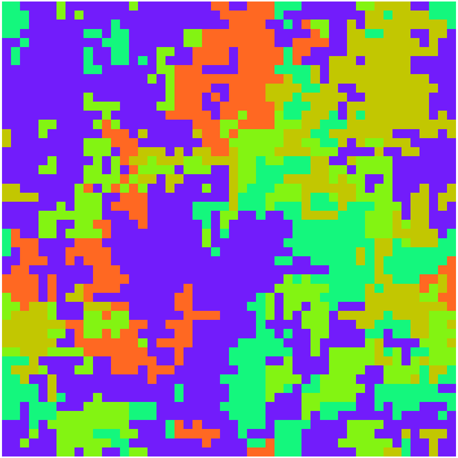
Time $t = 16$.

The voter model



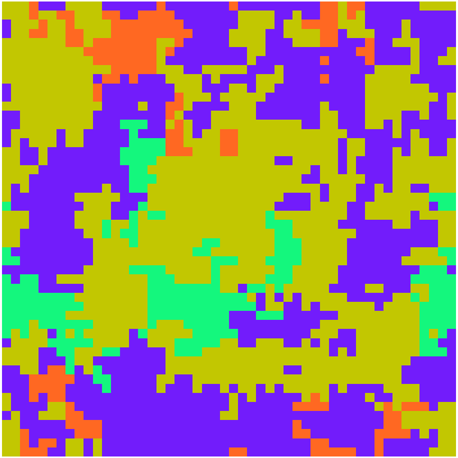
Time $t = 31.25$.

The voter model



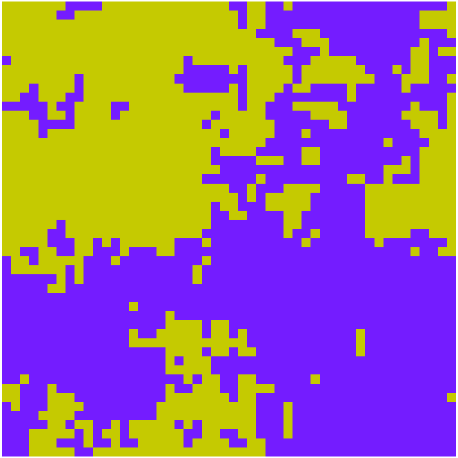
Time $t = 62.5$.

The voter model



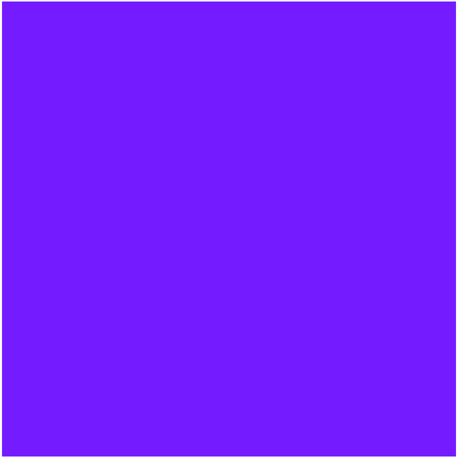
Time $t = 125.$

The voter model



Time $t = 250$.

The voter model



Time $t = 500$.

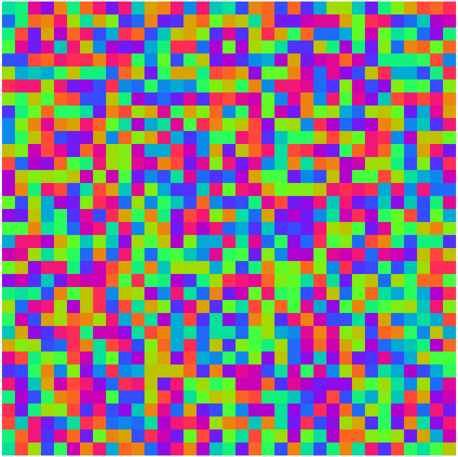
The voter model

The behavior of the voter model strongly depends on the dimension.

Clustering in dimensions $d = 1, 2$.

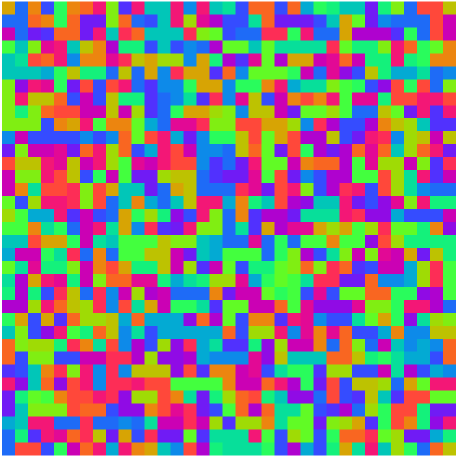
Stable behavior in dimensions $d \geq 3$.

The voter model



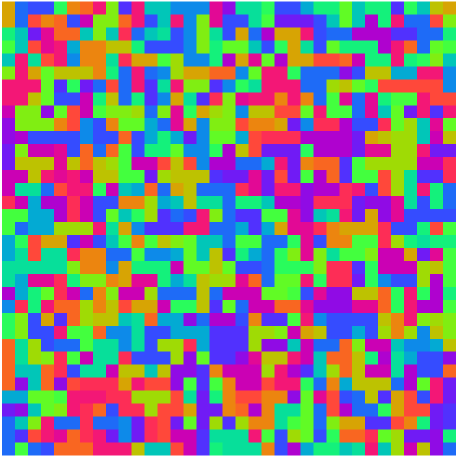
Cut of 3-dimensional model, time $t = 0$.

The voter model



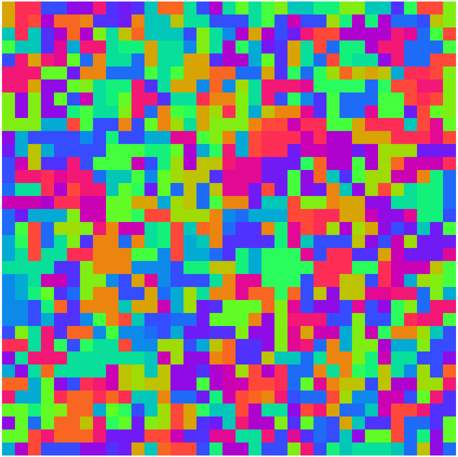
Cut of 3-dimensional model, time $t = 1$.

The voter model



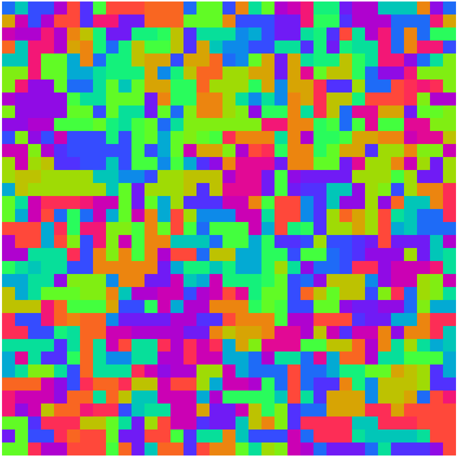
Cut of 3-dimensional model, time $t = 2$.

The voter model



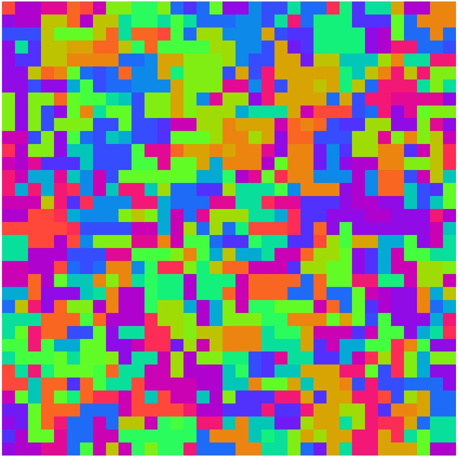
Cut of 3-dimensional model, time $t = 4$.

The voter model



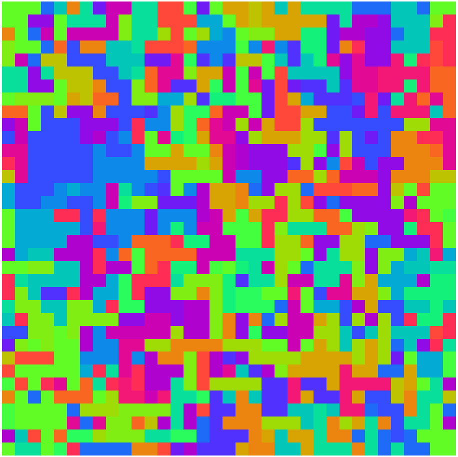
Cut of 3-dimensional model, time $t = 8$.

The voter model



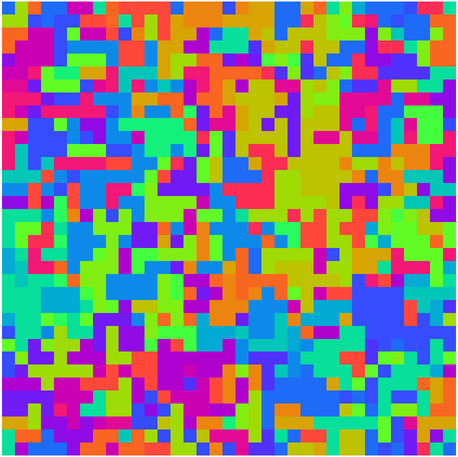
Cut of 3-dimensional model, time $t = 16$.

The voter model



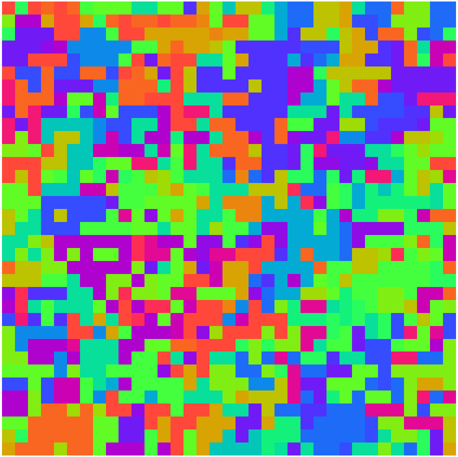
Cut of 3-dimensional model, time $t = 32$.

The voter model



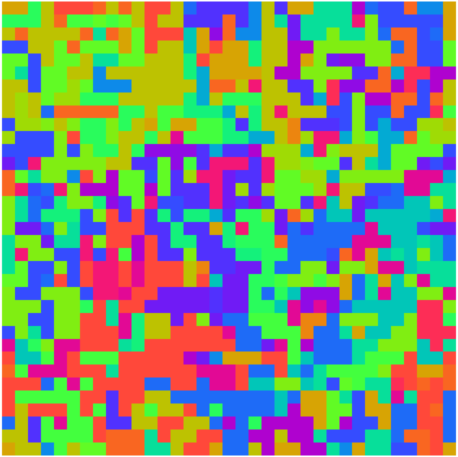
Cut of 3-dimensional model, time $t = 64$.

The voter model



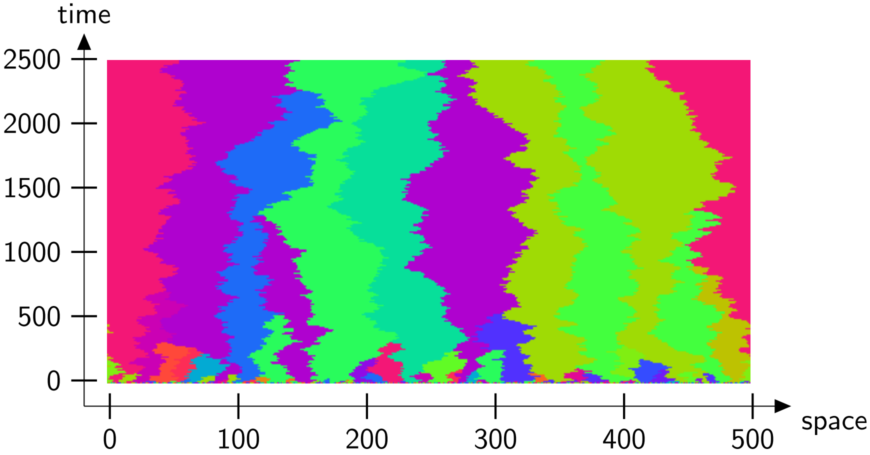
Cut of 3-dimensional model, time $t = 125$.

The voter model



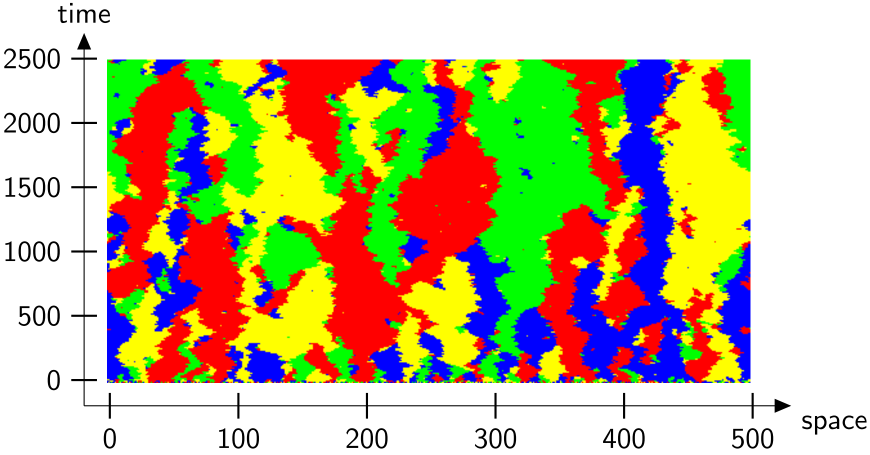
Cut of 3-dimensional model, time $t = 250$.

A one-dimensional voter model



A one-dimensional voter model.

A one-dimensional Potts model



In one-dimensional Potts models, the cluster size remains bounded in time even at very high β ($=$ low temperature).

The biased voter model

In the *biased* voter model with two states $\{0,1\}$, each organism i changes its type $X_t(i)$ with the rates

$$
\begin{aligned}
0 \mapsto 1 \qquad & \text{with rate } (1+s)\cdot \text{fraction of type 1 neighbors,}\\
1 \mapsto 0 \qquad & \text{with rate } 1\cdot \text{fraction of type 0 neighbors,}
\end{aligned}
$$

where $s > 0$ gives type 1 a (small) advantage.

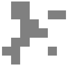
Contrary to the voter model, even if we start with just a single organism of type 1, there is a positive probability that type 1 never dies out.

Models spread of *advantageous mutation*.

The biased voter model

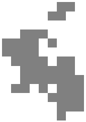
Biased voter model with $s = 0.2$. Time $t = 0$.

The biased voter model



Biased voter model with $s = 0.2$. Time $t = 10$.

The biased voter model

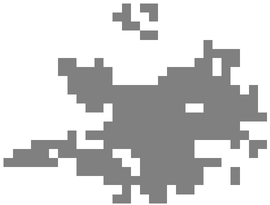


Biased voter model with $s = 0.2$. Time $t = 20$.

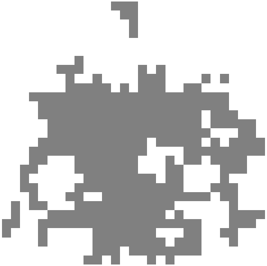
The biased voter model

Biased voter model with $s = 0.2$. Time $t = 30$.

The biased voter model

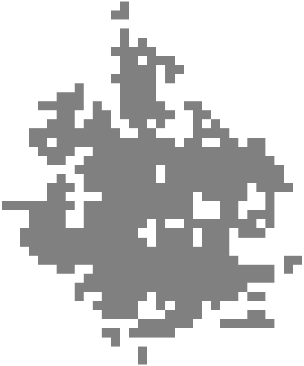


Biased voter model with $s = 0.2$. Time $t = 40$.



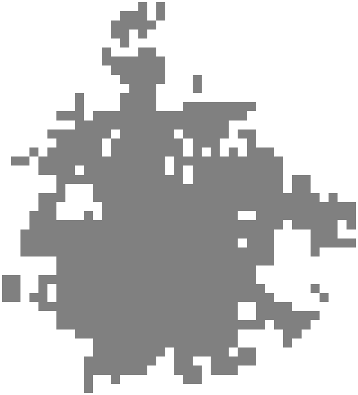
Biased voter model with $s = 0.2$. Time $t = 50$.

The biased voter model



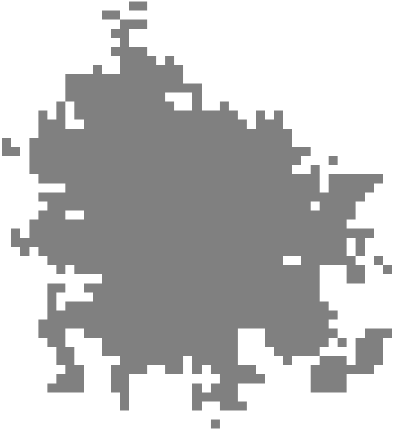
Biased voter model with $s = 0.2$. Time $t = 60$.

The biased voter model



Biased voter model with $s = 0.2$. Time $t = 70$.

The biased voter model



Biased voter model with $s = 0.2$. Time $t = 80$.

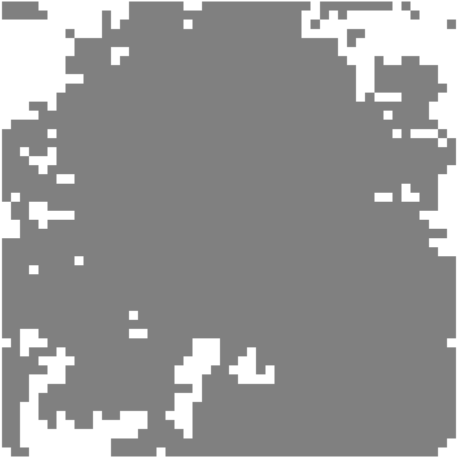
The biased voter model

Biased voter model with $s = 0.2$. Time $t = 90$.

The biased voter model

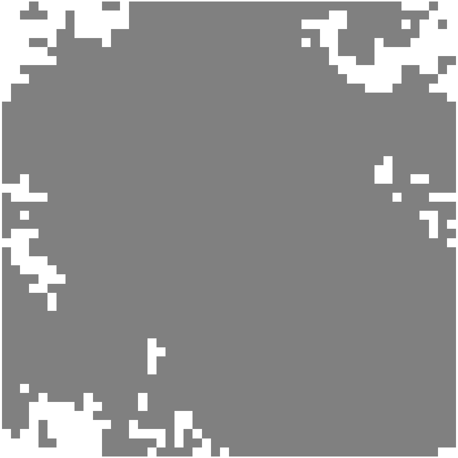
Biased voter model with $s = 0.2$. Time $t = 100$.

The biased voter model



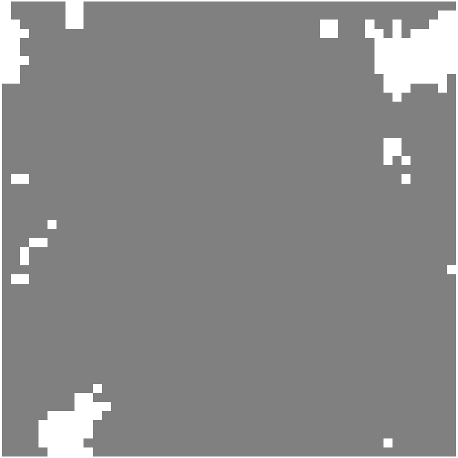
Biased voter model with $s = 0.2$. Time $t = 110$.

The biased voter model

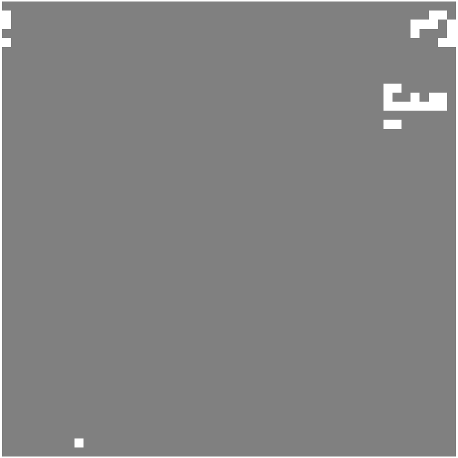


Biased voter model with $s = 0.2$. Time $t = 120$.

The biased voter model

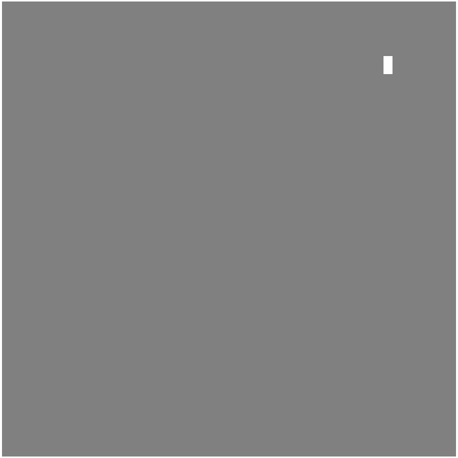


Biased voter model with $s = 0.2$. Time $t = 130$.



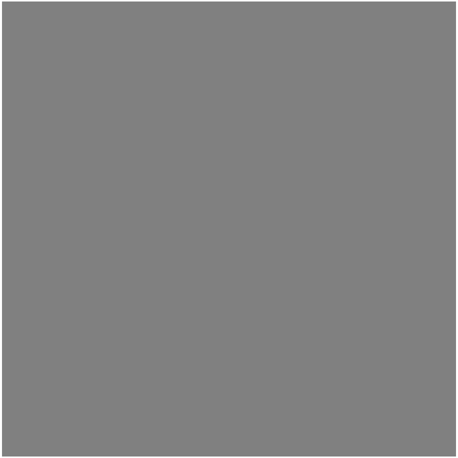
Biased voter model with $s = 0.2$. Time $t = 140$.

The biased voter model



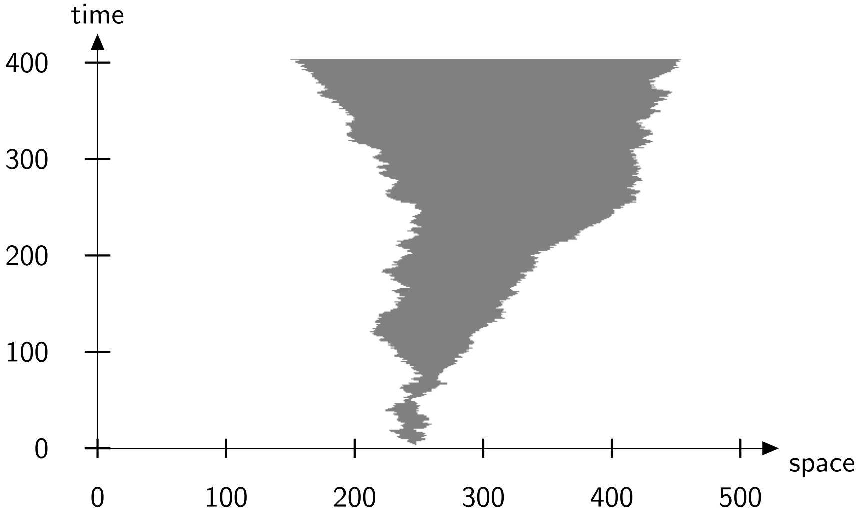
Biased voter model with $s = 0.2$. Time $t = 150$.

The biased voter model



Biased voter model with $s = 0.2$. Time $t = 160$.

The biased voter model



A one-dimensional biased voter model with bias $s = 0.2$.

The biased voter model

We can extend the biased voter model by also allowing spontaneous jumps from 1 to 0.

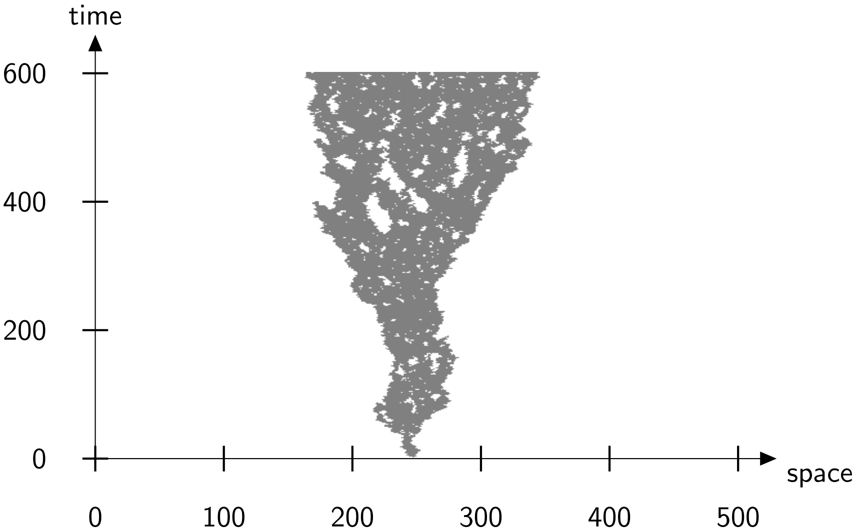
$$\begin{aligned} 0 \mapsto 1 \qquad & \text{with rate } (1+s)\cdot \text{fraction of type 1 neighbors},\\ 1 \mapsto 0 \qquad & \text{with rate } 1\cdot \text{fraction of type 0 neighbors}\\ & \qquad\qquad + d, \end{aligned}$$

where $s > 0$ gives type 1 an advantage and $d \geq 0$ is a *death rate*.

This models the fact that genes may become disfunctional due to deleterious mutations.

Whether 1's have a positive probability to survive now depends in a nontrivial way on s and d.

The biased voter model



Process with bias $s = 0.5$, death rate $d = 0.02$.

A rebellious voter model

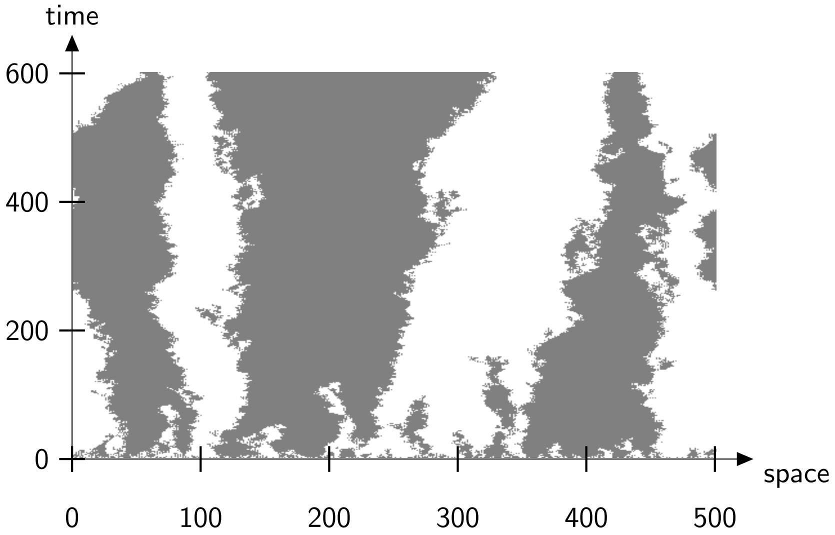
The *rebellious voter map* is defined as

$$\mathtt{rvot}_{kji}(x)(l) := \begin{cases} 1 - x(i) & \text{if } l = i \text{ and } x(k) \neq x(j), \\ x(l) & \text{otherwise.} \end{cases}$$

The *rebellious voter model* is the one-dimensional model with generator

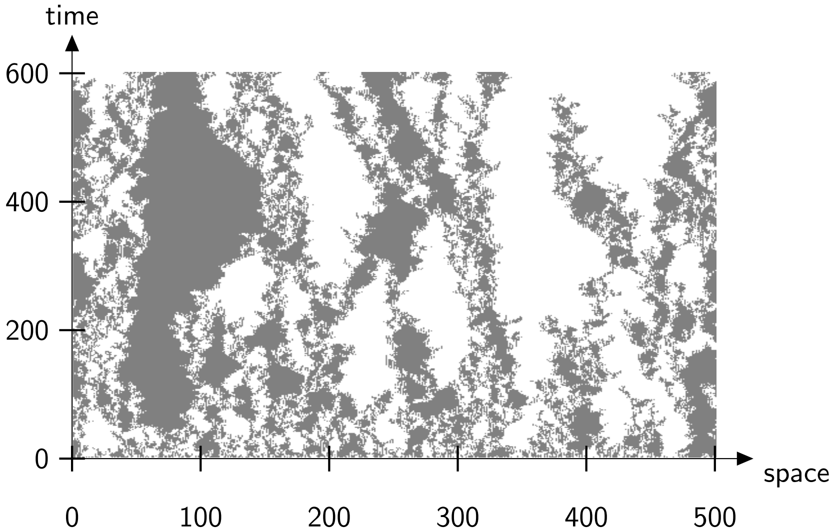
$$\begin{aligned} Gf(x) := {} & \alpha \sum_i \big\{ f\big(\mathtt{vot}_{i,i+1}(x)\big) - f\big(x\big) \big\} \\ & + \alpha \sum_i \big\{ f\big(\mathtt{vot}_{i,i-1}(x)\big) - f\big(x\big) \big\} \\ & + (1-\alpha) \sum_i \big\{ f\big(\mathtt{rvot}_{i-1,i,i+1}(x)\big) - f\big(x\big) \big\} \\ & + (1-\alpha) \sum_i \big\{ f\big(\mathtt{rvot}_{i+1,i,i-1}(x)\big) - f\big(x\big) \big\}. \end{aligned}$$

A rebellious voter model



Process with $\alpha = 0.8$ behaves more or less as a voter model.

A rebellious voter model



In the process with $\alpha = 0.3$, cluster size remains bounded in time.

Reaction diffusion models

Another rich class of models are *reaction diffusion models*.

These are systems of particles that perform independent random walks and interact when they are near to each other.

Let $X_t(i) = 1$ (resp. 0) signify the presence (resp. absence) of a particle and consider the maps $\mathtt{rw}_{ij} : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$

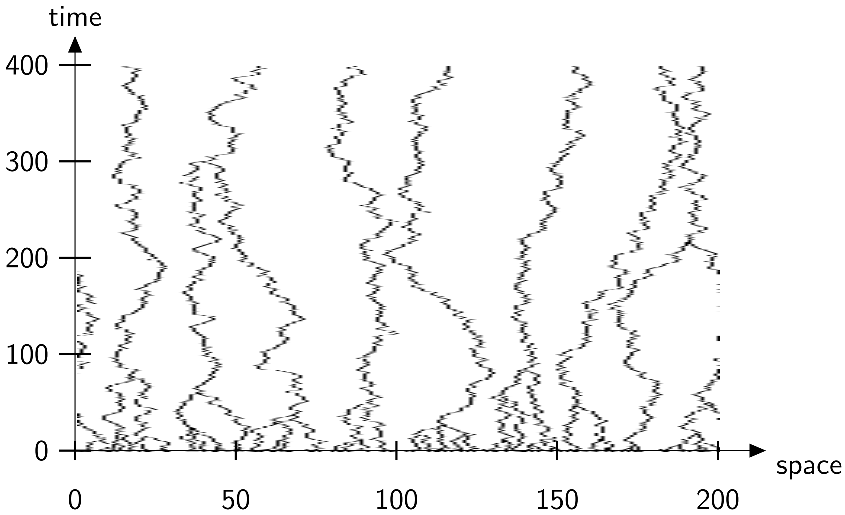
$$\mathtt{rw}_{i,j}x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

The process with generator

$$G = \tfrac{1}{2}\sum_{i\in\mathbb{Z}} \big\{ f\big(\mathtt{rw}_{i,i+1}x\big) - f\big(x\big) \big\} + \tfrac{1}{2}\sum_{i\in\mathbb{Z}} \big\{ f\big(\mathtt{rw}_{i,i-1}x\big) - f\big(x\big) \big\}$$

describes *coalescing random walks*.

Coalescing random walks



Reaction diffusion models

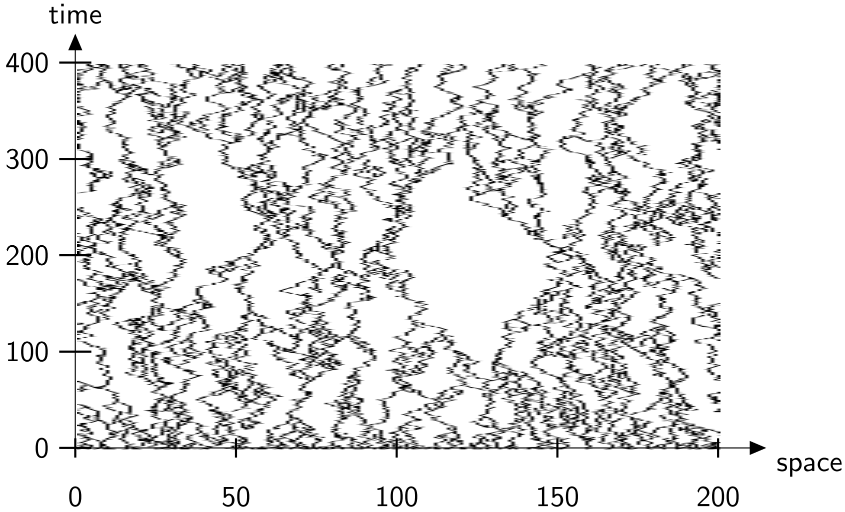
In analogy with the *branching map*

$$\texttt{bra}_{ij}x(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

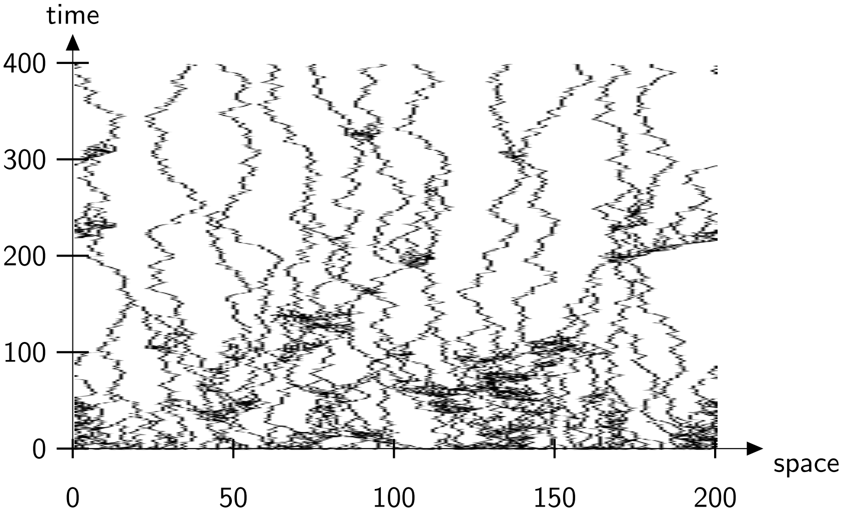
we can also define a *cooperative branching map*

$$\texttt{coop}_{ii'j}x(k) := \begin{cases} \big(x(i) \wedge x(i')\big) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

Branching and coalescing random walks

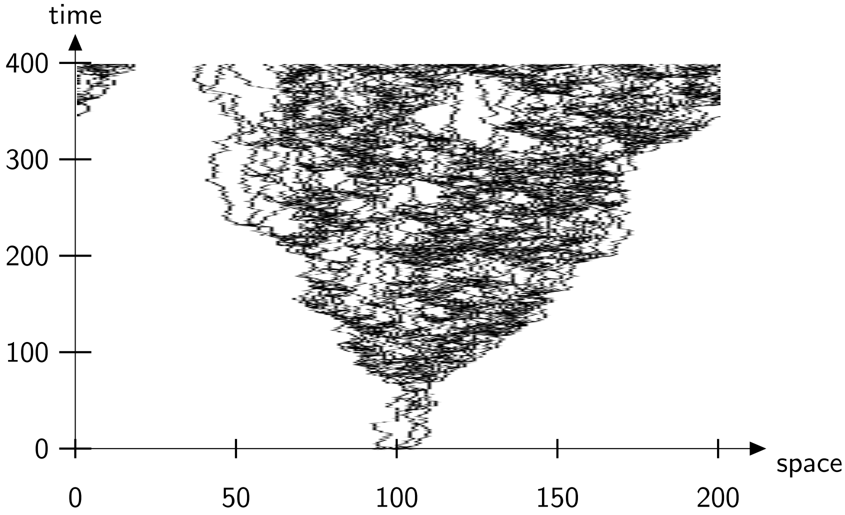


Cooperative branching and coalescence



Cooperative branching rate 2.2.

Cooperative branching



Cooperative branching rate 3.

A cancellative system

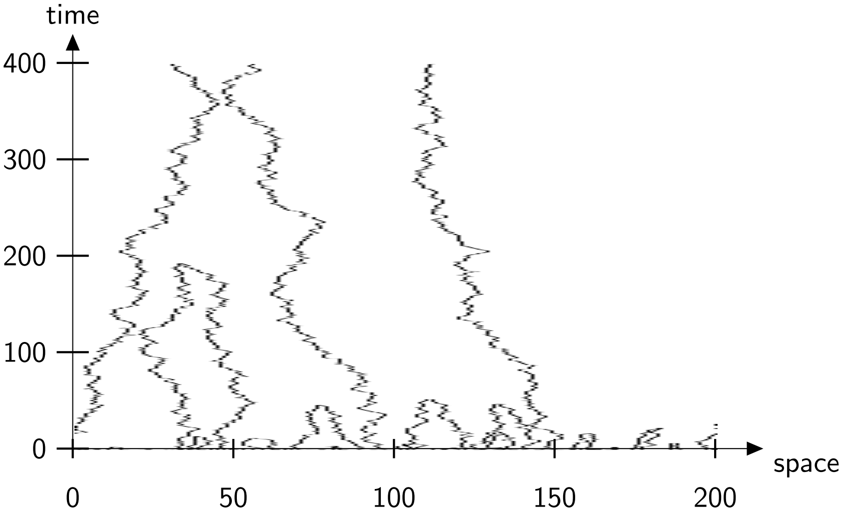
Two more maps of interest are the *annihilating random walk* map

$$\mathtt{arw}_{ij}x(k) := \begin{cases} 0 & \text{if } k = i, \\ x(i) + x(j) \bmod(2) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

and the *annihilating branching map*

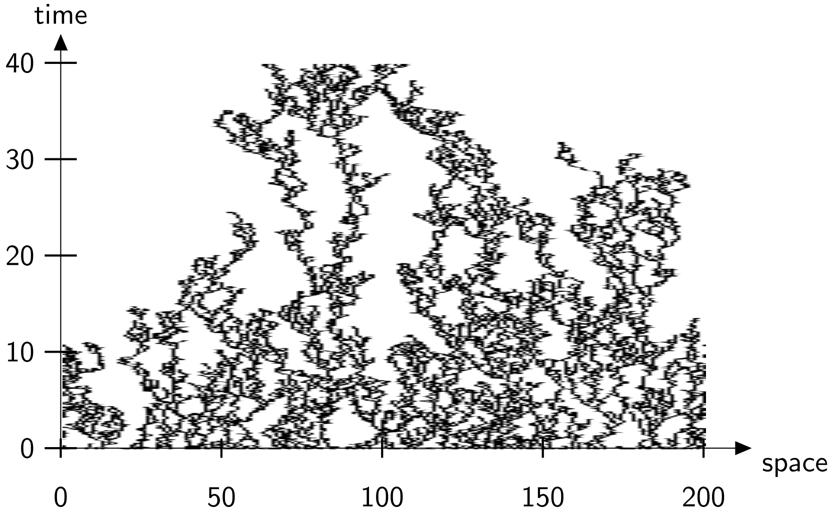
$$\mathtt{abra}_{ij}x(k) := \begin{cases} x(i) + x(j) \bmod(2) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

A cancellative system



Annihilating random walks.

A cancellative system



A system of branching annihilating random walks.

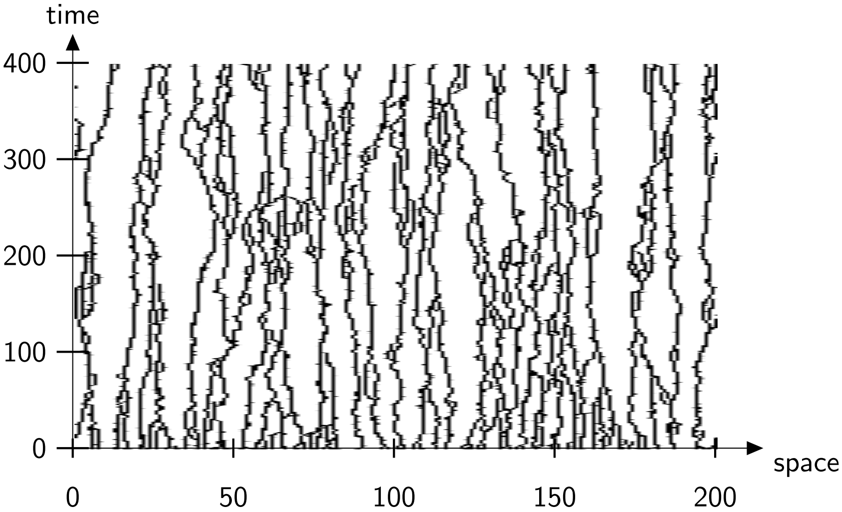
Killing

Define a killing map as

$$\texttt{kill}_{ij}x(k) := \begin{cases} \big(1 - x(i)\big) \wedge x(j) & \text{if } k = j, \\ x(k) & \text{otherwise,} \end{cases}$$

which says that the particle at i, if present, kills any particle at j.

Branching and killing



A system with branching and killing.