# II Spatial Models in Population Biology

Jan M. Swart

## Lecture 2: Duality

## Outline

- Monotone systems
- Self-duality of the contact process
- Duality, intertwining, and pathwise duality
- Cancellative duality
- Thinnings and Lloyd-Sudbury duality
- Additive duality
- General monotone duality

# Monotone Particle Systems

Let $S$, $T$ be partially ordered sets and $m : S \to T$. By definition, $m$ is *monotone* if

$$x \leq y \quad \text{implies} \quad m(x) \leq m(y).$$

Examples of monotone maps:

$$\mathtt{bra}_{ij}, \quad \mathtt{death}_i, \quad \mathtt{vot}_{ij}, \quad \mathtt{rw}_{ij}, \quad \mathtt{coop}_{ii'j}.$$

Examples of maps that are *not* monotone:

$$\mathtt{rvot}_{ij}, \quad \mathtt{arw}_{ij}, \quad \mathtt{abra}_{ij}, \quad \mathtt{kill}_{ij}.$$

# Monotone Particle Systems

*Examples of monotone particle systems:*

- The (ferromagnetic) Ising model ($\beta \geq 0$).
- The two-type voter model.
- The biased voter model.
- The contact process.
- Branching and coalescing random walks.
- Cooperative branching.

*Examples of particle systems that are not monotone:*

- The antiferromagnetic Ising model ($\beta < 0$).
- (Voter and Potts models with 3 or more types.)
- Rebellious voter models.
- Branching annihilating random walks.
- Systems with branching and killing.

# The stochastic order

We adopt the notation $\mu f := \int f \, \mathrm{d}\mu$. If $\mu, \nu$ satisfy the equivalent conditions below, then we say that they are *stochastically ordered* and write $\mu \leq \nu$.

**Stochastic Order** *Let $S$ be a compact metrizable space equipped with a partial order such that $\{(x, y) : x \leq y\}$ is a closed subset of $S \times S$. Let $\mu, \nu$ be probability measures on $S$. Then the following statements are equivalent:*

(i) $\mu f \leq \nu f$ *for every continuous monotone* $f : S \to \mathbb{R}$.

(ii) $\mu f \leq \nu f$ *for every bounded measurable monotone* $f : S \to \mathbb{R}$.

(iii) *Random variables $X, Y$ with laws $\mu, \nu$ can be coupled such that $X \leq Y$ a.s.*

**Proof** See Theorem II.2.4 in Liggett (1985). ∎

# The upper invariant law

Below, we consider an interacting particle system with transition kernels $(P_t)_{t\geq 0}$ and generator $G$ that is defined in terms of a set $\mathcal{G}$ of maps $m : S^\Lambda \to S^\Lambda$ and rates $(r_m)_{m\in\mathcal{G}}$.

**Upper invariant law** *Assume that $S$ partially ordered with minimal and maximal elements $0, 1$. Assume that all maps $m \in \mathcal{G}$ are monotone. Then there exist invariant laws $\underline{\nu}$ and $\overline{\nu}$ such that*

$$\delta_{\underline{0}} P_t \underset{t\to\infty}{\Longrightarrow} \underline{\nu} \quad \text{and} \quad \delta_{\underline{1}} P_t \underset{t\to\infty}{\Longrightarrow} \overline{\nu}.$$

*Moreover, any invariant law $\nu$ satisfies $\underline{\nu} \leq \nu \leq \overline{\nu}$.*

# The upper invariant law

**Proof** Since $\mathbf{X}_{s,t}$ is a concatenation of monotone maps, it is itself monotone. For any $s \leq t \leq u$,

$$\mathbf{X}_{s,t}(\underline{1}) \leq \underline{1} \quad \Rightarrow \quad \mathbf{X}_{s,u}(\underline{1}) = \mathbf{X}_{t,u} \circ \mathbf{X}_{s,t}(\underline{1}) \leq \mathbf{X}_{t,u}(\underline{1})$$

which shows that the decreasing limit

$$X_s := \lim_{t\to\infty} \mathbf{X}_{s-t,s}(\underline{1}) \qquad \text{a.s.}$$

exists for all $s \in \mathbb{R}$. One can check that $(X_s)_{s\in\mathbb{R}}$ is a stationary interacting particle system and hence $\overline{\nu} := \mathbb{P}[X_s \in \cdot\,]$ is an invariant law. Since $\mathbf{X}_{s-t,s}(\underline{1})$ has law $\delta_{\underline{1}} P_t$, this proves $\delta_{\underline{1}} P_t \Rightarrow \overline{\nu}$.

# The upper invariant law

If $\nu$ is another invariant law and $Y_0$ has law $\nu$ and is independent of the Poisson set $\omega$, then

$$\mathbf{X}_{0,t}(Y_0) \leq \mathbf{X}_{0,t}(\underline{1}),$$

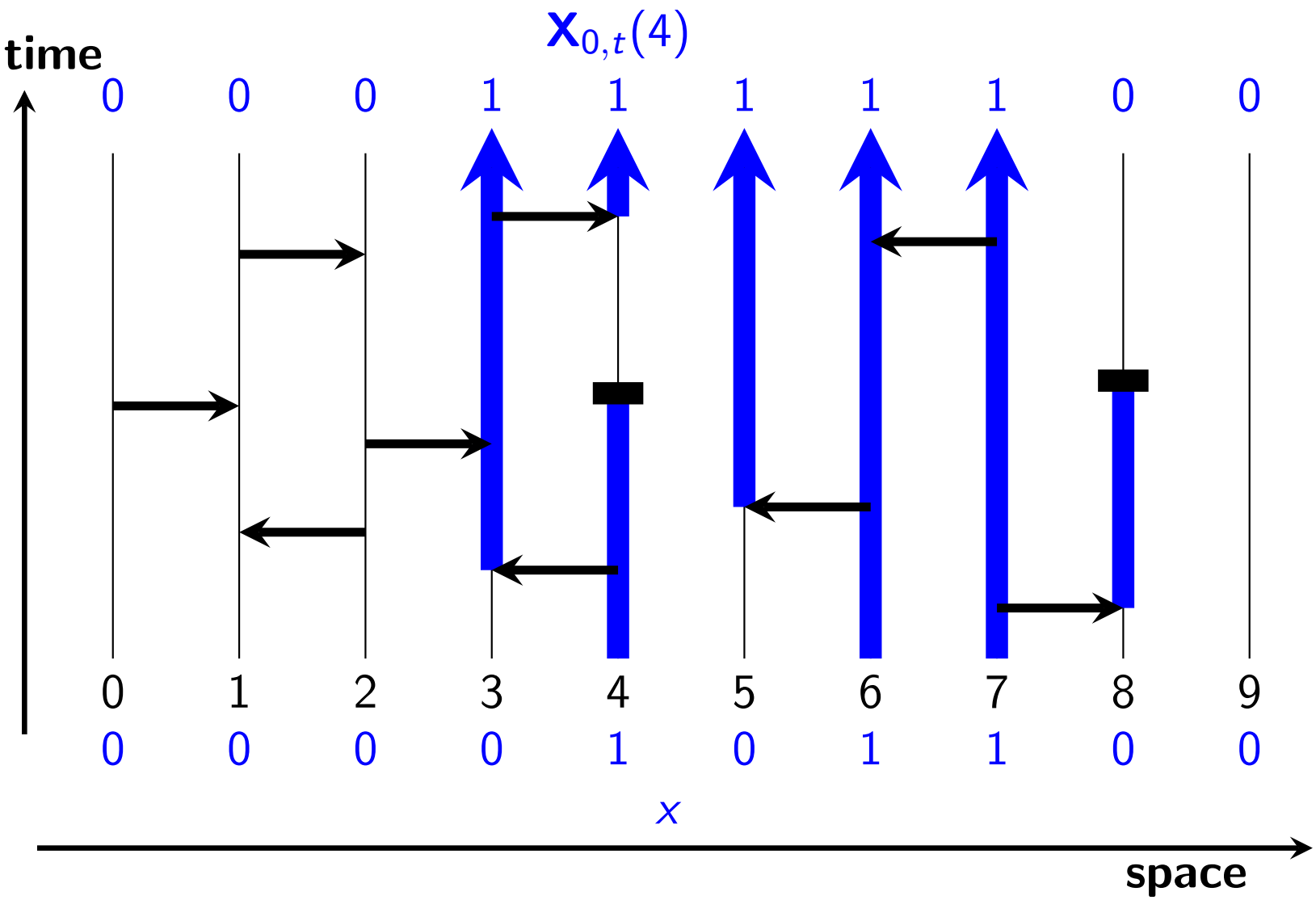
which proves that $\nu \leq \delta_{\underline{1}} P_t$ $(t \geq 0)$. Letting $t \to \infty$, we see that $\nu \leq \overline{\nu}$.

The proof for $\underline{\nu}$ is the same. ∎

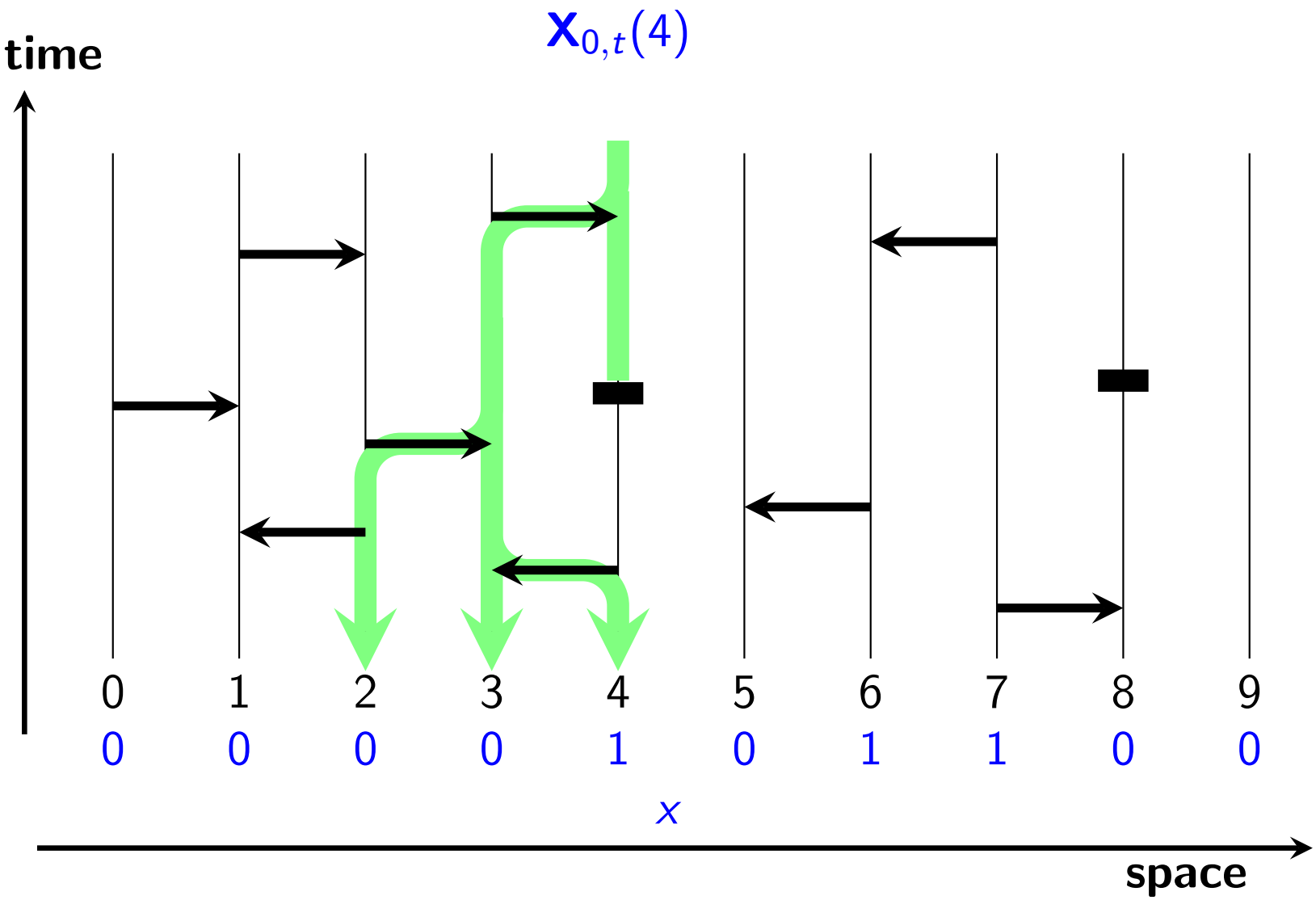
*Example 1* For the contact process on $\mathbb{Z}^d$, $\underline{\nu} = \delta_{\underline{0}}$, there exists a $0 < \lambda_{\rm c} < \infty$ such that $\overline{\nu} = \delta_{\underline{0}}$ for $\lambda < \lambda_{\rm c}$ and $\overline{\nu} \neq \delta_{\underline{0}}$ for $\lambda > \lambda_{\rm c}$.

*Example 2* For the Ising model on $\mathbb{Z}^d$, $d \geq 2$, there exists a $0 < \beta_{\rm c} < \infty$ such that $\underline{\nu} = \overline{\nu}$ for $\beta < \beta_{\rm c}$ and $\underline{\nu} \neq \overline{\nu}$ for $\beta > \beta_{\rm c}$.

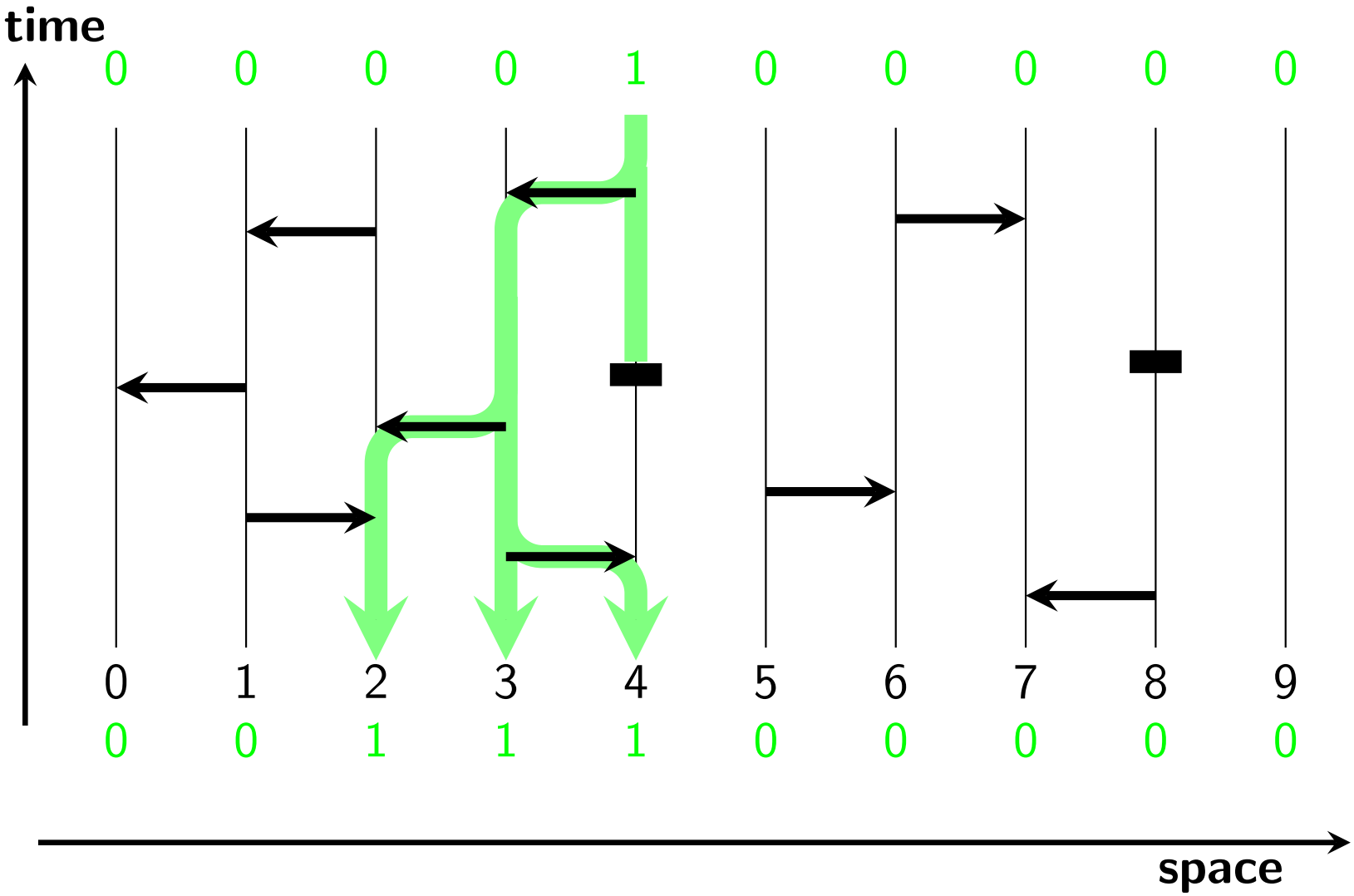
# The graphical representation of the contact process



# The graphical representation of the contact process



# The graphical representation of the contact process



# Self-duality of the contact process

The Poisson point process $\omega$ used to define the stochastic flow $(\mathbf{X}_{s,t})_{s\leq t}$ of the contact process is called the *graphical representation*.

Using the same graphical representation $\omega$, but reversing the direction of time and the direction of all branching arrows, we can define a *dual flow* $(\hat{\mathbf{X}}_{s,t})_{s\geq t}$ such that

$$\mathbf{X}_{s,t}(x) \wedge y = \underline{0} \quad \text{iff} \quad x \wedge \hat{\mathbf{X}}_{t,s}(y) = \underline{0} \qquad (s \leq t).$$

In particular, the contact process is *self-dual* in the sense that

$$\mathbb{P}^x\big[X_t \wedge y = \underline{0}\big] = \mathbb{P}^y\big[x \wedge X_t = \underline{0}\big] \qquad (t \geq 0).$$

# Survival

We say that the contact process on $\mathbb{Z}^d$ with infection rate $\lambda$ *survives* if $\mathbb{P}^{\delta_i}[X_t \neq \underline{0}\ \forall t \geq 0] > 0$ $(i \in \Lambda)$, and define

$$\lambda_{\rm c} := \inf\{\lambda \geq 0 : \overline{\nu} \neq \delta_{\underline{0}}\},$$
$$\lambda'_{\rm c} := \inf\{\lambda \geq 0 : X \text{ survives}\}.$$

**Lemma** $\lambda_{\rm c} = \lambda'_{\rm c}$.

**Proof** Set $\rho(x) := \mathbb{P}^x[X_t \neq \underline{0}\ \forall t \geq 0]$. Then

$$\mathbb{P}^{\underline{1}}\big[X_t \wedge x \neq \underline{0}\big] = \mathbb{P}^x\big[\underline{1} \wedge X_t \neq \underline{0}\big] = \mathbb{P}^x\big[X_t \neq \underline{0}\big] \underset{t\to\infty}{\longrightarrow} \rho(x),$$

which shows that

$$\int \overline{\nu}(\mathrm{d}y) 1_{\{y \wedge x \neq \underline{0}\}} = \rho(x),$$

In particular, $\int \overline{\nu}(\mathrm{d}y) y(i) = \rho(\delta_i)$ so $\overline{\nu} \neq \delta_{\underline{0}}$ if and only if the contact process survives. ∎

# Homogeneous initial laws

**Theorem** If the contact process on $\mathbb{Z}^d$ is started in a *translation invariant* initial law $\mu$ with $\mu(\{\underline{0}\}) = 0$, then $\mu P_t \underset{t\to\infty}{\Longrightarrow} \overline{\nu}$.

**Proof (sketch)** It suffices to show that

$$\mathbb{P}\big[X_t \wedge x \neq \underline{0}\big] = \mathbb{P}^x\big[X_0 \wedge X_t \neq \underline{0}\big] \underset{t\to\infty}{\longrightarrow} \rho(x)$$

for all finite particle configurations $x$. Since

$$\mathbb{P}^x\big[X_0 \wedge X_t \neq \underline{0}\big] = \mathbb{P}^x\big[X_0 \wedge X_t \neq \underline{0} \,|\, X_t \neq \underline{0}\big]\mathbb{P}^x\big[X_t \neq \underline{0}\big],$$

it suffices to show that

$$\mathbb{P}^x\big[X_0 \wedge X_t \neq \underline{0} \,|\, X_t \neq \underline{0}\big] \underset{t\to\infty}{\longrightarrow} 1,$$

which can be done. ∎

# Duality

$X = (X_t)_{t\geq 0}$ with state space $S$, generator $G$, semigroup $(P_t)_{t\geq 0}$.
$Y = (Y_t)_{t\geq 0}$ with state space $T$, generator $H$, semigroup $(Q_t)_{t\geq 0}$.

**Def** $X$ and $Y$ *dual* with *duality function* $\psi : S \times T \to \mathbb{R}$ iff

$$\mathbb{E}^x\big[\psi(X_t, y)\big] = \mathbb{E}^y\big[\psi(x, Y_t)\big] \qquad (t \geq 0).$$

This implies more generally that for any initial laws,

$$[0, t] \ni s \mapsto \mathbb{E}\big[\psi(X_{t-s}, Y_s))\big]$$

is constant when $X_{t-s}$ and $Y_s$ are independent.

*Example:* The contact process is *self-dual* with

$$\psi(x, y) = 1_{\{x \wedge y = \underline{0}\}}.$$

# Duality

Let $A^\dagger(x,y) := A(y,x)$ denote the adjoint of a matrix $X$. For finite state spaces, we can view the duality function $\psi$ as a matrix

$$\big(\psi(x,y)\big)_{x\in S,\ y\in T}.$$

If $S$ and $T$ are finite, then the following statements are equivalent:

(i) $X$ and $Y$ dual with duality function $\psi$,

(ii) $\sum_{x'} P_t(x,x')\psi(x',y) = \sum_{y'} Q_t(y,y')\psi(x,y')$ $(t \geq 0)$,

(iii) $P_t\psi = \psi Q_t^\dagger$ $(t \geq 0)$,

(iv) $G\psi = \psi H^\dagger$.

# Duality functions

**Examples of duality functions**

*Additive duality* $\psi_{\mathrm{add}}(x,y) = 1_{\{x \wedge y = \underline{0}\}}$

*Cancellative duality* $\psi_{\mathrm{canc}}(x,y) = \sum_i x(i)y(i) \bmod(2)$

*Lloyd-Sudbury duality* $\psi_q(x,y) = \prod_i (1-q)^{x(i)y(i)}$ with $q \in (0,2]$.

**Remark** With $0^0 := 1$, the special cases $q = 1, 2$ yield:

$$\psi_1(x,y) = \psi_{\mathrm{add}}(x,y),$$

$$\psi_2(x,y) = (-1)^{\psi_{\mathrm{canc}}(x,y)} = 1 - 2\psi_{\mathrm{canc}}(x,y).$$

# Intertwining

$X = (X_t)_{t\geq 0}$ with state space $S$, generator $G$, semigroup $(P_t)_{t\geq 0}$.
$Y = (Y_t)_{t\geq 0}$ with state space $T$, generator $H$, semigroup $(Q_t)_{t\geq 0}$.

**Def** $Y$ is an *intertwined* Markov process *on top* of $X$ with *intertwining kernel* $K(x, \mathrm{d}y)$ iff

$$\mu K = \nu \quad \text{implies} \quad \mu P_t K = \nu Q_t.$$

For finite matrices, the following statements are equivalent:

(i) $Y$ is an intertwined process on top of $X$ with kernel $K$.

(ii) $\sum_{x'} P_t(x, x')K(x', y) = \sum_{y'} K(x, y')Q_t(y', y)$ $(t \geq 0)$,

(iii) $P_t K = K Q_t$ $(t \geq 0)$,

(iv) $GK = KH$.

# Intertwining

$$\begin{tikzcd}
Y_0 \arrow[r, "Q_t"] & Y_t \\
X_0 \arrow[u, "K"] \arrow[r, "P_t"] & X_t \arrow[u, "K"']
\end{tikzcd}$$

**Example** *Thinning kernel* $K_p$ on $\{0,1\}^\Lambda$ defined as

$$K_p(x,y) := \prod_{i\in\Lambda} \kappa_p(x(i), y(i))$$

with

$$\kappa_p(1,1) := p, \quad \kappa_p(1,0) := 1-p, \quad \text{and} \quad \kappa_p(0,0) := 1.$$

# Pathwise duality

**Def** Two maps $m : S \to S$ and $\hat{m} : T \to T$ are *dual* with *duality function* $\psi$ iff

$$\psi\big(m(x), y\big) = \psi\big(x, \hat{m}(y)\big) \qquad (x \in S,\ y \in T).$$

*Additive duals* with $\psi_{\mathrm{add}}(x, y) = 1_{\{x \wedge y = \underline{0}\}}$:

$$m = \mathtt{bra}_{ij} \qquad \hat{m} = \mathtt{bra}_{ji},$$

$$m = \mathtt{death}_i \qquad \hat{m} = \mathtt{death}_i,$$

$$m = \mathtt{vot}_{ij} \qquad \hat{m} = \mathtt{rw}_{ji}.$$

*Cancellative duals:* with $\psi_{\mathrm{canc}}(x, y) = \sum_i x(i)y(i) \bmod(2)$:

$$m = \mathtt{abra}_{ij} \qquad \hat{m} = \mathtt{abra}_{ji},$$

$$m = \mathtt{death}_i \qquad \hat{m} = \mathtt{death}_i,$$

$$m = \mathtt{vot}_{ij} \qquad \hat{m} = \mathtt{arw}_{ji}.$$

# Pathwise duality

Let $(\mathbf{X}_{s,t})_{s\leq t}$ be a stochastic flow defined in terms of a Poisson point set $\omega$ whose elements are pairs $(m,t)$ with $m \in \mathcal{G}$.

Fix some duality function $\psi$ and assume that each $m \in \mathcal{G}$ has some dual $\hat{m}$ w.r.t. $\psi$.
Then we can define a *dual flow* $(\hat{\mathbf{X}}_{t,s})_{t\geq s}$ by

$$\mathbf{X}_{s,t} := m_n \circ \cdots \circ m_1 \quad \text{and} \quad \hat{\mathbf{X}}_{t,s} := \hat{m}_1 \circ \cdots \circ \hat{m}_n,$$
$$\text{with} \quad \omega_{s,t} := \omega \cap \mathcal{G} \times (s,t] = \big\{(m_1,t_1),\ldots,(m_n,t_n)\big\}$$

and $t_1 < \cdots < t_n$. Then $\mathbf{X}_{s,t}$ and $\hat{\mathbf{X}}_{t,s}$ are dual maps, i.e.,

$$\psi\big(\mathbf{X}_{s,t}(x), y\big) = \psi\big((x), \hat{\mathbf{X}}_{t,s}(y)\big).$$

# Pathwise duality

Let $X_0$ and $Y_0$ be independent of $\omega$. For fixed $s \in \mathbb{R}$, setting

$$X_t := \mathbf{X}_{s,s+t}(X_0) \qquad (t \geq 0)$$

defines a process $(X_t)_{t\geq 0}$ with generator

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m\big\{f\big(m(x)\big) - f\big(x\big)\big\}.$$

Also, setting

$$Y_t := \hat{\mathbf{X}}_{s,s-t}(Y_0) \qquad (t \geq 0)$$

defines a process $(Y_t)_{t\geq 0}$ with generator

$$Gf(y) = \sum_{m\in\mathcal{G}} r_m\big\{f\big(\hat{m}(y)\big) - f\big(y\big)\big\}.$$

*Slight complication:* $(Y_t)_{t\geq 0}$ has *left-continuous* sample paths.

# Pathwise duality

Pathwise duality implies duality:

$$\begin{aligned}\mathbb{E}^x\big[\psi(X_t, y)\big] &= \mathbb{E}\big[\psi\big(\mathbf{X}_{0,t}(x), y\big)\big] \\ &= \mathbb{E}\big[\psi\big(x, \hat{\mathbf{X}}_{t,0}(y)\big)\big] = \mathbb{E}^y\big[\psi(x, Y_t)\big] \qquad (t \geq 0).\end{aligned}$$

# Cancellative duality

A *field* is a set $F$ on which a sum and product are defined such that

(i) $(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$,

(ii) $a+b = b+a$ and $ab = ba$,

(iii) $\exists 0, 1$ s.t. $a+0 = a$, $a1 = 1$,

(iv) $\forall a \; \exists -a$ s.t. $a + -a = 0$,

(v) $\forall a \neq 0 \; \exists a^{-1}$ s.t. $aa^{-1} = 1$,

(vi) $a(b+c) = ab + ac$.

If $p$ is prime, then $\{0, \ldots, p-1\} \cong \mathbb{Z} \bmod(p)$ is a finite field.

*Linear spaces* can be defined over any field.

# Cancellative duality

We may view $\{0,1\}^\Lambda$ as a *linear space* over the *finite field* $\{0,1\}$ (with calculation modulo 2).

In this picture, a map $m : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ is linear iff it is of the form

$$mx(i) = \sum_j m(i,j)x(j) \mod(2)$$

for some matrix $(m(i,j))_{i,j\in\Lambda}$ with entries $m(i,j) \in \{0,1\}$.

Introducing the "inner product"

$$\langle\langle x, y\rangle\rangle := \sum_i x(i)y(i) \mod(2),$$

and letting $m^\dagger(i,j) := m(j,i)$, we observe that

$$\langle\langle mx, y\rangle\rangle = \langle\langle x, m^\dagger y\rangle\rangle,$$

i.e., $m$ and $m^\dagger$ are dual w.r.t. $\psi_{\mathrm{canc}}(x,y) = \langle\langle x, y\rangle\rangle$.

# Cancellative duality

A cancellative map $m$ is *local* iff

$$\{(i,j) : i \neq j,\ m(i,j) = 1\} \quad \text{and} \quad \{(i,i) : m(i,i) = 0\}$$

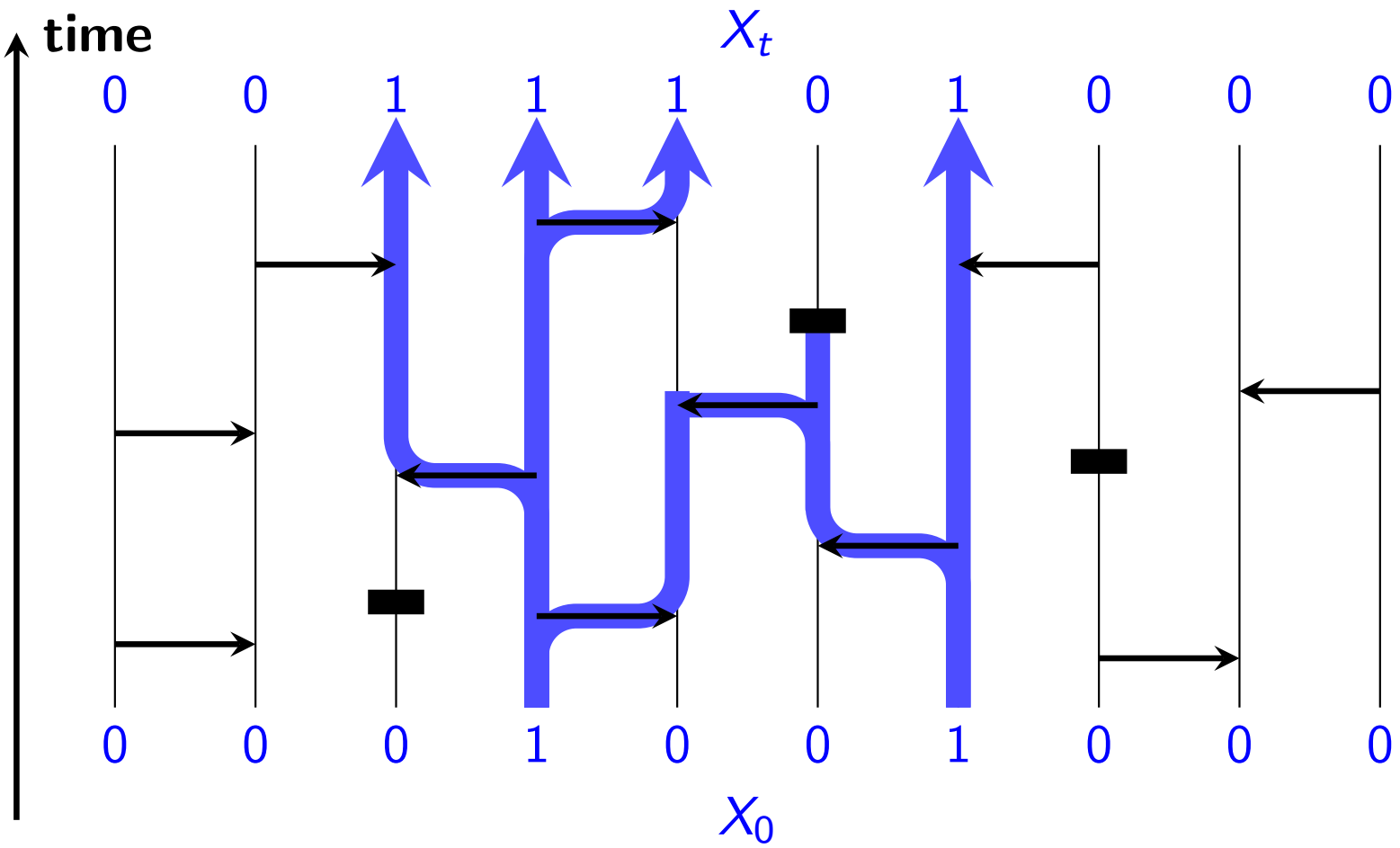
are finite sets.

We can draw the matrix $(m(i,j))_{i,j\in\Lambda}$ in terms of arrows and blocking symbols:

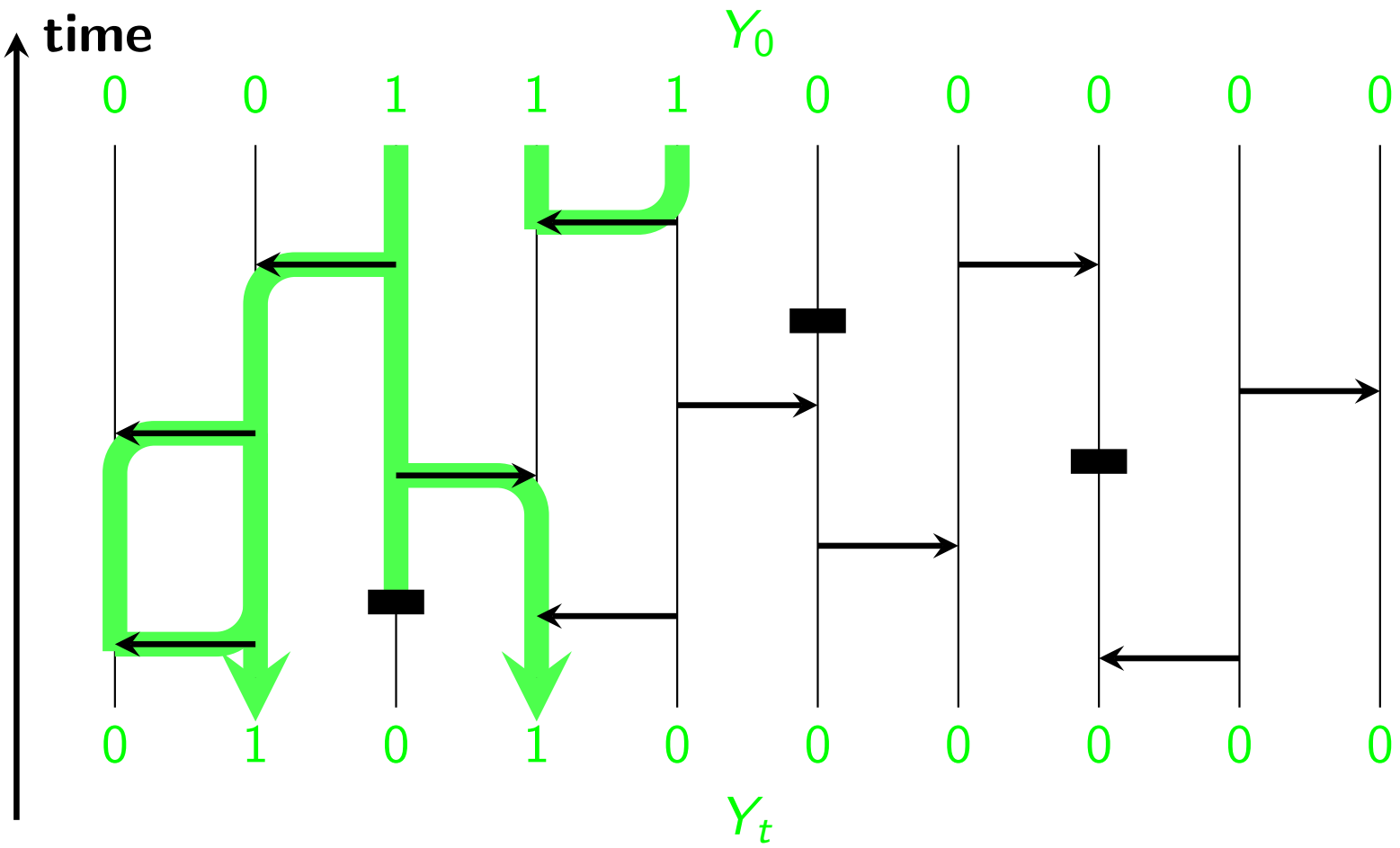
- An arrow from $i$ to $j$ for each $i \neq j$ such that $m(i,j) = 1$.
- A blocking symbol at each $i$ such that $m(i,i) = 0$.

To draw $m^\dagger$, we reverse the arrows of $m$ and keep the blocking symbols where they are.

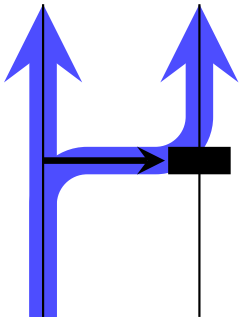
# Annihilating branching and deaths



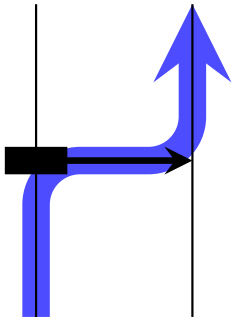
# Annihilating branching and deaths



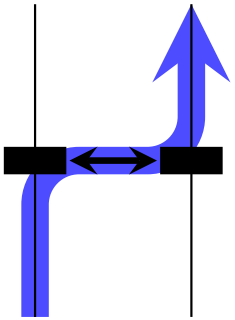
# Cancellative duality



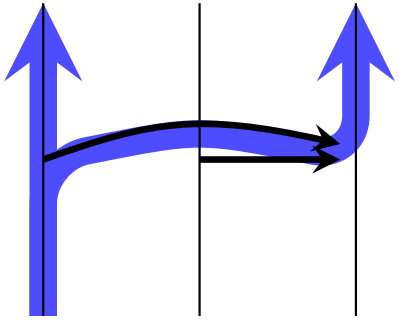
$\mathtt{vot}_{1,2}$



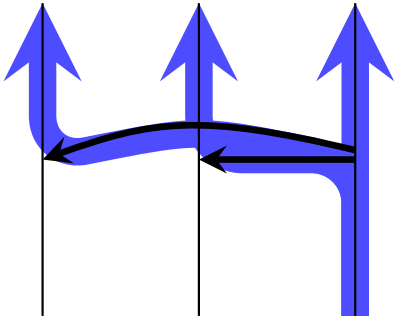
$\mathtt{arw}_{1,2}$



$\mathtt{excl}_{1,2}$



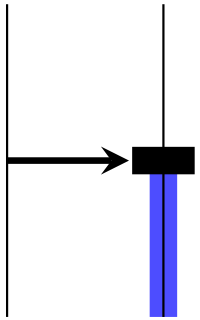
$\mathtt{rvot}_{1,2,3}$



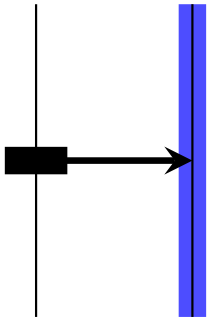
$\mathtt{adbr}_{3,2,1}$

Graphical representations of cancellative maps.

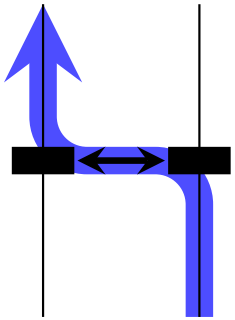
# Cancellative duality



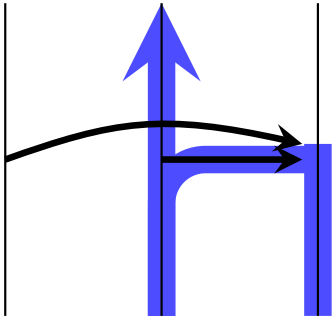
$\mathtt{vot}_{1,2}$



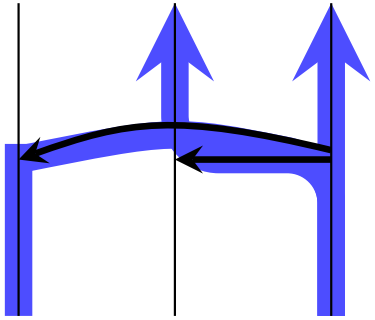
$\mathtt{arw}_{1,2}$



$\mathtt{excl}_{1,2}$



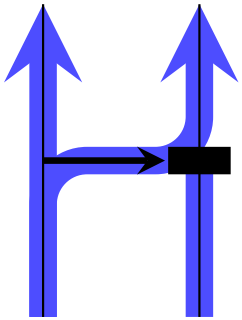
$\mathtt{rvot}_{1,2,3}$



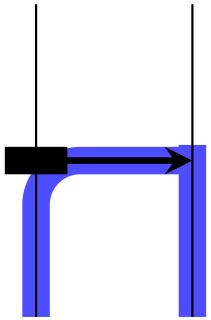
$\mathtt{adbr}_{3,2,1}$

Graphical representations of cancellative maps.

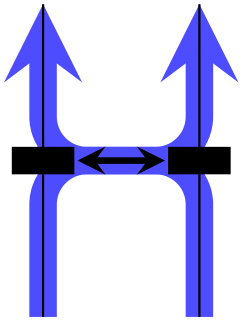
# Cancellative duality



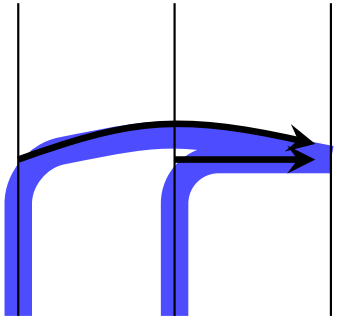
$\mathtt{vot}_{1,2}$



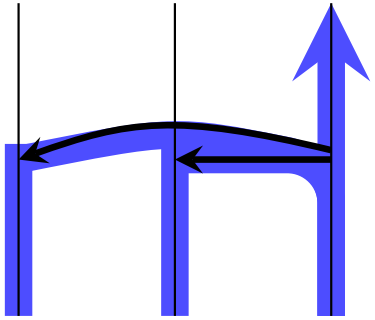
$\mathtt{arw}_{1,2}$



$\mathtt{excl}_{1,2}$



$\mathtt{rvot}_{1,2,3}$



$\mathtt{adbr}_{3,2,1}$

Graphical representations of cancellative maps.

# Thinnings and Lloyd-Sudbury duality

Recall the Lloyd-Sudbury duality function

$$\psi_q(x,y) := \prod_i (1-q)^{x(i)y(i)} \quad \text{with} \quad q \in (0,2].$$

In particular

$$\psi_1(x,y) = 1_{\{x \wedge y = \underline{0}\}},$$
$$\psi_2(x,y) = (-1)^{\langle\langle x,y \rangle\rangle}.$$

Recall that $K_p$ is a *thinning kernel* that independently keeps particles with probability $p$ and throws them away with probability $1-p$.

**Lemma** $K_p K_r = K_{pr}$ and $K_p \psi_q = \psi_{pq}$.

# Thinnings and Lloyd-Sudbury duality

**Proof** The relation $K_p K_r = K_{pr}$ is trivial, since thinning first with $p$ and then with $r$ means that each particle independently has a probability $pr$ to survive both thinnings.

To check that $K_p \psi_q = \psi_{pq}$, we start with the case that $\Lambda$ consists of a single site. Then

$$K_p = \begin{pmatrix} K_p(0,0) & K_p(0,1) \\ K_p(1,0) & K_p(1,1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1-p & p \end{pmatrix}$$

and

$$\psi_q = \begin{pmatrix} \psi_q(0,0) & \psi_q(0,1) \\ \psi_q(1,0) & \psi_q(1,1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1-q \end{pmatrix}$$

which gives

$$K_p \psi_q = \begin{pmatrix} 1 & 0 \\ 1-p & p \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1-q \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1-pq \end{pmatrix} = \psi_{pq}.$$

# Thinnings and Lloyd-Sudbury duality

The space of all real functions on $\{0,1\}^\Lambda$ can be written as a tensor product

$$\mathbb{R}^{\{0,1\}^\Lambda} \cong \bigotimes_{i\in\Lambda} \mathbb{R}^{\{0,1\}}.$$

A linear operator $A$ acting on $\mathbb{R}^{\{0,1\}}$ with matrix $(A(x,y))_{x,y\in S}$ can be "lifted" to a linear operator $A^{\{i\}}$ acting on $\mathbb{R}^{\{0,1\}^\Lambda}$ with matrix

$$A^{\{i\}}(x,y) = A\big(x(i),y(i)\big) \prod_{j:\,j\neq i} 1\big(x(j),y(j)\big).$$

Then $A^{\{i\}}$ "acts only on the $i$-th coordinate":

$$\begin{aligned} &A^{\{i\}} f\big(x(1),\ldots,x(n)\big) \\ &\quad = \sum_{y\in S} A\big(x(i),y\big) f\big(x(1),\ldots,x(i-1),y,x(i+1),\ldots,x(n)\big). \end{aligned}$$

# Thinnings and Lloyd-Sudbury duality

Operators acting on different coordinates commute:

$$[A^{\{i\}}, B^{\{j\}}] = 0 \qquad (i \neq j).$$

We can write $K_p$ as a (commuting) product of operators that thin only a single site:

$$K_p = K_p^{\{1\}} \cdots K_p^{\{n\}} \qquad \text{with } \Lambda = \{1, \ldots, n\}.$$

Since $\psi_q(x, y) := \prod_i (1-q)^{x(i)y(i)}$, we see that likewise,

$$\psi_q = \prod_{i \in \Lambda} \psi_q^{\{i\}}.$$

Now $K_p^{\{i\}} \psi_q^{\{i\}} = \psi_{pq}^{\{i\}}$ for each $i \in \Lambda$ implies $K_p \psi_q = \psi_{pq}$. ∎

# Thinnings and Lloyd-Sudbury duality

**Example**

$$1_{\{x=\underline{0}\}} = 1_{\{x \wedge \underline{1} = \underline{0}\}} = \psi_1(x,\underline{1}) = \sum_y K_{1/2}(x,y)\psi_2(y,\underline{1})$$
$$= \mathbb{E}\big[\psi_2\big(\mathrm{Thin}_{1/2}(x),\underline{1}\big)\big] = \mathbb{E}\big[(-1)^{\sum_i \mathrm{Thin}_{1/2}(x)(i)}\big].$$

# Thinnings and Lloyd-Sudbury duality

**Lemma** Let $(P_t)_{t\geq 0}$, $(Q_t)_{t\geq 0}$, and $(R_t)_{t\geq 0}$, be transition kernels on $\{0,1\}^\Lambda$, and let $p \in (0,1]$, $q \in (0,2]$. Then of the following relations, any two imply the third one:

(i) $$\sum_{x'} P_t(x,x')\psi_{pq}(x',z) = \sum_{z'} R_t(z,z')\psi_{pq}(x,z') \qquad \forall x,z,t,$$

(ii) $$\sum_{y'} Q_t(y,y')\psi_q(y',z) = \sum_{z'} R_t(z,z')\psi_q(y,z') \qquad \forall y,z,t,$$

(iii) $$\sum_{x'} P_t(x,x')K_p(x',y) = \sum_{y'} K_p(x,y')Q_t(y',y) \qquad \forall x,y,t.$$

In words:

(i) $(P_t)_{t\geq 0}$ and $(R_t)_{t\geq 0}$ are dual w.r.t. $\psi_{pq}$,

(ii) $(Q_t)_{t\geq 0}$ and $(R_t)_{t\geq 0}$ are dual w.r.t. $\psi_q$,

(iii) $(Q_t)_{t\geq 0}$ is interwined on top of $(P_t)_{t\geq 0}$ w.r.t. $K_p$.

In (iii), we also say that $(Q_t)_{t\geq 0}$ is a *p-thinning* of $(P_t)_{t\geq 0}$.

# Thinnings and Lloyd-Sudbury duality

**Proof** We use that $K_p$ and $\psi_q$ are invertible matrices and $K_p\psi_q = \psi_{pq}$. Let $F, G, H$ denote the generators of $(P_t)_{t\geq 0}$, $(Q_t)_{t\geq 0}$, and $(R_t)_{t\geq 0}$. Then our relations say:

$$\text{(i)}\ FK_p\psi_q = K_p\psi_q H^\dagger, \quad \text{(ii)}\ G\psi_q = \psi_q H^\dagger, \quad \text{(iii)}\ FK_p = K_p G.$$

(i)&(ii)⇒(iii): $FK_p\psi_q = K_p\psi_q H^\dagger = K_p G\psi_q$, multiply by $\psi_q^{-1}$.

(ii)&(iii)⇒(i): $FK_p\psi_q = K_p G\psi_q = K_p\psi_q H^\dagger$.

(i)&(iii)⇒(ii): $K_p G\psi_q = FK_p\psi_q = K_p\psi_q H^\dagger$, multiply by $K_p^{-1}$. ∎

**Remark** We have not used that $F, G, H$ are Markov generators. E.g., if $F, G$ are Markov generators but $H$ is not, then (i)&(ii)⇒(iii) remains valid.

# Thinnings and Lloyd-Sudbury duality

Let $G$ be the generator of a Markov process in $\{0,1\}^2$ for which 00 is a trap (i.e., $G(00,x) = 0 \ \forall x \neq 00$) and

$$\begin{aligned} G(11,00) &= a && \text{(annihilation)}, \\ G(01,11) = G(10,11) &= b && \text{(branching)}, \\ G(11,01) = G(11,10) &= c && \text{(coalescence)}, \\ G(01,00) = G(10,00) &= d && \text{(death)}, \\ G(01,10) = G(10,01) &= e && \text{(exclusion)}. \end{aligned}$$

Let $(\Lambda, E)$ be an undirected graph and set

$$G = \sum_{\{i,j\} \in E} G^{\{i,j\}},$$

where we "lift" an operator acting only on two sites to the larger space as before.

# Thinnings and Lloyd-Sudbury duality

**[Lloyd and Sudbury ('95, '97, '00)]** Let $G$ be given by rates $a, b, c, d, e \geq 0$. Fix $q \in (0, 2]$, set

$$\gamma := q^{-1}(a + c - d + (1 - q)b),$$

and define $G'$ by

$$a' := a + 2(1 - q)\gamma, \qquad b' := b + \gamma,$$
$$c' := c - (2 - q)\gamma, \qquad d' := d + \gamma,$$
$$e' := e - \gamma.$$

If $a', b', c', d', e' \geq 0$, then $G'$ is a Markov generator and the processes $X$ and $Y$ are dual with duality function $\psi_q$:

$$\mathbb{E}^x\left[\psi_q(X_t, y)\right] = \mathbb{E}^y\left[\psi_q(x, Y_t)\right] \qquad (t \geq 0).$$

# Thinnings and Lloyd-Sudbury duality

**Remark** One can check that

$$\gamma' := q^{-1}(a' + c' - d' + (1-q)b')$$

satisfies $\gamma' = -\gamma$. The relations between $a, b, c, d, e$ and $a', b', c', d', e'$ can be written in a more symmetric form as:

$$\begin{aligned} \tfrac{1}{2}a + c - e &= \tfrac{1}{2}a' + c' - e', & b + e &= b' + e', \\ \tfrac{1}{2}a + (1-q)e &= \tfrac{1}{2}a' + (1-q)e', & d + e &= d' + e', \\ \big[a + c - d + (1-q)b\big] &= -\big[a' + c' - d' + (1-q)b'\big]. \end{aligned}$$

# Pathwise Lloyd-Sudbury duality

Recall the Lloyd-Sudbury duality function

$$\psi_q(x,y) = (1-q)^{\sum_i x(i)y(i)} \quad \text{with} \quad q \in (0,2],$$
$$\psi_1(x,y) = 1_{\{x \wedge y = \underline{0}\}},$$
$$\psi_2(x,y) = (-1)^{\langle\langle x,y \rangle\rangle}.$$

A map $m : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ is:

dual w.r.t. $\psi_2$ $\Leftrightarrow$ $m$ is *cancellative:*

$$m(x + y \bmod(2)) = m(x) + m(y) \bmod(2),$$

dual w.r.t. $\psi_1$ $\Leftrightarrow$ $m$ is *additive:* $m(x \vee y) = m(x) \vee m(y)$.

Very few maps are dual w.r.t. $\psi_q$ for $q \neq 1, 2$, though the exclusion map is an example.

# A bit of order theory

A partially ordered set $S$ is *bounded from below* resp. *above* if there exists an element 0 resp. 1 such that

$$0 \leq x \quad (x \in S) \quad \text{resp. } x \leq 1 \quad (x \in S).$$

The "upset" and "downset" of $A \subset S$ are defined as

$$A^{\uparrow} := \{x \in S : x \geq a \text{ for some } a \in A\},$$
$$A^{\downarrow} := \{x \in S : x \leq a \text{ for some } a \in A\}.$$

A set $A \subset S$ is *increasing* (resp. *decreasing*) if $A^{\uparrow} = A$ (resp. $A^{\downarrow} = A$). We let

$$\mathcal{P}_{\mathrm{inc}}(S) := \{A \subset S : A \text{ is increasing}\},$$
$$\mathcal{P}_{\mathrm{dec}}(S) := \{A \subset S : A \text{ is decreasing}\}.$$

# A bit of order theory

A *lattice* is a partially ordered set such that for every $x, y \in S$ there exist $x \vee y \in S$ and $x \wedge y \in S$ called the *supremum* or *join* and *infimum* or *meet* of $x$ and $y$, respectively, such that

$$\{x\}^{\uparrow} \cap \{y\}^{\uparrow} = \{x \vee y\}^{\uparrow} \quad \text{and} \quad \{x\}^{\downarrow} \cap \{y\}^{\downarrow} = \{x \wedge y\}^{\downarrow}.$$

Finite lattices are bounded from below and above.

By definition, a lattice $S$ is *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad (x, y, z \in S).$$

If $\Lambda$ is a partially ordered set, then $S := \mathcal{P}_{\mathrm{dec}}(\Lambda)$ with the order of set inclusion is a distributive lattice. *Birkhoff's representation theorem* says that every distributive lattice is of this form.

# A bit of order theory

Let $S$ be a partially ordered set. A *dual* of $S$ is a partially ordered set $S'$ together with a bijection $S \ni x \mapsto x' \in S'$ such that

$$x \leq y \quad \text{if and only if} \quad x' \geq y'.$$

*Example 1:* For any partially ordered set $S$, we may take $S' := S$ but equipped with the reversed order, and $x \mapsto x'$ the identity map.

*Example 2:* If $\Lambda$ is a partially ordered set and $S := \mathcal{P}_{\mathrm{dec}}(\Lambda)$ with the order of set inclusion, then we may take $S' := \mathcal{P}_{\mathrm{inc}}(\Lambda)$ and $x' := \Lambda \backslash x$ the complement of $x$.

# Additive duality

Let $S$ be a lattice and let $S'$ be its dual. A map $m : S \to S$ is *additive* if

$$m(0) = 0 \quad \text{and} \quad m(x \vee y) = m(x) \vee m(y) \qquad (x, y \in S).$$

**(Additive duality)** *A map $m : S \to S$ has a dual $m' : S' \to S'$ w.r.t.*

$$\psi(x, y) = 1_{\{x \leq y'\}} = 1_{\{y \leq x'\}} \qquad (x \in S,\ y \in S').$$

*if and only if $m$ is additive. The dual map $m'$ is unique and also an additive map.*

# Additive duality

If $S$ is a distributive lattice, then w.l.o.g. $S = \mathcal{P}_{\rm dec}(\Lambda)$, $S' = \mathcal{P}_{\rm inc}(\Lambda)$, and

$$\psi(x,y) = 1_{\{x \leq y'\}} = 1_{\{x \cap y = \emptyset\}} \qquad (x \in S,\ y \in S').$$

An additive map $m : S \to S$ can be represented in terms of a "matrix" $M \subset \Lambda \times \Lambda$ as

$$m(x) = \{j \in \Lambda : (i,j) \in M \text{ for some } i \in x\} \qquad (x \in S).$$

We can choose $M$ increasing in its first and decreasing in its second argument:

$$\begin{aligned} &\text{(i)} \quad (i,j) \in M \text{ and } i \leq \tilde{\imath} \text{ implies } (\tilde{\imath},j) \in M, \\ &\text{(ii)} \quad (i,j) \in M \text{ and } j \geq \tilde{\jmath} \text{ implies } (i,\tilde{\jmath}) \in M, \end{aligned}$$

and with these conventions it is unique.

# Additive duality

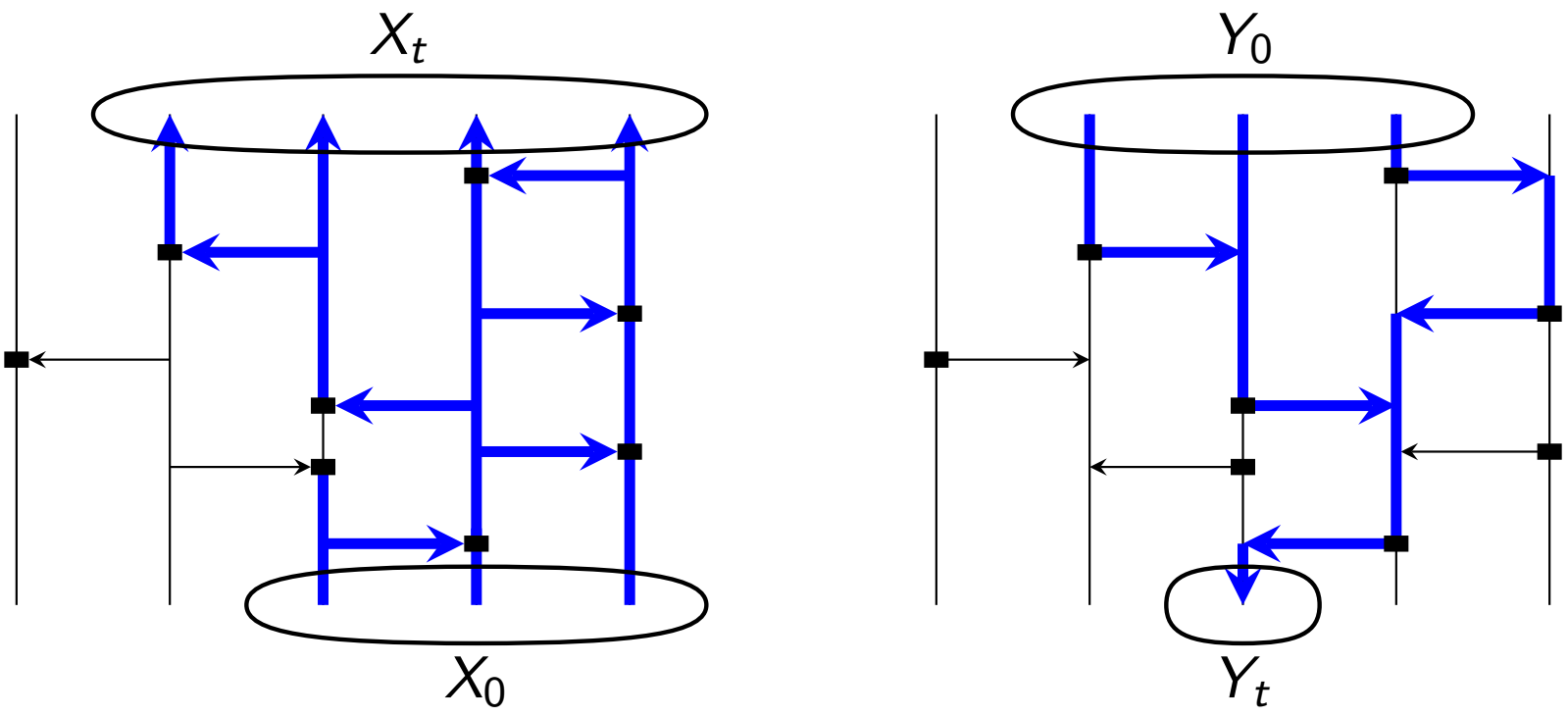
We can draw the matrix $M$ in terms of arrows and blocking symbols:

- An arrow from $i$ to $j$ for each $i \neq j$ such that $(i, j) \in M$.
- A blocking symbol at each $i$ such that $(i, i) \notin M$.

The dual map $m'$ is given by $M' := \{(j, i) : (i, j) \in M\}$.

This corresponds to reversing the direction of all arrows and keeping the blocking symbols where they are.

# Percolation representations




Graphical representation of a voter model $X$ and its dual $Y$, a system of coalescing random walks.

# Percolation representations

In the previous example, we equip $\Lambda$ with the *trivial order* $i \not\leq j$ for all $i \neq j$. With respect to this order,

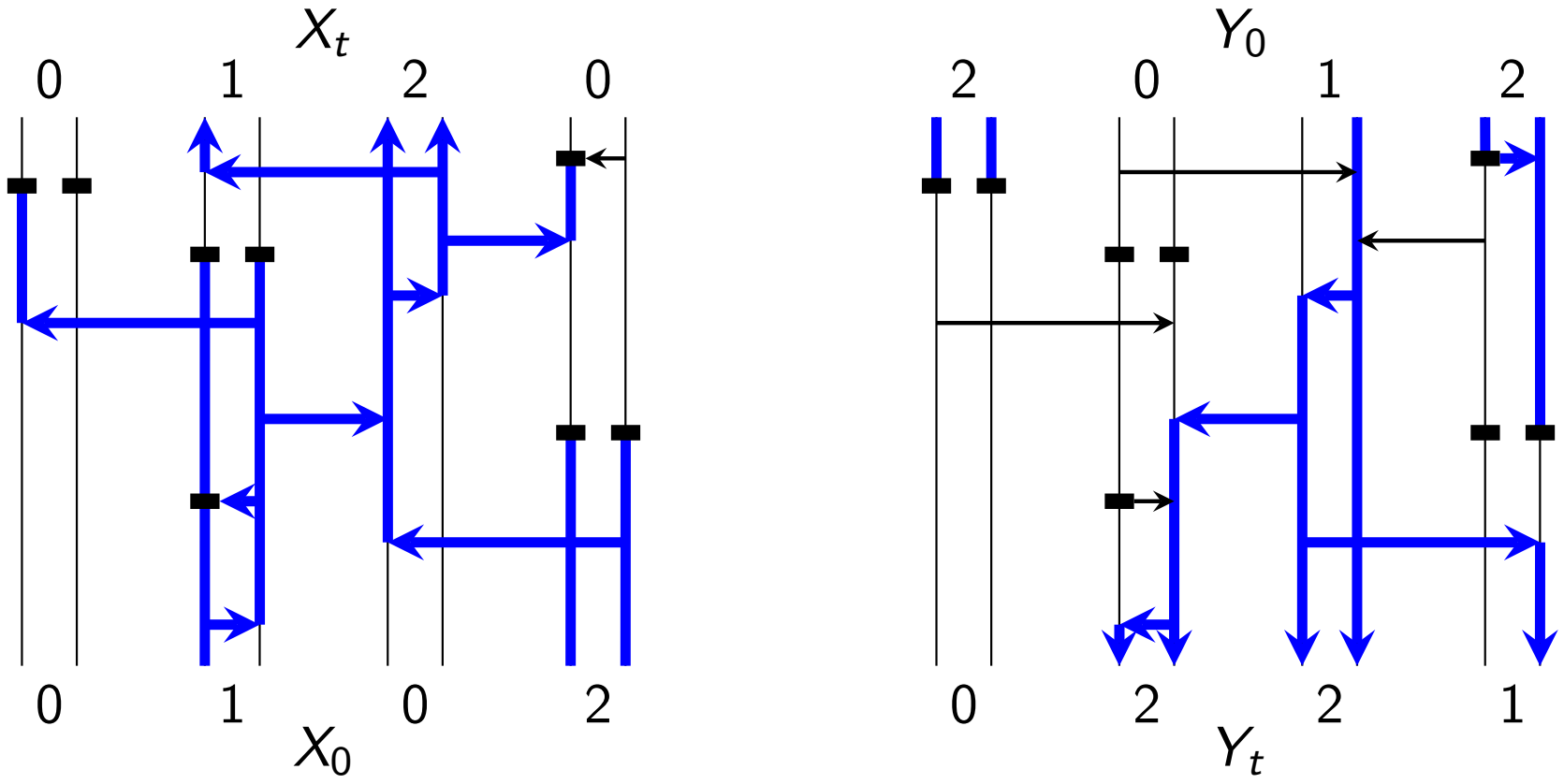
$$S = \mathcal{P}_{\rm dec}(\Lambda) = \mathcal{P}(\Lambda) = \mathcal{P}_{\rm inc}(\Lambda) = S'.$$

Steve Krone [AAP 1999] has studied a two-stage contact process, with state space of the form $S = \{0,1,2\}^\Lambda$, where $x(i) = 0, 1$, or 2 are interpreted as an empty site, young, or adult organism.
We view

$$S \cong \mathcal{P}_{\rm dec}\big(\Lambda \times \{1,2\}\big)$$

where $\Lambda$ is equipped with the trivial order, $\{1,2\}$ with the natural order (i.e., $1 \leq 2$), and $\Lambda \times \{1,2\}$ with the product order.

# Percolation representations




A percolation representation of Krone's 2-stage contact process and its dual.

# Krone's duality

The two-stage contact process and its dual are defined in terms of the maps

| | | |
|---|---|---|
| grow up | $a_i$ | $\cdots 1 \cdots\cdot \mapsto \cdots 2 \cdots\cdot$ |
| give birth | $b_{ij}$ | $\cdots 20 \cdots \mapsto \cdots 21 \cdots$ |
| young dies | $c_i$ | $\cdots 1 \cdots\cdot \mapsto \cdots 0 \cdots\cdot$ |
| death | $d_i$ | $\cdots ? \cdots\cdot \mapsto \cdots 0 \cdots\cdot$ |
| grow younger | $e_i$ | $\cdots 2 \cdots\cdot \mapsto \cdots 1 \cdots\cdot$ |

where in all cases not mentioned, the maps have no effect.

**(Krone's dual)** *The maps $a_i, b_{ij}, c_i, d_i, e_i$ are all additive and their duals are given by*

$$a_i' = a_i, \quad b_{ij}' = b_{ji}, \quad c_i' = e_i, \quad d_i' = d_i, \quad e_i' = c_i.$$

# Invariant subspaces

Let $\mathcal{P}(S)$ be the set of all subsets of $S$.
Let $m^{-1} : \mathcal{P}(S) \to \mathcal{P}(S)$ denote the *inverse image map*

$$m^{-1}(A) := \{x \in S : m(x) \in A\}.$$

**Observation** $m^{-1}$ is dual to $m$ w.r.t. to the duality function

$$\psi(x, A) := 1_{\{x \in A\}}.$$

**Consequence** Each Markov process $X$ with state space $S$ (and given random mapping representation) has a pathwise dual $Y$ with state space $\mathcal{P}(S)$ and generator

$$Hf(A) := \sum_{m \in \mathcal{G}} r_m \big(f(m^{-1}(A)) - f(A)\big)$$

*In practice*, this dual is not very useful since the space $\mathcal{P}(S)$ is very big. *Useful* duals are associated with *invariant subspaces* of $\mathcal{P}(S)$.

# Monotone systems

Let $S$ be a finite lattice and let $m : S \to S$ be monotone. Then $m$ is automatically *superadditive:*

$$m(x \vee y) \geq m(x) \vee m(y)$$

For monotone maps that are not additive, this inequality is strict. A good example is the *cooperative branching map*

$$\mathrm{coop}_{ii'j}x(k) := \begin{cases} \big(x(i) \wedge x(i')\big) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

**Observation** $m : S \to S$ is monotone if and only if

$$m^{-1}(A) \in \mathcal{P}_{\mathrm{inc}}(S) \text{ for all } A \in \mathcal{P}_{\mathrm{inc}}(T).$$

By taking complements, we can replace $\mathcal{P}_{\mathrm{inc}}(S)$ by $\mathcal{P}_{\mathrm{dec}}(S)$. Additive maps have the stronger property that $m^{-1}(\{x'\}^{\downarrow})$ is again of the form $\{y'\}^{\downarrow}$, where in fact $y = m'(x)$.

# Monotone systems duality

Let $\Lambda$ countable and $S = \{0,1\}^\Lambda$, equipped with the product order and topology. Set $S_{\rm fin} := \{y \in S : |y| < \infty\}$ with $|y| := \sum_i y(i)$. We can encode open increasing sets $A \subset S$ by their set of *minimal elements*

$$A^\circ := \big\{y \in A : z = y \ \forall A \ni z \leq y\big\}.$$

Set

$$\mathcal{I}(\Lambda) := \{Y : Y \text{ is open and } Y^\uparrow = Y\},$$

$$\mathcal{H}(\Lambda) := \{Y : Y \subset S_{\rm fin} \text{ and } Y^\circ = Y\}.$$

It is easy to see that $(Y^\uparrow)^\uparrow = Y^\uparrow$ and $(Y^\circ)^\circ = Y^\circ$.

**(Encoding open increasing sets)** *The map $Y \mapsto Y^\uparrow$ is a bijection from $\mathcal{H}(\Lambda)$ to $\mathcal{I}(\Lambda)$, and $Y \mapsto Y^\circ$ is its inverse.*

# Monotone systems duality

**(Monotone systems duality)** Let $(\mathbf{X}_{s,t})_{s\leq t}$ be the stochastic flow of an interacting particle system defined in terms of monotone maps.
Then there exists a *dual flow* $(\hat{\mathbf{X}}_{s,t})_{s\geq t}$ that is dual to $(\mathbf{X}_{s,t})_{s\leq t}$ w.r.t. the duality function

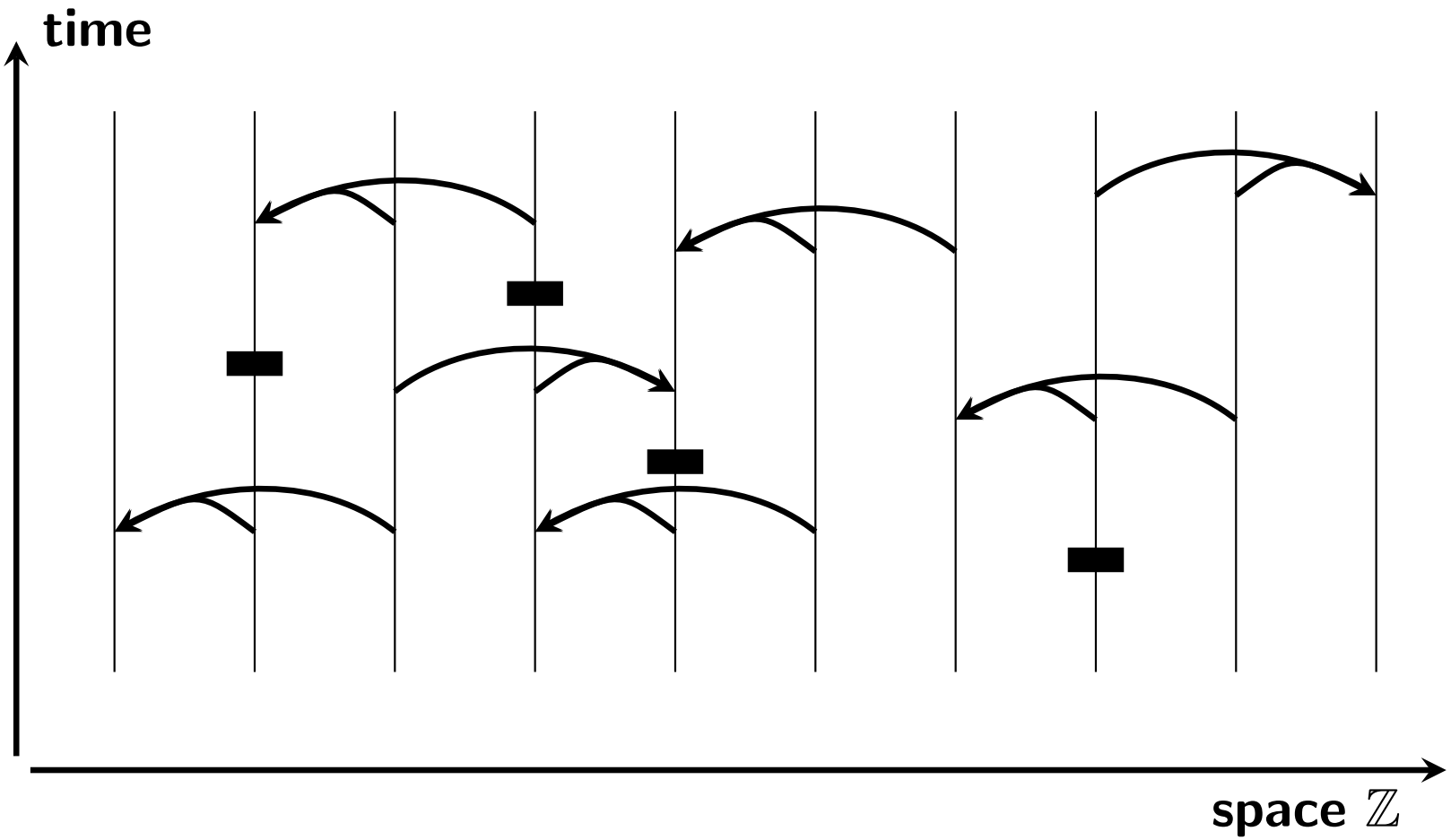
$$\psi(x, Y) = 1_{\{x \geq y \text{ for some } y \in Y\}} \quad \big(x \in \{0,1\}^\Lambda,\ Y \in \mathcal{H}(\Lambda)\big).$$

In particular,

$$\mathbf{X}_{s,t}(x) \geq y \text{ for some } y \in Y \quad \text{iff} \quad x \geq y \text{ for some } y \in \hat{\mathbf{X}}_{t,s}(Y)$$

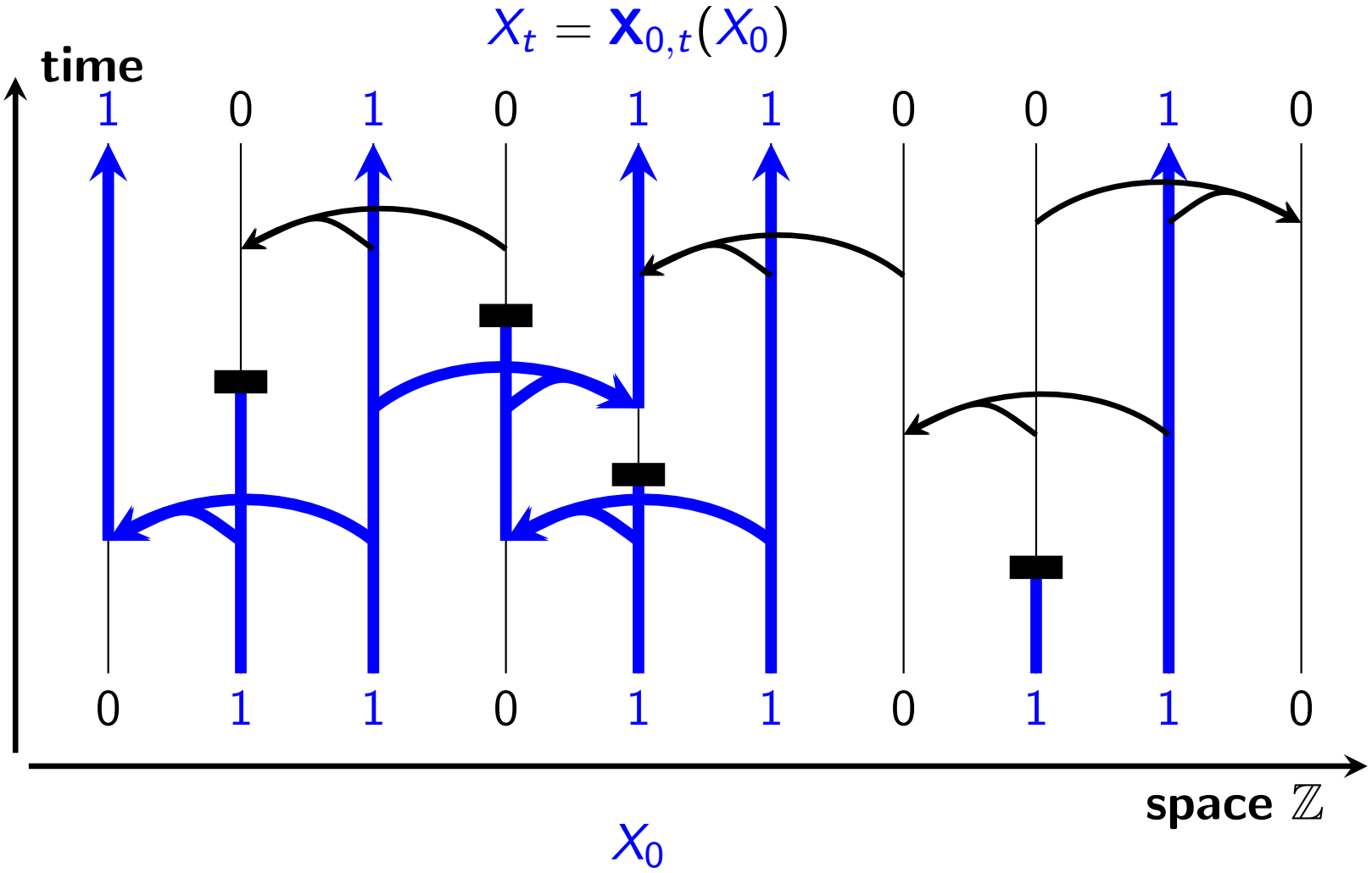
**Proof** Since $\mathbf{X}_{s,t}^{-1}(A) \in \mathcal{I}(\Lambda)$ for all $A \in \mathcal{I}(\Lambda)$, this follows from the trivial duality between $\mathbf{X}_{s,t}$ and $\mathbf{X}_{s,t}^{-1}$ and the bijection between $\mathcal{I}(\Lambda)$ and $\mathcal{H}(\Lambda)$. ∎

# A graphical representation



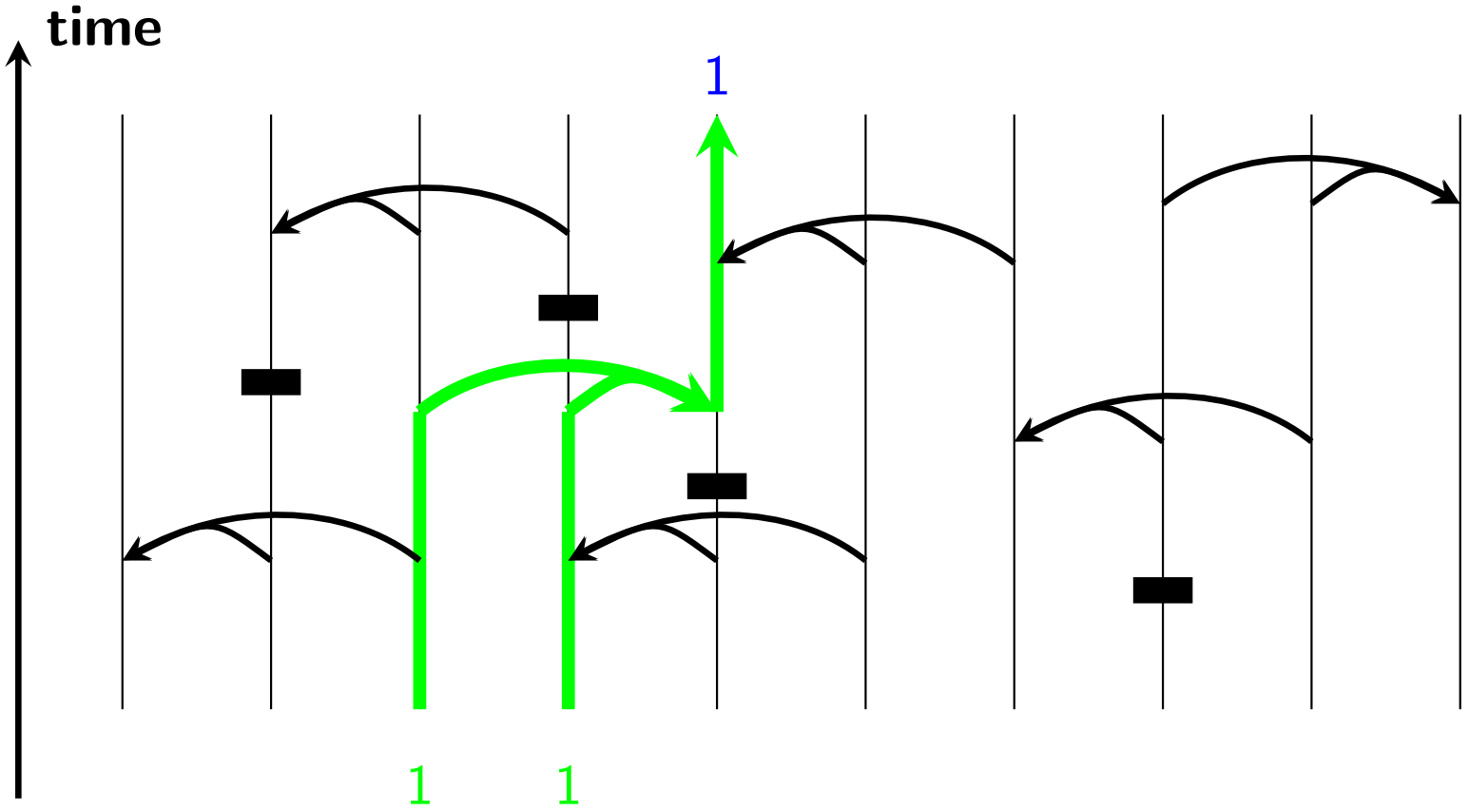

We denote $\mathtt{coop}_{ijk}$ by a suitable symbol and denote $\mathtt{dth}_i$ as before.

# A graphical representation

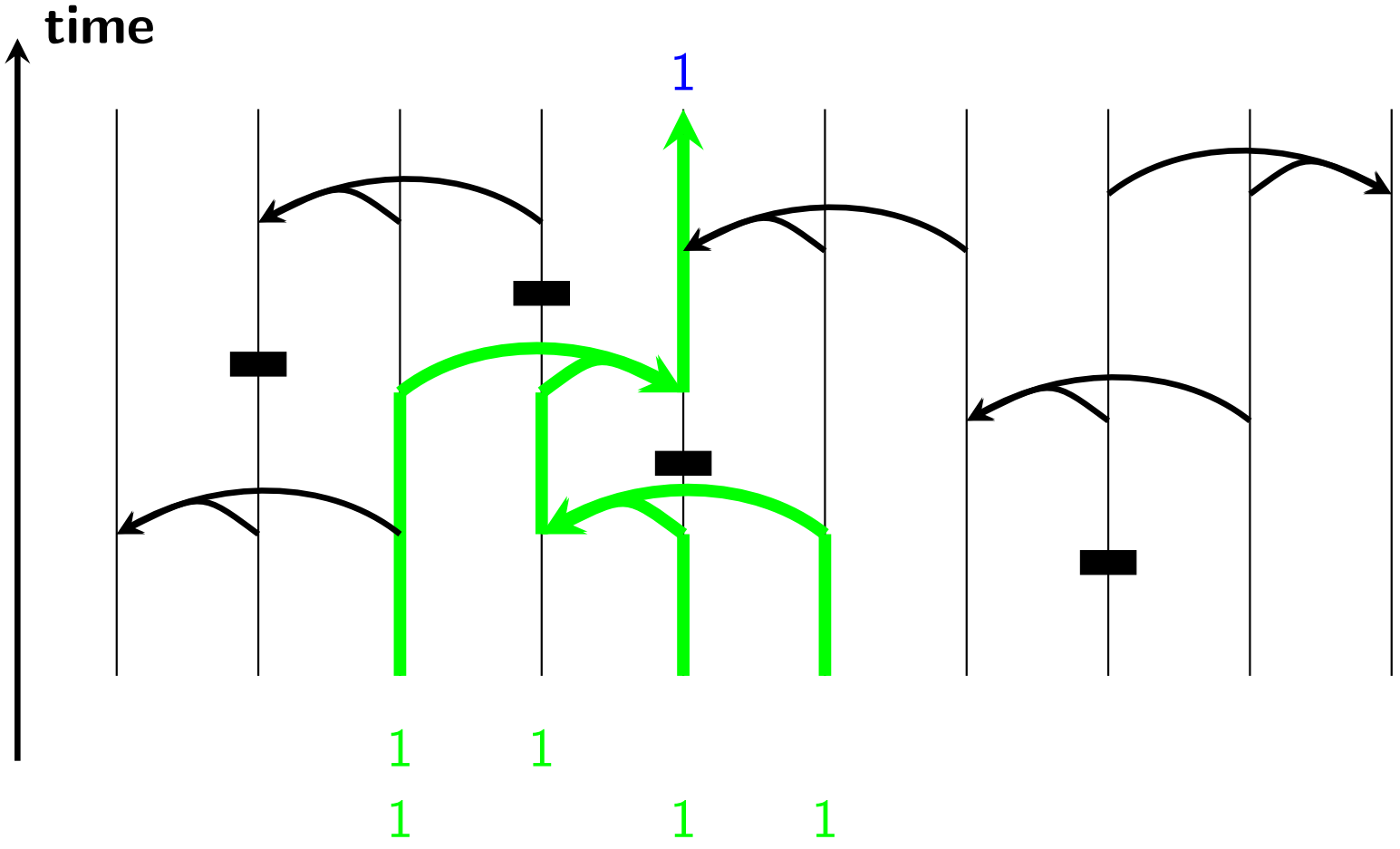



The map $x \mapsto \mathbf{X}_{0,t}(x)$.

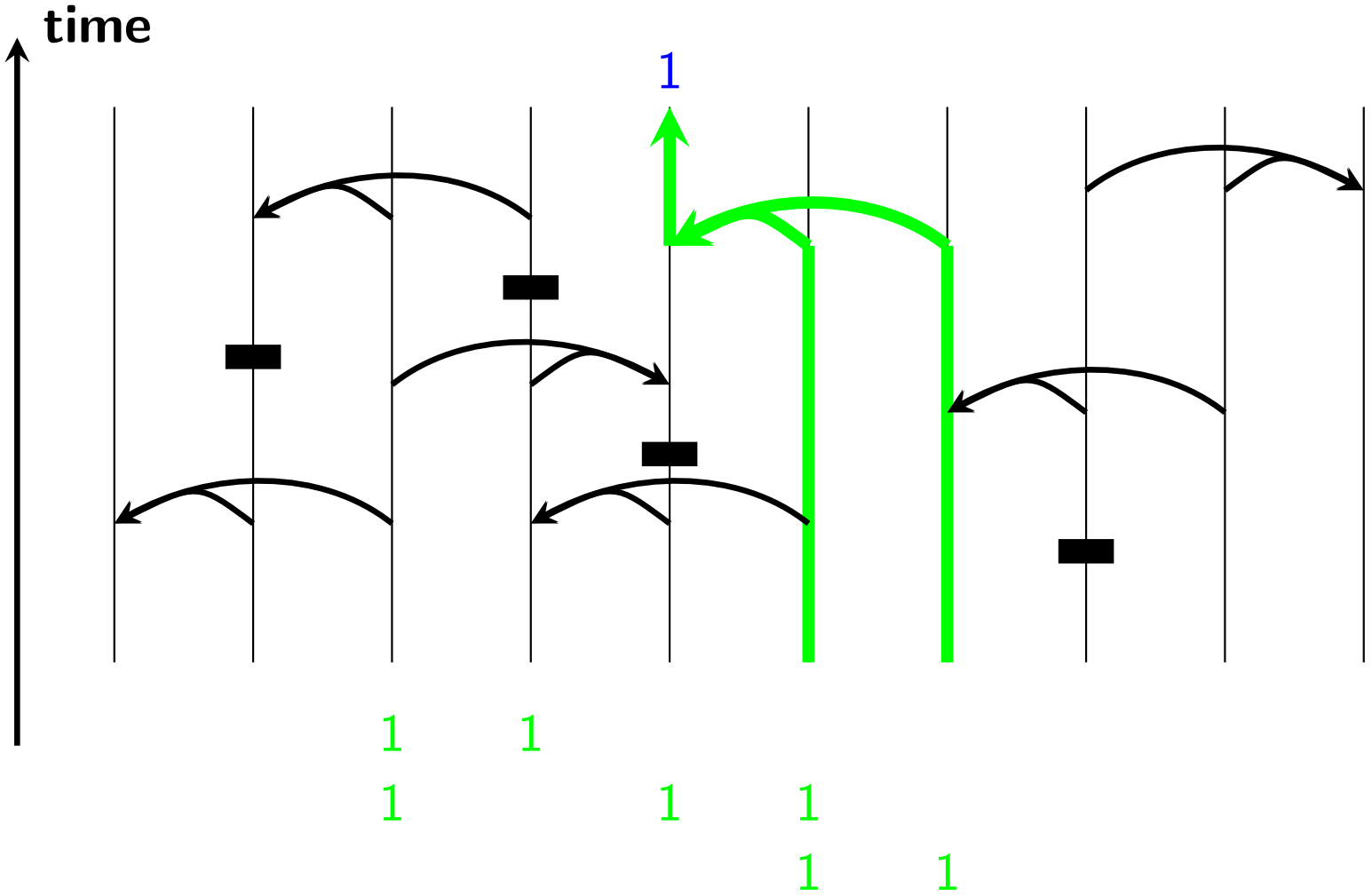
# Open paths



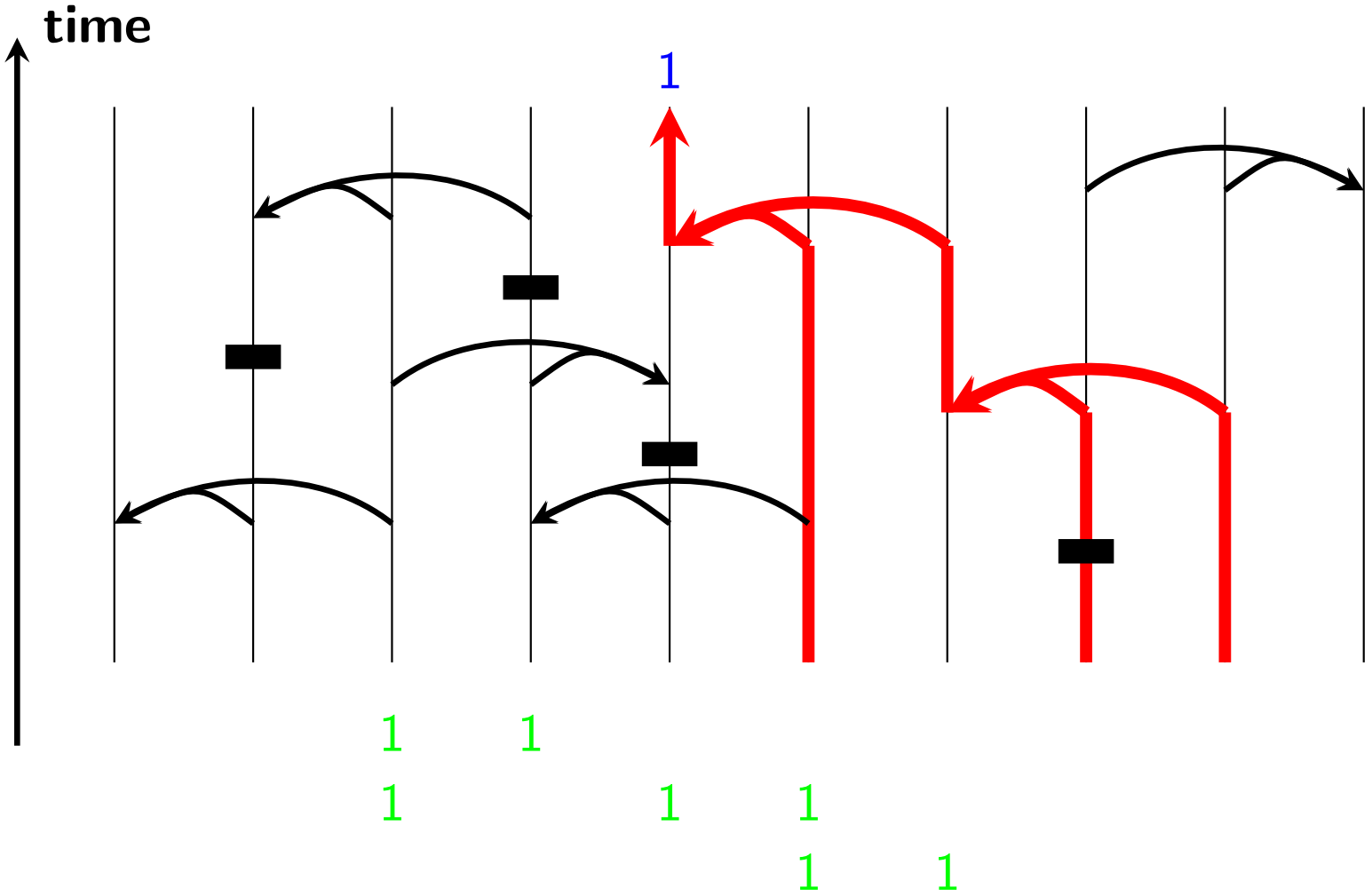
# Open paths



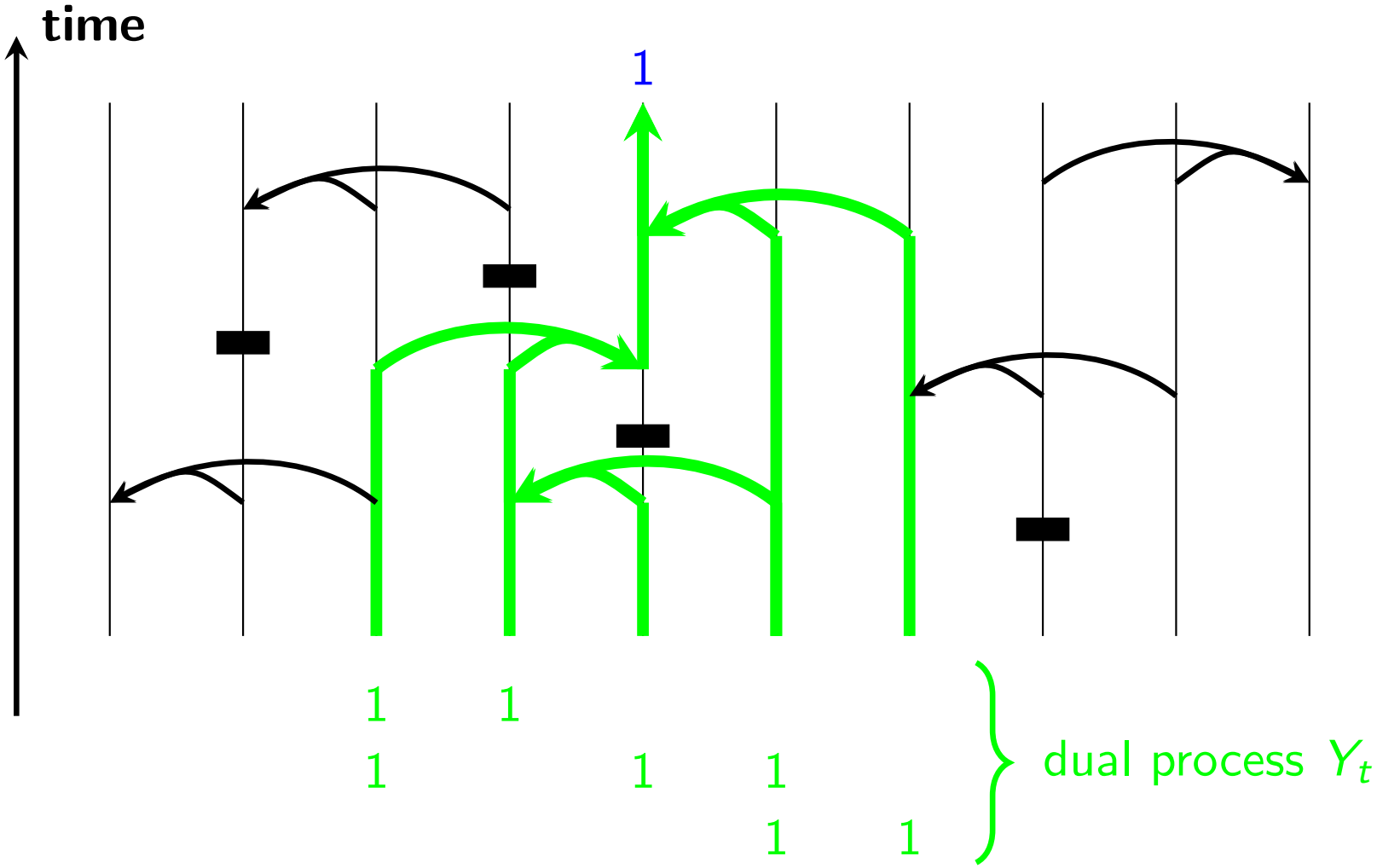
# Open paths



# Open paths



# Open paths




$$\mathbb{P}\big[X_t \geq y \text{ for some } y \in Y_0\big] = \mathbb{P}\big[X_0 \geq y \text{ for some } y \in Y_t\big].$$

# Random mapping representations

Let $S, T$ be finite sets, let $\mathcal{F}(S, T)$ denote the set of all functions $f : S \to T$, and let $K$ be a probability kernel from $S$ to $T$.

A *random mapping representation* of $K$ is an $\mathcal{F}(S, T)$-valued random variable $M$ such that

$$K(x, y) = \mathbb{P}[M(x) = y] \qquad (x \in S,\ y \in T).$$

We say that $K$ is *representable* in $\mathcal{G} \subset \mathcal{F}(S, T)$ if $M$ can be chosen so that it takes values in $\mathcal{G}$.

# Monotone probability kernels

For partially ordered sets $S, T$, let $\mathcal{F}_{\mathrm{mon}}(S, T)$ be the set of all monotone maps $m : S \to T$, i.e., those for which $x \leq x'$ implies $m(x) \leq m(x')$.

A probability kernel $K$ is called *monotone* if

$$Kf \in \mathcal{F}_{\mathrm{mon}}(S, \mathbb{R}) \quad \forall f \in \mathcal{F}_{\mathrm{mon}}(T, \mathbb{R}),$$

and *monotonically representable* if $K$ is representable in $\mathcal{F}_{\mathrm{mon}}(S, T)$.

Monotonical representability implies monotonicity:

$$f \in \mathcal{F}_{\mathrm{mon}}(T, \mathbb{R}) \quad \text{and} \quad x \leq x' \quad \Rightarrow$$
$$Kf(x) = \mathbb{E}\big[f\big(M(x)\big)\big] \leq \mathbb{E}\big[f\big(M(x')\big)\big] = Kf(x').$$

# Monotone probability kernels

J.A. Fill & M. Machida (AOP 2001) (and also D.A. Ross (unpublished)) discovered that the converse does not hold. There are counterexamples with $S = T = \{0,1\}^2$.

On the positive side, Kamae, Krengel & O'Brien (1977) and Fill & Machida (2001) have shown that:

**(Sufficient conditions for monotone representability)**
*Let $S, T$ be finite partially ordered sets and assume that at least one of the following conditions is satisfied:*

(i) *$S$ is totally ordered.*

(ii) *$T$ is totally ordered.*

*Then any monotone probability kernel from $S$ to $T$ is monotonically representable.*

# Stochastic order

In particular, setting $S = \{1, 2\}$, this proves that if $\mu_1, \mu_2$ are probability laws on $T$ such that

$$\mu_1 f \leq \mu_2 f \quad \forall f \in \mathcal{F}_{\mathrm{mon}}(T, \mathbb{R}),$$

then it is possible to couple random variables $M_1, M_2$ with laws $\mu_1, \mu_2$ such that $M_1 \leq M_2$.