# III Spatial Models in Population Biology

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## Lecture 3: Some concrete models

# Some concrete models

Outline

- The contact process
- The voter model
- The biased voter model
- Rebellious voter models

# The contact process

Let $(\Lambda, E)$ be an undirected graph with countable vertex set $\Lambda$ and set of undirected edges $E$. We sometimes view $(\Lambda, E)$ as a directed graph with set of directed edges

$$\vec{E} := \big\{(i,j) \in \Lambda^2 : \{i,j\} \in E\big\}.$$

$(\Lambda, E)$ is *locally finite* if $\{j : \{i,j\} \in E\}$ is finite for all $i \in \Lambda$.
By definition, an *automorphism* of $(\Lambda, E)$ is a bijection $\phi : \Lambda \to \Lambda$ such that

$$\big\{\phi(i), \phi(j)\big\} \in E \quad \text{iff} \quad \big\{i,j\big\} \in E.$$

A *transitive* graph is a graph $(\Lambda, E)$ such that for all $i, j \in \Lambda$, there exists an automorphism $\phi$ such that $\phi(i) = j$. This means that:

*As seen from each point, the space looks the same.*

# The contact process

Let $(\Lambda, E)$ be a locally finite, transitive graph. The *contact process* with *infection rate* $\lambda \geq 0$ is the interacting particle system $(X_t)_{t\geq 0}$ with state space $\{0,1\}^\Lambda$ and generator

$$\begin{aligned} Gf(x) := \lambda \sum_{(i,j)\in\vec{E}} & \big\{ f\big(\mathtt{bra}_{ij}x\big) - f(x)\big\} \\ & + \sum_{i\in\Lambda} \big\{ f\big(\mathtt{death}_i x\big) - f(x)\big\}. \end{aligned}$$

This is the *basic model for the spread of an organism*.
We call $i \in \Lambda$ *sites*. Sites with $x(i) = 0, 1$ are *empty* (or *healthy*) resp. *occupied* (or *infected*).

We could be more general and allow infection rates $\lambda(i,j)$ that depend, e.g., on the (graph) distance. In such a setting, we usually assume that $\forall i,j,\ \exists$ bijection $\phi$ s.t. $\lambda(\phi(i),\phi(j)) = \lambda(i,j)$.

# The mean-field limit

Let $X^N$ be a contact process with infection rate $\lambda_N$ on the *complete graph* $(\Lambda_N, E_N)$ with $N$ vertices. Then

$$\overline{X}^N_t := \frac{1}{N} \sum_{i \in \Lambda_N} X^N_t(i)$$

is a Markov process that jumps

$$\overline{x} \mapsto \begin{cases} \overline{x} + \frac{1}{N} & \text{with rate } \lambda N^2 \overline{x}(1-\overline{x}), \\ \overline{x} - \frac{1}{N} & \text{with rate } N\overline{x}. \end{cases}$$

We can find a nontrivial limit by letting $\lambda_N := \lambda N^{-1}$ for some fixed constant $\lambda \geq 0$. Since

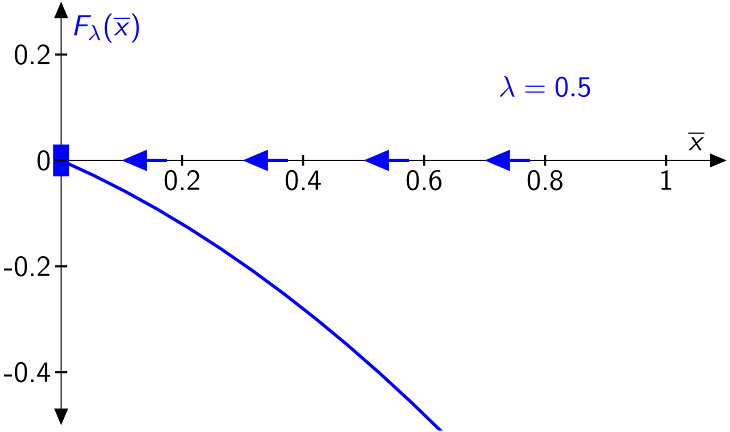
$$\mathbb{E}^x\big[\overline{X}^N_t - \overline{x}\big] = \big[\lambda\overline{x}(1-\overline{x}) - \overline{x}\big]t + O(t^2),$$

$$\mathbb{E}^x\big[(\overline{X}^N_t - \overline{x})^2\big] = O(t^2),$$

$X^N_t$ can be approximated by a solution to the *mean-field ODE*

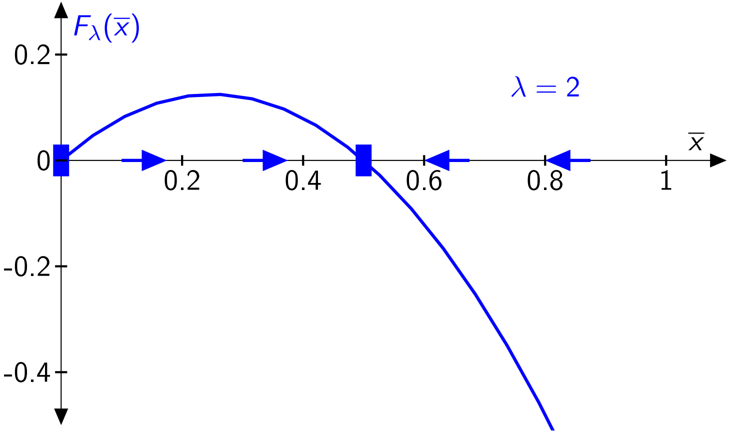
$$\frac{\partial}{\partial t}\overline{X}_t = \lambda \overline{X}_t(1-\overline{X}_t) - \overline{X}_t =: F_\lambda(\overline{X}_t) \quad (t \geq 0).$$

# The mean-field limit of the contact process



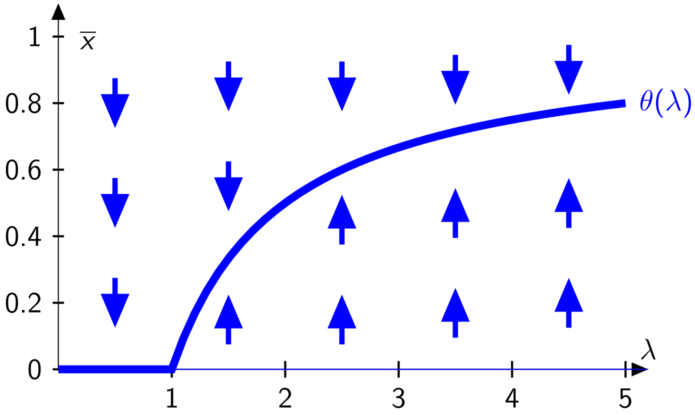

For $\lambda \leq 1$, the equation $\frac{\partial}{\partial t}\overline{X}_t = F_\lambda(\overline{X}_t)$ has a single, stable fixed point $\overline{x} = 0$.

# The mean-field limit of the contact process

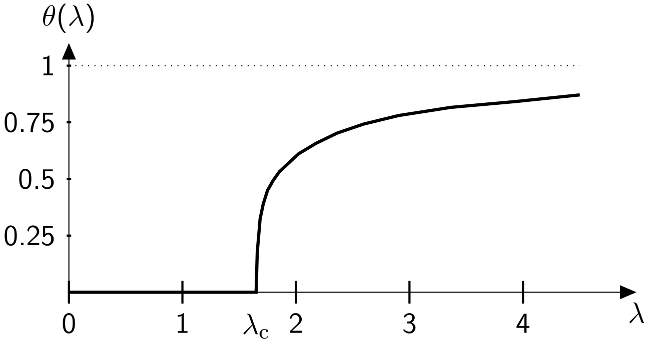



For $\lambda > 1$, the fixed point at 0 becomes unstable and a new, stable fixed point appears.

# The mean-field limit of the contact process




Fixed points of $\frac{\partial}{\partial t}\overline{X}_t = F_\lambda(\overline{X}_t)$ for different values of $\lambda$.




The probability $\theta(\lambda) := \int \overline{\nu}_\lambda(\mathrm{d}x)x(0)$ of the origin being occupied for the upper invariant law of the one-dimensional contact process.

# The Contact Process

By transitivity, every vertex $i \in \Lambda$ has the same degree $D := |\{j : \{i,j\} \in E\}|$.

In Lecture 1, we have seen that the contact process is uniquely ergodic for $\lambda < 1/D$ and hence $\lambda_{\rm c} \geq 1/D$.

Note that in the mean-field limit, we find a critical point at $\lambda \sim 1/N$, where the degree of each vertex is $N-1$.

For the contact process on $\mathbb{Z}^d$, it is known that

$$\lambda_{\rm c}(d) \sim \frac{1}{2d} \quad \text{as } d \to \infty$$

where $f(d) \sim g(d)$ means $f(d)/g(d) \to 1$.

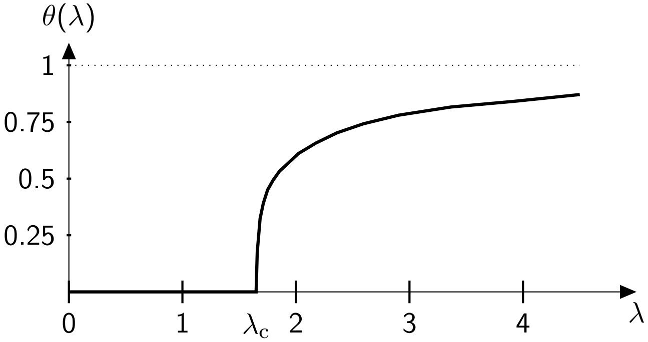
# The Contact Process

Sharp lower bounds on $\lambda_c$ can be obtained by finding bounded stopping times $\tau$ such that

$$\mathbb{E}^{\delta_i}\big[|X_\tau|\big] < 1.$$

Rigorous upper bounds are harder to obtain, especially in low dimensions. The best result in dimension 1 is $\lambda_c \leq 1.942$ [Liggett 1995].

# The Contact Process




Numerically $\lambda_c(1) \approx 1.649$ and

$$\theta(\lambda) \propto (\lambda - \lambda_c)^\beta \qquad \text{as } \lambda \downarrow \lambda_c,$$

with a critical exponent $\beta \approx 0.27648$.

# The Contact Process

For the contact process, one observes that

$$\theta(\lambda) \propto (\lambda - \lambda_{\rm c})^c \qquad \text{as } \lambda \downarrow \lambda_{\rm c},$$

with a critical exponent

$$c \approx 0.276 \text{ in dim } 1, \quad c \approx 0.583 \text{ in dim } 2,$$
$$c \approx 0.813 \text{ in dim } 3, \quad \text{and} \quad c = 1 \text{ in dim} \geq 4.$$

In theoretical physics, (nonrigorous) *renormalization group theory* is used to explain these critical exponents and calculate them.

The *lace expansion* has been used to prove that $c = 1$ in very large dimensions or for long-range models in dim $\geq 4$, in line with the mean-field prediction.

# The Contact Process

By self-duality, $\theta(\lambda)$ is not only the density of the upper invariant law, but also the survival probability started with a single occupied site.

It is easy to see that $\lambda \mapsto \theta(\lambda)$ is right-continuous and nondecreasing.

Duality can be used to prove that for $\lambda > \lambda_{\rm c}$, the contact process has a unique translation-invariant stationary law. This in turn can be seen to imply that $\lambda \mapsto \theta(\lambda)$ is left-continuous on $(\lambda_{\rm c}, \infty)$.

Proving left-continuity at $\lambda_{\rm c}$ amounts to showing that the critical process with $\lambda = \lambda_{\rm c}$ dies out. This has been proved in a celebrated paper by Bezuidenhout and Grimmett (1990).

The analogue question for unoriented percolation in 3 dimensions is still open.

# Local survival

The critical infection rate for *local survival* is defined as

$$\lambda_{\rm loc} := \sup\big\{\lambda \geq 0 : \lim_{t\to\infty} \mathbb{P}^{\delta_i}[X_t(i) = 1] = 0\big\}.$$

It is known that

$$\lambda_{\rm c} = \lambda_{\rm loc} \quad \text{on } \mathbb{Z}^d, \quad \text{but} \quad \lambda_{\rm c} < \lambda_{\rm loc} \quad \text{on regular trees.}$$

For $\lambda > \lambda_{\rm loc}$, all invariant laws are convex combinations of $\delta_{\underline{0}}$ and $\overline{\nu}$, but for $\lambda_{\rm c} < \lambda \leq \lambda_{\rm loc}$, the set of invariant laws is much larger.

# The Voter Model

Let $(\Lambda, E)$ be a locally finite transitive graph in which each vertex has degree $D$ as before and let $S$ be any set.
The voter model is the interacting particle system with state space $S^\Lambda$ and generator

$$Gf(x) := \sum_{(i,j)\in\vec{E}} \big\{ f\big(\mathtt{vot}_{ij}x\big) - f(x) \big\},$$

where $\mathtt{vot}_{ij}$ is the *voter map*

$$\mathtt{vot}_{ji}(x)(k) := \begin{cases} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

This is the *basic model for neutral genetic drift*. We interpret $x(i)$ as the (genetic) *type* of the organism living at the site $i$.

With rate $D$, the organism at site $i$ dies and is replaced by the offspring of a random parent, chosen uniformly from its $D$ neighbors.

# The mean-field model

We will mostly focus on the two-type model with $S = \{0, 1\}$.

The model on the *complete graph* $(\Lambda_N, E_N)$ with $N$ vertices is called the *Moran model*.[1] The fraction of sites of type 1

$$\overline{X}^N_t := \frac{1}{N} \sum_{i \in \Lambda_N} X^N_t(i)$$

is a Markov process that jumps

$$\overline{x} \mapsto \begin{cases} \overline{x} + \frac{1}{N} & \text{with rate } N^2 \overline{x}(1 - \overline{x}), \\ \overline{x} - \frac{1}{N} & \text{with rate } N^2 \overline{x}(1 - \overline{x}). \end{cases}$$

This gives

$$\mathbb{E}^x \big[ \overline{X}^N_t - \overline{x} \big] = 0,$$
$$\mathbb{E}^x \big[ (\overline{X}^N_t - \overline{x})^2 \big] = 2\overline{x}(1 - \overline{x})t + O(t^2).$$

[1] More precisely, the Moran model is the embedded Markov chain associated with the continuous-time process $X^N_t$.

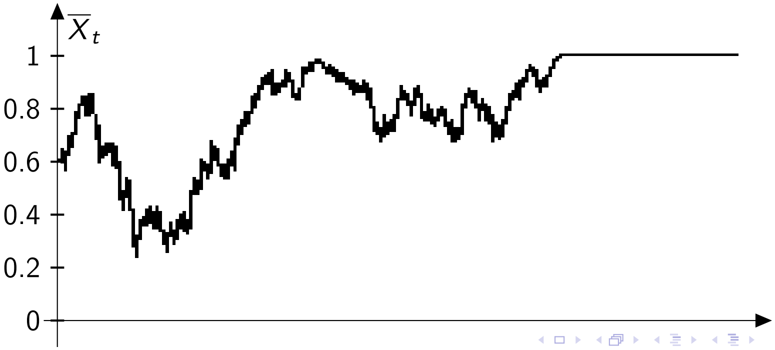
# The mean-field model

It can be shown that $X_t^N$ converges in law to the *Wright-Fisher diffusion* with generator

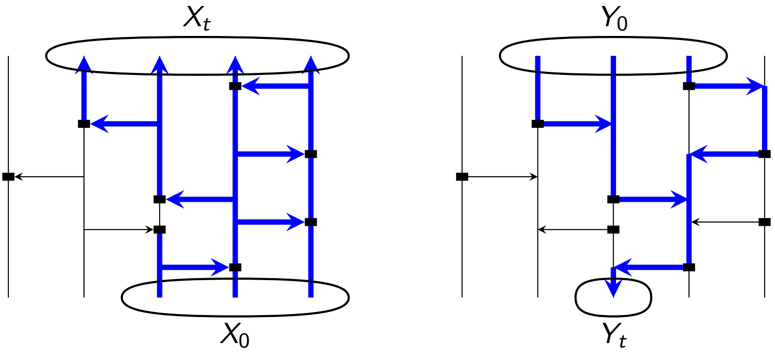
$$Gf(\overline{x}) = \overline{x}(1-\overline{x})\frac{\partial^2}{\partial \overline{x}^2} f(\overline{x}),$$

which is given (in law) by solutions to the SDE

$$\mathrm{d}\overline{X}_t = \sqrt{2\overline{X}_t(1-\overline{X}_t)}\mathrm{d}B_t \quad (t \geq 0).$$



# Duality to coalescing random walks




We recall that the two-type voter model is additive and dual to a system of coalescing random walks.

# Duality to coalescing random walks

This duality works even for voter models with more than two types. For each space-time point $(i, u)$, we can define a random walk $\xi^{(i,u)}_{t\geq 0}$ that traces back where the (unique) ancestor of $(i, u)$ lived at time $u - t$ $(t \geq 0)$. Then

$$\mathbf{X}_{s,u}(x)(i) = x(\xi^{(i,u)}_{u-s}) \qquad (s \leq u).$$

Random walks behave independently until the first time when they meet. At that time, they coalesce and continue as a single walker.

# The Kingman coalescent

Let $Y_t^N$ denote the dual process on the complete graph with $N$ vertices. Then

$$\overline{Y}_t^N := \sum_{i \in \Lambda_N} Y_t^N(i)$$

is a Markov process with state space $\{0, \ldots, N\}$ that jumps

$$\overline{y} \mapsto \overline{y} - 1 \quad \text{with rate } \overline{y}(\overline{y} - 1).$$

Note that the rates do not depend on $N$. The limiting process with state space $\mathbb{N}$ is the *Kingman coalescent*.

The Kingman coalescent is dual to the Wright-Fisher diffusion:

$$\mathbb{E}^{\overline{x}}\big[(1 - \overline{X}_t)^{\overline{y}}\big] = \mathbb{E}^{\overline{y}}\big[(1 - x)^{\overline{Y}_t}\big].$$

We interpret $\overline{x}$ is the frequency of type 1 individuals in an infinite population. Then $(1 - \overline{x})^{\overline{y}}$ is the probability that $\overline{y}$ individuals drawn from this population are all of type 0, corresponding to the duality function $\psi(x, y) = 1_{\{x \wedge y = \underline{0}\}}$.

# Coming down from infinity

We can couple processes $\overline{Y}^n_t$ with initial states $n = 1, 2, \ldots$ such that $\overline{Y}^n_t \leq \overline{Y}^{n+1}_t$ for each $n$. Then the a.s. increasing limit

$$\overline{Y}^\infty_t := \lim_{n\to\infty} \overline{Y}^n_t \qquad (t \geq 0)$$

exists in $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ for each $t \geq 0$.

It turns out that the Kingman coalescent *comes down from infinity*. Indeed, $\overline{Y}^\infty_t < \infty$ a.s. $(t > 0)$ and

$$\begin{aligned}\mathbb{E}^\infty\big[(1-x)^{\overline{Y}^\infty_t}\big] &= \lim_{n\to\infty} \mathbb{E}^{\overline{x}}\big[(1-\overline{X}_t)^n\big]\\ &= \mathbb{P}^{\overline{x}}\big[\overline{X}_t = 0\big] > 0 \qquad (\overline{x} < 1,\ t > 0),\end{aligned}$$

where $(\overline{X}_t)_{t\geq 0}$ is the Wright-Fisher diffusion.

# Mean-field behavior

Let $X^N$ be voter models on $\Lambda_N = \{0, \ldots, N-1\}^d$ with edges connecting nearest neighbors and periodic boundary conditions, and set

$$\overline{X}^N_t := \frac{1}{N^d} \sum_{i \in \Lambda_N} X^N_t(i).$$

Define constants $s_N$ by

$$s_N := \begin{cases} N^2 \frac{1}{2\pi} \log N & \text{if } d = 2, \\ N^d G_d & \text{if } d \geq 3, \end{cases}$$

where $G_d$ is the expected time spent at the origin by a random walk that jumps with rate $D$ and starts in 0. Then Cox (1989) has proved that, provided the initial states converge,

$$\mathbb{P}\big[\big(\overline{X}^N_{s_N t}\big)_{t \geq 0} \in \cdot\,\big] \underset{N \to \infty}{\Longrightarrow} \mathbb{P}\big[\big(\overline{X}_t\big)_{t \geq 0} \in \cdot\,\big],$$

where $(\overline{X}_t)_{t \geq 0}$ is the Wright-Fisher diffusion.

# Mean-field behavior

This result says that finite, but large populations in dimensions $d \geq 2$ behave essentially as well-mixing populations.
The reason for this is a *separation of time scales:* if we start two random walks on a large torus in dimensions $d \geq 2$, then the time till coalescence is in the limit $N \to \infty$ much larger than the time scale governing the movement of the walkers.
The result of this is that any finite number of coalescing random walks, started sufficienty far from each other, converges in the right time scale to the Kingman coalescent.

This result is not true in dimensions $d < 2$.

# Infinite lattices

Since $\delta_{\underline{0}}$ and $\delta_{\underline{1}}$ are two different invariant laws, the voter model is never uniquely ergodic.

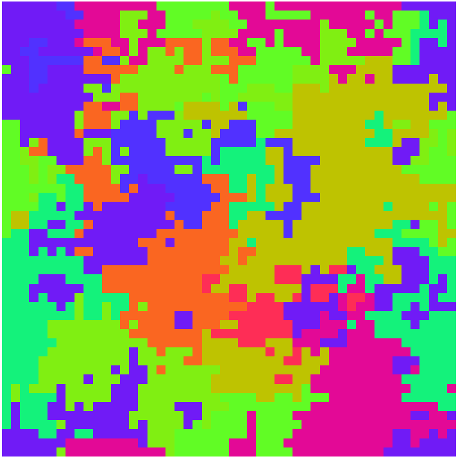
**Long-time behavior** Let $(X_t)_{t\geq 0}$ be a voter model started in an initial law such that $(X_0(i))_{i\in\mathbb{Z}^d}$ are i.i.d. with $\mathbb{P}[X_0(i)=1]=\theta$. Then

$$\mathbb{P}\big[X_t\in\cdot\,\big]\underset{t\to\infty}{\Longrightarrow}\begin{cases}(1-\theta)\delta_{\underline{0}}+\theta\delta_{\underline{1}} & \text{if } d=1,2,\\ \nu_\theta & \text{if } d\geq 3,\end{cases}$$

where in dimensions $d\geq 3$, there exists a one-parameter family $(\nu_\theta)_{\theta\in[0,1]}$ of mutually singular, translation-invariant stationary measures.

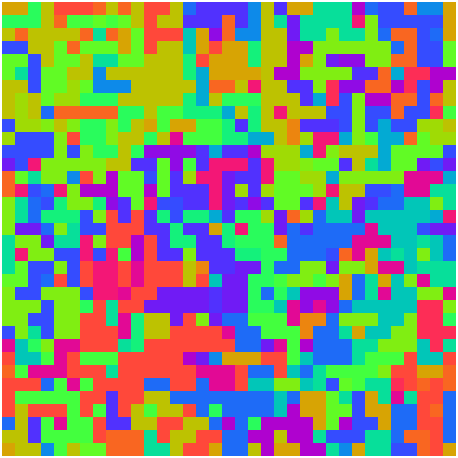
The behavior in $d=1,2$ is called *clustering* and in $d\geq 3$ *stable behavior*.

# Infinite lattices



Clustering of a two-dimensional voter model.

# Infinite lattices



Stable behavior of a three-dimensional voter model.

# Infinite lattices

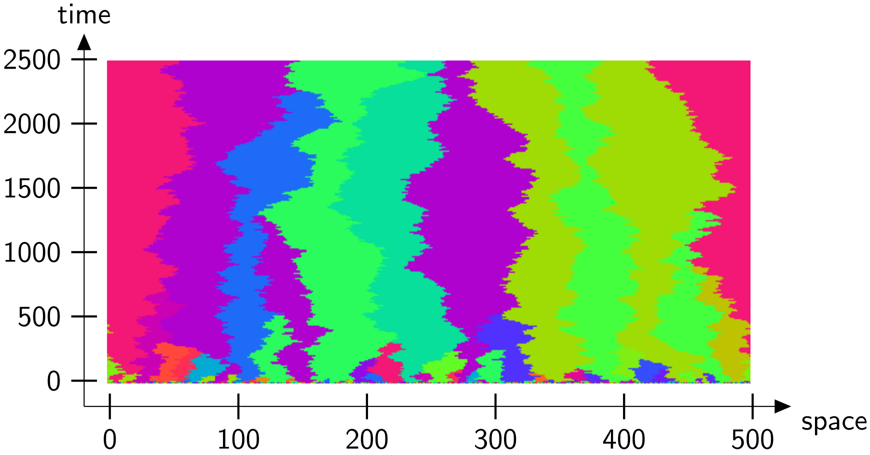
**Proof idea** Let $(\xi^i_t)_{t\geq 0}$ and $(\xi^j_t)_{t\geq 0}$ be independent random walks, started from $\xi^i_0 = i$ and $\xi^j_0 = j$. By duality

$$\mathbb{P}\big[X_t(i) = X_t(j)\big] \underset{t\to\infty}{\longrightarrow} \mathbb{P}\big[\exists t \geq 0 \text{ s.t. } \xi^i_t = \xi^j_t\big] \begin{cases} = 1 & \text{if } d = 1, 2, \\ < 1 & \text{if } d \geq 3. \end{cases}$$

**Remark** The statement holds more generally for voter models with any number of types and for general transitive graphs $\Lambda$. If random walk on $\Lambda$ is recurrent, then two independent random walks meet a.s., while on transient graphs there is[2] a positive probability they never meet.

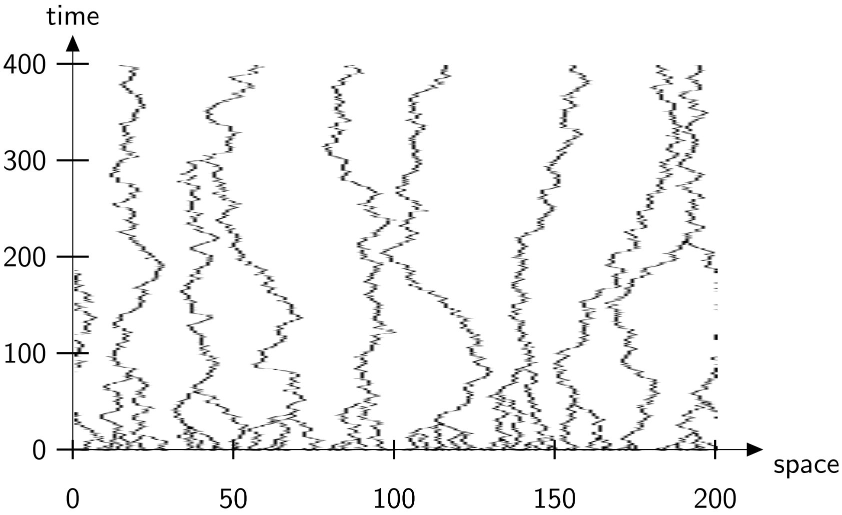
[2] Usually? always?

# Infinite lattices



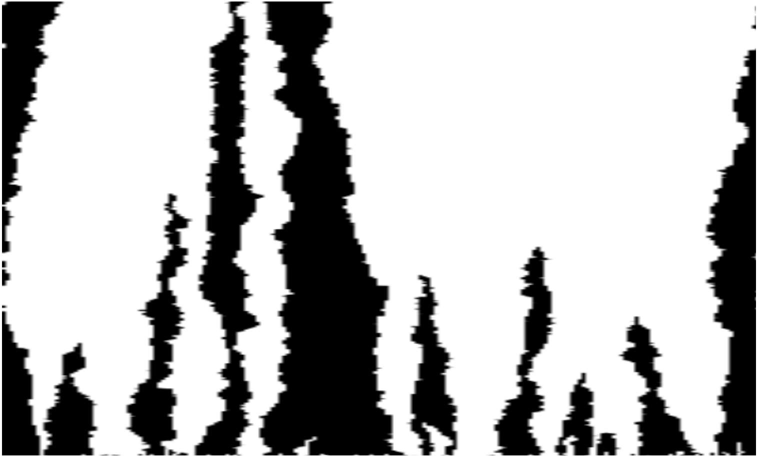

In one dimension, spatial movement and coalescence happen on the same time scale, and there is a nontrivial *scaling limit*.

# Boundaries



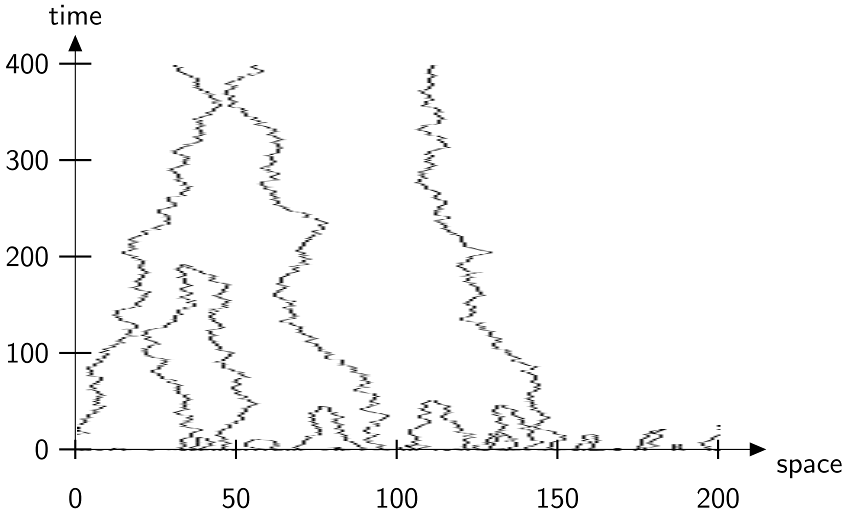

If we start with each site occupied by a different type, then the boundaries between types form *coalescing random walks*.

# Boundaries



For the two-type model, we obtain *annihilating random walks*.

# Boundaries




For the two-type model, we obtain *annihilating random walks*.

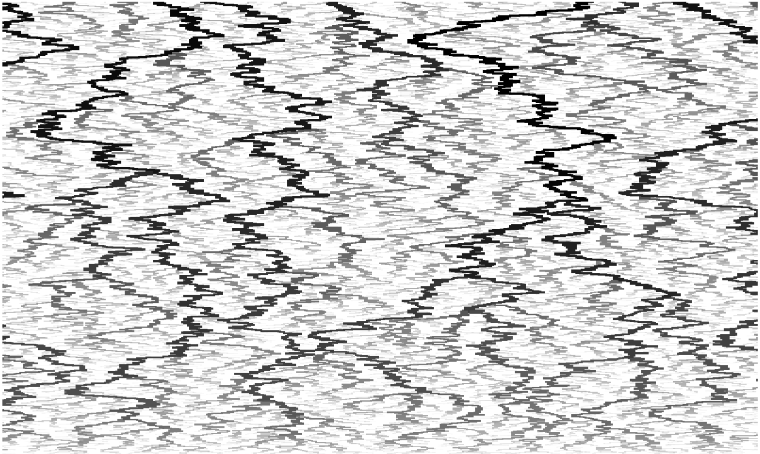
# Coalescing Brownian motions

The diffusive scaling limit of coalescing random walks are *coalescing Brownian motions*.

Coalescing Brownian motions *come down from infinity*. It is possible to start with Brownian motions "everywhere" and still have a finite particle density at each $t > 0$.

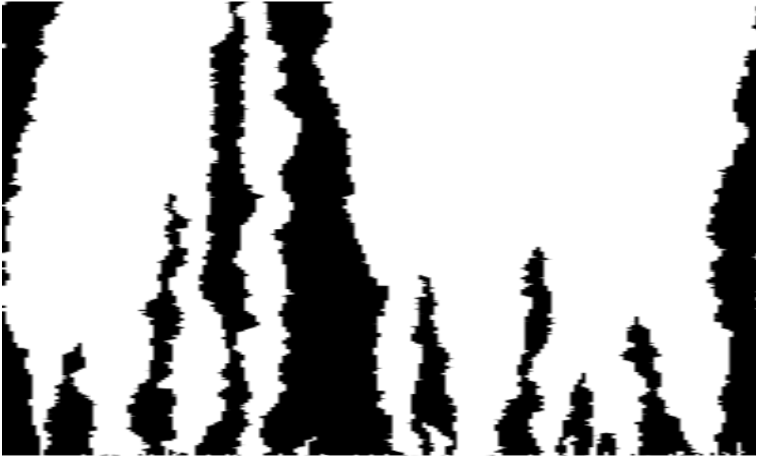
For the process coming down from infinity, the particle density is $\sim ct^{-1/2}$ as $t \to 0$, different from the $\sim ct^{-1}$ of Kingman's coalescent.
It is even possible to start Brownian motions from every point in space *and* time. The resulting collection of paths is known as the *Brownian web*.

# The Brownian web



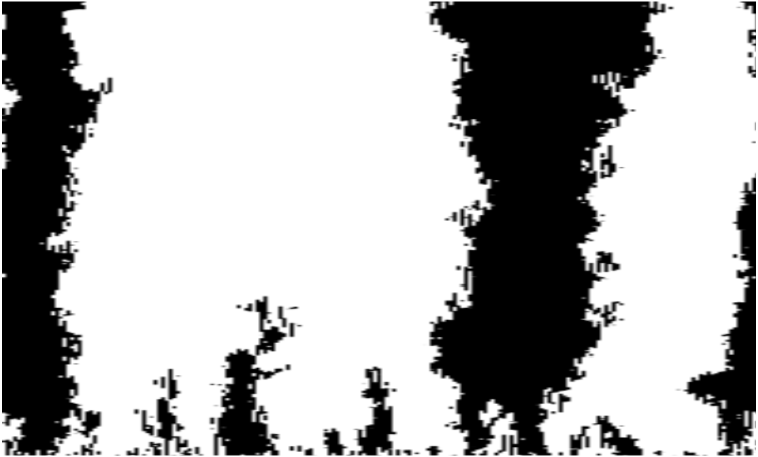
Artist's impression of the Brownian web.

# The boundary process



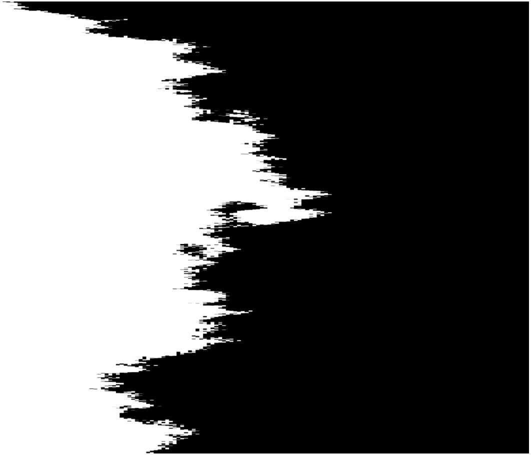
The boundaries of the voter model are annihilating random walks only for the *nearest neighbor model*

# The boundary process



Nevertheless, the picture for the *range two voter model* looks roughly similar.

# Interface tightness



The boundary between two infinite populations of type 0 and 1 remains rather sharp.

# Interface tightness

Consider a voter model $X$ where $\mathtt{vot}_{ij}$ is applied with rate $\lambda(j-i)$. Let

$$S^{01}_{\rm int} := \big\{x \in \{0,1\}^{\mathbb{Z}} : \lim_{i\to-\infty} x(i) = 0,\ \lim_{i\to\infty} x(i) = 1\big\}$$

be the set of configuration describing the interface between two infinite populations of 0's and 1's.

**Lemma** If $\sum_k \lambda(k)|k| < \infty$, then $X_0 \in S^{01}_{\rm int}$ implies $X_t \in S^{01}_{\rm int}$ for all $t \geq 0$ a.s.

Define an equivalence relation

$$x \sim y \text{ iff } \exists k \text{ s.t. } x(i) = y(i+k) \quad (i \in \mathbb{Z}).$$

Let $\tilde{x} := \{y : x \sim y\}$ and $\tilde{S}^{01}_{\rm int} := \{\tilde{x} : x \in S^{01}_{\rm int}\}$.

**Def** $X$ exhibits *interface tightness* if the *process modulo translations* $(\tilde{X}_t)_{t\geq 0}$ is positive recurrent.

# Interface tightness

**Cox & Durrett 1995** If $\sum_k \lambda(k)|k|^3 < \infty$, then interface tightness holds.

**Belhaouari, Mountford & Valle 2007** Interface tightness holds when $\sum_k \lambda(k)k^2 < \infty$, but not when $\sum_k \lambda(k)k^{2-\varepsilon} = \infty$ for some $\varepsilon > 0$.

The width of the interface is $W(X_t) := R(X_t) - L(X_t)$ with

$$L(x) := \sup\{i \in \mathbb{Z} + \tfrac{1}{2} : x(j) = 0 \forall j < i\},$$
$$R(x) := \inf\{i \in \mathbb{Z} + \tfrac{1}{2} : x(j) = 1 \forall j > i\}.$$

**[B, M, Sun & V 2006]** If $\sum_k \lambda(k)|k|^{3+\varepsilon} < \infty$, then in equilibrium,

$$\mathbb{P}[W(X_\infty) \geq N] \asymp N^{-1} \quad \text{as } N \to \infty.$$

# Interface tightness

*Some intuition:* Let $\xi^1, \ldots, \xi^n$ be independent nearest-neighbor random walks started in $\xi^1_0 < \cdots < \xi^n_0$.
Let $\tau_{i,i+1} := \inf\{t \geq 0 : \xi^i_t = \xi^{i+1}\}$ and

$$\tau_n := \tau_{1,2} \wedge \cdots \wedge \tau_{n-1,n}.$$

It is known[3] that

$$\mathbb{P}[\tau_n > t] \sim c_n h_n(\xi^1_0, \ldots, \xi^n_0) t^{-\beta_n} \quad \text{as } t \to \infty,$$

with $c_n$ an explicit constant, $\beta_n := \frac{1}{4} n(n-1)$, and

$$h_n(\xi^1_0, \ldots, \xi^n_0) := \prod_{1 \leq k < m \leq n} (x_m - x_k).$$

The harmonic function $h_n$ is the *Vandermonde determinant*.

[3] See Grabiner 1999 for the case of $n$ Brownian motions.

# Interface tightness

The interface process modulo translations makes i.i.d. excursions away from the simplest heaviside interface.

Since $\beta_3 < \beta_5 < \cdots$, we expect the tail of $W(X_\infty)$ to be dominated by excursions where the interface effectively splits into three random walks that do not meet for a long time.
In view of this, the duration $\sigma$ of an excursion away from the heaviside state should have tail distribution

$$\mathbb{P}[\sigma > t] \approx t^{-\beta_3} = t^{-3/2} \quad \text{as } t \to \infty.$$

In equilibrium, the duration of the excursion that is going on at time zero has a size-biased law, which should satisfy

$$\hat{\mathbb{P}}[\sigma > t] \approx t^{-1/2} \quad \text{as } t \to \infty.$$

Since the width of an excursion should roughly be the root of its duration, this implies

$$\mathbb{P}[W(X_\infty) \geq N] \approx N^{-1} \quad \text{as } N \to \infty.$$

# Interface tightness

In **[CD95]** and **[BMV07]**, duality is used to show that the laws of the *number of inversions*

$$f(X_t) := \sum_{i<j} 1_{\{X_t(i)=1,\ X_t(j)=0\}} \qquad (x \in S^{01}_{\mathrm{int}})$$

are tight as $t \to \infty$, which implies interface tightness.

**Sturm & S 2008** show that $f$ can be used as a "pseudo Lyapunov function". More precisely, if interface tightness would not hold, then on sufficiently long time intervals $f(X_t)$ would decrease linearly in time, which is not possible since $f \geq 0$.

Let $b(x) := \sum_i 1_{\{x(i) \neq x(i+1)\}}$ denote the *number of boundaries*.

**Open Problem** $\mathbb{P}[b(X_\infty) \geq N] \asymp e^{-\varepsilon N}$ as $N \to \infty$?

# The biased voter model

The *biased voter model* with *selection rate* $\lambda$ and *deleterious mutation rate* $\mu$ on a locally finite, transitive graph $(\Lambda, E)$, is the interacting particle system with state space $S^\Lambda$ and generator

$$Gf(x) := \sum_{(i,j)\in\vec{E}} \big\{f\big(\mathtt{vot}_{ij}x\big) - f(x)\big\} + \lambda \sum_{(i,j)\in\vec{E}} \big\{f\big(\mathtt{bra}_{ij}x\big) - f(x)\big\} + \mu \sum_{i\in\Lambda} \big\{f\big(\mathtt{death}_i x\big) - f(x)\big\}.$$

Alternatively, we can view this as a mixture between *voter model* and *contact process* dynamics.

We interpret the type 1 as *fitter* individuals that carry a gene that gives them a selective advantage in reproduction.
On the other hand, *deleterious mutations* can cause the advantageous gene to become dysfunctional.

# The biased voter model

Since the biased voter model has pairwise interactions and $\underline{0}$ is a trap, it falls in the general class of models studied by Lloyd and Sudbury. Recall that

$$\begin{array}{llll} 11 \mapsto 00 & \text{with rate} & a & \text{(annihilation)}, \\ 01 \mapsto 11 & \text{with rate} & b & \text{(branching)}, \\ 11 \mapsto 01 & \text{with rate} & c & \text{(coalescence)}, \\ 01 \mapsto 00 & \text{with rate} & d & \text{(death)}, \\ 01 \mapsto 10 & \text{with rate} & e & \text{(exclusion)}. \end{array}$$

In particular, setting

$$a = 0, \quad b = 1 + \lambda, \quad c = \mu, \quad d = 1 + \mu, \quad e = 0$$

yields a biased voter model with selection parameter $\lambda$ and mutation parameter $\mu$.

# The biased voter model

For any $q \in (0,2]$, setting

$$\begin{aligned}\gamma &:= q^{-1}(a + c - d + (1-q)b) \\ &= q^{-1}\big(0 + \mu - (1+\mu) + (1-q)(1+\lambda)\big) = \frac{1-q}{q}\lambda - 1\end{aligned}$$

and

$$\begin{aligned} a' &:= 2(1-q)\gamma, & b' &:= \lambda/q, \\ c' &:= \mu - (2-q)\gamma, & d' &:= \mu + \tfrac{1-q}{q}\lambda, \end{aligned}$$
$$e' := -\gamma.$$

yields a dual process w.r.t. $\psi_q$, provided $a', b, c', d', e' \geq 0$.

# Duals of the biased voter model

Let $X$ be a biased voter model with parameters $\lambda, \mu$. For $\varepsilon \in [0,1]$ and $\rho, \beta \geq 0$, let $Y$ be the process with generator

$$
\begin{aligned}
G' f(y) := \sum_{(i,j)\in\vec{E}} \Big[ & (1-\varepsilon)\rho\big\{f(\mathtt{rw}_{ij}y) - f(y)\big\} + \varepsilon\rho\big\{f(\mathtt{arw}_{ij}y) - f(y)\big\} \\
& + (1-\varepsilon)\beta\big\{f(\mathtt{bra}_{ij}y) - f(y)\big\} + \varepsilon\beta\big\{f(\mathtt{abra}_{ij}y) - f(y)\big\}\Big] \\
& + \sum_{i\in\Lambda} \delta\big\{f(\mathtt{death}_i y) - f(y)\big\}
\end{aligned}
$$

**Lloyd-Sudbury duals of biased voter models** The biased voter model $X$ has a dual w.r.t. the duality function $\psi_q$ with $q \in (0,2]$ precisely in the following cases.

**(a)** If $\lambda > 0$, then $X$ is self-dual with $q = \lambda/(1+\lambda)$.

**(b)** $X$ is dual with $q = 1+\varepsilon$ to the process $Y$ with parameters $\rho = 1 + \frac{\varepsilon}{1+\varepsilon}\lambda$, $\beta = \frac{1}{1+\varepsilon}\lambda$, and $\delta = \mu - \frac{\varepsilon}{1+\varepsilon}\lambda$, for any $\varepsilon \in [0,1]$ such that $\delta \geq 0$.

# Thinning relations

Branching and coalescing random walks $Y$ with jump rate $\rho = 1$, branching rate $\beta = \lambda$, and death rate $\delta = \mu$ are the additive dual of the biased voter model with bias $\lambda$ and mutation rate $\mu$.

Recall that if $(P_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ are both dual to the same $(R_t)_{t\geq 0}$, with duality functions $\psi_q$ and $\psi_{q'}$, respectively, then $P_t K_{q/q'} = K_{q/q'} Q_t$, i.e., $(Q_t)_{t\geq 0}$ is a $q/q'$-thinning of $(P_t)_{t\geq 0}$.

**Thinnings of biased voter models**

**(a)** The process $Y$ with parameters $\rho = 1$, $\beta = \lambda$, and $\delta = \mu$ is a $\lambda/(1+\lambda)$-thinning of the biased voter model with parameters $\lambda$ and $\mu$.

**(b)** The process $Y$ with parameters $\rho = 1 + \frac{\varepsilon}{1+\varepsilon}\lambda$, $\beta = \frac{1}{1+\varepsilon}\lambda$, and $\delta = \mu - \frac{\varepsilon}{1+\varepsilon}\lambda$ is a $1/(1+\varepsilon)$-thinning of the process $Y$ with parameters $\rho = 1$, $\beta = \lambda$, and $\delta = \mu$.

# The skeletal process

Let $(\mathbf{X}_{s,t})_{s\leq t}$ the stochastic flow of a biased voter model with parameters $\lambda > 0$ and $\mu = 0$. Set

$$\overline{Y}_t(i) := 1_{\{\mathbf{X}_{t,u}(\delta_i) \neq \underline{0} \ \forall u \geq t\}}.$$

Then $(\overline{Y}_t)_{t\in\mathbb{R}}$ is a stationary system of branching and coalescing random walks with branching parameter $\lambda$.
For each $t \in \mathbb{R}$, the random variables $(\overline{Y}_t(i))_{i\in\Lambda}$ are i.i.d. with $\mathbb{P}[\overline{Y}_t(i) = 1] = \lambda/(1+\lambda)$.

Let $X_0$ be independent of the graphical representation and

$$X_t = \mathbf{X}_{0,t}(X_0) \quad \text{and} \quad Y_t(i) := X_t(i) \wedge \overline{Y}_t(i) \qquad (t \geq 0).$$

Then $(Y_t)_{t\in\mathbb{R}}$ is a $\lambda/(1+\lambda)$-thinning of $(X_t)_{t\geq 0}$.

# Interpretation of the self-duality

Let $(\mathbf{X}_{s,t})_{s\leq t}$ be the stochastic flow of a biased voter model, and let $(\mathbf{X}_{t,s})_{t\geq s}$ be the stochastic flow of another such biased voter model, with time running backwards.
Let $q = \lambda/(1+\lambda)$. Self-duality says that

$$\mathbb{E}\big[\psi_q\big(\mathbf{X}_{s,t}(x), x'\big)\big] = \mathbb{E}\big[\psi_q\big(x, \mathbf{X}_{t,s}(x')\big)\big],$$

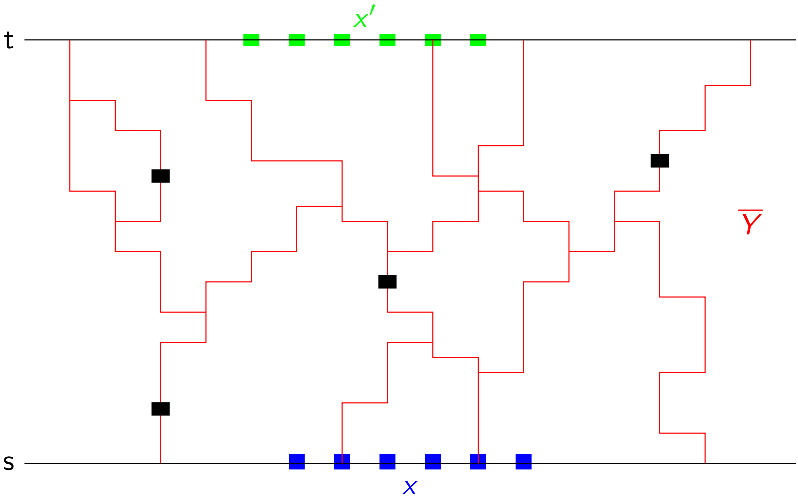
but this is not a pathwise duality.

**Problem** Can we couple $(\mathbf{X}_{s,t})_{s\leq t}$ and $(\mathbf{X}_{t,s})_{t\geq s}$ such that they have the same death events and skeletal process $(\overline{Y}_t)_{t\in\mathbb{R}}$?

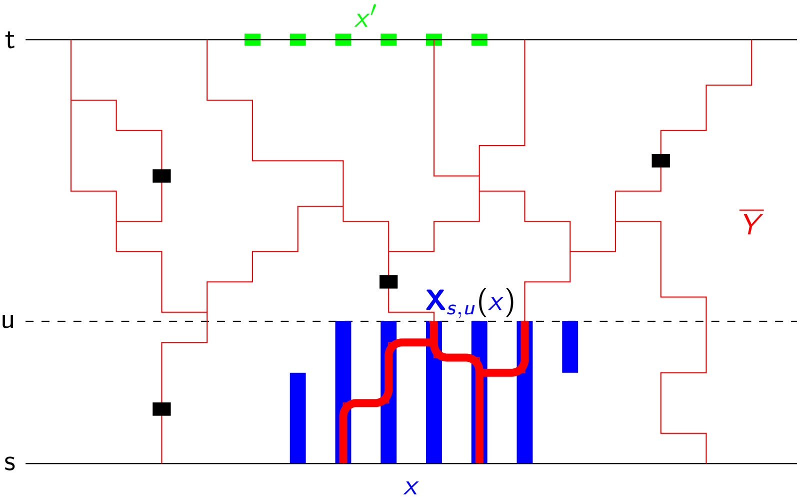
If we can, then $\mathbb{E}\big[\psi_q\big(\mathbf{X}_{s,t}(x), x'\big)\big] = \mathbb{P}\big[\mathbf{X}_{s,t}(x) \wedge x' \wedge \overline{Y}_t = \underline{0}\big]$ and

$$\begin{aligned}1_{\{\mathbf{X}_{s,t}(x) \wedge x' \wedge \overline{Y}_t = \underline{0}\}} &= 1_{\{x \wedge \mathbf{X}_{t,s}(x') \wedge \overline{Y}_s = \underline{0}\}} \\ &= 1_{\{\text{there is a path through } \overline{Y} \text{ from } x\times\{s\} \text{ to } x'\times\{t\}\}}.\end{aligned}$$

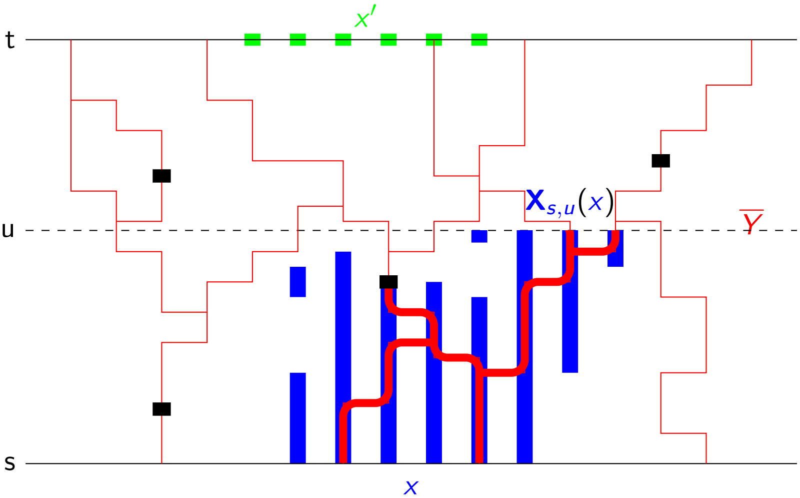
# Interpretation of the self-duality



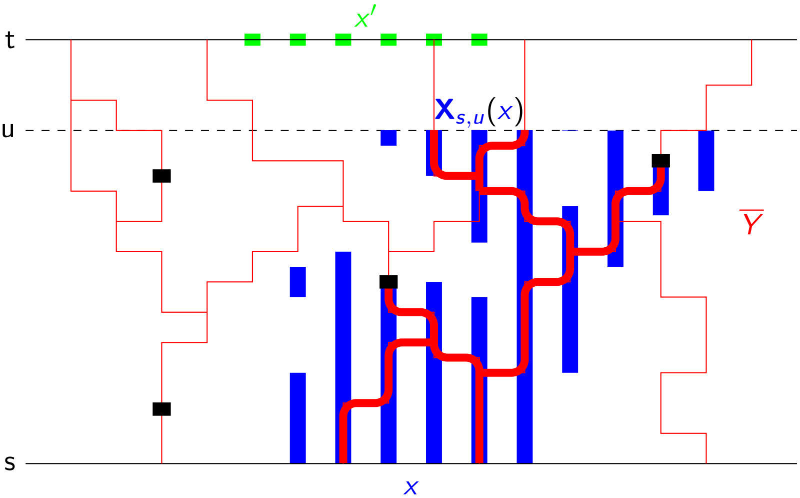
# Interpretation of the self-duality



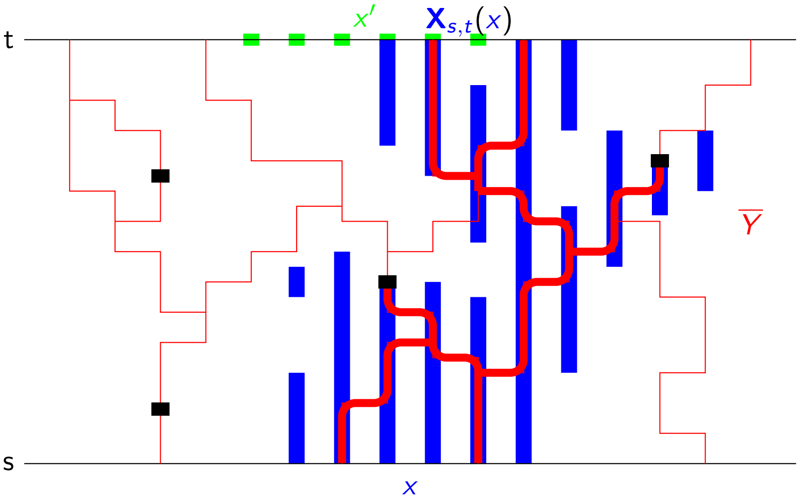
# Interpretation of the self-duality



# Interpretation of the self-duality



# Interpretation of the self-duality



# The biased voter model

Since the maps $\mathtt{vot}_{ij}$, $\mathtt{bra}_{ij}$, and $\mathtt{death}_i$ are monotone, the biased voter model has an upper invariant law $\overline{\nu}$.
We say that the process *survives* if

$$\mathbb{P}^{\delta_i}\big[X_t \neq \underline{0}\ \forall t \geq 0\big] > 0.$$

For any $\mu > 0$, we define

$$\lambda_{\rm c}(\mu) := \inf\{\lambda \geq 0 : \overline{\nu} \neq \delta_{\underline{0}}\},$$
$$\lambda'_{\rm c}(\mu) := \inf\{\lambda \geq 0 : X \text{ survives}\}.$$

Our basic result about ergodicity of particle systems implies

$$\lambda_{\rm c}(\mu) \geq \mu/D \quad \text{and} \quad \lambda'_{\rm c}(\mu) \geq \mu/D,$$

where $D$ is the degree of any vertex in $(\Lambda, E)$. On the other hand, one can prove that $\lambda_{\rm c}(\mu), \lambda'_{\rm c}(\mu) < \infty$.

**Proposition** $\lambda_{\rm c}(\mu) = \lambda'_{\rm c}(\mu)$.

**Proof** We use self-duality. As already observed, if $X, X', Y$ are independent $\{0,1\}^\Lambda$-valued r.v.'s and $(Y(i))_{i\in\Lambda}$ are i.i.d. with $\mathbb{P}[Y(i) = 1] = q = \lambda/(1+\lambda)$, then

$$\mathbb{E}\big[\psi_q(X, X')\big] = \mathbb{P}\big[X \wedge X' \wedge Y = \underline{0}\big]$$

Now

$$\begin{aligned} q\mathbb{P}^{\underline{1}}\big[X_t(i) = 1\big] &= \mathbb{P}^{\underline{1}}\big[X_t \wedge \delta_i \wedge Y \neq \underline{0}\big] \\ &= \mathbb{P}^{\delta_i}\big[\underline{1} \wedge X'_t \wedge Y \neq \underline{0}\big] \underset{t\to\infty}{\longrightarrow} \mathbb{P}^{\delta_i}\big[X'_t \neq \underline{0}\ \forall t \geq 0\big], \end{aligned}$$

where in the last step, we have used

*extinction versus unbounded growth*.

# Extinction versus unbounded growth

**Lemma** If $\mu > 0$, then the biased voter model $X$ started in any finite initial state $X_0$ satisfies

$$\exists t < \infty \text{ s.t. } X_t = \underline{0} \quad \text{or} \quad \lim_{t\to\infty} |X_t| = \infty \quad \text{a.s.}$$

**Proof** W.l.o.g. $|X_0| \leq N$. Set $\tau_0 := 0$ and

$$\tau_{n+1} := \inf\big\{t \geq \tau_n + 1 : |X_t| \leq N\big\}.$$

The assumption $\mu > 0$ implies

$$\varepsilon := \inf_{x:\,|x|\leq N} \mathbb{P}^x[X_1 = \underline{0}] > 0,$$

and hence, by the strong Markov property

$$\mathbb{P}[X_{\tau_n} \neq \underline{0}] \leq (1-\varepsilon)^n.$$

# The additive dual

The additive, pathwise dual of the biased voter model $X$ with generator

$$G f(x) := \sum_{(i,j)\in\vec{E}} \big\{f\big(\mathtt{vot}_{ij}x\big) - f(x)\big\} + \lambda \sum_{(i,j)\in\vec{E}} \big\{f\big(\mathtt{bra}_{ij}x\big) - f(x)\big\} + \mu \sum_{i\in\Lambda} \big\{f\big(\mathtt{death}_i x\big) - f(x)\big\},$$

are systems of branching and coalescing random walks $Y$, with generator

$$G' f(y) := \sum_{(i,j)\in\vec{E}} \big\{f(\mathtt{rw}_{ij}y) - f(y)\big\} + \lambda \sum_{(i,j)\in\vec{E}} \big\{f(\mathtt{bra}_{ij}y) - f(y)\big\} + \sum_{i\in\Lambda} \mu \big\{f(\mathtt{death}_i y) - f(y)\big\}.$$

# Interpretation of the additive dual

We can interpret the particles of $Y$ as *potential ancestors*. Start $Y$ with a single particle at $i$. The pathwise duality relation

$$1_{\{\mathbf{X}_{s,t}(x) \wedge \delta_i = \underline{0}\}} = 1_{\{x \wedge \mathbf{Y}_{t,s}(\delta_i) = \underline{0}\}}$$

says that the site $i$ is at time $t$ occupied by a fit organism if and only if at least one of the occupied sites of $Yb_{t,s}(\delta_i)$ contains a fit organism.

In the forward picture, a branching map $\texttt{bra}_{ij}$ represents a *selection event*, i.e., a reproduction event that can only be used by fit individuals.

In the backward picture, this translates into a branching map $\texttt{bra}_{ji}$ that says we must follow both potential ancestors back in time, and if one of them is fit, then the organism at $j$ will be fit.

Likewise, deleterious mutations in the forward picture mean that these individuals stop being potential ancestors, since their type is certainly 0.

# The mean-field model

Let $X^N$ be a biased voter model on the complete graph $(\Lambda_N, E_N)$ with $N$ vertices and with selection rate $\lambda N^{-1}$ and mutation rate $\mu$. Then

$$\overline{X}^N_t := \frac{1}{N}\sum_{i\in\Lambda_N} X^N_t(i)$$

is a Markov process that jumps

$$\overline{x} \mapsto \begin{cases} \overline{x} + \frac{1}{N} & \text{with rate } (N^2 + \lambda N)\overline{x}(1-\overline{x}), \\ \overline{x} - \frac{1}{N} & \text{with rate } N^2\overline{x}(1-\overline{x}) + \mu N\overline{x}. \end{cases}$$

Since

$$\mathbb{E}^x\big[\overline{X}^N_t - \overline{x}\big] = \lambda\overline{x}(1-\overline{x}) - \mu\overline{x}t + O(t^2),$$

$$\mathbb{E}^x\big[(\overline{X}^N_t - \overline{x})^2\big] = O(N^{-1}) + O(t^2),$$

$X^N_t$ converges in law to the diffusion with generator

$$G(\overline{x}) = \big[\overline{x}(1-\overline{x})\tfrac{\partial^2}{\partial\overline{x}^2} + \lambda\overline{x}(1-\overline{x})\tfrac{\partial}{\partial\overline{x}} - \mu\overline{x}\tfrac{\partial}{\partial\overline{x}}\big]f(\overline{x}).$$

# The mean-field model

The diffusion with generator

$$G(\overline{x}) = \Big[\overline{x}(1-\overline{x})\tfrac{\partial^2}{\partial \overline{x}^2} + \lambda\overline{x}(1-\overline{x})\tfrac{\partial}{\partial \overline{x}} - \mu\overline{x}\tfrac{\partial}{\partial \overline{x}}\Big] f(\overline{x}).$$

is dual to a generalization of the Kingman coalescent that jumps

$$\overline{y} \mapsto \begin{cases} \overline{y}+1 & \text{with rate } \lambda\overline{y}(\overline{y}-1), \\ \overline{y}-1 & \text{with rate } \overline{y}(\overline{y}-1)+\mu\overline{y}. \end{cases}$$

When $\overline{y}$ is large, the coalescence dominates the branching. In line with this, the process still comes down from infinity.

The diffusion with generator $G$ is also self-dual with duality function

$$\psi(x, x') = e^{-\lambda^{-1} x x'}.$$

# The Brownian net

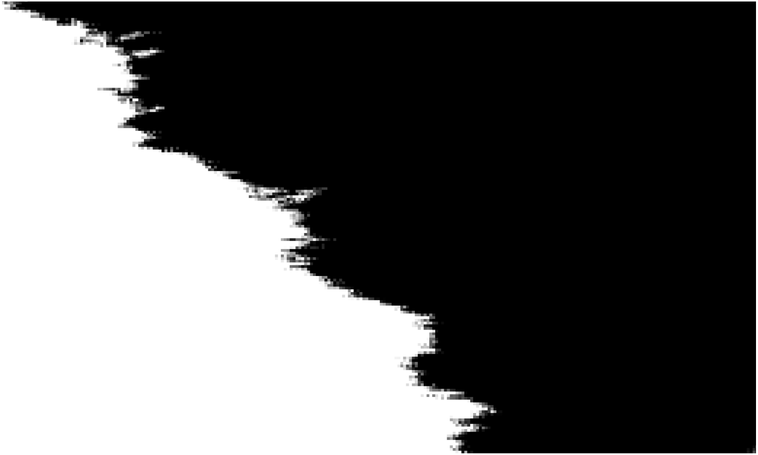
Consider a one-dimensional system of branching and coalescing random walks with small branching rate $\varepsilon$. Sun & S. (2008) have shown that in the *diffusive scaling limit*, when space is rescaled by $\varepsilon$ and time by $\varepsilon^2$, such a system converges to a nontrivial limit process, the *branching coalescing point set*.

It is even possible to start particle from each point in space and time. This yields an object called the *Brownian net*.

This has been extended to systems with deaths by Newman, Ravishankar & Schertzer (2015). They have also shown that with the help of the Brownian web and net, it is possible to describe the scaling limits of a large class of one-dimenional systems, including low-temperature Potts models.

In dimensions $d \geq 2$, the time scales for the motion of particles and for coalescence separate, leading to different scaling limits.

# Interface tightness

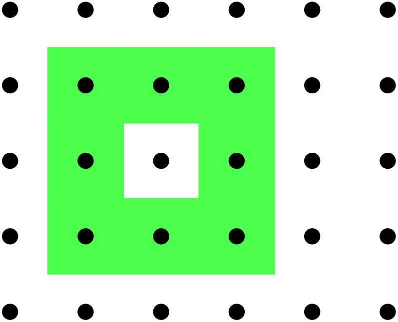


Interface tightness for biased voter models has recently been proved by Sun, S., & Yu, based on the "pseudo Lyapunov function" technique of Sturm & S. (2008).

# The Neuhauser-Pacala model

Denote a point in $\mathbb{Z}^d$ by $i = (i_1, \ldots, i_d)$.

**Def** *neighborhood* of a site $\mathcal{N}_i := \{j \in \mathbb{Z}^d : 0 < \|i - j\|_\infty \leq R\}$.



(Here $R = 1$, $d = 2$).

# The Neuhauser-Pacala model

**Def** *local frequency* $f_\tau(i) := |\mathcal{N}_i|^{-1}|\{j \in \mathcal{N}_i : x(j) = \tau\}|.$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 |

Here $f_0(i) = 3/8,\ f_1(i) = 5/8.$

# The Neuhauser-Pacala model

Fix rates $\alpha_{01}, \alpha_{10} \geq 0$.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | † | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 |

With rate $f_0 + \alpha_{01} f_1$ an organism of type 0 dies...

# The Neuhauser-Pacala model

| | | | | | |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 |

. . . and is replaced by a random type from the neighborhood.

# The Neuhauser-Pacala model

**Neuhauser & Pacala (1999):** Markov process in the space $\{0,1\}^{\mathbb{Z}^d}$ of spin configurations $x = (x(i))_{i\in\mathbb{Z}^d}$, where spin $x(i)$ flips:

$$0 \mapsto 1 \text{ with rate } f_1(f_0 + \alpha_{01} f_1),$$
$$1 \mapsto 0 \text{ with rate } f_0(f_1 + \alpha_{10} f_0),$$

with

$$f_\tau(i) := \frac{|\{j \in \mathcal{N}_i : x(j) = \tau\}|}{|\mathcal{N}_i|} \quad \mathcal{N}_i := \{j : 0 < \|i - j\|_\infty \leq R\}.$$

the local frequency of type $\tau = 0, 1$.

**Interpretation:** *Interspecific competition rates* $\alpha_{01}, \alpha_{10}$. Organism of type 0 dies with rate $f_0 + \alpha_{01} f_1$ and is replaced by type sampled at random from distance $\leq R$.

# The Neuhauser-Pacala model

Parameter $\alpha_{01}$ measures the strength of competition felt by type 0 from type 1 (compared to strength 1 from its own type).
If $\alpha_{01} < 1$, then type 0 dies *less* often due to competition from type 1 than from competition with its own type: *balancing selection*.
If $\alpha_{01} > 1$, then type 0 dies *more* often due to competition from type 1 than from competition with its own type, i.e., type 1 is an *agressive species*.

By definition, type 0 *survives* if starting from a single organism of type 0 and all other organisms of type 1, there is a positive probability that the organisms of type 0 never die out.

By definition, one has *coexistence* if there exists an invariant law concentrated on states where both types are present.

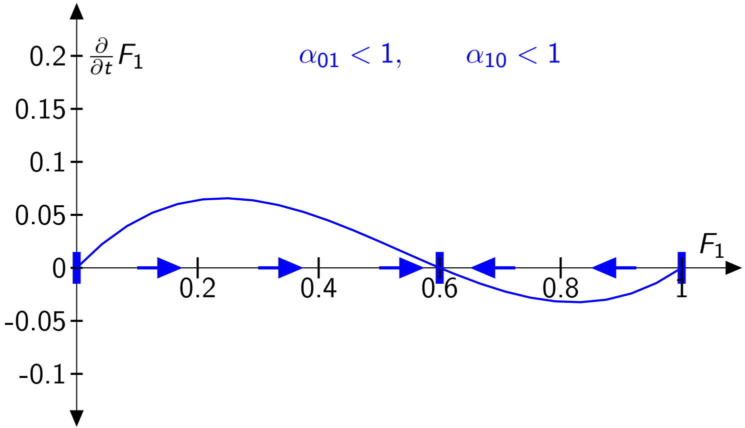
# Mean field model

In the *mean field model*, the lattice $\mathbb{Z}^d$ is replaced by a complete graph with $N$ vertices. In this case, the neighborhood $\mathcal{N}_i$ of a vertex $i$ is simply all sites except $i$.

In the limit $N \to \infty$, the frequencies $F_\tau(t)$ of type $\tau = 0, 1$ satisfy a differential equation:

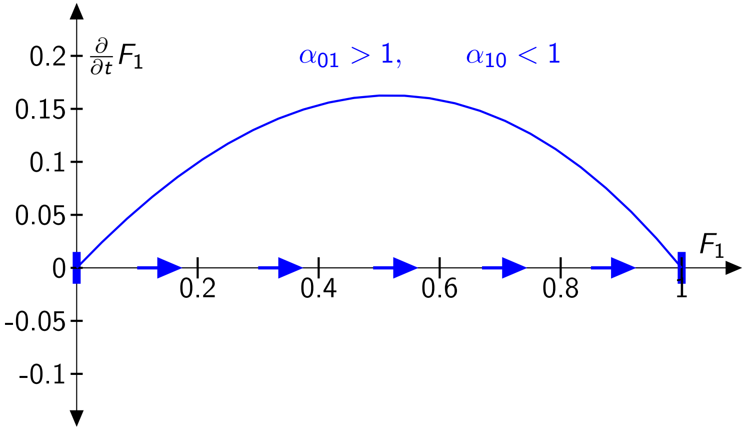
$$\begin{aligned}\tfrac{\partial}{\partial t} F_1(t) = {} & F_1(t)\big(F_0(t) + \alpha_{01} F_1(t)\big) F_0(t) \\ & - F_0(t)\big(F_1(t) + \alpha_{10} F_0(t)\big) F_1(t).\end{aligned}$$

with $F_0 = 1 - F_1$.



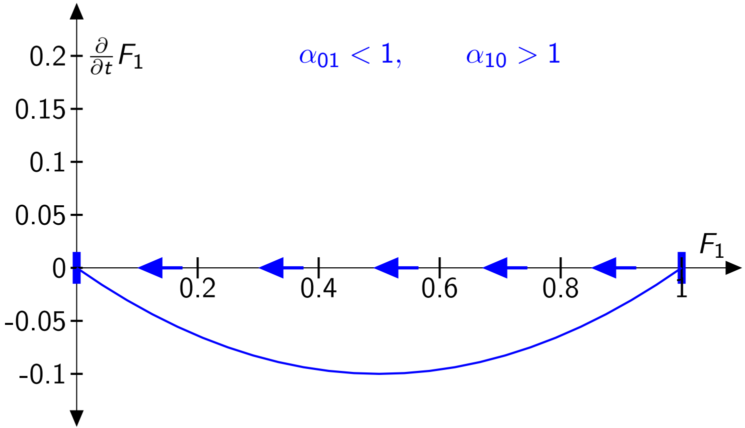

Balancing selection ($\alpha_{01} = 0.6, \alpha_{10} = 0.4$).

# Mean field model



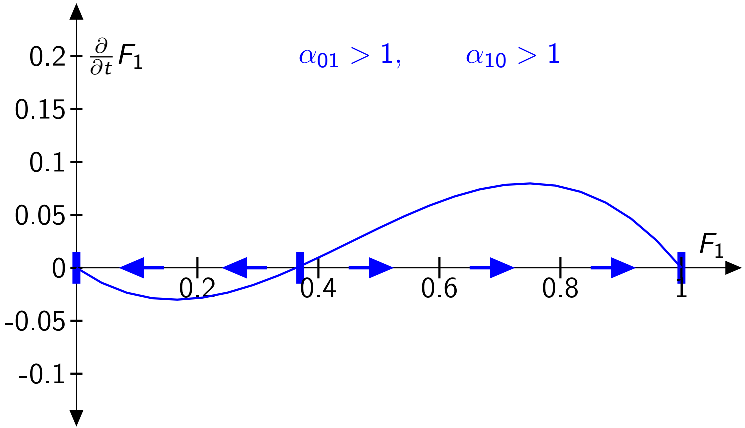

Type 1 is an agressive species ($\alpha_{01} = 1.7, \alpha_{10} = 0.4$).

# Mean field model



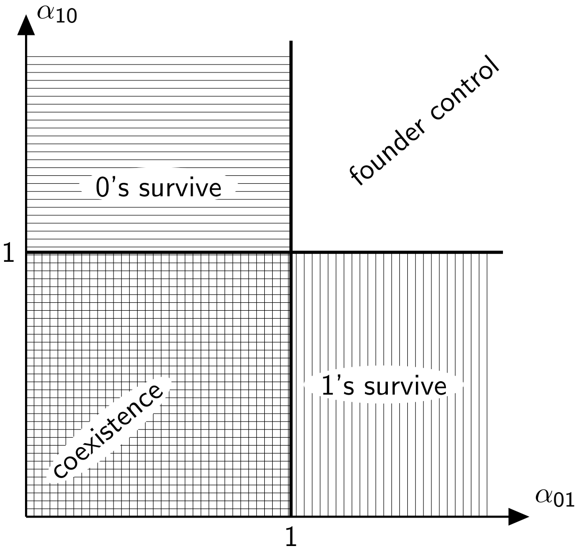

Type 0 is an agressive species ($\alpha_{01} = 0.6, \alpha_{10} = 1.4$).

# Mean field model

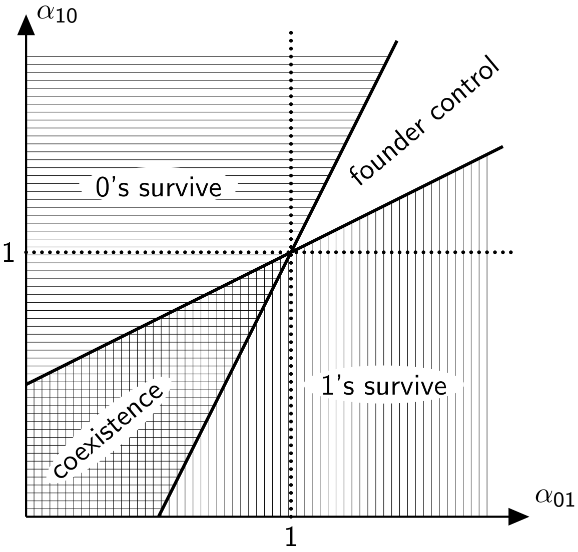



Both types are agressive species ($\alpha_{01} = 1.7, \alpha_{10} = 1.4$).

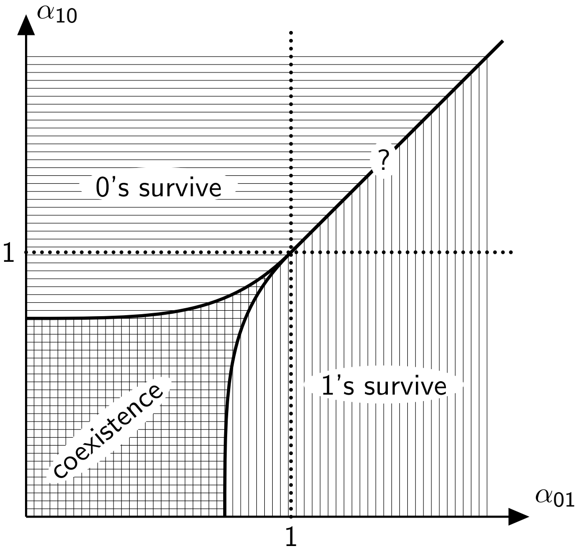
# Mean field model



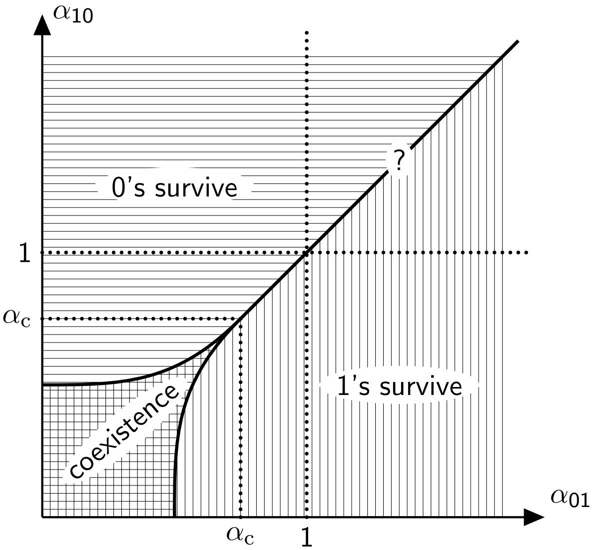
# Dimension $d \geq 3$



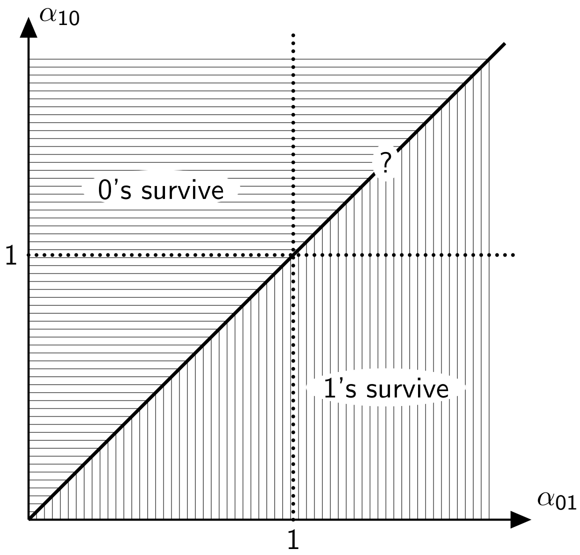
# Dimension $d = 2$



# Dimension $d = 1$, range $R \geq 2$



# Dimension $d = 1$, range $R = 1$



# Rigorous results

Sudbury, AOP, 1990

Neuhauser & Pacala, AAP, 1999

Cox & Perkins, AOP, 2005

Cox & Perkins, PTRF, 2007

Cox & Perkins, AAP, 2008

Sturm & S., AAP, 2008

Sturm & S., ECP, 2008

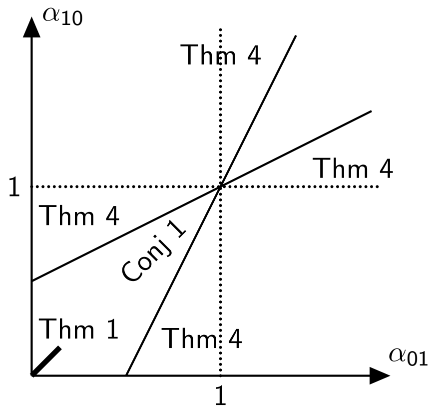
Cox, Merle, & Perkins, EJP, 2010

S., ECP, 2013

Cox, Durrett, & Perkins, Astérisque, 2013

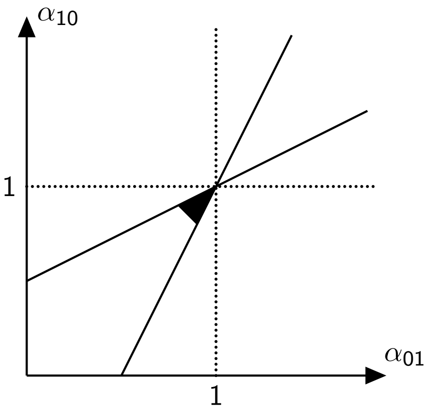
Cox & Perkins, AAP, 2014

# Rigorous results




Neuhauser & Pacala (1999) have proved that in the spatial model, the regions of coexistence and founder control are reduced. Except when $d = 1 = R$, coexistence is possible for $\alpha_{01} = \alpha_{10} = \alpha$ small enough. They conjectured that this is true for all $\alpha < 1$.

# Rigorous results




Cox & Perkins (2007) have proved coexistence in a cone near $(1,1)$ for dimensions $d \geq 3$. Cox, Merle & Perkins (2010) have an analogue result for $d = 2$. The statement is believed to be false in dimension $d = 1$.

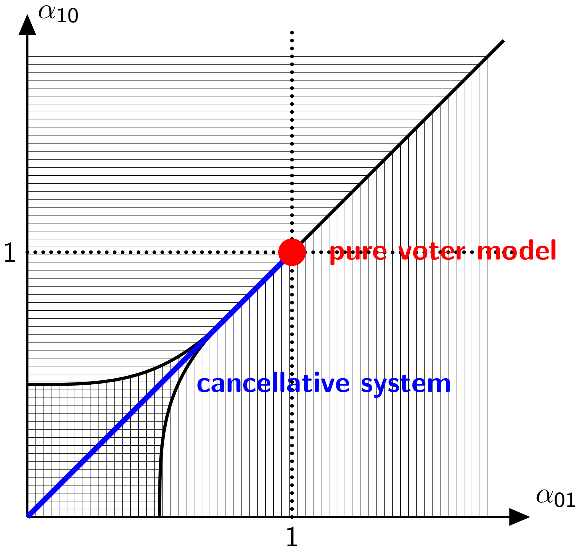
# Voter model perturbations

For $(\alpha_{01}, \alpha_{10}) = (1, 1)$ we have a classical voter model.

In dimensions $d \geq 2$, Cox, Merle and Perkins prove that it is possible to send $\alpha_{01}, \alpha_{10} \to 1$ through a cone ($d \geq 3$) or cusp ($d = 2$) such that rescaled sparse models converge to supercritical *super Brownian motion*.

Using this, for $(\alpha_{01}, \alpha_{10})$ very close to $(1, 1)$, they can set up a comparison with oriented percolation and prove survival of the ones. By symmetry, the same holds for the zeros and one can conclude coexistence.

# Special models



# A cancellative representation

Recall that the *rebellious voter map* is defined as

$$\mathtt{rvot}_{kji}(x)(l) := \begin{cases} 1 - x(i) & \text{if } l = i \text{ and } x(k) \neq x(j), \\ x(l) & \text{otherwise.} \end{cases}$$

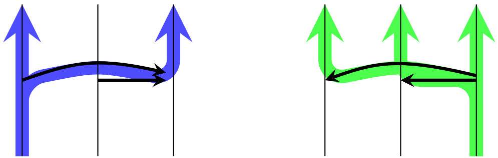
**Lemma** The generator of the symmetric Neuhauser-Pacala model with $\alpha_{01} = \alpha_{10} = \alpha \in [0, 1]$ can be represented in cancellative maps as

$$\begin{aligned} Gf(x) = \alpha \sum_{i \in \mathbb{Z}^d} |\mathcal{N}_i|^{-1} \sum_{j \in \mathcal{N}_i} \big\{ f\big(\mathtt{vot}_{ji}(x)\big) - f\big(x\big) \big\} \\ + \tfrac{1}{2}(1 - \alpha) \sum_{i \in \mathbb{Z}^d} |\mathcal{N}_i|^{-2} \sum_{k,j \in \mathcal{N}_i} \big\{ f\big(\mathtt{rvot}_{kji}(x)\big) - f\big(x\big) \big\}. \end{aligned}$$

**Proof** Since the probability that two sites sampled at random from $\mathcal{N}_i$ are of different types is given by $2f_0 f_1$, we see that the site $i$ jumps $0 \mapsto 1$ with rate

$$\alpha f_1 + \tfrac{1}{2}(1 - \alpha) 2 f_0 f_1 = f_1 \big[\alpha(f_0 + f_1) + (1 - \alpha) f_0\big] = f_1(f_0 + \alpha f_1).$$

# A cancellative representation

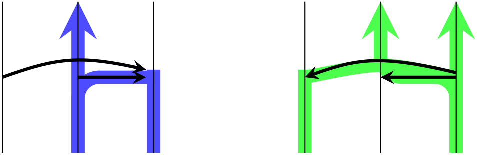


$\mathtt{rvot}_{1,2,3}$ $\mathtt{adbr}_{3,2,1}$

The rebellious voter model map $\mathtt{rvot}_{kji}$ is dual to the *annihilating double branching map*

$$\mathtt{adbr}_{ijk}(y)(l) = \begin{cases} y(l) + y(i) \mod(2) & \text{if } l = j, k, \\ y(l) & \text{otherwise.} \end{cases}$$

# A cancellative representation



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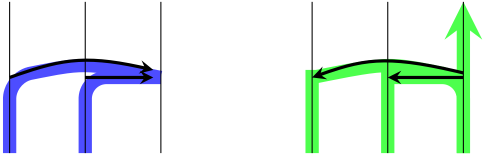
# A cancellative representation



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$$\texttt{adbr}_{ijk}(y)(l) = \begin{cases} y(l) + y(i) \mod(2) & \text{if } l = j, k, \\ y(l) & \text{otherwise.} \end{cases}$$

# Parity preserving branching

The cancellative dual of the symmetric Neuhauser-Pacala model is a system of *parity preserving branching and annihilating random walks*.

$$
\begin{aligned}
G' f(y) &= \alpha \sum_{i \in \mathbb{Z}^d} |\mathcal{N}_i|^{-1} \sum_{j \in \mathcal{N}_i} \big\{ f\big(\mathtt{arw}_{ij}(y)\big) - f\big(y\big) \big\} \\
&\quad + \tfrac{1}{2}(1-\alpha) \sum_{i \in \mathbb{Z}^d} |\mathcal{N}_i|^{-2} \sum_{k,j \in \mathcal{N}_i} \big\{ f\big(\mathtt{adbr}_{ijk}(y)\big) - f\big(y\big) \big\}.
\end{aligned}
$$

# Parity preserving branching

**Def** A cancellative system $X$ is *type symmetric* if the transition $x \mapsto x'$ has the same rate as $(1-x) \mapsto (1-x')$.

**Def** A cancellative system $Y$ is *parity preserving* if a.s. $|Y_t|$ is odd iff $|Y_0|$ is odd $(t \geq 0)$.

**Lemma** A cancellative system $X$ is type symmetric iff its dual $Y$ is parity preserving.

**Proof** Let $m(i,j)$ be the matrix of a cancellative map $m$. Then:

- $m$ is type symmetric iff $\sum_j m(i,j)$ is even for each $i$,
- $m$ is parity preserving iff $\sum_i m(i,j)$ is even for each $j$.

In particular, $m$ is type symmetric iff $m^\dagger$ is parity preserving. ∎

# Interfaces

In the one-dimensional case, we have an extra tool available.

Let $\mathbb{Z}+\frac{1}{2} := \{k+\frac{1}{2} : k \in \mathbb{Z}\}$ and let $\mathbb{I} = \mathbb{Z}$ or $= \mathbb{Z}+\frac{1}{2}$.
Define a gradient operator $\nabla : \{0,1\}^{\mathbb{I}} \to \{0,1\}^{\mathbb{I}+\frac{1}{2}}$ by

$$\nabla x(i) := x(i-\tfrac{1}{2}) \oplus x(i+\tfrac{1}{2}).$$

If $(X_t)_{t\geq 0}$ is type symmetric, then $(\nabla X_t)_{t\geq 0}$ is a Markov process: the *interface model* of $X$.

$$\begin{array}{ccccccccccccccccc}
X & 0 & & 1 & & 1 & & 1 & & 0 & & 0 & & 1 & & 0 \\
\nabla X & & 1 & & 0 & & 0 & & 1 & & 0 & & 1 & & 1 &
\end{array}$$

Interface models are always parity preserving.

# Interfaces and duality

**[S. '13]** The interface model of a type symmetric cancellative spin system is a parity preserving cancellative spin system. Conversely, every parity preserving cancellative spin system is the interface model of a unique type symmetric cancellative spin system. Moreover, the following commutative diagram holds:

$$\begin{tikzcd} X \arrow[r, "\text{interface}"] \arrow[d, leftrightarrow, "\text{dual}"'] & Y' \arrow[d, leftrightarrow, "\text{dual}"] \\ Y & X' \arrow[l, "\text{interface}"] \end{tikzcd}$$

Here $X, X'$ are type symmetric and $Y, Y'$ are parity preserving. $X$ and $X'$ are dual with the duality function $\psi(x, x') = \langle\!\langle x, \nabla x' \rangle\!\rangle$.

**Proof (sketch)** Recall the duality function

$$\langle\!\langle x, y \rangle\!\rangle = \sum_i x(i)y(i) \bmod(2).$$

Then

$$\langle\!\langle x, \nabla y \rangle\!\rangle = \langle\!\langle \nabla x, y \rangle\!\rangle \qquad (x \in \{0,1\}^{\mathbb{I}},\ y \in \{0,1\}^{\mathbb{I}+\frac{1}{2}}).$$

If $m$ is type symmetric and $m^\dagger$ is the dual map, then $\nabla m \nabla^{-1}$ is the corresponding map on interfaces. Now

$$(\nabla m \nabla^{-1})^\dagger = \nabla^{-1} m^\dagger \nabla$$

correspond to the dual of the interface model resp. the model whose interface model is the dual.

(Some care is needed to define $\nabla^{-1}$ but this is the basic idea.)

# Classification of behavior

Let $Y$ be parity preserving.

**Def** An invariant law $\nu$ is *nontrivial* if $\nu(\{\underline{0}\}) = 0$. We say that $Y$ *persists* if it has a nontrivial invariant law.

**Def** $Y$ *survives* if $\mathbb{P}^y[Y_t \neq \underline{0} \ \forall t \geq 0] > 0$ for some *even* initial state $y$.

**Def** $Y$ is *stable* if the state with a single particle is positively recurrent for the *process modulo translations* $\tilde{Y}$.

**Def** $Y$ is *strongly stable* if $Y$ is stable and $\mathbb{E}\big[|\tilde{Y}_\infty|\big] < \infty$ in equilibrium.

# Classification of behavior

Let $X$ be type symmetric.

**Def** An invariant law $\nu$ is *coexisting* if $\nu(\{\underline{0},\underline{1}\}) = 0$. We say that $X$ exhibits *coexistence* if it has a coexisting invariant law.

**Def** $X$ *survives* if $\mathbb{P}^x[X_t \neq \underline{0}\ \forall t \geq 0] > 0$ for some *finite* initial state $x$.

**Def** $X$ exhibits *(strong) interface tightness* if its interface model is (strongly) stable.

# The odd upper invariant law

**Odd upper invariant law** Let $X$ be a cancellative spin system, started in an initial law such that $\mathbb{P}[(X_0(i))_{i\in\Lambda} \in \cdot\,] = \pi_{1/2}$, the product measure with intensity $\frac{1}{2}$. Then there exists an invariant law $\nu^{1/2}$ such that

$$\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \nu^{1/2}.$$

**Proof** Let $Y$ be the cancellative dual of $X$. Then

$$\mathbb{E}\big[\langle\!\langle X_t, y\rangle\!\rangle\big] = \mathbb{E}^y\big[\langle\!\langle X_0, Y_t\rangle\!\rangle\big] = \tfrac{1}{2}\mathbb{P}^y\big[Y_t \neq \underline{0}\big] \underset{t\to\infty}{\longrightarrow} \tfrac{1}{2}\mathbb{P}^y\big[Y_t \neq \underline{0}\ \forall t \geq 0\big].$$

Since this holds for each $y \in \{0,1\}^\Lambda$ with $|y| < \infty$, using the compactness of $\{0,1\}^\Lambda$, it follows that $\pi_{1/2}P_t \underset{\nu\to\infty}{\Longrightarrow}{}^{1/2}$ for some probability measure $\nu^{1/2}$. Using the continuity of $P_s$ we see that $\nu^{1/2}P_s = \lim_{t\to\infty}\pi_{1/2}P_tP_s = \nu^{1/2}$ $(s \geq 0)$, so $\nu^{1/2}$ is invariant. ∎

# The odd upper invariant law

**Lemma** Let $X$ be a type symmetric cancellative spin system with odd upper invariant law $\nu^{1/2}$, and let $Y$ be its dual partity preserving cancellative spin system with odd upper invariant law ${\nu'}^{1/2}$. Then:

**(a)** The following statements are equivalent: (i) $X$ has a coexisting invariant law, (ii) $\nu^{1/2}(\{\underline{0},\underline{1}\}) < 1$, (iii) $Y$ survives.

**(b)** The following statements are equivalent: (i) $Y$ has a nontrivial invariant law, (ii) ${\nu'}^{1/2}(\{\underline{0}\}) < 1$, (iii) $X$ survives.

# The odd upper invariant law

**Proof of (a)** Recall that $Y$ survives if $\mathbb{P}^y[Y_t \neq \underline{0}\ \forall t \geq 0] > 0$ for some *even* initial state $y$. Let $X_\infty$ have law $\nu^{1/2}$. Then

$$\mathbb{E}\big[\langle\langle X_\infty, y\rangle\rangle\big] = \tfrac{1}{2}\mathbb{P}^y\big[Y_t \neq \underline{0}\ \forall t \geq 0\big] > 0$$

for some even $y$, which is only possible if $\nu^{1/2}(\{\underline{0}, \underline{1}\}) < 1$. Subtracting a linear combination of $\delta_{\underline{0}}$ and $\delta_{\underline{1}}$ if necessary, we find a coexisting invariant law.

Conversely, if $X_\infty$ is distributed according to some coexisting invariant law, then, there exists $i, j$ such that $\mathbb{P}[X_\infty(i) \neq X_\infty(j)] > 0$ and hence

$$\mathbb{P}^{\delta_i + \delta_j}\big[Y_t \neq \underline{0}\big] \geq \mathbb{E}\big[\langle\langle X_\infty, Y_t\rangle\rangle\big] = \mathbb{E}\big[\langle\langle X_\infty, \delta_i + \delta_j\rangle\rangle\big] > 0$$

uniformly in $t$, proving that $Y$ survives. ■

# Strong interface tightness

**Theorem [S.'13]** Strong interface tightness implies noncoexistence.

**Proof (sketch)** Assume that strong interface tightness holds for $X$. Let the law of $Y'_\infty$ be invariant modulo shifts and let $\tilde{Y}'_\infty + i$ denote the configuration $\tilde{Y}'_\infty$ shifted by $i$. Then

$$h(x) := \sum_{i \in \mathbb{Z}+\frac{1}{2}} \mathbb{E}\big[\langle\!\langle x, Y'_\infty + i \rangle\!\rangle\big]$$

is a harmonic function for the process $X'$ (dual of interface model of $X$). Moreover, there exist constants $0 < c \leq C < \infty$ s.t.

$$c|x| \leq h(x) \leq C|x|.$$

By martingale convergence, $h(X'_t)$ converges a.s., which implies that $X'$ dies out a.s. The same holds for its interface model $Y$ which is dual to $X$, so by duality $X$ exhibits noncoexistence. ∎

# Invariant laws and harmonic functions

The previous proof exploited a general principle that is sometimes useful:

Let $X$ and $Y$ be Markov processes that are dual w.r.t. some duality function $\psi$ and let $\nu$ be an invariant law of $X$. Then

$$h(y) := \int \nu(\mathrm{d}x)\psi(x, y)$$

defines a harmonic function for $Y$, as follows by writing

$$\mathbb{E}^y\big[h(Y_t)\big] = \mathbb{E}^y\big[\psi(Y_t, X_0)\big] = \mathbb{E}\big[\psi(y, X_t)\big] = h(y),$$

where $(X_t)_{t\geq 0}$ has initial law $\nu$ and is independent of $Y$.

# The odd upper invariant law

**Theorem [Sturm & S. '08]** Let $X$ be the Neuhauser-Pacala model and let $Y$ be its cancellative dual.
**(a)** If $\alpha < 1$ and $Y$ survives, then $\nu^{1/2}$ is the unique translation invariant coexisting invariant law of $X$.
**(b)** If $0 < \alpha < 1$, and $Y$ survives and is not stable, then the Neuhauser-Pacala model started in any translation invariant coexisting initial law satisfies

$$\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \nu^{1/2}.$$

**Note** We have seen that survival of $Y$ is equivalent to the existence of a coexisting invariant law. In one dimension, by [S. '13], survival of $Y$ implies that $Y$ is not strongly stable.

**Proof idea** Extinction versus unbounded growth.

# The rebellious voter model

The *rebellious voter model* is defined by the generator:

$$\begin{aligned} Gf(x) := {} & \alpha \sum_i \{ f(\mathtt{vot}_{i,i+1}(x)) - f(x) \} \\ & + \alpha \sum_i \{ f(\mathtt{vot}_{i,i-1}(x)) - f(x) \} \\ & + (1-\alpha) \sum_i \{ f(\mathtt{rvot}_{i-1,i,i+1}(x)) - f(x) \} \\ & + (1-\alpha) \sum_i \{ f(\mathtt{rvot}_{i+1,i,i-1}(x)) - f(x) \}. \end{aligned}$$

The *one-sided rebellious voter model* is defined by:

$$\begin{aligned} Gf(x) := {} & \alpha \sum_i \{ f(\mathtt{vot}_{i,i+1}(x)) - f(x) \} \\ & + (1-\alpha) \sum_i \{ f(\mathtt{rvot}_{i-1,i,i+1}(x)) - f(x) \}. \end{aligned}$$

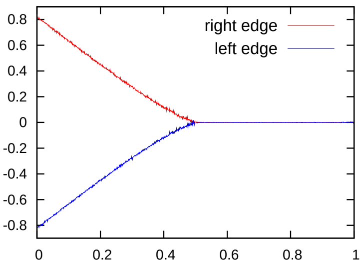
# The rebellious voter model

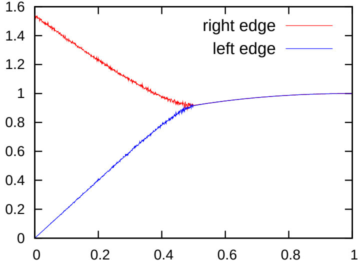
The rebellious voter model is *self-dual* in the sense that it is equal to the dual of its interface model, or more simply:

$$\begin{tikzcd} X \arrow[d, bend right, leftrightarrow, "\text{dual}"'] \arrow[d, bend left, "\text{interface}"] \\ Y \end{tikzcd}$$

**Consequence** Survival equivalent to coexistence.

# The rebellious voter model




Edge speeds for the rebellious voter model (left) and its one-sided counterpart (right) [S. & Vrbenský '10].

# The rebellious voter model

Define the *survival probability*

$$\rho(\alpha) := \mathbb{P}^{\delta_0}[X_t \neq 0 \ \forall t \geq 0].$$

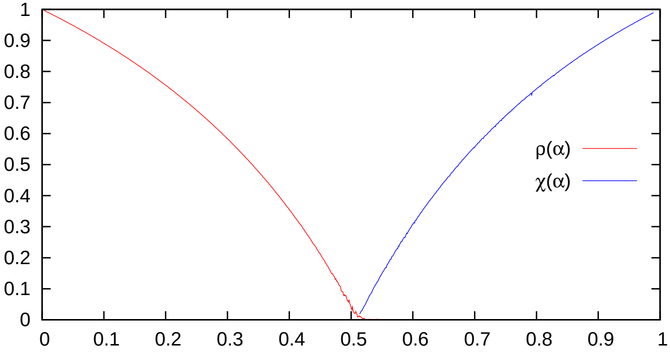
- coexistence $\Leftrightarrow$ $\rho(\alpha) > 0$.

Define the *fraction of time spent with a single interface*

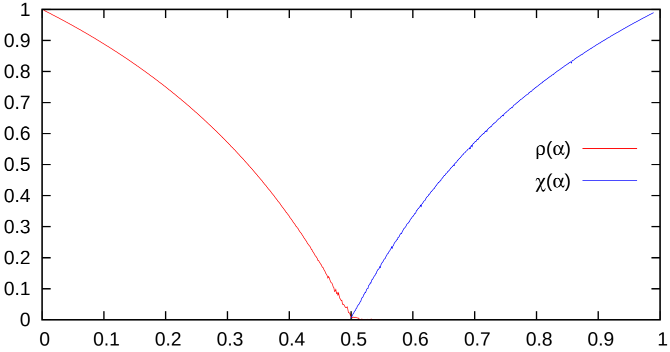
$$\chi(\alpha) := \mathbb{P}\big[|Y'_\infty| = 1\big].$$

- interface tightness $\Leftrightarrow$ $\chi(\alpha) > 0$.

# The rebellious voter model




The functions $\rho$ and $\chi$ for the two-sided rebelious voter model.




The functions $\rho$ and $\chi$ for the one-sided rebelious voter model.

# The rebellious voter model

**[S. & Vrbenský '10]** It seems that for the one-sided model, the functions $\rho$ and $\chi$ are described by the explicit formulas:

$$\rho(\alpha) = 0 \vee \frac{1-2\alpha}{1-\alpha} \quad \text{and} \quad \chi(\alpha) = 0 \vee \big(2 - \frac{1}{\alpha}\big).$$

In particular, one has the symmetry $\rho(1-\alpha) = \chi(\alpha)$ and the critical parameter seems to be given by $\alpha_{\rm c} = 1/2$.

**Open problem** Prove (strong) interface tightness for some $\alpha < 1$.

# Cooperative branching

Recall the *cooperative branching map*

$$\mathrm{coop}_{ii'j}x(k) := \begin{cases} \big(x(i) \wedge x(i')\big) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

We will be interested in systems with cooperative branching and deaths. We start with a mean-field model. Let $[N] := \{1, \ldots, N\}$ and

$$[N]^{\langle 3 \rangle} := \big\{(i_1, i_2, i_3) \in [N]^k : i_m \neq i_n \ \forall n \neq m\big\}.$$

Consider the generator

$$Gf(x) := \frac{\alpha}{(N-1)(N-2)} \sum_{(i,i',j) \in [N]^{\langle 3 \rangle}} \big\{ f\big(\mathrm{coop}_{ii'j}x\big) - f\big(x\big) \big\} + \sum_{i \in [N]} \big\{ f\big(\mathtt{death}_i x\big) - f\big(x\big) \big\}.$$

# Cooperative branching

Then

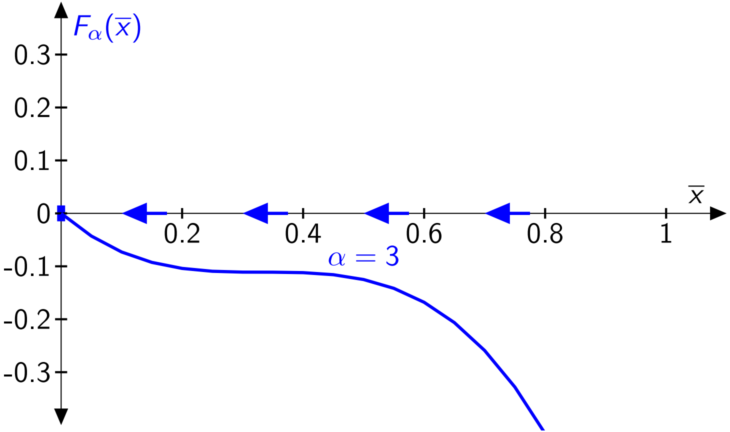
$$\overline{X}_t^N := \frac{1}{N} \sum_{i \in \Lambda_N} X_t^N(i)$$

is a Markov process that jumps

$$\overline{x} \mapsto \begin{cases} \overline{x} + \frac{1}{N} & \text{with rate } \approx \alpha N \overline{x}^2 (1 - \overline{x}), \\ \overline{x} - \frac{1}{N} & \text{with rate } N \overline{x}. \end{cases}$$

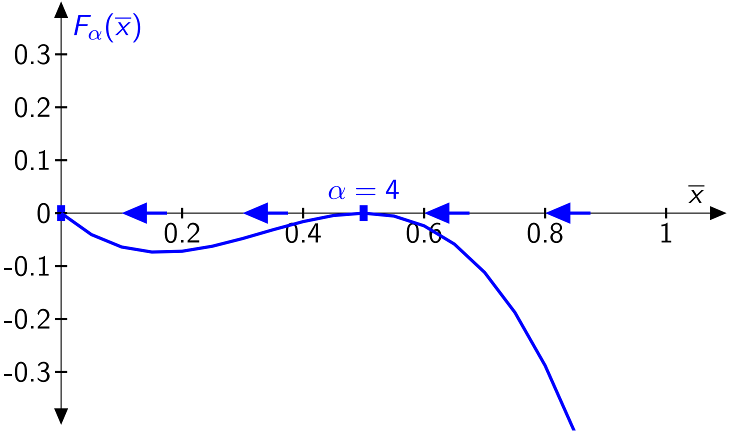
Letting $N \to \infty$, $X_t^N$ can be approximated by a solution to the mean-field ODE

$$\frac{\partial}{\partial t} \overline{X}_t = \alpha \overline{X}_t^2 (1 - \overline{X}_t) - \overline{X}_t =: F_\alpha(\overline{X}_t) \quad (t \geq 0).$$



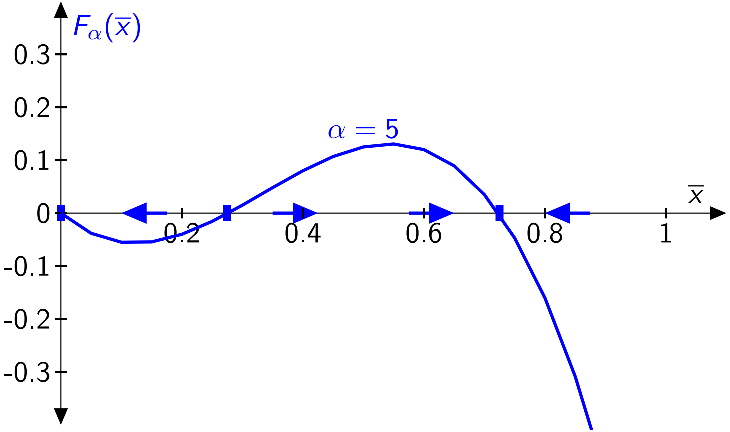

For $\alpha < 4$, the equation $\frac{\partial}{\partial t}\overline{X}_t = F_\alpha(\overline{X}_t)$ has a single, stable fixed point $\overline{x} = 0$.

# Cooperative branching

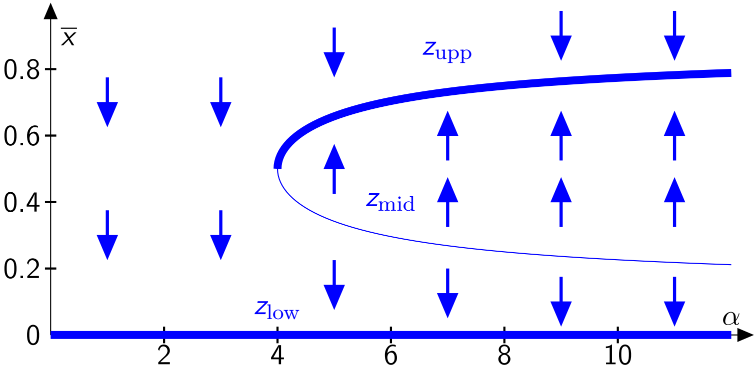



For $\alpha = 4$, a second fixed point appears at $\overline{x} = 0.5$.

# Cooperative branching




For $\alpha > 4$, there are two stable fixed points and one unstable fixed point, which separates the domains of attraction of the other two.




Fixed points of $\frac{\partial}{\partial t}\overline{X}_t = F_\alpha(\overline{X}_t)$ for different values of $\alpha$.

# Cooperative branching

In physics notation for reaction-diffusion models, cooperative branching is denoted as $2A \mapsto 3A$. This sort of dynamics, together with $3A \mapsto 2A$, was already considered by *F. Schlögl* [Z. Phys. 1972]. *Lebowiz, Presutti and Spohn* [JSP 1988] call this *binary reproduction*.

*C. Noble* [AOP 1992], *R. Durrett* [JAP 1992], and *C. Neuhauser and S.W. Pacala* [AAP 1999] call a model with cooperative branching and deaths the *sexual reproduction process*.

The unstable fixed point says that in well-mixing populations, once the population drops below a critical level, it becomes so hard for organisms to find a partner that the population dies out.
This effect is also responsible for the first order (discontinuous) phase transition - at least in well-mixing populations.

# A spatial model

Recall the *exclusion map*

$$\mathtt{excl}_{ij}x(k) := \begin{cases} x(i) & \text{if } k = j, \\ x(j) & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Consider the one-dimensional model with generator

$$\begin{aligned} Gf(x) := {} & \tfrac{1}{2}\alpha \sum_i \big\{ f\big(\mathtt{coop}_{i-2,i-1,i}x\big) - f\big(x\big) \big\} \\ & + \tfrac{1}{2}\alpha \sum_i \big\{ f\big(\mathtt{coop}_{i+2,i+1,i}x\big) - f\big(x\big) \big\} \\ & + \sum_i \big\{ f\big(\mathtt{death}_i x\big) - f\big(x\big) \big\} \\ & + \varepsilon^{-1} \sum_i \big\{ f\big(\mathtt{excl}_{i,i+1}x\big) - f\big(x\big) \big\}. \end{aligned}$$

# Fast stirring

Set

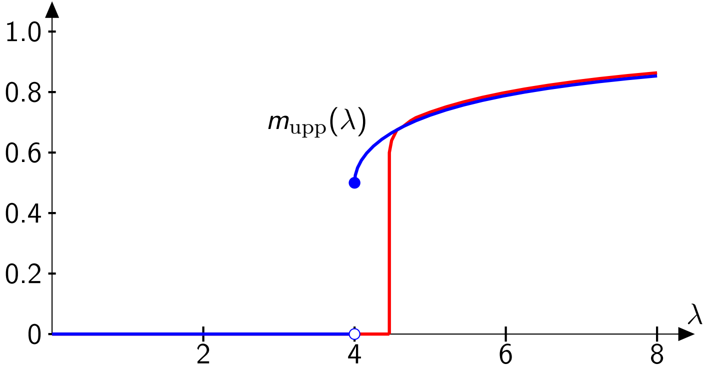
$$m_\varepsilon(x,t) := \mathbb{P}[X_{\varepsilon^{-2}t}(\lfloor \varepsilon x \rfloor) \qquad (x \in R,\ t \geq 0).$$

**[DeMasi, Ferrari & Lebowitz '86]** In the fast stirring limit $\varepsilon \downarrow 0$, the particle density $m_\varepsilon$ converges to a solution of the PDE

$$\frac{\partial}{\partial t} m = \frac{\partial^2}{\partial x^2} m + \alpha m^2 (1-m) - m.$$

**Note** if we start with a constant density, then $m_t(x) \equiv m_t$ solves the mean-field ODE.

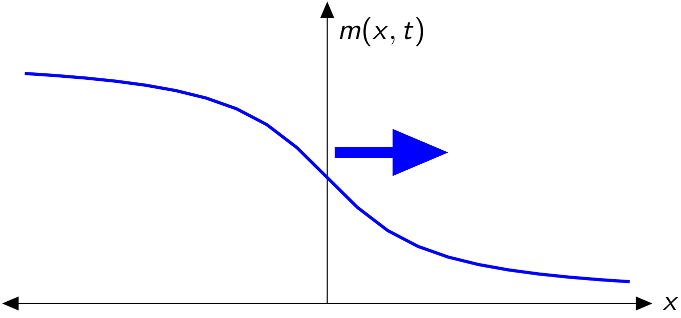
# The upper invariant law




**[Noble '92]** For small $\varepsilon > 0$, the density of the upper invariant law is at least $z_{\mathrm{upp}}(\lambda)$ for $\lambda > 4.5$ and close to zero for $\lambda < 4.5$.

# Travelling waves

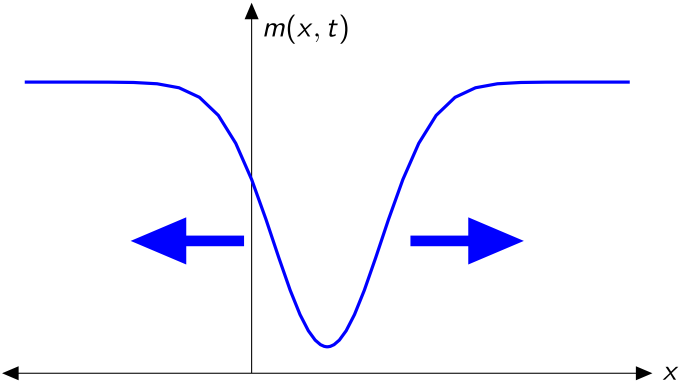
For $\lambda > 4$, the equation $\frac{\partial}{\partial t} m = \frac{\partial^2}{\partial x^2} m + \lambda m^2 (1 - m) - m$ has travelling wave solutions.




**[DeMasi, Ianiro, Pellegrinotti, & Presutti '84]** The propagation speed is positive for $\lambda > 4.5$, and negative for $4 < \lambda < 4.5$.

# Metastability

For $4 < \lambda < 4.5$ and $\varepsilon$ small, rare random events bring the local particle density below a critical value.




The interval of low population density spreads in both directions.

# The critical values

**Def** The process *survives* if $\mathbb{P}^x[X_t \neq \underline{0}\ \forall t \geq 0] > 0$ for some, and hence for all finite nonzero initial states $x$.

Monotonicity implies that there exist $\lambda_c, \lambda'_c$ such that

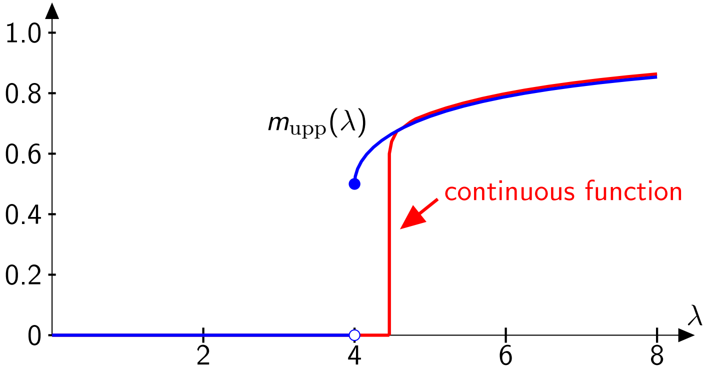
- The upper invariant law satisfies $\overline{\nu}(\{\underline{0}\}) = 0$ for $\lambda > \lambda_c$ and $\overline{\nu} = \delta_{\underline{0}}$ for $\lambda < \lambda_c$.
- The process survives for $\lambda > \lambda'_c$ and dies out for $\lambda < \lambda'_c$.

**Conjecture 1** $\lim_{\varepsilon \downarrow 0} \lambda_c(\varepsilon) = 4.5$.

**Conjecture 2** $\lambda'_c = \lambda_c$.

**[Noble '92]** $2 \leq \lambda_c(\varepsilon)$ for all $\varepsilon > 0$ and $\limsup_{\varepsilon \downarrow 0} \lambda_c(\varepsilon) \leq 4.5$.

# Order of the phase transition




**Conjecture** For fixed $\varepsilon > 0$, the phase transition is second order and in the same universality class as the contact process.

# Amenability

Let $(\Lambda, E)$ be a transitive graph. For each $A \subset \Lambda$, set

$$\partial A := \big\{ i \notin A : \exists j \in A \text{ s.t. } \{i,j\} \in E \big\}$$

**Def** $(\Lambda, E)$ is *amenable* if for every $\varepsilon > 0$ there exists a finite nonzero $A \subset \Lambda$ such that

$$\frac{|\partial A|}{|A|} \leq \varepsilon.$$

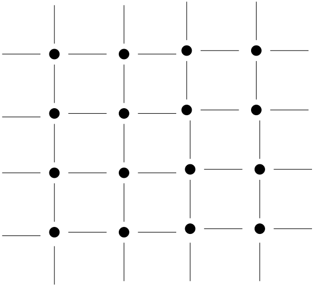
$(\Lambda, E)$ is said to have *exponential growth* (resp. *subexponential growth*) if the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \big| \{ j \in \Lambda : d(i,j) \leq n \} \big|$$

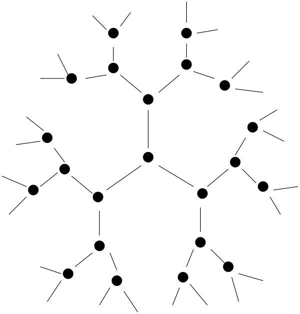
is positive (resp. zero), where $d(\,\cdot\,,\,\cdot\,)$ denotes the usual graph distance.

$$\text{subexponential growth} \quad \begin{array}{c}\Rightarrow \\ \not\Leftarrow\end{array} \quad \text{amenability.}$$

# Amenability



$\mathbb{Z}^2$



$\mathbb{T}_2$

$\mathbb{Z}^d$ with nearest-neighbor edges is amenable and of subexponential growth, but regular trees are nonamenable and of exponential growth.

The *lamplighter group* gives through its *Cayley graphs* rise to transitive graphs that have exponential growth but are amenable.

# Some conjectures and questions

Recall that $\lambda_c, \lambda'_c$ are the critical values for nontriviality of the upper invariant law and for survival, respectively.

**Conjecture** $\lambda_c \leq \lambda'_c$, with $=$ on $\mathbb{Z}^d$ and $<$ on regular trees.

**Motivation** On nonamenable lattices, in any finite population, a positive fraction of the population lives on the boundary of the population, where it is harder to find a partner.

# Some conjectures and questions

**Conjecture** On $\mathbb{Z}^d$, the upper invariant is the limit law started from any nontrivial initial law.

**Conjecture** On regular trees, there exist two mutually singular translation invariant stationary laws, that roughly correspond to the fixed points $z_{\rm mid}$ and $z_{\rm upp}$ of the mean-field ODE.

**Motivation** By our previous arguments, on nonamenable lattices, if one starts in a translation invariant initial law with a very low density, then the process should converge to $\delta_{\underline{0}}$.
But then, by monotonicity, there should be some set in the space of all translation invariant laws equipped with the stochastic order that separates the domains of attraction of $\delta_{\underline{0}}$ and $\overline{\nu}$.