IV Spatial Models in Population Biology

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Lecture 4: Mean-field duality

Mean-field duality

Plan:

- The mean-field ODE
- A recursive tree representation
- Endogeny
- The multivariate ODE
- A higher-level ODE

With *cooperative branching* as a running example.

The mean-field ODE

Let S be a Polish space, let $\mathcal{G}$ me a collection of measurable maps $g : S^k \to S$ with $k = k_g \geq 0$, and let $(r_g)_{g\in\mathcal{G}}$ be nonnegative rates. We view S^0 as a set with a single element, i.e., $k_g = 0$ means the function g is constant.

Let $[N] := \{1, \ldots, N\}$ and

$$[N]^{\langle k\rangle} := \big\{\mathbf{i} = (i_1, \ldots, i_k) \in [N]^k : i_m \neq i_n \ \forall n \neq m\big\},$$

which has $N^{\langle k\rangle} := N(N-1)\cdots(N-k+1)$ elements.
For $g : S^k \to S$, $N \geq k$, $\mathbf{i} \in [N]^{(\langle k\rangle}$, and $j \in [N]$, define $g^{\mathbf{i},j} : S^N \to S^N$ by

$$g^{\mathbf{i},j}x(j') := \begin{cases} g\big(x(i_1), \ldots, x(i_k)\big) & \text{if } j' = j, \\ x(j') & \text{otherwise.} \end{cases}$$

The mean-field ODE

The Markov process $(X_t^N)_{t\geq 0}$ with state space S^N and generator

$$Gf(x) := \sum_{g\in\mathcal{G}} r_g \sum_{j\in[N]} \frac{1}{N^{\langle k\rangle}} \sum_{\mathbf{i}\in[N]^{\langle k\rangle}} \big\{ f\big((g^{\mathbf{i},j}x\big) - f\big((x\big) \big\}$$

can be constructed in a Poissonian way as before, leading to a stochastic flow $(\mathbf{X}_{s,t}^N)_{s\leq t}$. As before, the *empirical process*

$$\mu_t^N := \mu\big[X_t^N\big] \quad (t \geq 0) \quad \text{with} \quad \mu[x] := \frac{1}{N} \sum_{i\in[N]} \delta_{x(i)}$$

is a Markov process. In the mean-field limit $N \to \infty$, we expect $(\mu_t^N)_{t\geq 0}$ to be close to the solution of an ODE.

The mean-field ODE

Let $\mathcal{M}^1(S)$ denote the space of all probability measures on S, equipped with the topology of weak convergence and the Borel-σ-algebra.
For each measurable map $g : S^k \to S$, we define a measurable map $\check{g} : \mathcal{M}^1(S)^k \to \mathcal{M}^1(S)$ by

$$\check{g}(\mu_1, \ldots, \mu_k) := \mathbb{P}\big[g(X_1, \ldots, X_k) \in \cdot\,\big]$$
$$\text{where} \quad X_1, \ldots, X_k \text{ are indep. with } \mathbb{P}[X_i \in \cdot\,] = \mu_i.$$

We also define $T_g : \mathcal{M}^1(S) \to \mathcal{M}^1(S)$ by

$$T_g(\mu) := \check{g}(\mu, \ldots, \mu).$$

Note that T_g is in general nonlinear, unless $k = 1$.

The mean-field ODE

Theorem [Mach, Sturm & S. '18] Assume that

$$\sum_{g\in\mathcal{G}} r_g k_g < \infty.$$

Then, for each initial state $\mu_0 \in \mathcal{M}^1(S)$, the *mean-field ODE*

$$\frac{\partial}{\partial t}\mu_t = \sum_{g\in\mathcal{G}} \big\{ T_g(\mu_t) - \mu_t \big\} \qquad (t \geq 0) \tag{1}$$

has a unique solution. Here, writing $\langle \mu, \phi \rangle := \int \phi \, \mathrm{d}\mu$, we interpret (1) in a weak sense: for each bounded measurable $\phi : S \to \mathbb{R}$, the function $t \mapsto \langle \mu_t, \phi \rangle$ is continuously differentiable and

$$\frac{\partial}{\partial t}\langle \mu_t, \phi \rangle = \sum_{g\in\mathcal{G}} \big\{ \langle T_g(\mu_t), \phi \rangle - \langle \mu_t, \phi \rangle \big\} \qquad (t \geq 0).$$

The mean-field ODE

Theorem [Mach, Sturm & S. '18] Assume that $\sum_{g\in\mathcal{G}} r_g k_g < \infty$ and let S be finite. Let $(\mu_t^N)_{t\geq 0}$ be empirical processes such that $\mu_0^N \to \mu_0$ for some $\mu_0 \in \mathcal{M}^1(S)$. Then

$$\mathbb{P}\Big[\sup_{0\leq t\leq T} \|\mu^N(t) - \mu_t\| \geq \varepsilon\Big] \underset{N\to\infty}{\longrightarrow} 0 \qquad \forall \varepsilon > 0,\ T < \infty,$$

where $(\mu_t)_{t\geq 0}$ solves the mean-field ODE (1).

Note Something similar should hold for infinite (even uncountable) S, at least when $\big(X_0^N(1), \ldots, X_0^N(N)\big)$ are i.i.d. with law μ_0.

The mean-field ODE

Remark We could be more general and also consider maps

$$S^k \ni (x_1, \ldots, x_k) \mapsto \big(g_1(x_1, \ldots, x_k), g_2(x_1, \ldots, x_k)\big) \in S^2,$$

and similarly with S^2 replaced by S^m $(m \geq 1)$. However, applying such a map with rate r has for the mean-field ODE the same effect as applying the maps g_1 and g_2 each with rate r.

Also, in our definition of $g^{\mathbf{i},j}$, we could have chosen $j \in \{i_1, \ldots, i_k\}$, e.g. $j = i_1$ always. Again, although this yields a different Markov process, for the mean-field ODE this has no effect.

Cooperative branching

Let $S = \{0,1\}$ and $\mathcal{G} = \{\mathtt{coop}, \mathtt{death}\}$, where

$$\mathtt{coop} : \{0,1\}^3 \to \{0,1\} \quad \text{and} \quad \mathtt{death} : \{0,1\}^0 \to \{0,1\}$$

are defined as

$$\mathtt{coop}(x_1, x_2, x_3) := x_1 \vee (x_2 \wedge x_3) \quad \text{and} \quad \mathtt{death}(\,) := 0.$$

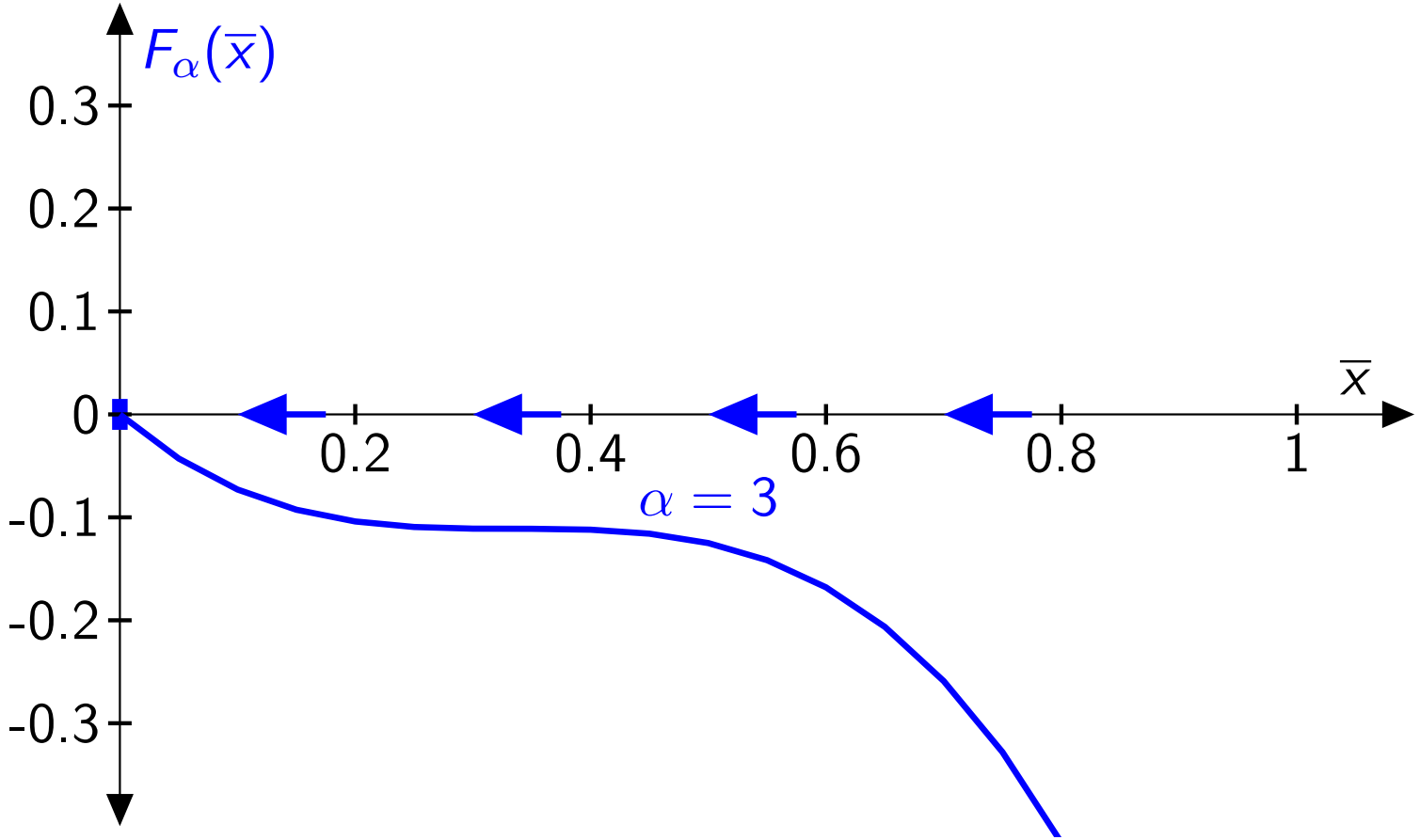
A probability measure μ on $\{0,1\}$ is uniquely determined by $\mu(\{1\})$. Setting $\overline{X}_t := \mu_t(\{1\})$ and choosing the rates

$$r_{\mathtt{coop}} := \alpha \quad \text{and} \quad r_{\mathtt{death}} := 1,$$

we find the mean-field ODE

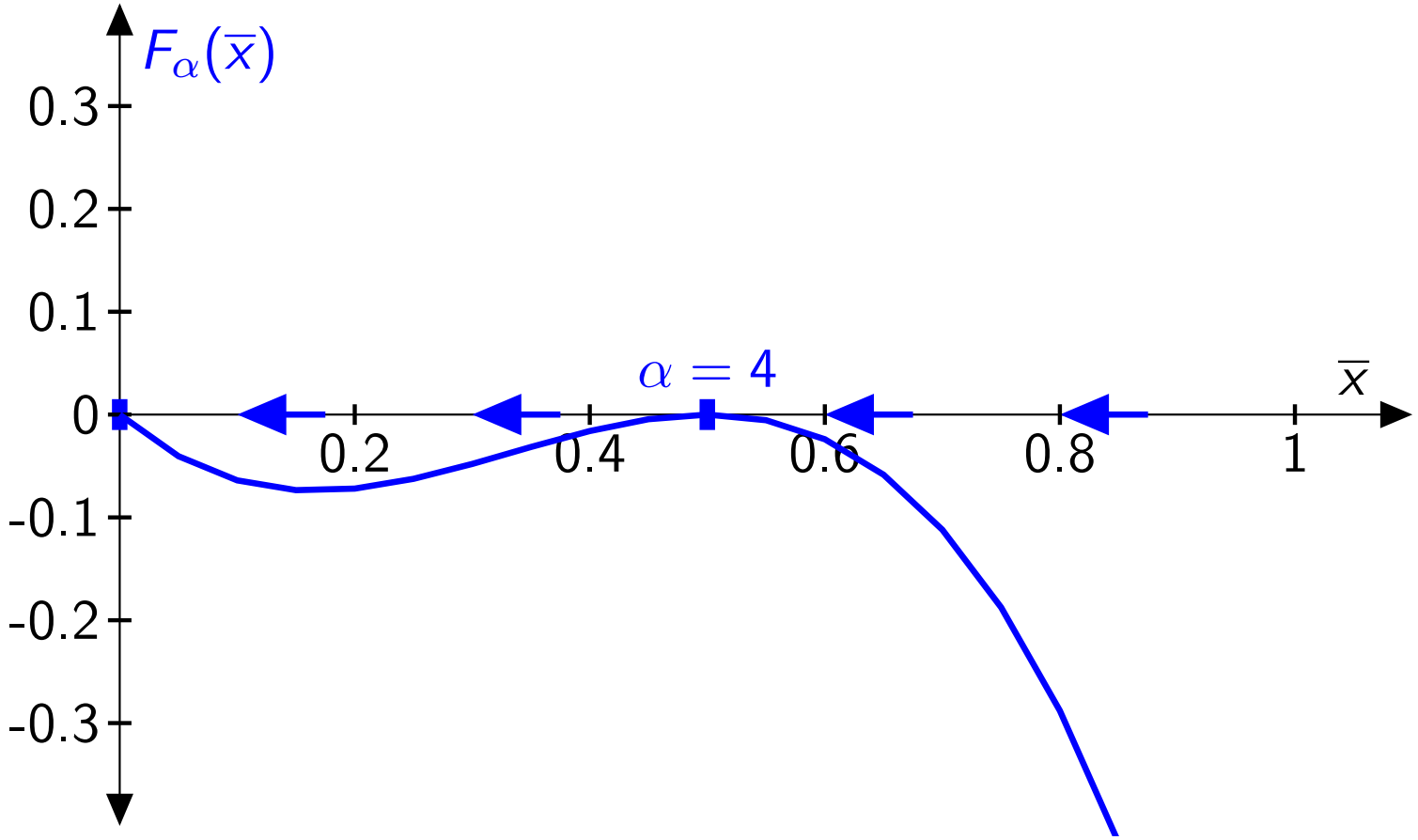
$$\tfrac{\partial}{\partial t}\overline{X}_t = \alpha \overline{X}_t^2 (1 - \overline{X}_t) - \overline{X}_t =: F_\alpha(\overline{X}_t).$$

Cooperative branching



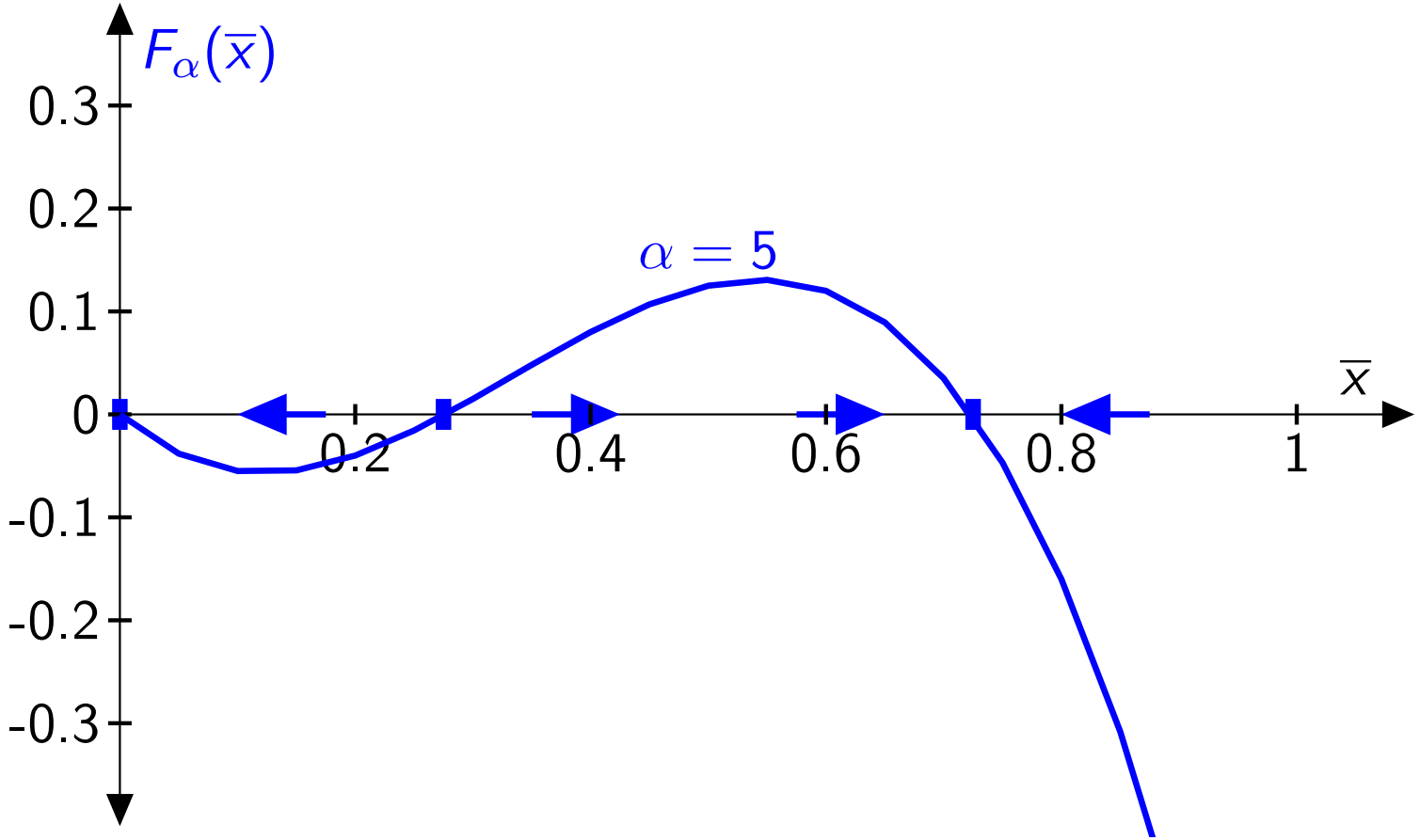
For $\alpha < 4$, the equation $\frac{\partial}{\partial t}\overline{X}_t = F_\alpha(\overline{X}_t)$ has a single, stable fixed point $\overline{x} = 0$.

Cooperative branching



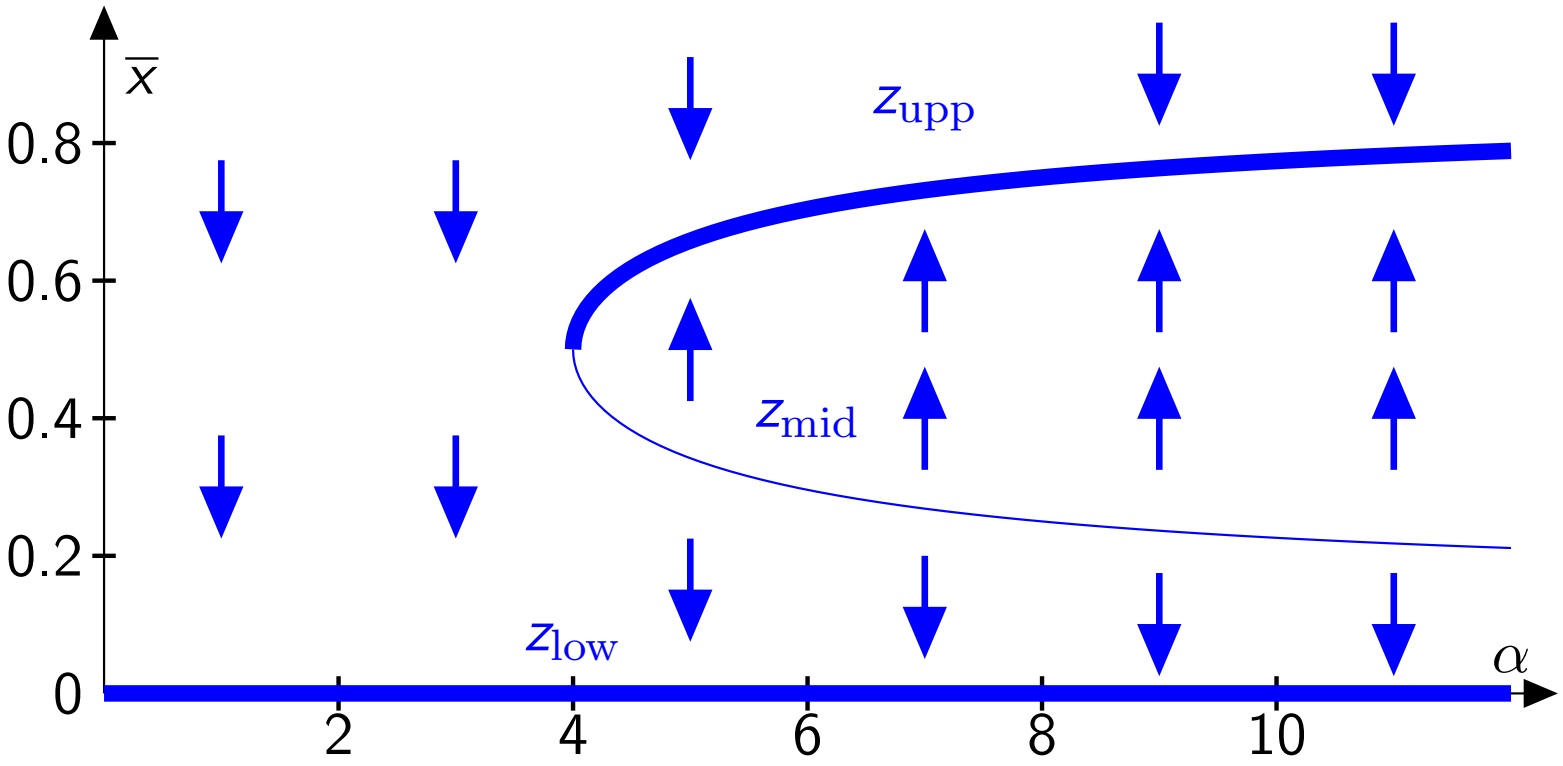
For $\alpha = 4$, a second fixed point appears at $\overline{x} = 0.5$.

Cooperative branching



For $\alpha > 4$, there are two stable fixed points and one unstable fixed point, which separates the domains of attraction of the other two.

Cooperative branching



Fixed points of $\frac{\partial}{\partial t}\overline{X}_t = F_\alpha(\overline{X}_t)$ for different values of α.

Cooperative branching

In physics notation for reaction-diffusion models, cooperative branching is denoted as $2A \mapsto 3A$. This sort of dynamics, together with $3A \mapsto 2A$, was already considered by *F. Schlögl* [Z. Phys. 1972]. *Lebowiz, Presutti and Spohn* [JSP 1988] call this *binary reproduction*.

C. Noble [AOP 1992], *R. Durrett* [JAP 1992], and *C. Neuhauser and S.W. Pacala* [AAP 1999] call a model with cooperative branching and deaths the *sexual reproduction process*.

The unstable fixed point says that in well-mixing populations, once the population drops below a critical level, it becomes so hard for organisms to find a partner that the population dies out.
This effect is also responsible for the first order (discontinuous) phase transition - at least in well-mixing populations.

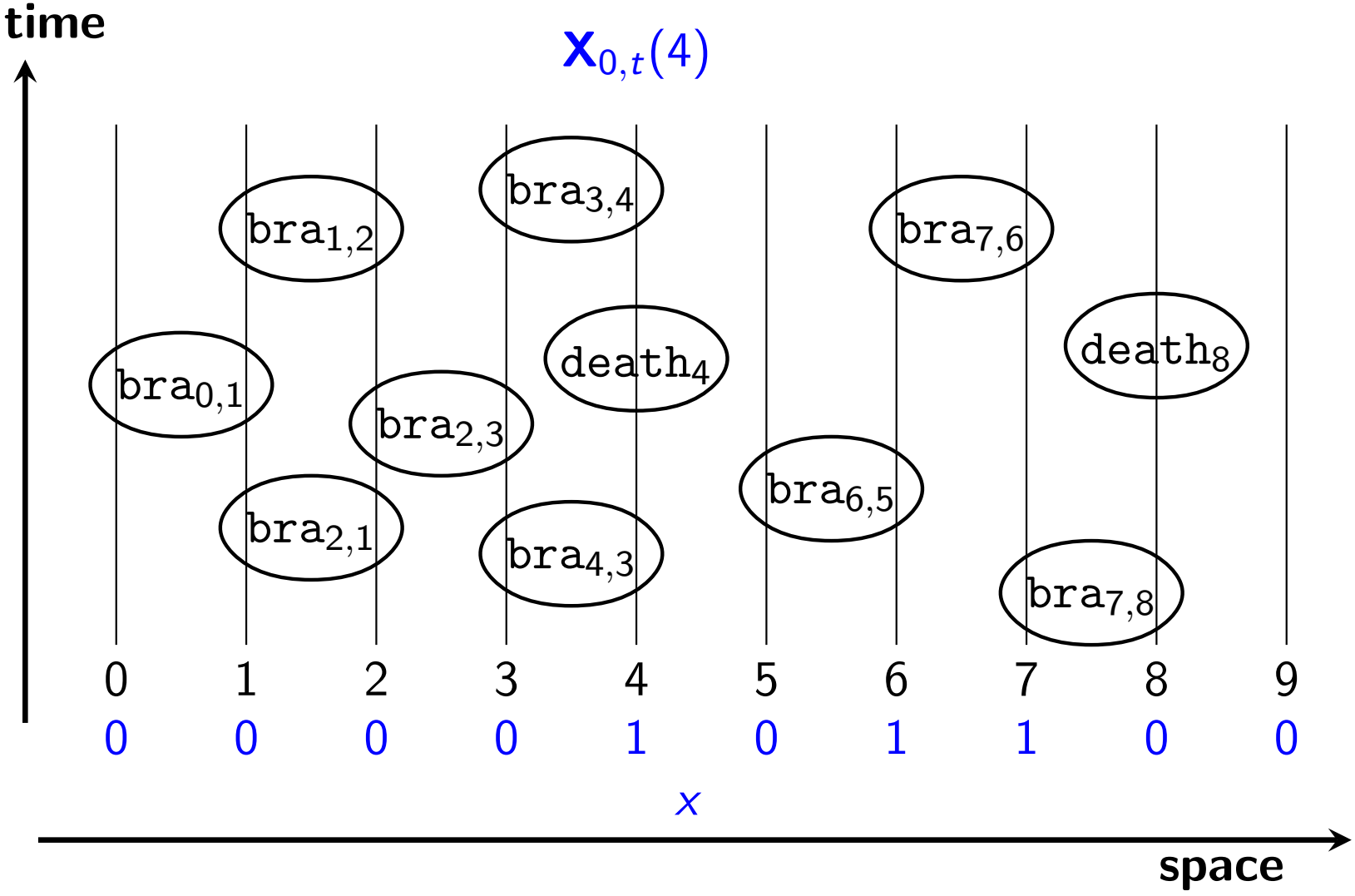
A recursive tree representation

Recall the Markov process

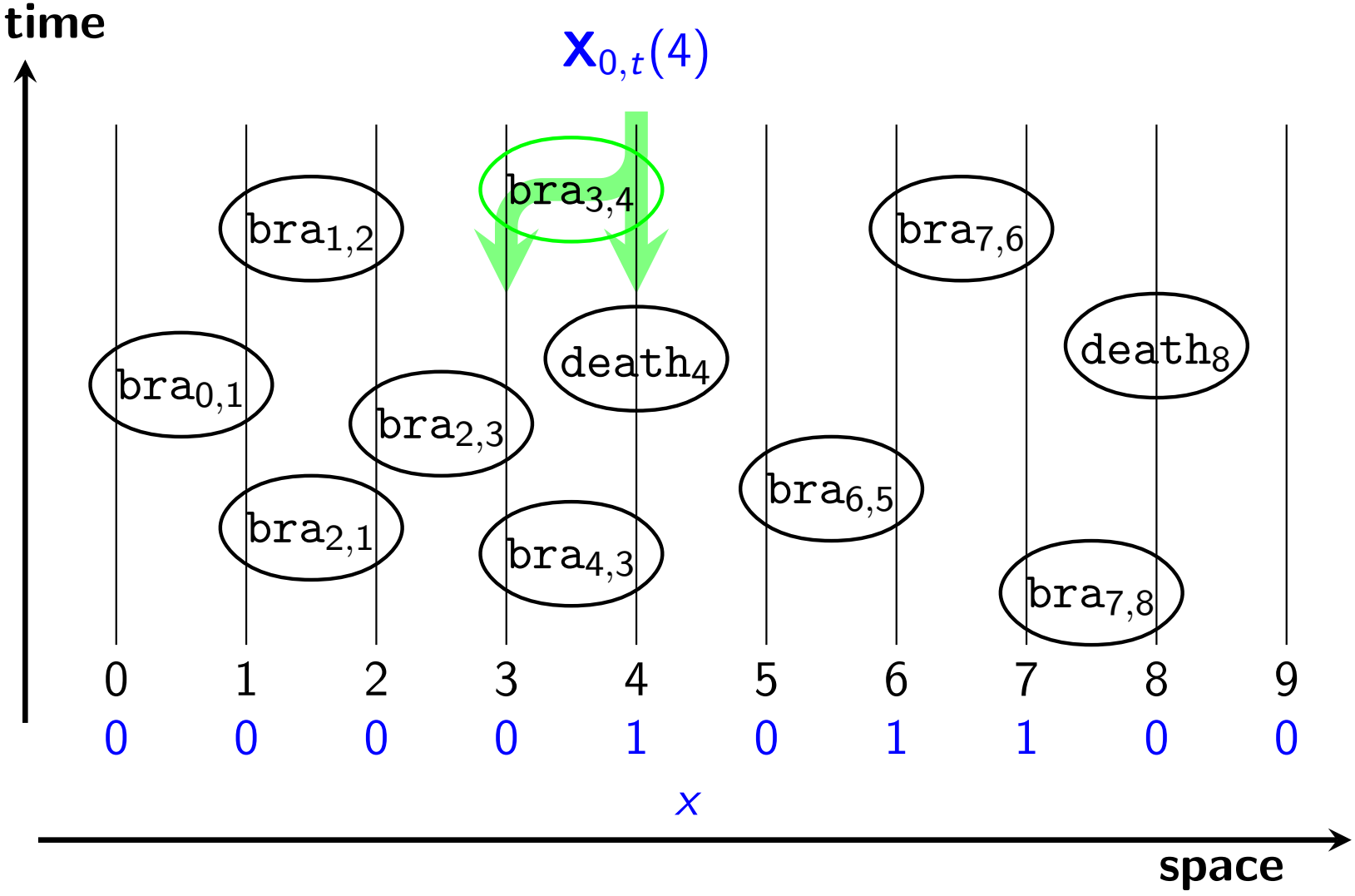
$$\big(\mathcal{R}_i(\mathbf{X}_{s-t,t})\big)_{t\geq 0}$$

that traces back in time all sites at time $s-t$ that are relevant for the state at the site i at time s.

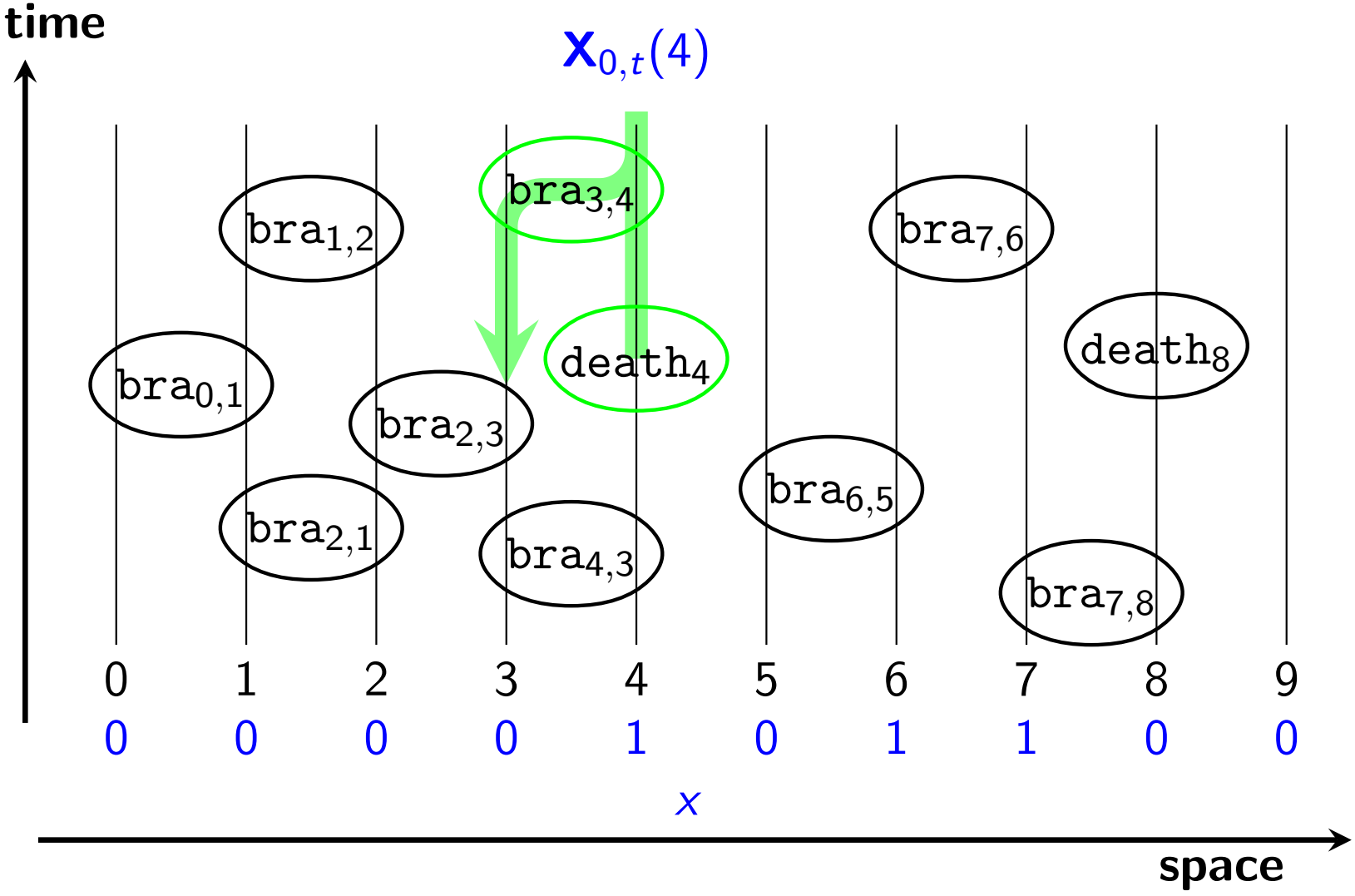
A recursive tree representation



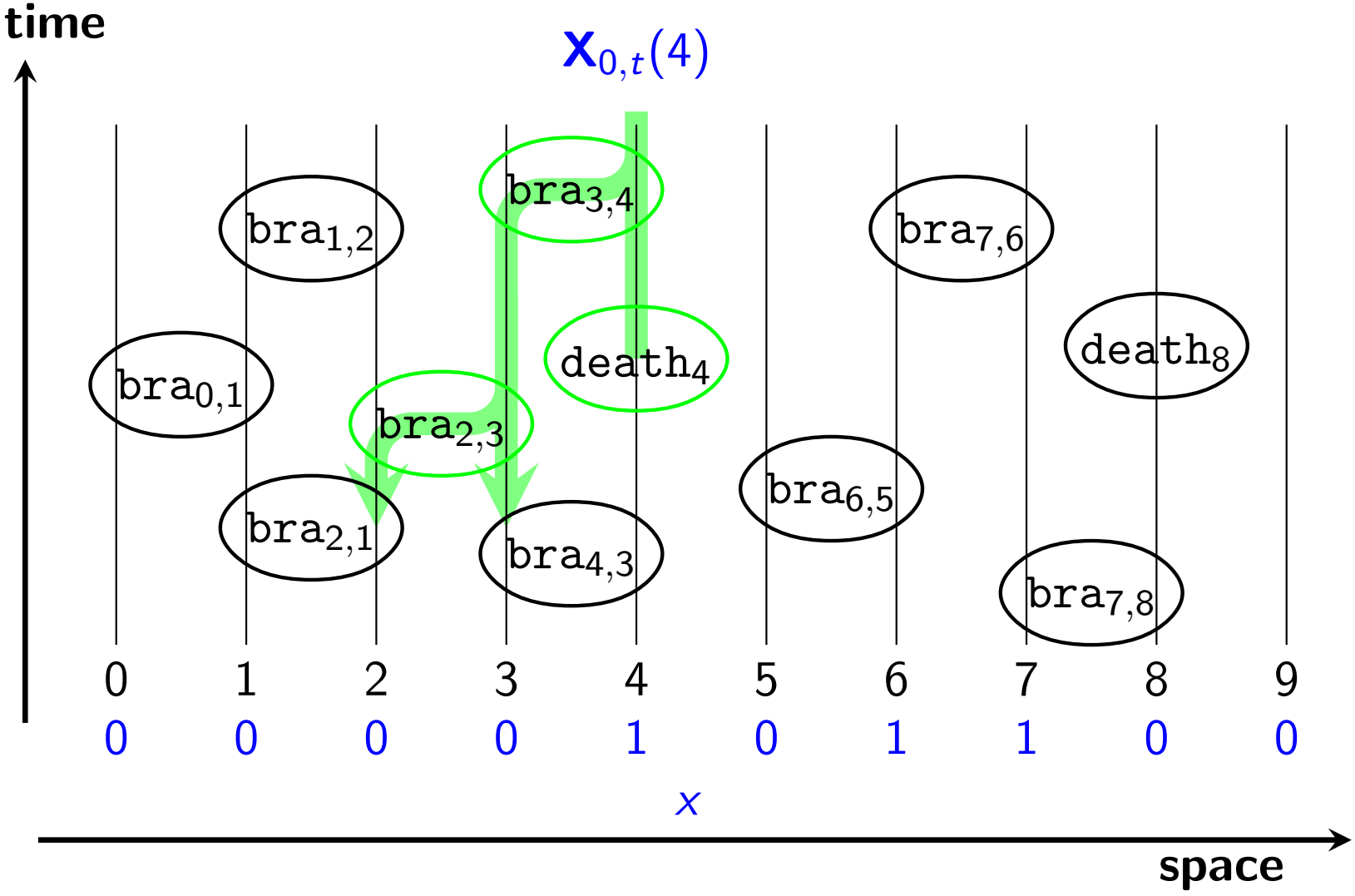
A recursive tree representation



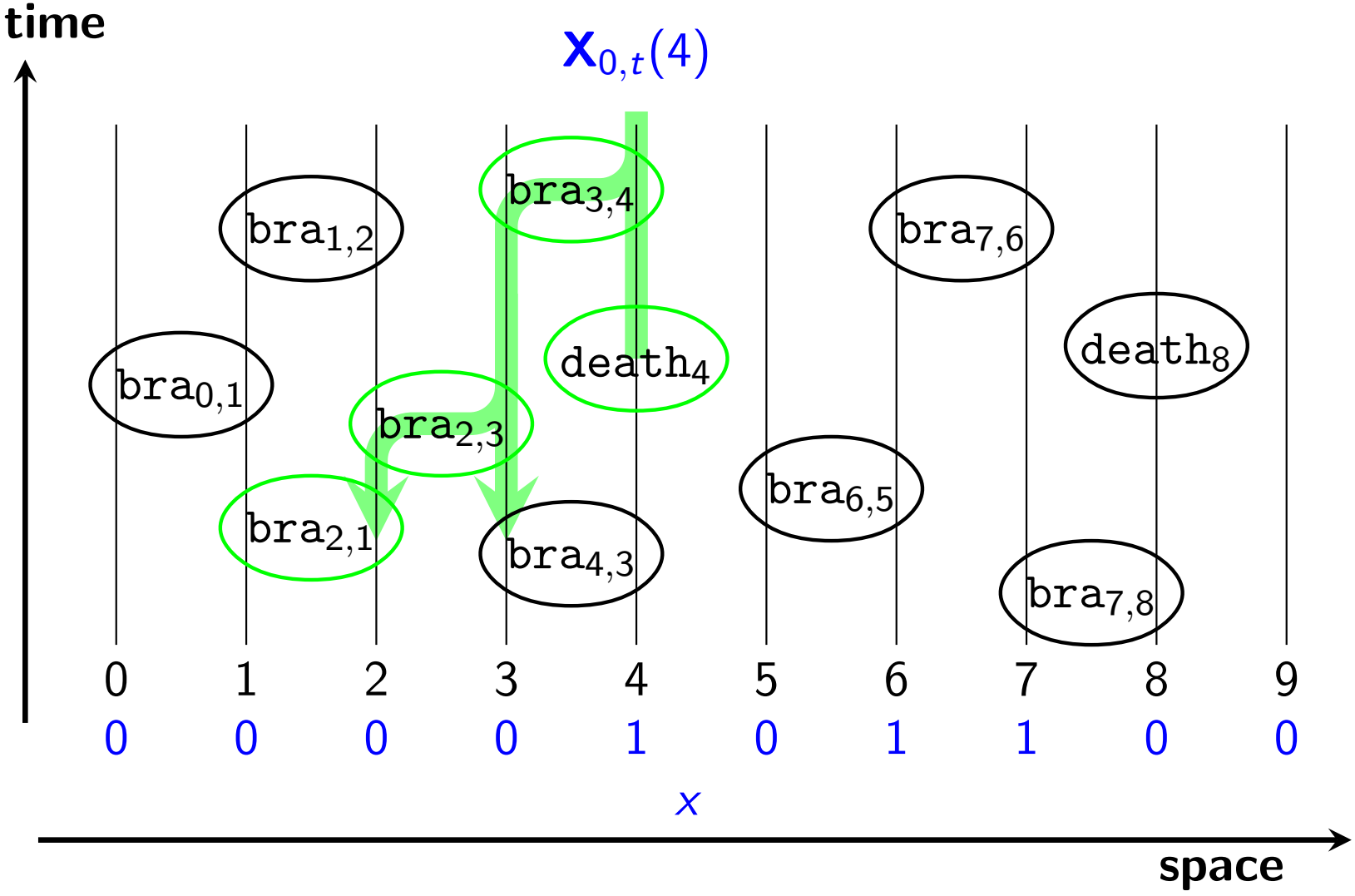
A recursive tree representation



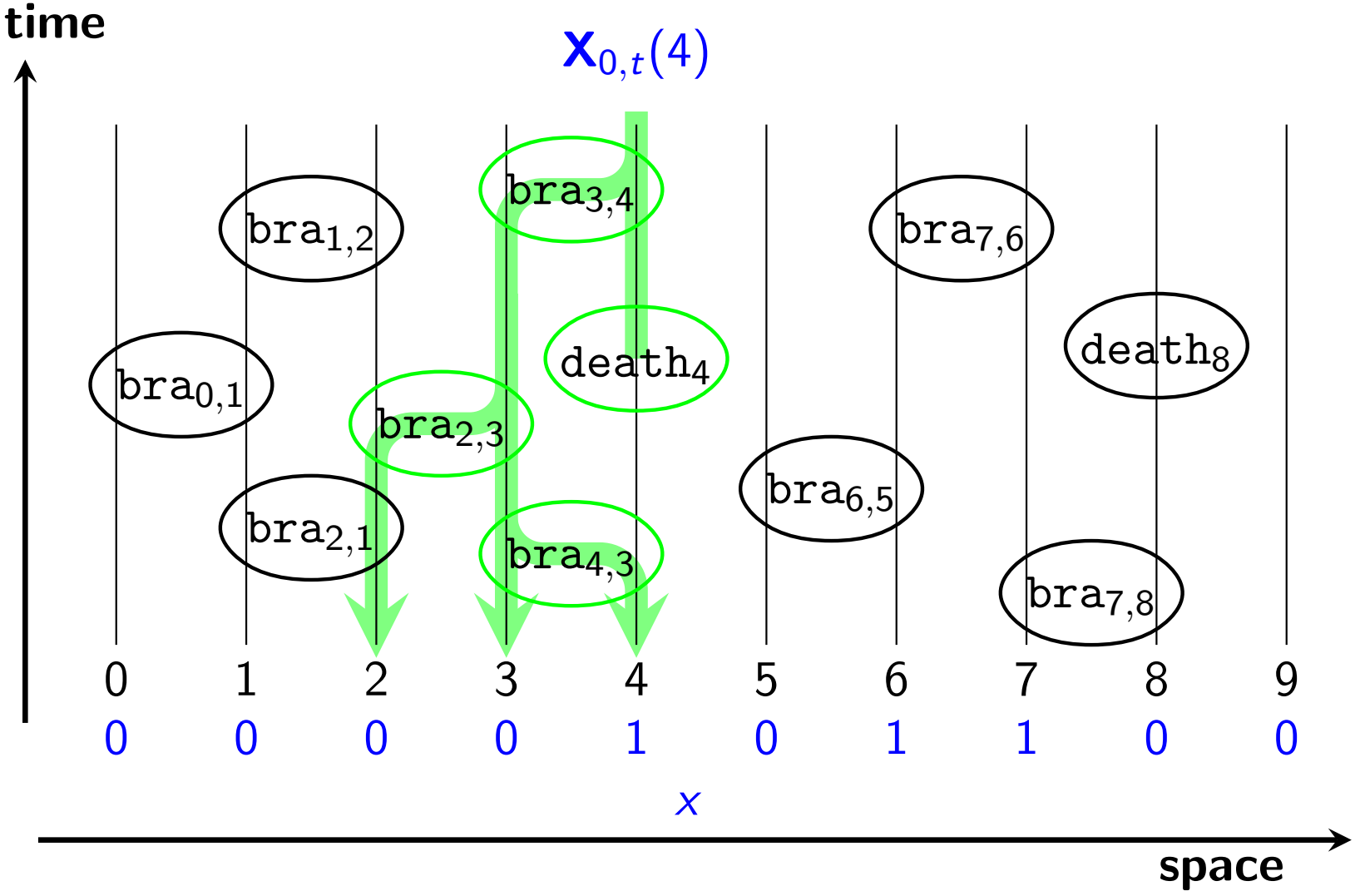
A recursive tree representation



A recursive tree representation



A recursive tree representation



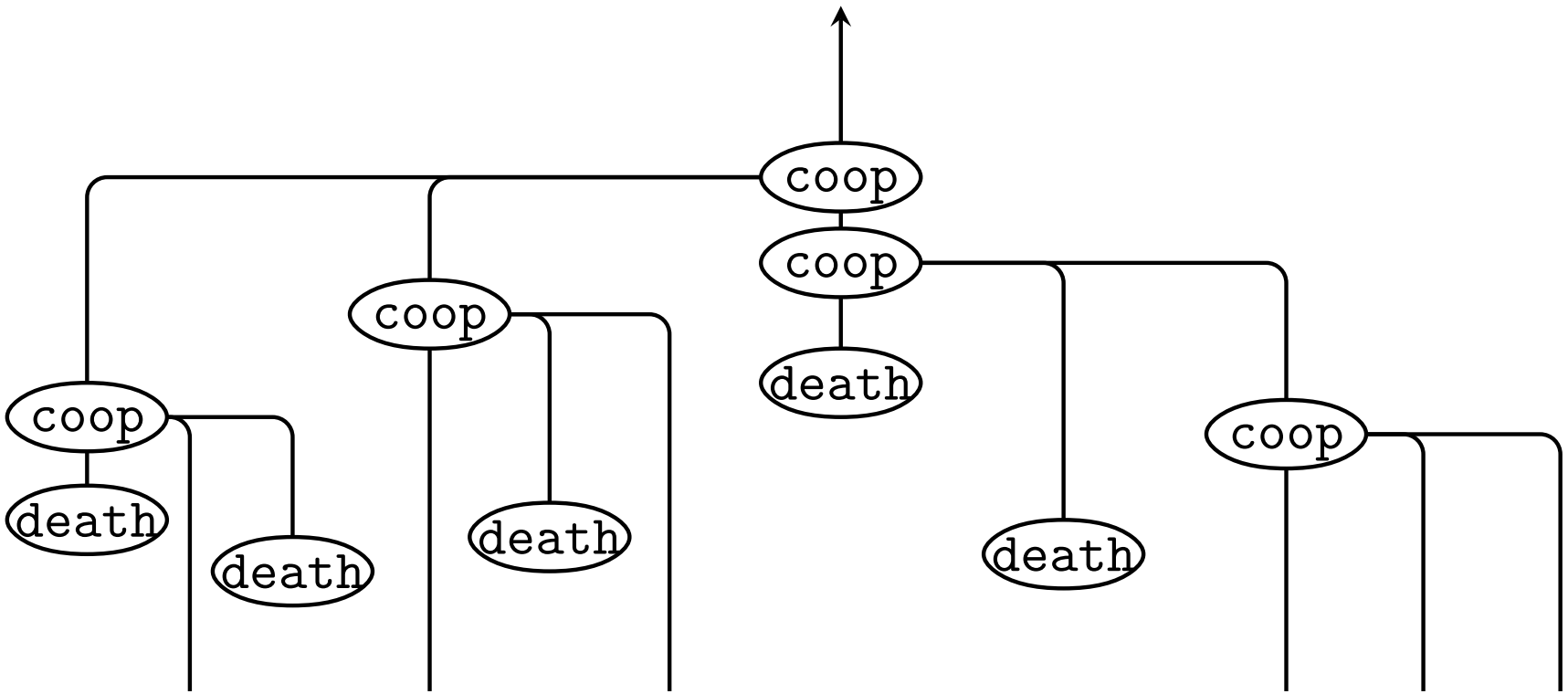
A recursive tree representation

In the mean-field limit,

$$\big(\mathcal{R}_i(\mathbf{X}_{s-t,t})\big)_{t\geq 0}$$

converges to a branching process.

A recursive tree representation



A recursive tree representation

Let $\overline{\mathbb{T}}$ be the set of all finite words $\mathbf{i} = i_1 \cdots i_n$ $(n \geq 0)$ made up from the alphabet $\mathbb{N}_+ = \{1, 2, \ldots\}$. We view $\overline{\mathbb{T}}$ as a tree with root $\emptyset$, the word of length zero, in which each individual $\mathbf{i}$ has infinitely many offspring $\mathbf{i}1, \mathbf{i}2, \ldots$

Let $|r| := \sum_{g \in \mathcal{G}} r_g$ and let $(\gamma_{\mathbf{i}})_{\mathbf{i} \in \overline{\mathbb{T}}}$ be i.i.d. with law

$$\mathbb{P}[\gamma_{\mathbf{i}} = g] = |r|^{-1} r_g \qquad (g \in \mathcal{G}).$$

We inductively define a random subtree $\mathbb{T} \subset \overline{\mathbb{T}}$ which contains the root and satisfies

$$\mathbf{i}j \in \mathbb{T} \quad \text{iff} \quad \mathbf{i} \in \mathbb{T} \quad \text{and} \quad j \leq k,$$

where $k = k_{\mathbf{i}} := k_{\gamma_{\mathbf{i}}}$ is the integer such that $\gamma_{\mathbf{i}} : S^k \to S$.
Then $\mathbb{T}$ is the family tree of a branching process with maps $(\gamma_{\mathbf{i}})_{\mathbf{i} \in \overline{\mathbb{T}}}$ attached to its vertices, such that the individual $\mathbf{i}$ has $k_{\mathbf{i}}$ offspring.

A recursive tree representation

For any subtree $\mathbb{U} \subset \mathbb{T}$ that contains the root, we write

$$\partial\mathbb{U} := \{\mathbf{i}j \in \mathbb{T}\backslash\mathbb{U} : \mathbf{i} \in \mathbb{U}\}.$$

Let $(\sigma_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}$ be i.i.d. exponentially distributed random variables with mean $|r|^{-1}$, independent of $(\gamma_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}$. We interpret $\sigma_{\mathbf{i}}$ as the *lifetime* of $\mathbf{i}$ and let

$$\tau^*_{i_1\cdots i_n} := \sigma_\emptyset + \sigma_{i_1} + \cdots + \sigma_{i_1\cdots i_{n-1}} \quad \text{and} \quad \tau^\dagger_{\mathbf{i}} := \tau^*_{\mathbf{i}} + \sigma_{\mathbf{i}}$$

denote its birth and death time. Then

$$\mathbb{T}_t := \{\mathbf{i} \in \mathbb{T} : \tau^\dagger_{\mathbf{i}} \leq t\} \quad \text{and} \quad \partial\mathbb{T}_t$$

denote the set of individuals that have died before time t resp. are alive at time t. In particular,

$$\left(\partial\mathbb{T}_t\right)_{t\geq 0}$$

is a branching process where each individual $\mathbf{i}$ gives with rate r_g birth to k_g offspring, for each $g \in \mathcal{G}$.

A recursive tree representation

Given a finite subtree $\mathbb{U} \subset \overline{\mathbb{T}}$ that contains the root, we define a map $G_{\mathbb{U}} : S^{\partial \mathbb{U}} \to S$ by

$$G_{\mathbb{U}} := x_\emptyset \quad \text{where} \quad (x_{\mathbf{i}})_{\mathbf{i} \in \mathbb{U}} \text{ satisfy } x_{\mathbf{i}} = \gamma_{\mathbf{i}}(x_{\mathbf{i}1}, \ldots, x_{\mathbf{i}k_{\mathbf{i}}}) \quad (\mathbf{i} \in \mathbb{U}).$$

In particular, we set $G_t := G_{\mathbb{T}_t}$.

Theorem [Mach, Sturm & S. '18] Assume that $\sum_{g \in \mathcal{G}} r_g k_g < \infty$. Then the solution to the mean-field equation (1) is given by

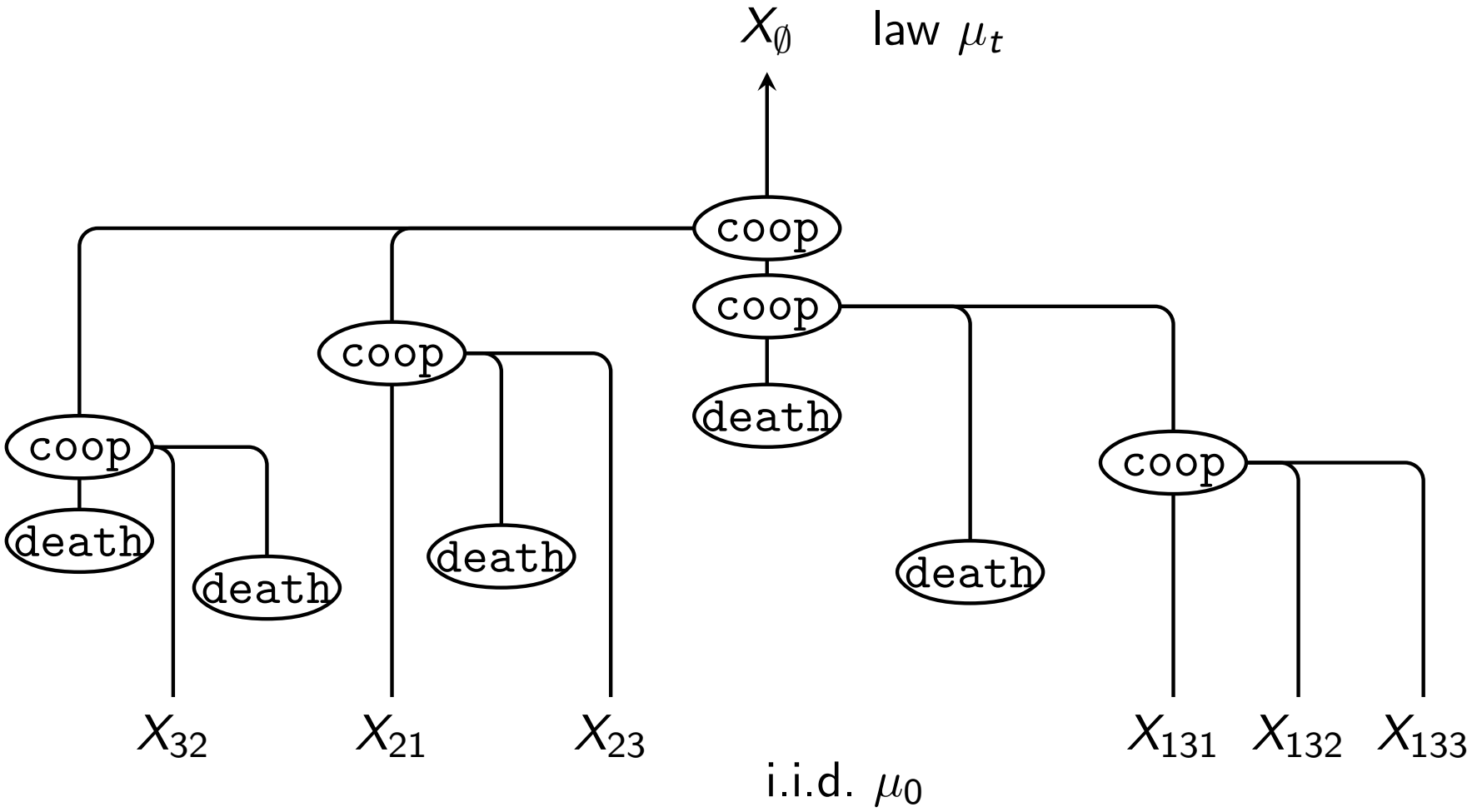
$$\mu_t = \mathbb{E}\big[T_{G_t}(\mu_0)\big] \qquad (t \geq 0),$$

i.e., $\mu_t = \mathbb{P}[X_\emptyset \in \cdot\,]$ where $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}_t \cup \partial \mathbb{T}_t}$ satisfy

(i) $(X_{\mathbf{i}})_{\mathbf{i} \in \partial \mathbb{T}_t}$ are i.i.d. with law μ_0 and independent of $(\gamma_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}_t}$.

(ii) $X_{\mathbf{i}} = \gamma_{\mathbf{i}}(X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}k_{\mathbf{i}}}) \quad (\mathbf{i} \in \mathbb{T}_t)$.

A recursive tree representation



A recursive tree representation

In the special case that $k_g = 1$ for each $g \in \mathcal{G}$, the mean-field ODE (1) is just the backward equation of a continuous-time Markov chain where each map $g \in \mathcal{G}$ is applied with Poisson rate r_g.

We can think of the collection of random variables $(\gamma_{\mathbf{i}}, \sigma_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ as a generalization of the Poisson construction of a continuous-time Markov chain, where "time" now has a tree-like structure.

We let

$$\mathcal{F}_t := \sigma\big((\partial \mathbb{T}_s)_{0 \leq s \leq t},\ (\gamma_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}_t}\big)$$

denote the filtration generated by the branching process $(\partial \mathbb{T}_t)_{t \geq 0}$ as well as the maps attached to the particles that have died by time t.

Unique ergodicity

Lemma Assume that

$$R := \sum_{g \in \mathcal{G}} r_g (k_g - 1)$$

satisfies $R < 0$. Then the mean-field ODE (1) has a unique fixed point ν and solutions started in an arbitrary initial law μ_0 satisfy

$$\|\mu_t - \nu\| \underset{t \to \infty}{\longrightarrow} 0,$$

where $\| \cdot \|$ denotes the total variation norm.

Proof The condition $R < 0$ guarantees that $(\partial \mathbb{T}_t)_{t \geq 0}$ is a subcritical branching process and hence the tree $\mathbb{T}$ is a.s. finite. Now $\partial \mathbb{T} = \emptyset$ and $G_{\mathbb{T}} : S^0 \to S$ is a random constant that depends on the random finite tree $\mathbb{T}$. Setting $\nu := \mathbb{P}\big[G_{\mathbb{T}} \in \cdot\,\big]$, the statement follows by observing that

$$G_t = G_{\mathbb{T}_t} \underset{t \to \infty}{\longrightarrow} G_{\mathbb{T}} \quad \text{a.s.}$$

Unique ergodicity

For our process with cooperative branching and deaths,

$$R = \alpha \cdot (3-1) + 1 \cdot 0 = 2\alpha,$$

which implies that the mean-field ODE has a unique attractive fixed point for $\alpha < 1/2$.
This is not very good compared to the necessary and sufficient condition $\alpha < 4$ that came out of our earlier analysis of the ODE, but the proof of the previous lemma actually works more generally:
Lemma Assume that

$$\mathbb{P}\big[\exists t < \infty \text{ s.t. } G_t \text{ is constant}\big] = 1.$$

Then the mean-field ODE (1) has a unique fixed point ν and solutions started in an arbitrary initial law μ_0 satisfy

$$\|\mu_t - \nu\| \underset{t\to\infty}{\longrightarrow} 0,$$

where $\|\cdot\|$ denotes the total variation norm.

Unique ergodicity

For our process with cooperative branching and deaths, say that $\mathbb{S} \subset \mathbb{T}$ is a *good subtree* if $\emptyset \in \mathbb{S}$ and

(i) $\gamma_{\mathbf{i}} \neq \texttt{death}$ for all $\mathbf{i} \in \mathbb{S}$

(ii) $\forall \mathbf{i} \in \mathbb{S}$, $\{\mathbf{i}1, \mathbf{i}2, \mathbf{i}3\} \cap \mathbb{S}$ is either $\{\mathbf{i}1\}$ or $\{\mathbf{i}2, \mathbf{i}3\}$.

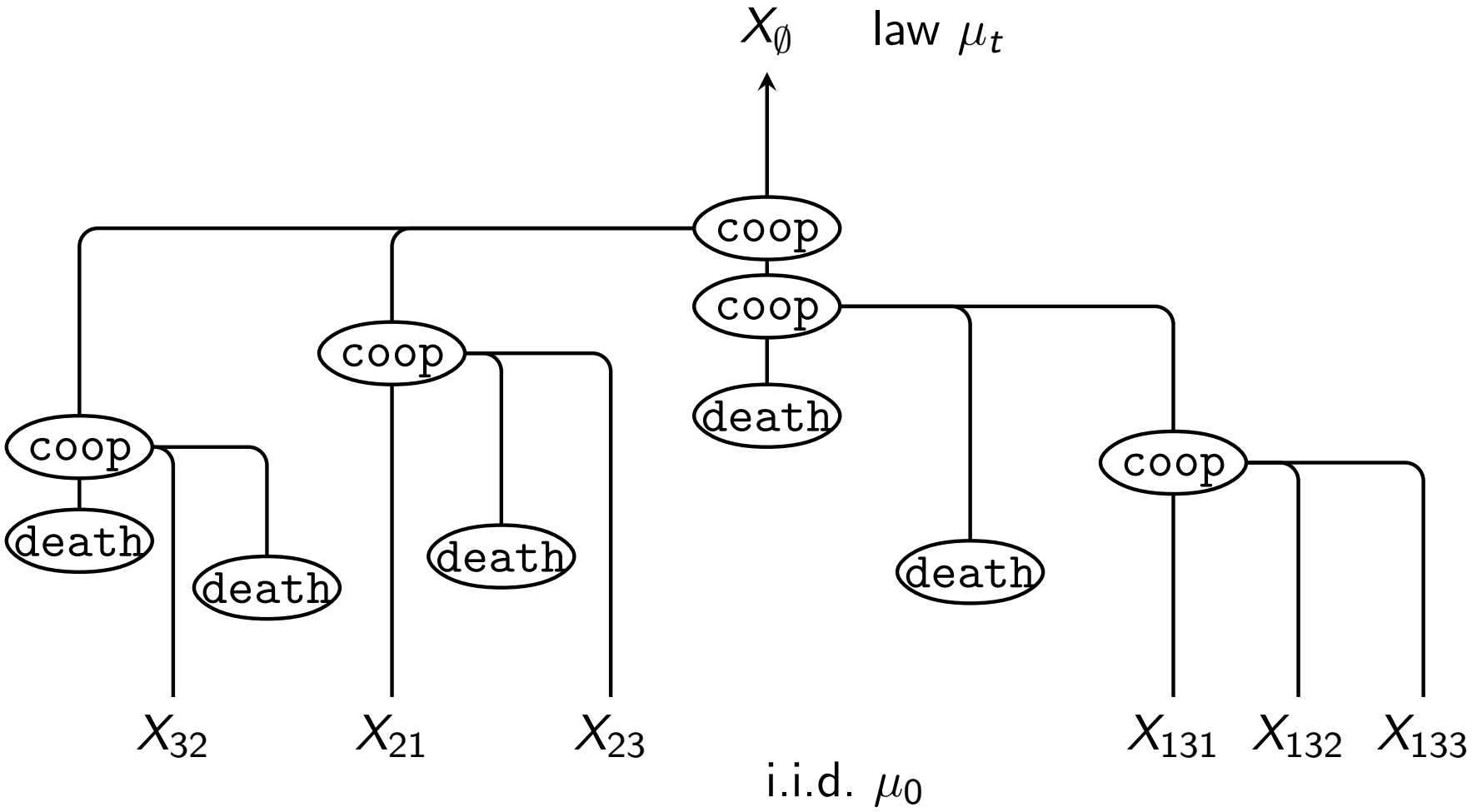
Lemma The following events are a.s. equal:

(i) $\mathbb{T}$ contains a good subtree.

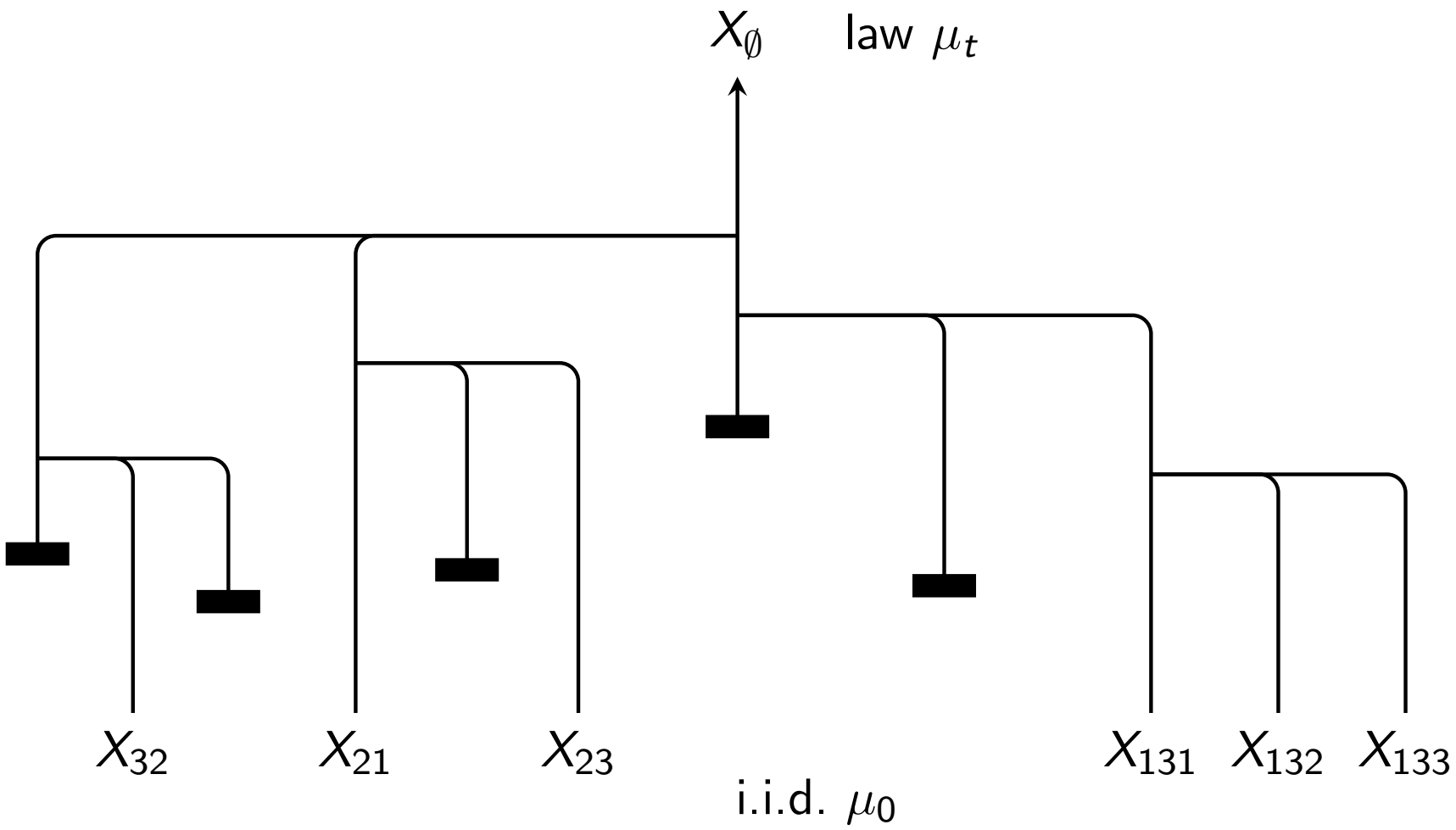
(ii) G_t is constant for some $t < \infty$.

Moreover, $\mathbb{P}[\mathbb{T}$ contains a good subtree$] > 0$ iff $\alpha \geq 4$.

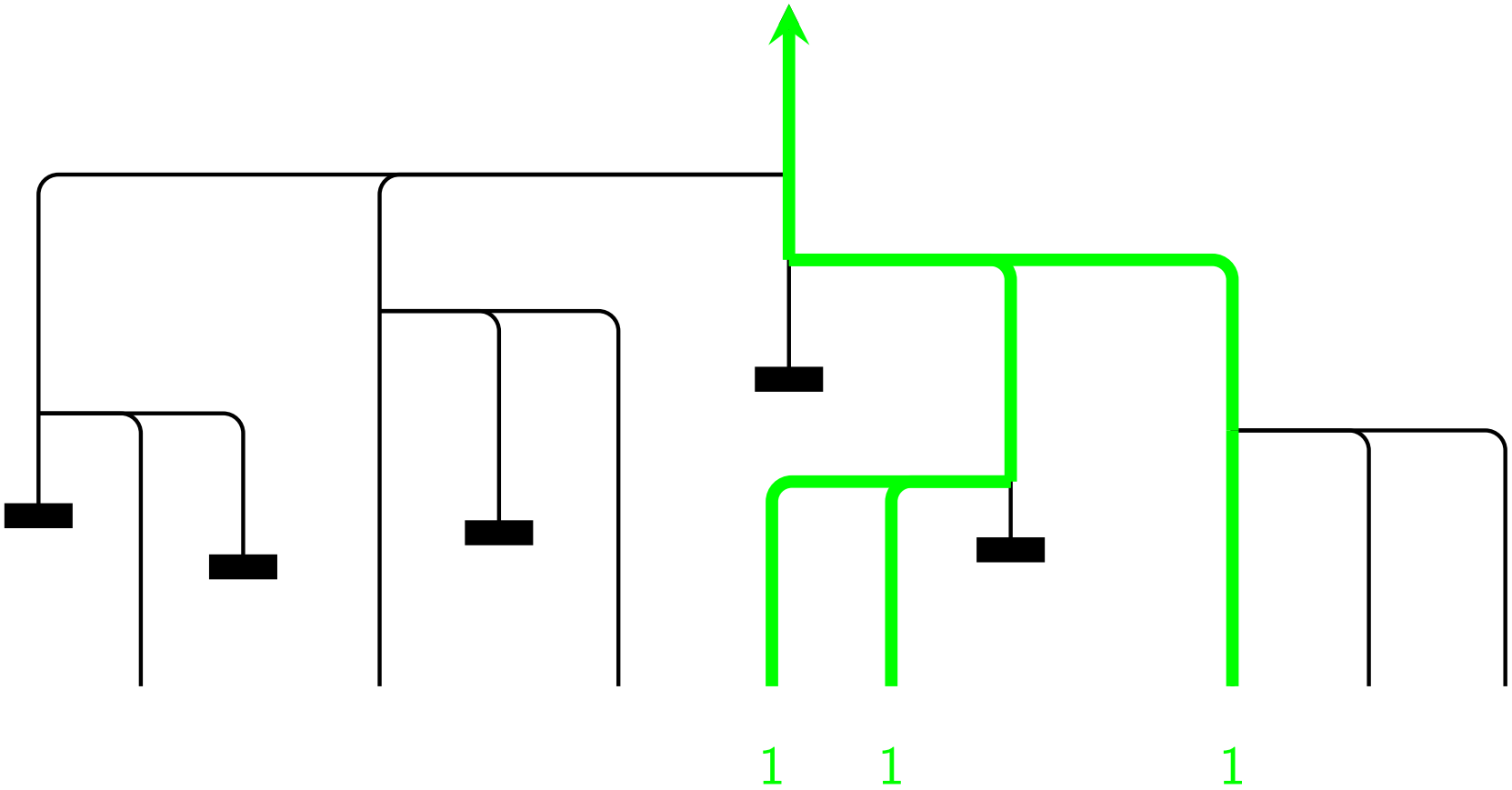
Good subtrees



Good subtrees

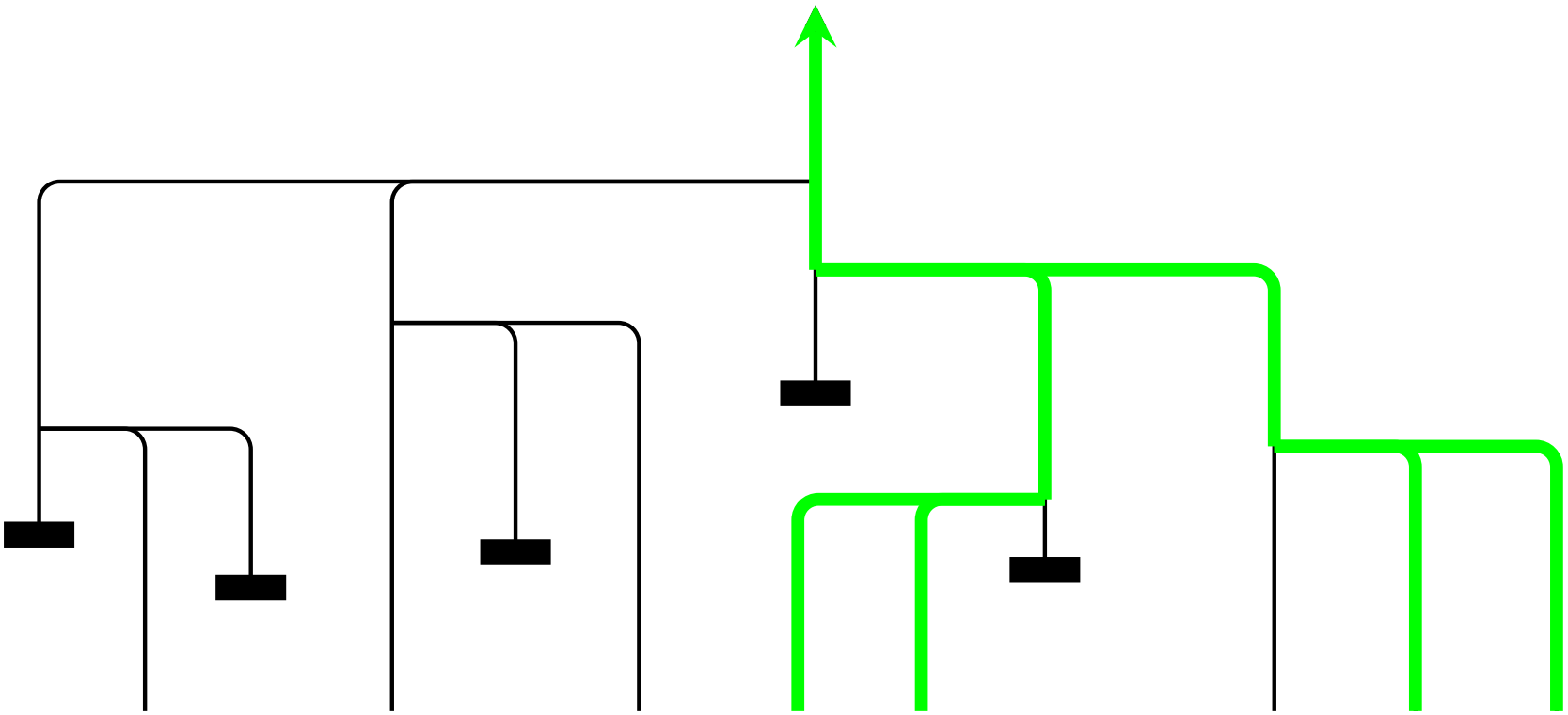


Good subtrees



a good subtree

Good subtrees



dual process $\overline{Y}_t$ $\left\{ \begin{array}{cccccc} 1 & 1 & & 1 & & \\ 1 & 1 & & & 1 & 1 \end{array} \right.$

A Recursive Distributional Equation

Fixed points μ of the mean-field ODE (1) solve the *Recursive Distributional Equation (RDE)*

$$\mu = |r|^{-1} \sum_{g \in \mathcal{G}} T_g(\mu). \tag{2}$$

For each solution μ to the RDE, it is possible to define a collection of random variables $(\gamma_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i} \in \overline{\mathbb{T}}}$ such that:

(i) $(\gamma_{\mathbf{i}})_{\mathbf{i} \in \overline{\mathbb{T}}}$ is an i.i.d. collection of $\mathcal{G}$-valued random variables with law $\mathbb{P}[\gamma_{\mathbf{i}} = g] = |r|^{-1} r_g$ $(g \in \mathcal{G})$.

(ii) For each finite subtree $\mathbb{U} \subset \overline{\mathbb{T}}$ that contains the root, $(X_{\mathbf{i}})_{\mathbf{i} \in \partial \mathbb{U}}$ are i.i.d. with common law μ and independent of $(\gamma_{\mathbf{i}})_{\mathbf{i} \in \mathbb{U}}$.

(iii) $X_{\mathbf{i}} = \gamma_{\mathbf{i}}(X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}k_{\mathbf{i}}})$ $(\mathbf{i} \in \overline{\mathbb{T}})$.

Following Aldous and Bandyopadhyay (2005), we call such a collection of r.v.'s a *Recursive Tree Process (RTP)*.

A Recursive Distributional Equation

We can think of fixed points μ of the mean-field ODE (1) as a generalization of the invariant law of a (continuous-time) Markov chain.

Then a Recursive Tree Process (RTP) is a generalization of a stationary (continuous-time) Markov chain.

If we add independent exponentially distributed lifetimes $(\sigma_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}$ as before, then for each $t \geq 0$:

> $(X_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{T}_t}$ *are i.i.d. with common law* μ *and independent of the* σ*-field* $\mathcal{F}_t$ *of events measurable before time* t*.*

The n-variate ODE

Recall that the mean-field ODE (1) describes the Markov process $(X_t^N)_{t\geq 0}$ on the complete graph in the mean-field limit $N \to \infty$.

The Markov process $(X_t^N)_{t\geq 0}$ is defined in terms of a stochastic flow $(\mathbf{X}_{s,t}^N)_{s\leq t}$.

The stochastic flow $(\mathbf{X}_{s,t}^N)_{s\leq t}$ contains more information than the Markov process $(X_t^N)_{t\geq 0}$ alone; in particular, the stochastic flow provides us with a natural way of coupling processes with different initial states.

We would like to understand this coupling in the mean-field limit $N \to \infty$.

The n-variate ODE

For each measurable map $g : S^k \to S$ and $n \geq 1$, we define an *n-variate map* $g^{(n)} : (S^n)^k \to S^n$ by

$$g^{(n)}(x^1, \ldots, x^n) := \big(g(x^1), \ldots, g(x^n)\big) \qquad (x^1, \ldots, x^n \in S^k).$$

Let $\mathcal{G}$ and $(r_g)_{g \in \mathcal{G}}$ be as before. We will be interested in the *n-variate ODE*

$$\tfrac{\partial}{\partial t}\mu_t^{(n)} = \sum_{g \in \mathcal{G}} r_g \big\{ T_{g^{(n)}}(\mu_t^{(n)}) - \mu_t^{(n)} \big\} \qquad (t \geq 0),$$

that describes the mean-field limit of n coupled Markov processes, that are constructed from the same stochastic flow but have different initial states.

The n-variate ODE

Let $\mathcal{M}^1_{\mathrm{sym}}(S^n)$ be the space of all probability measures on S^n that are symmetric under a permutation of the coordinates.
For any $\mu \in \mathcal{M}^1(S)$, let $\mathcal{M}^1_{\mathrm{sym}}(S^n)_\mu$ be the set of all symmetric $\mu^{(n)}$ whose one-dimensional marginals are given by μ.
Let $S^n_{\mathrm{diag}} := \{(x^1, \ldots, x^n) \in S^n : x^1 = \cdots = x^n\}$.

Observations

- If $(\mu^{(n)}_t)_{t\geq 0}$ solves the n-variate ODE, then its m-dimensional marginals solve the m-variate ODE.
- $\mu^{(n)}_0 \in \mathcal{M}^1_{\mathrm{sym}}(S^n)$ implies $\mu^{(n)}_t \in \mathcal{M}^1_{\mathrm{sym}}(S^n)$ $(t \geq 0)$.
- If $\mu^{(n)}$ solves the n-variate ODE, then $\mu^{(n)}_0 \in \mathcal{M}^1_{\mathrm{sym}}(S^n)_\mu$ implies $\mu^{(n)}_t \in \mathcal{M}^1_{\mathrm{sym}}(S^n)_\mu$ $(t \geq 0)$.
- If $\mu^{(n)}_0$ is concentrated on S^n_{diag} then so is $\mu^{(n)}_t$ $(t \geq 0)$.

The n-variate ODE

In particular, these observations show that if $\mu^{(n)}$ solves the n-variate RDE, then its marginals must solve the RDE (2). Conversely, if μ solves the RDE (2) and X is a random variable with law μ, then

$$\overline{\mu}^{(n)} := \mathbb{P}\big[(X, \ldots, X) \in \cdot\,\big]$$

solves the n-variate RDE.

Question Are all fixed points of the n-variate RDE of this form?

The bivariate ODE for cooperative branching

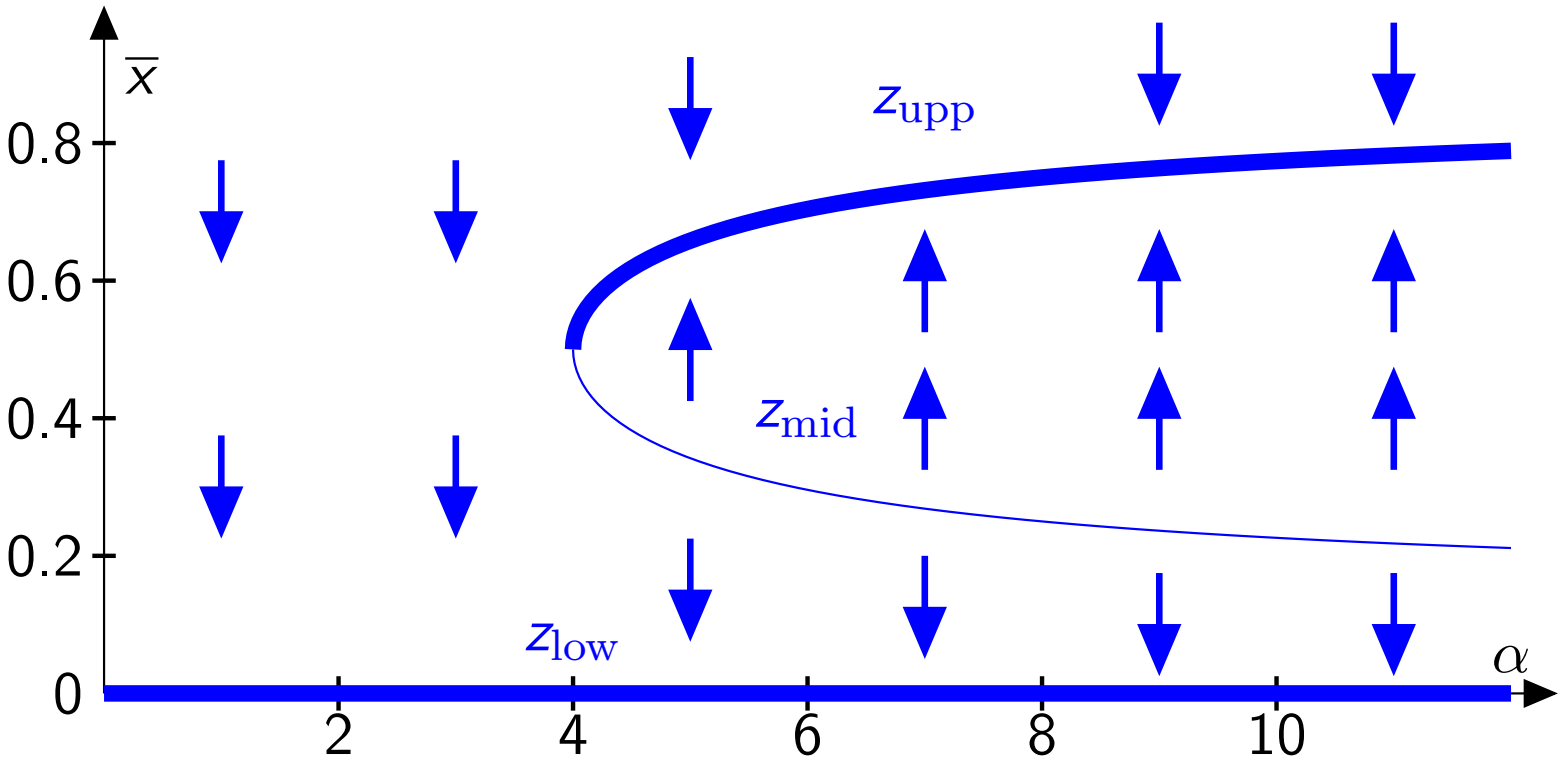
For our system with cooperative branching and deaths, set

$$z_{\rm low} := 0, \quad z_{\rm mid} := \tfrac{1}{2} - \sqrt{\tfrac{1}{4} - \tfrac{1}{\alpha}}, \quad \text{and} \quad z_{\rm upp} := \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - \tfrac{1}{\alpha}},$$

where $z_{\rm mid}$ and $z_{\rm upp}$ are only defined for $\alpha \geq 4$ and satisfy $z_{\rm mid} = z_{\rm upp}$ for $\alpha = 4$ and $z_{\rm mid} < z_{\rm upp}$ for $\alpha > 4$.

Let $\mu_{\rm low}, \mu_{\rm mid}, \mu_{\rm upp}$ be the measures on $\{0, 1\}$ with these intensities. Then $\mu_{\rm low}, \mu_{\rm mid}, \mu_{\rm upp}$ are all fixed points of the mean-field ODE (1).

The bivariate ODE for cooperative branching



Fixed points of $\frac{\partial}{\partial t}\overline{X}_t = F_\alpha(\overline{X}_t)$ for different values of α.

The bivariate ODE for cooperative branching

Proposition The measures $\overline{\mu}^{(2)}_{\rm low}, \overline{\mu}^{(2)}_{\rm mid}, \overline{\mu}^{(2)}_{\rm upp}$ defined as

$$\overline{\mu}^{(2)}_{\rm low} := \mathbb{P}\big[(X,X) \in \,\cdot\,\big] \quad \text{with} \quad \mathbb{P}[X \in \,\cdot\,] = \mu_{\rm low}$$

etc. are fixed points of the bivariate ODE. In addition, for $\alpha > 4$, there exists one more fixed point $\underline{\mu}^{(2)}_{\rm mid} \in \mathcal{M}^1_{\rm sym}(\{0,1\}^2)$ that has marginals $\mu_{\rm mid}$ but differs from $\overline{\mu}^{(2)}_{\rm mid}$.

Any solution $(\mu^{(2)}_t)_{t\geq 0}$ to the bivariate ODE with $\mu^{(2)}_0 \in \mathcal{M}^1_{\rm sym}(\{0,1\})_{\mu_{\rm mid}}$ and $\mu^{(2)}_0 \neq \overline{\mu}^{(2)}_{\rm mid}$ satisfies

$$\mu^{(2)}_t \underset{t\to\infty}{\Longrightarrow} \underline{\mu}^{(2)}_{\rm mid}.$$

The bivariate ODE for cooperative branching

Interpretation For large N, let $x, x' \in \{0,1\}^N$ be initial states such that

$$\frac{1}{N}\sum_{i=1}^{N} x(i) = z_{\mathrm{mid}} = \sum_{i=1}^{N} \frac{1}{N} x'(i),$$

and

$$\frac{1}{N}\sum_{i=1}^{N} 1_{\{x(i) \neq x'(i)\}} > 0,$$

but arbitrarily small. Then

$$\frac{1}{N}\sum_{i=1}^{N} 1_{\{\mathbf{X}^N_{0,t}(x)(i) \neq \mathbf{X}^N_{0,t}(x')(i)\}}$$

converges in probability as $N \to \infty$ and then $t \to \infty$ to $\underline{\mu}^{(2)}_{\mathrm{mid}}(\{(0,1),(1,0)\}) > 0$. In particular, the evolution under the stochastic flow is *unstable* in the sense that small differences in the initial states are multiplied, provided the initial density is z_{mid}.

Endogeny

Aldous and Bandyopadhyay (2005) call a Recursive Tree Process (RTP) $(\gamma_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}$ *endogenous* if

$X_\emptyset$ is measurable w.r.t. the σ-field generated by $(\gamma_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}$.

Theorem [AB '05, MSS '18] For any solution μ to the RDE (2), the following statements are equivalent.

(i) The RTP corresponding to μ is endogenous.

(ii) The measure $\overline{\mu}^{(2)}$ is the only solution of the bivariate RDE in the space $\mathcal{M}^1_{\mathrm{sym}}(S^2)_\mu$.

(iii) Solutions to the n-variate ODE satisfy $\mu_t^{(n)} \underset{t\to\infty}{\Longrightarrow} \overline{\mu}^{(2)}$ for all $\mu_0^{(n)} \in \mathcal{M}^1_{\mathrm{sym}}(S^2)_\mu$ and $n \geq 1$.

Endogeny

In our example of a system with cooperative branching and deaths, the RTPs corresponding to $\mu_{\rm low}$ and $\mu_{\rm upp}$ are endogenous, but for $\alpha > 4$, the RTP corresponding to $\mu_{\rm mid}$ is *not endogenous*.

Proposition [AB '05] Let S be a finite partially ordered set with minimal and maximal elements $0, 1$. Assume that all maps $m \in \mathcal{G}$ are monotone. Let $(\mu_t^0)_{t\geq 0}$ and $(\mu_t^1)_{t\geq 0}$ be solutions to the mean-field ODE with initial states $\mu_0^0 = \delta_0$ and $\mu_0^1 = \delta_1$. Then there exist solutions $\underline{\nu}$ and $\overline{\nu}$ to the RDE (2) such that

$$\mu_t^0 \underset{t\to\infty}{\Longrightarrow} \underline{\nu} \quad \text{and} \quad \mu_t^1 \underset{t\to\infty}{\Longrightarrow} \overline{\nu}.$$

Moreover, the RTPs corresponding to $\underline{\nu}$ and $\overline{\nu}$ are endogenous.

Moment measures

Let ξ be a $\mathcal{M}^1(S)$-valued random variable, i.e., a random probability measure on S, and let $\rho \in \mathcal{M}^1(\mathcal{M}^1(S))$ denote its law. Conditional on ξ, let $X^1, \ldots, X^n$ be independent with law ξ. Then

$$\rho^{(n)} := \mathbb{P}\big[(X^1, \ldots, X^n) \in \cdot\,\big] = \mathbb{E}\big[\underbrace{\xi \otimes \cdots \otimes \xi}_{n \text{ times}}\big]$$

is called the *n-th moment measure* of ξ.

Then $\rho^{(n)} \in \mathcal{M}^1_{\mathrm{sym}}(S^n)$ for each $\rho \in \mathcal{M}^1(\mathcal{M}^1(S))$.
For $n = \infty$, De Finetti's theorem says that each element of $\mathcal{M}^1_{\mathrm{sym}}(S^n)$ is of the form $\rho^{(n)}$ for some $\rho \in \mathcal{M}^1(\mathcal{M}^1(S))$.

Using this idea, we seek to define a *higher level* ODE that corresponds to the n-variate ODE with $n = \infty$.

The higher-level ODE

Recall that for each measurable map $g : S^k \to S$, we have defined a measurable map $\check{g} : \mathcal{M}^1(S)^k \to \mathcal{M}^1(S)$ by

$$\check{g}(\mu_1, \ldots, \mu_k) := \mathbb{P}\big[g(X_1, \ldots, X_k) \in \cdot\,\big]$$
$$\text{where} \quad X_1, \ldots, X_k \text{ are indep. with } \mathbb{P}[X_i \in \cdot\,] = \mu_i.$$

In particular, $T_g : \mathcal{M}^1(S) \to \mathcal{M}^1(S)$ is defined by

$$T_g(\mu) := \check{g}(\mu, \ldots, \mu).$$

The *higher level ODE* is the equation

$$\tfrac{\partial}{\partial t}\rho_t = \sum_{g \in \mathcal{G}} \big\{ T_{\check{g}}(\rho_t) - \rho_t \big\} \qquad (t \geq 0).$$

This differs from the mean-field ODE (1) in the sense that T_g is replaced by $T_{\check{g}}$ and ρ_t takes values in $\mathcal{M}^1(\mathcal{M}^1(S))$.

The higher-level ODE

Lemma If $(\rho_t)_{t\geq 0}$ solves the higher level ODE, then its n-th moment measures solve the n-variate ODE.

Below, we equip the space $\mathcal{M}^1(\mathcal{M}^1(S))_\mu$ of all $\rho \in \mathcal{M}^1(\mathcal{M}^1(S))$ with first moment measure $\rho^{(1)} = \mu$ with the *convex order* $\rho_1 \leq_{\rm cv} \rho_2$, defined as

$$\int \phi \, {\rm d}\rho_1 \leq \int \phi \, {\rm d}\rho_2 \quad \forall \text{ convex bounded contin. } \phi : \mathcal{M}^1(S) \to \mathbb{R}.$$

Theorem [MSS '18] Let μ be a fixed point of the mean-field ODE (1). Then the higher-level ODE has fixed points $\underline{\mu}, \overline{\mu} \in \mathcal{M}^1(\mathcal{M}^1(S))_\mu$ that are minimal and maximal with respect to the convex order.

Remark One has $\rho_1 \leq_{\rm cv} \rho_2$ iff there exists an S-valued random variable X on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and sub-σ-fields $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$ such that $\rho_i = \mathbb{P}\big[\mathbb{P}[X \in \cdot \,|\mathcal{F}_i] \in \cdot\,\big]$ $(i = 1, 2)$.

The higher-level ODE

Theorem [MSS '18] Let μ be a fixed point of the mean-field ODE (1). Let $(\gamma_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}$ be the RTP corresponding to μ. Set

$$\xi_{\mathbf{i}} := \mathbb{P}\big[X_{\mathbf{i}} \in \cdot \,|\, (\gamma_{\mathbf{ij}})_{\mathbf{j}\in\mathbb{T}}\big].$$

Then

$$(\check{\gamma}_{\mathbf{i}}, \xi_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}} \quad \text{and} \quad (\check{\gamma}_{\mathbf{i}}, \delta_{X_{\mathbf{i}}})_{\mathbf{i}\in\overline{\mathbb{T}}}$$

are RTPs corresponding to the fixed points $\underline{\mu}, \overline{\mu}$ of the higher-level ODE. The original RTP is endogenous if and only if $\underline{\mu} = \overline{\mu}$.

Remark 1 $\underline{\mu}$ and $\overline{\mu}$ correspond to minimal and maximal knowledge about $X_\emptyset$. The former describes the knowledge contained in $(\gamma_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}$, the latter represents perfect knowledge.

Remark 2 $\overline{\mu}^{(n)} := \mathbb{P}\big[(X, \ldots, X) \in \cdot\,\big]$ in line with earlier notation.

The higher-level ODE

In our example of a system with cooperative branching and deaths, we identify

$$\mathcal{M}^1(\{0,1\}) \ni \mu \mapsto \mu(\{1\}) \in [0,1]$$

and correspondingly $\mathcal{M}^1(\mathcal{M}^1(\{0,1\})) \cong \mathcal{M}^1[0,1]$. If $g = \text{coop}$, then $\check{g} : [0,1]^3 \to [0,1]$ is given by

$$\check{g}(\omega_1, \omega_2, \omega_3) := \mathbb{P}[X_1 \vee (X_2 \wedge X_3) = 1] \quad \text{with} \quad \mathbb{P}[X_i = 1] = \omega_i,$$

which gives

$$\check{g}(\omega_1, \omega_2, \omega_3) = \omega_1 + (1 - \omega_1)\omega_2\omega_3$$

and, for any $\rho \in \mathcal{M}^1[0,1]$,

$$T_{\check{g}}(\rho) = \mathbb{P}\big[\omega_1 + (1-\omega_1)\omega_2\omega_3 \in \cdot\,\big] \quad \text{with} \quad \omega_1, \omega_2, \omega_3 \text{ i.i.d. with law } \rho.$$

The higher-level ODE

Proposition For our system with cooperative branching and deaths, for $\alpha > 4$, the higher-level ODE

$$\frac{\partial}{\partial t}\rho_t = \alpha\big\{T_{\check{\mathbf{g}}}(\rho) - \rho\big\} + \big\{\delta_0 - \rho\big\}$$

has precisely four fixed points. Three trivial fixed points of the form

$$\rho = (1-z)\delta_0 + z\delta_1 \quad \text{with} \quad z = z_{\rm low}, z_{\rm mid}, z_{\rm upp},$$

and a nontrivial fixed point $\underline{\mu}_{\rm mid}$ which is the law of the $[0,1]$-valued random variable

$$\mathbb{P}\big[X_\emptyset = 1 \,\big|\, (\gamma_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}\big]$$

where $(\gamma_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\overline{\mathbb{T}}}$ is the RTP with $\mathbb{P}[X_\emptyset = 1] = z_{\rm mid}$.

Numerical results

The measure $\underline{\mu}_{\mathrm{mid}}$ should be the limit of measures μ_n inductively defined as $\mu_0 = \delta_{z_{\mathrm{mid}}}$ and

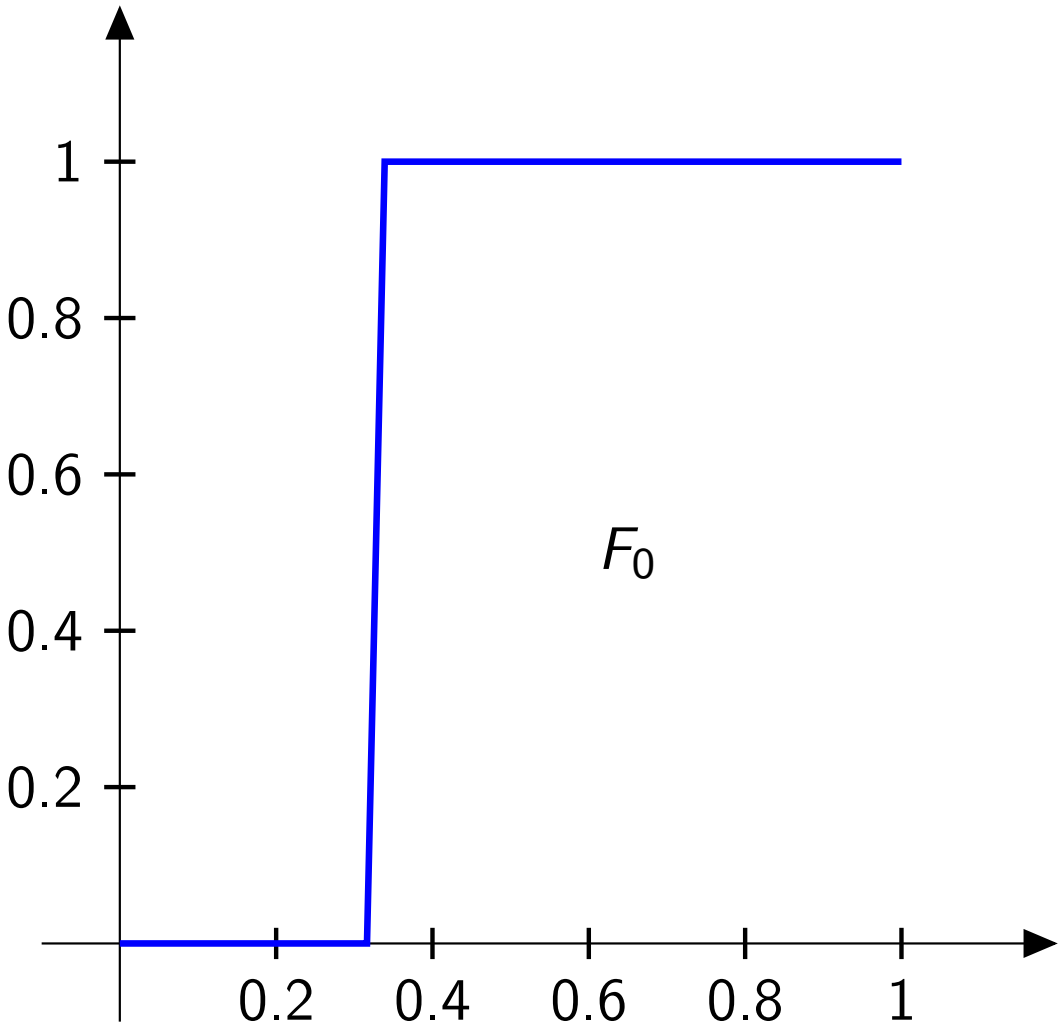
$$\mu_n = \frac{\alpha}{\alpha+1} T_{\check{g}}(\mu_{n-1}) + \frac{1}{\alpha+1}\delta_0.$$

We plot the distribution functions

$$F_n(s) := \mu_n\big([0,s]\big) \qquad \big(s \in [0,1]\big)$$

for the parameters $\alpha = 9/2$, $z_{\mathrm{mid}} = 1/3$, $z_{\mathrm{upp}} = 2/3$ and various values of n.

Numerical results



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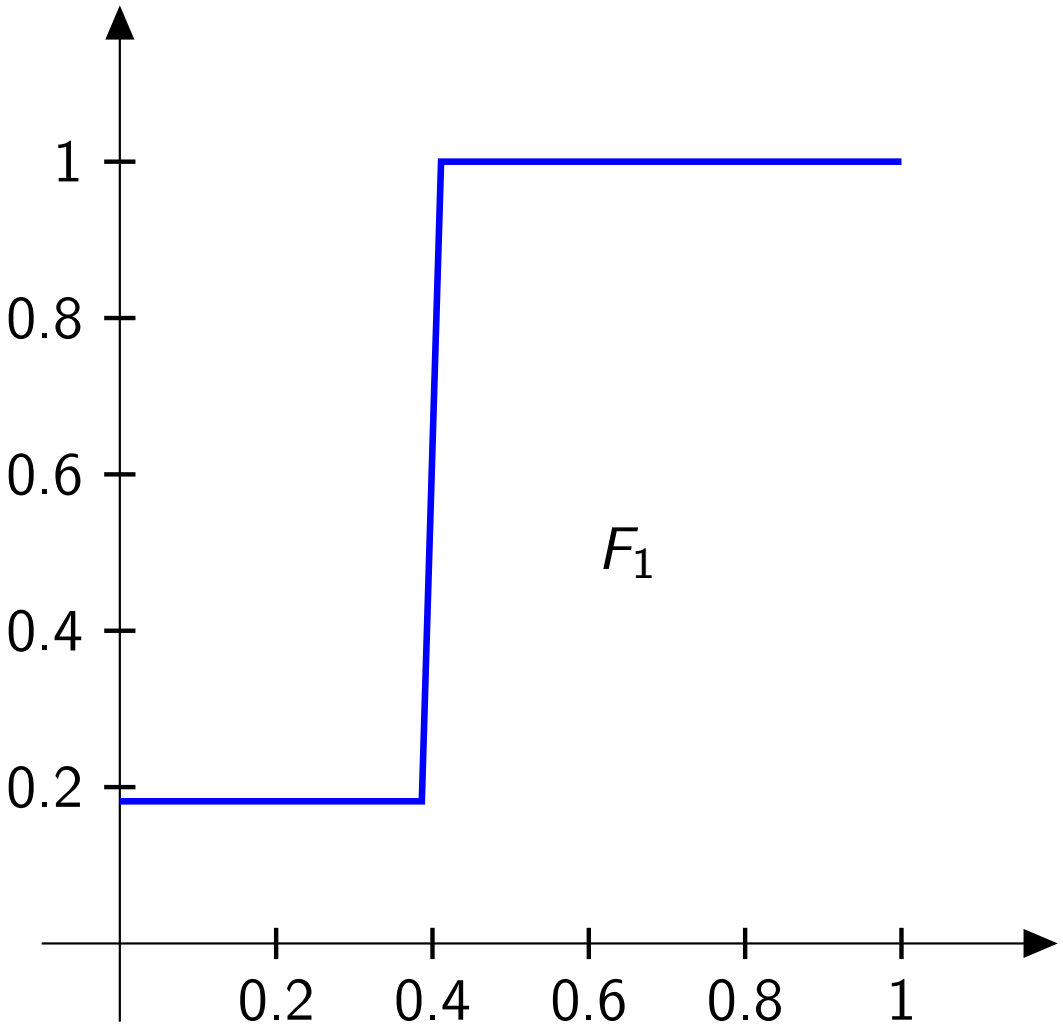

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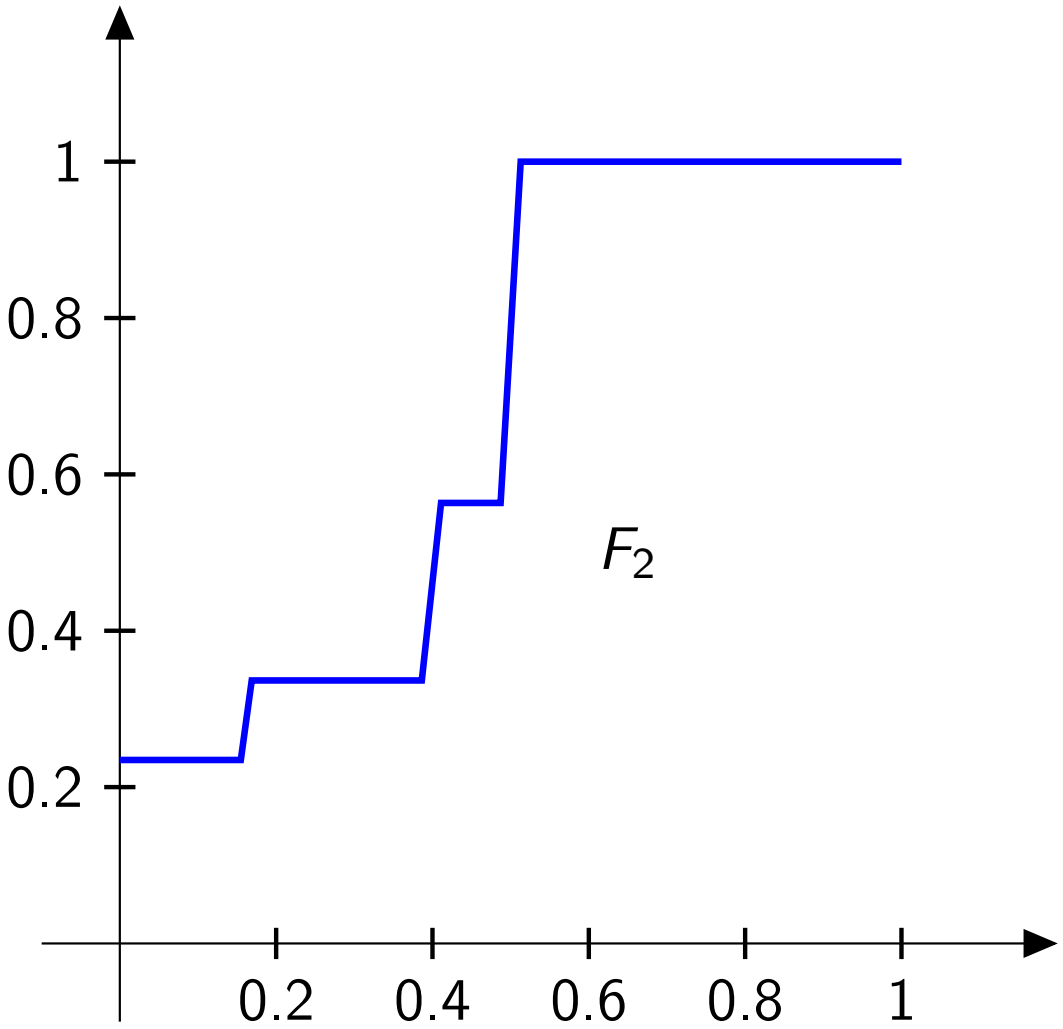

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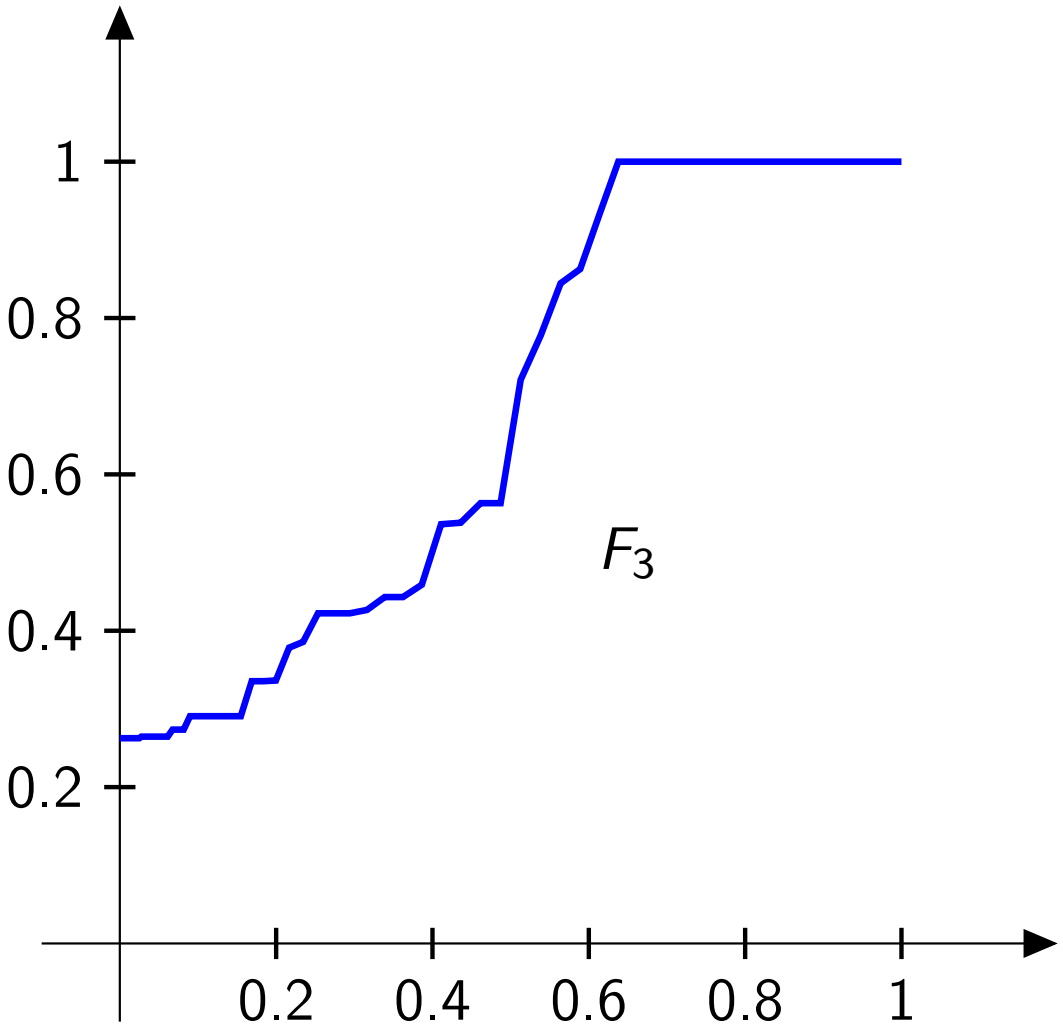

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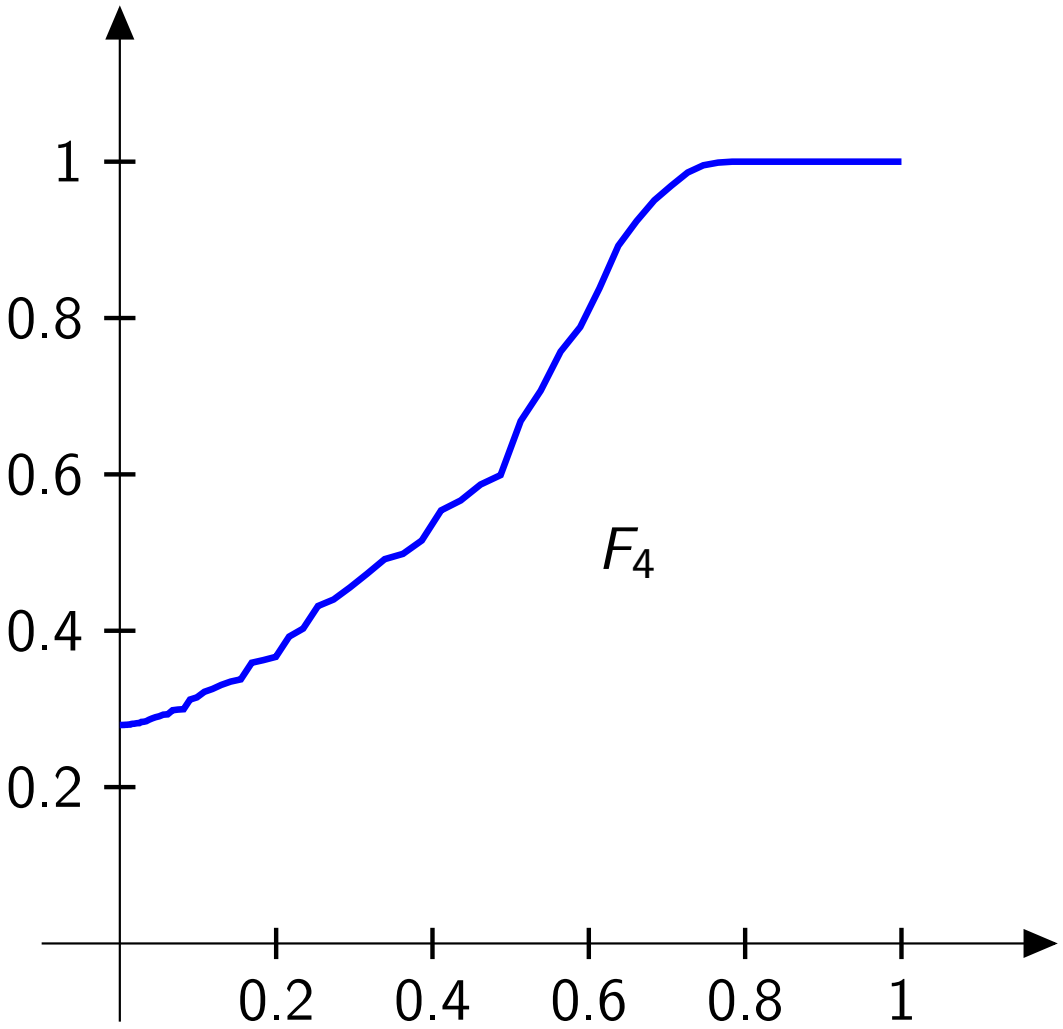

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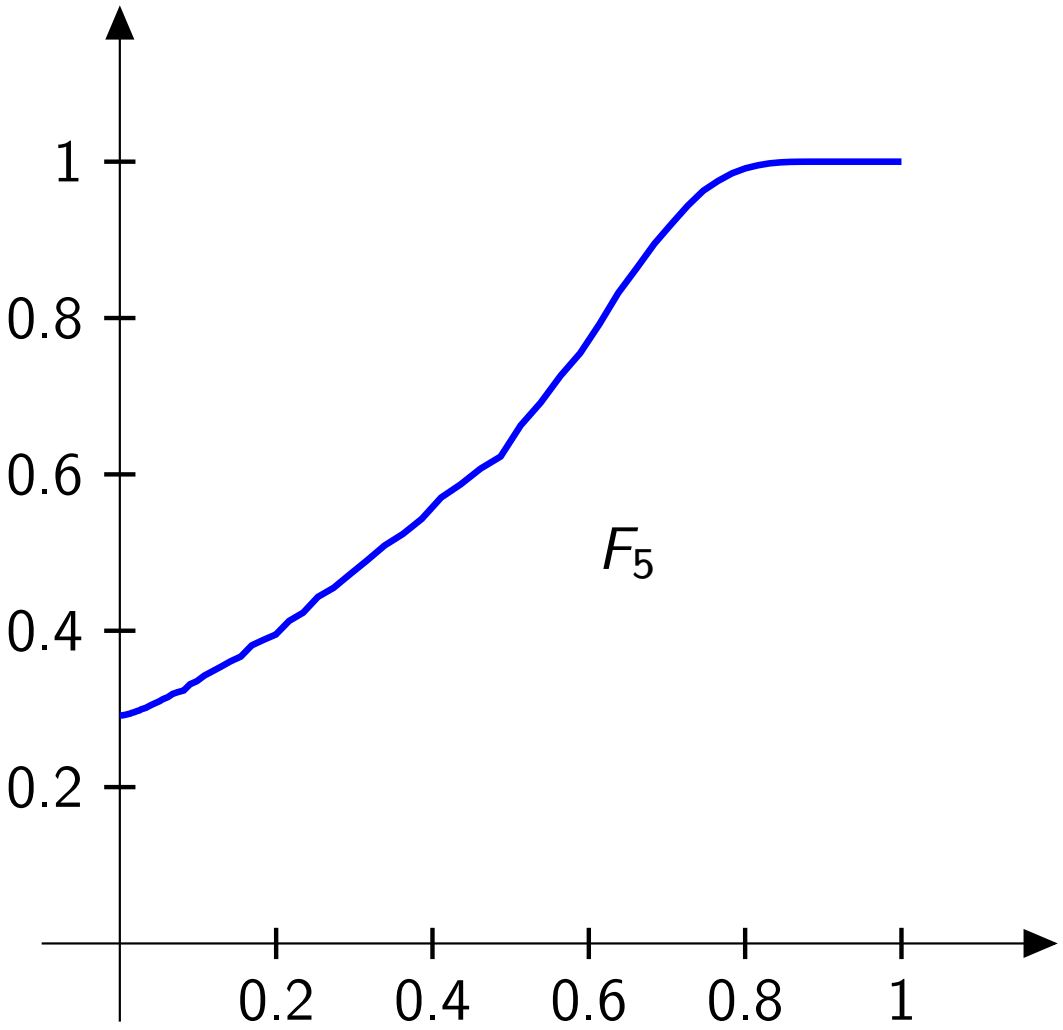

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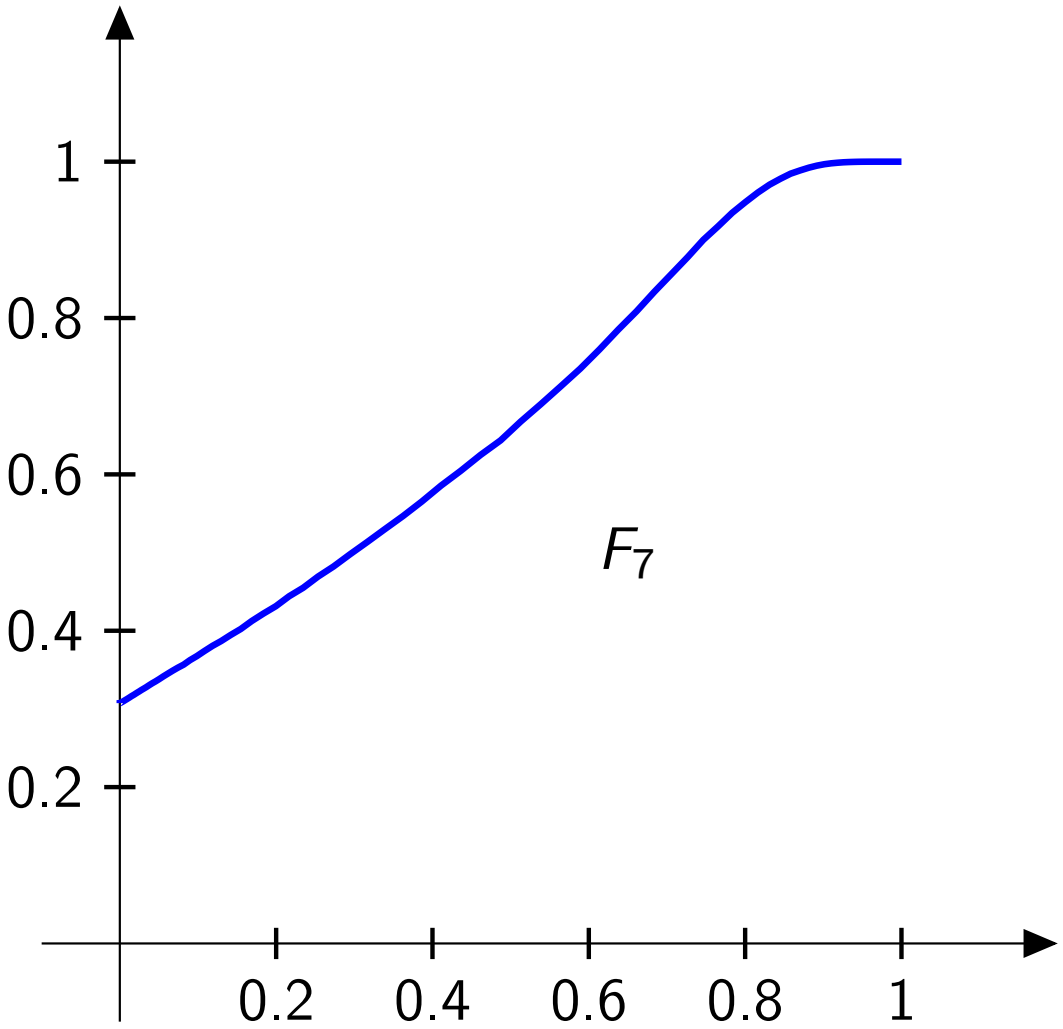

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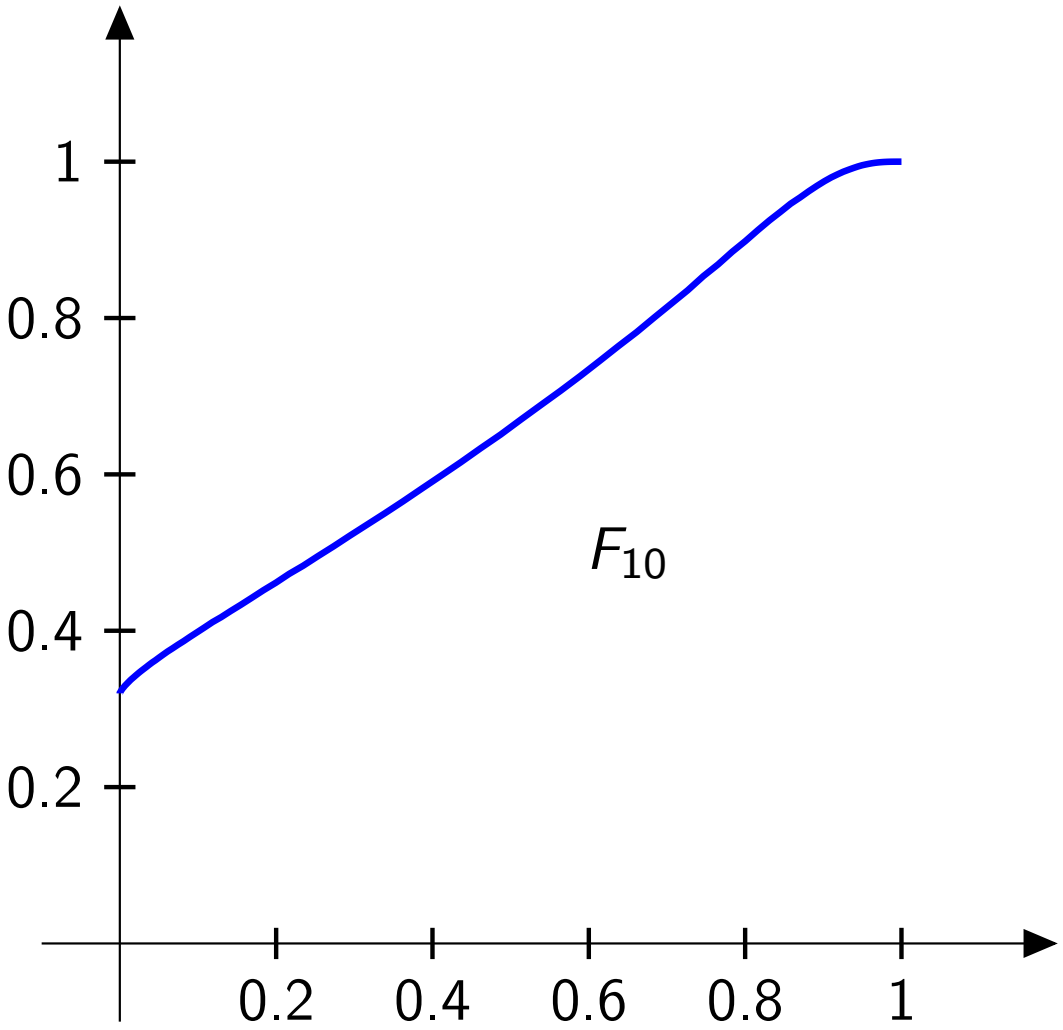

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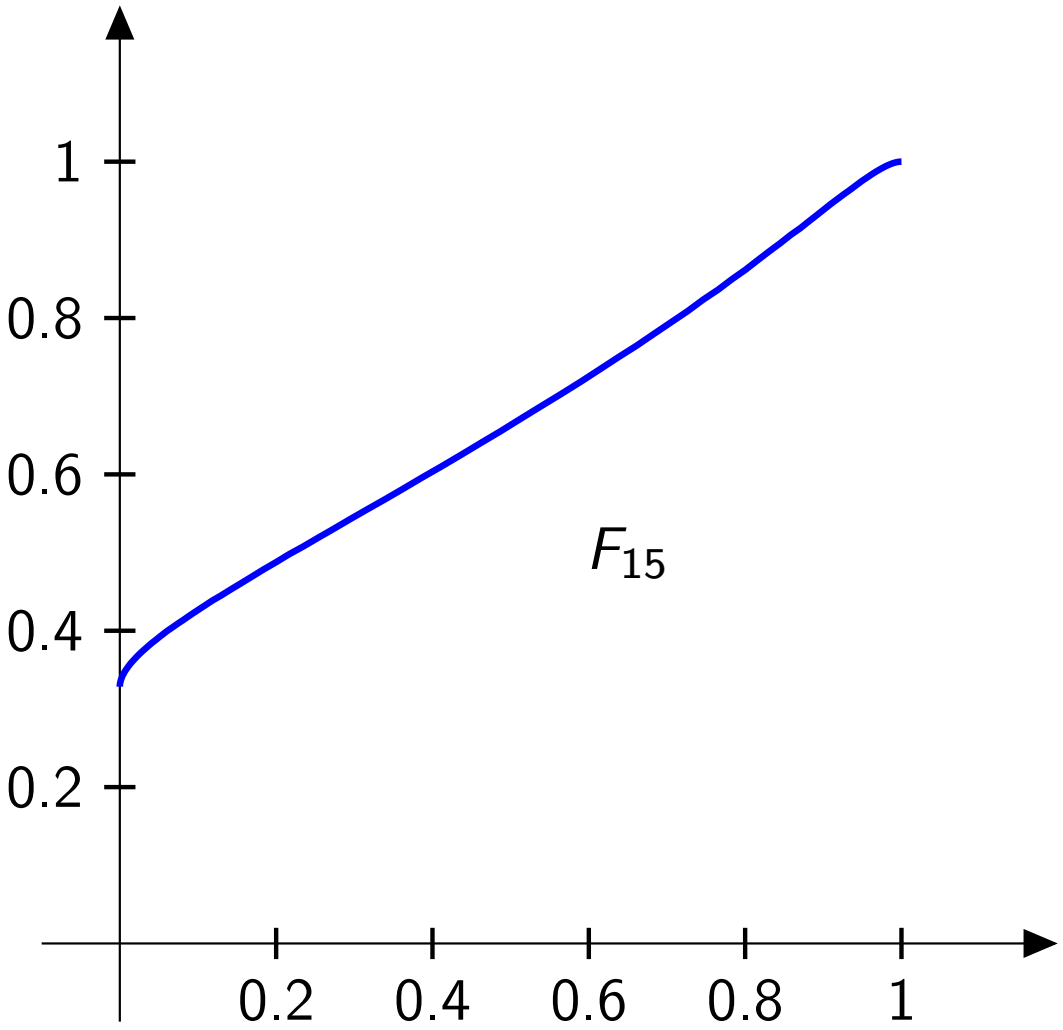

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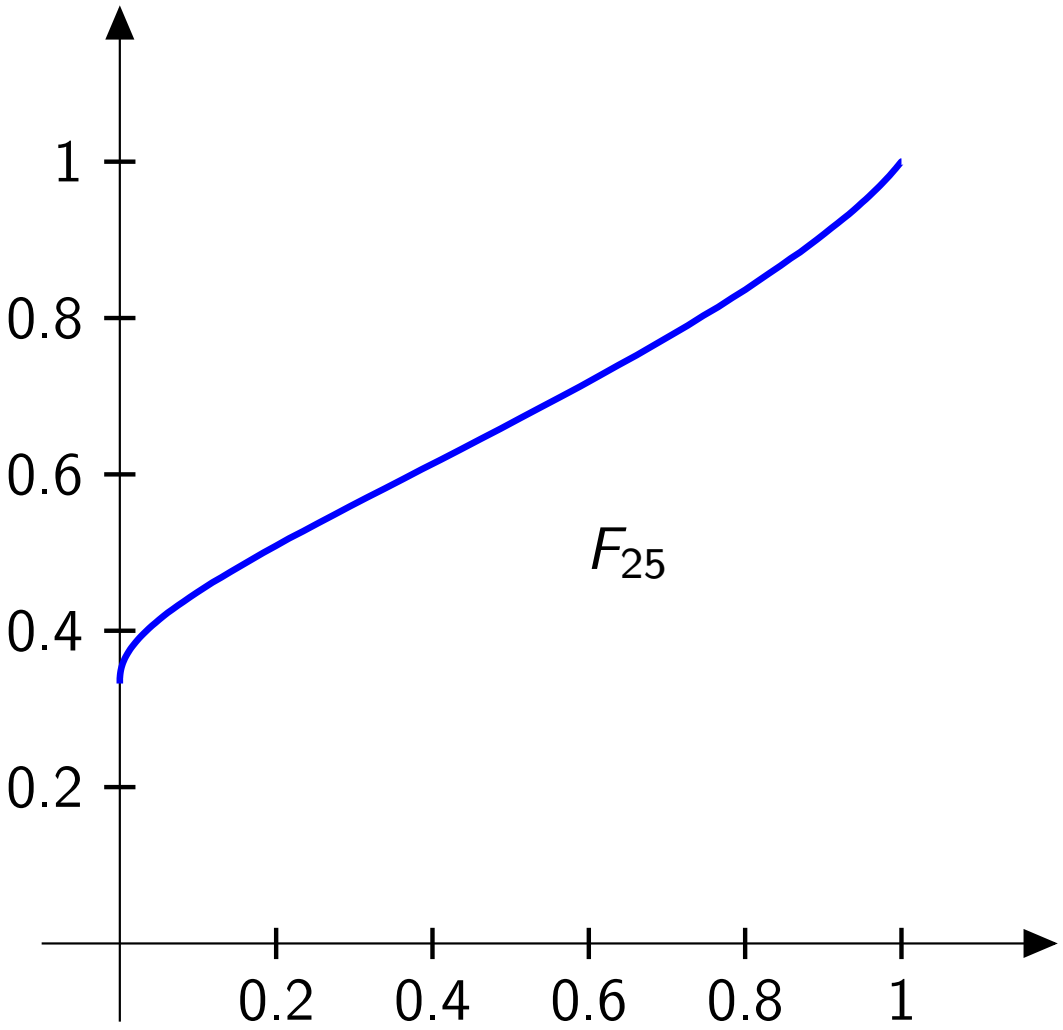

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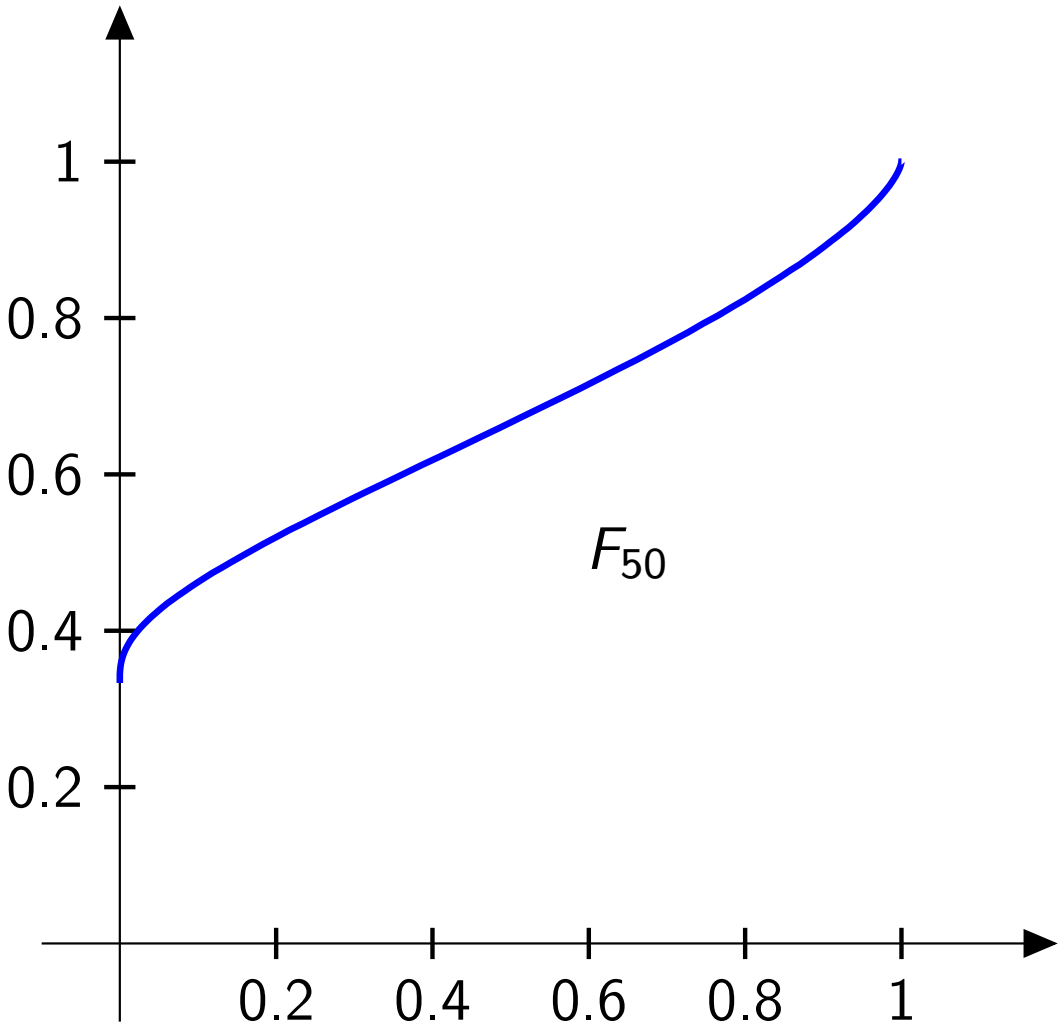

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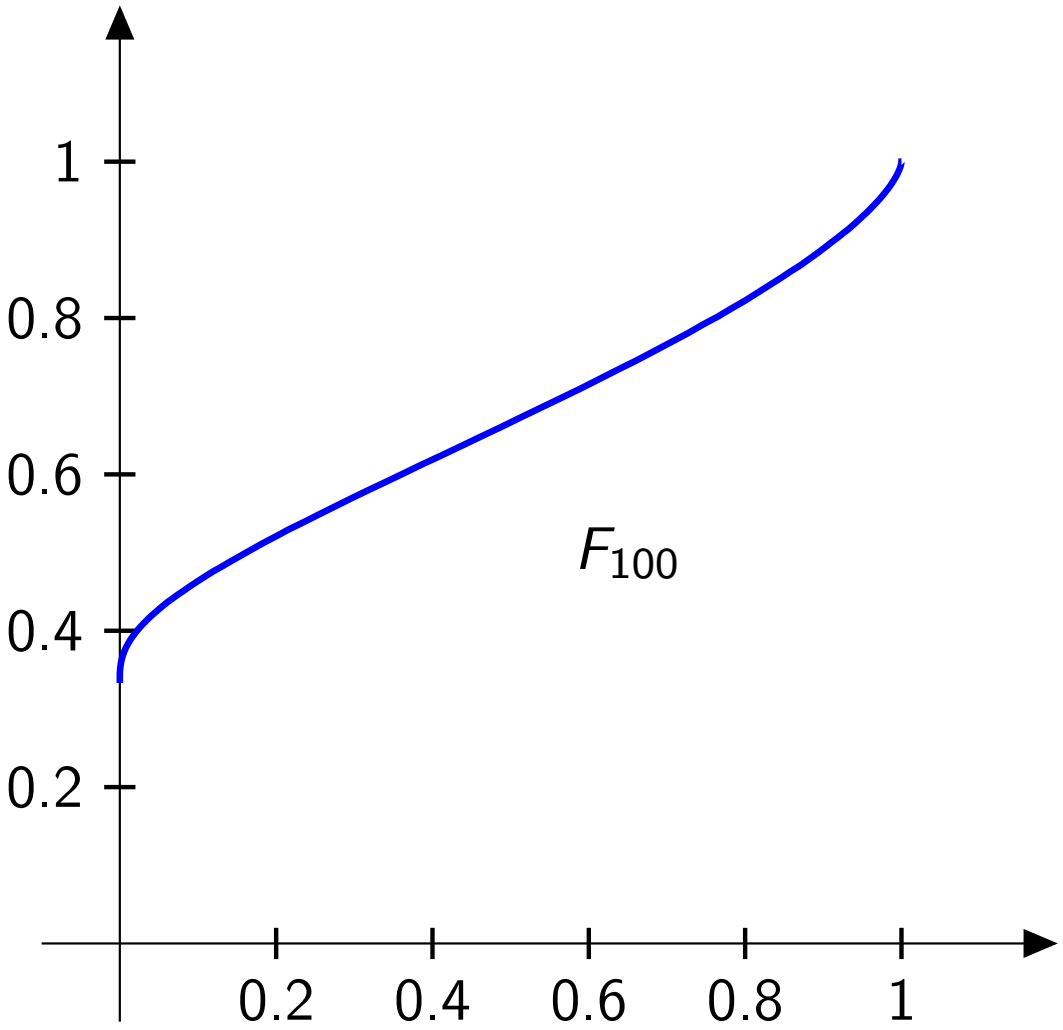

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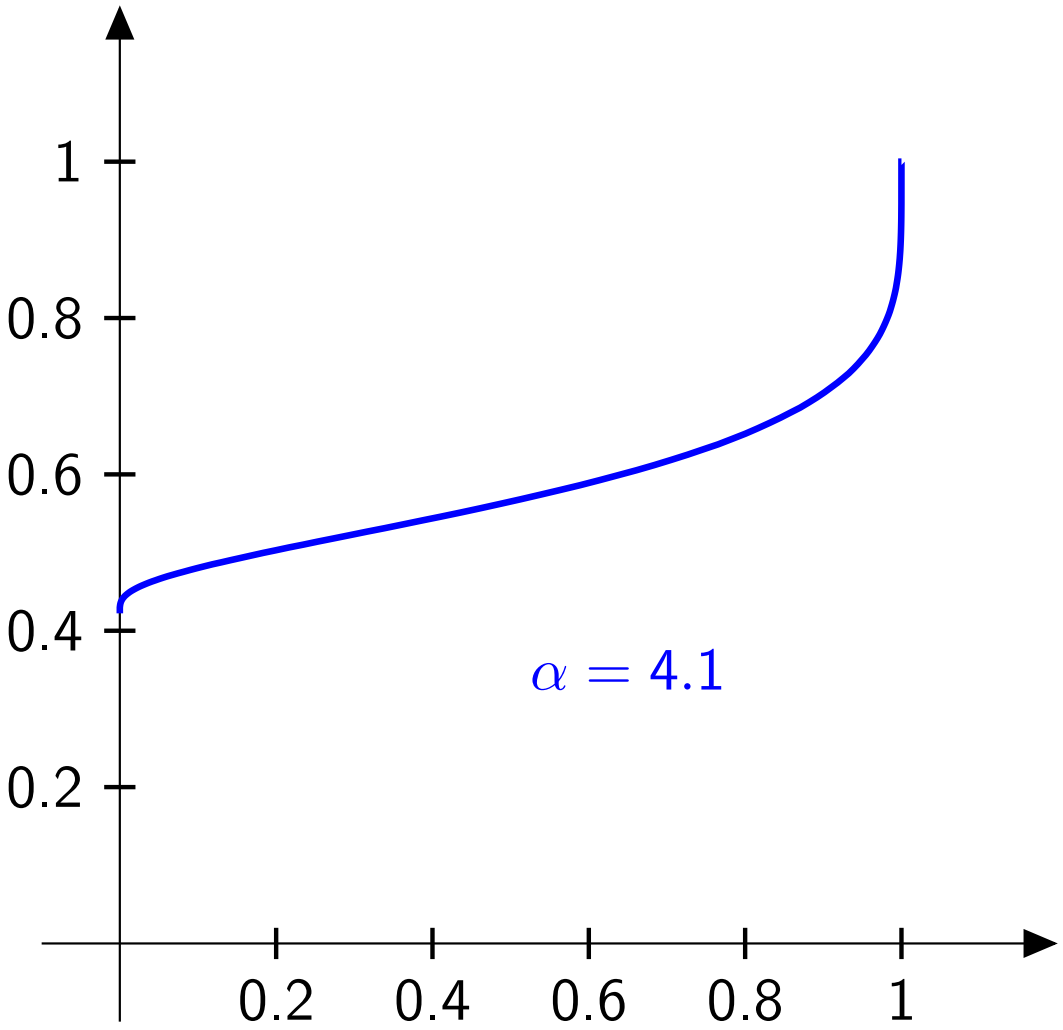

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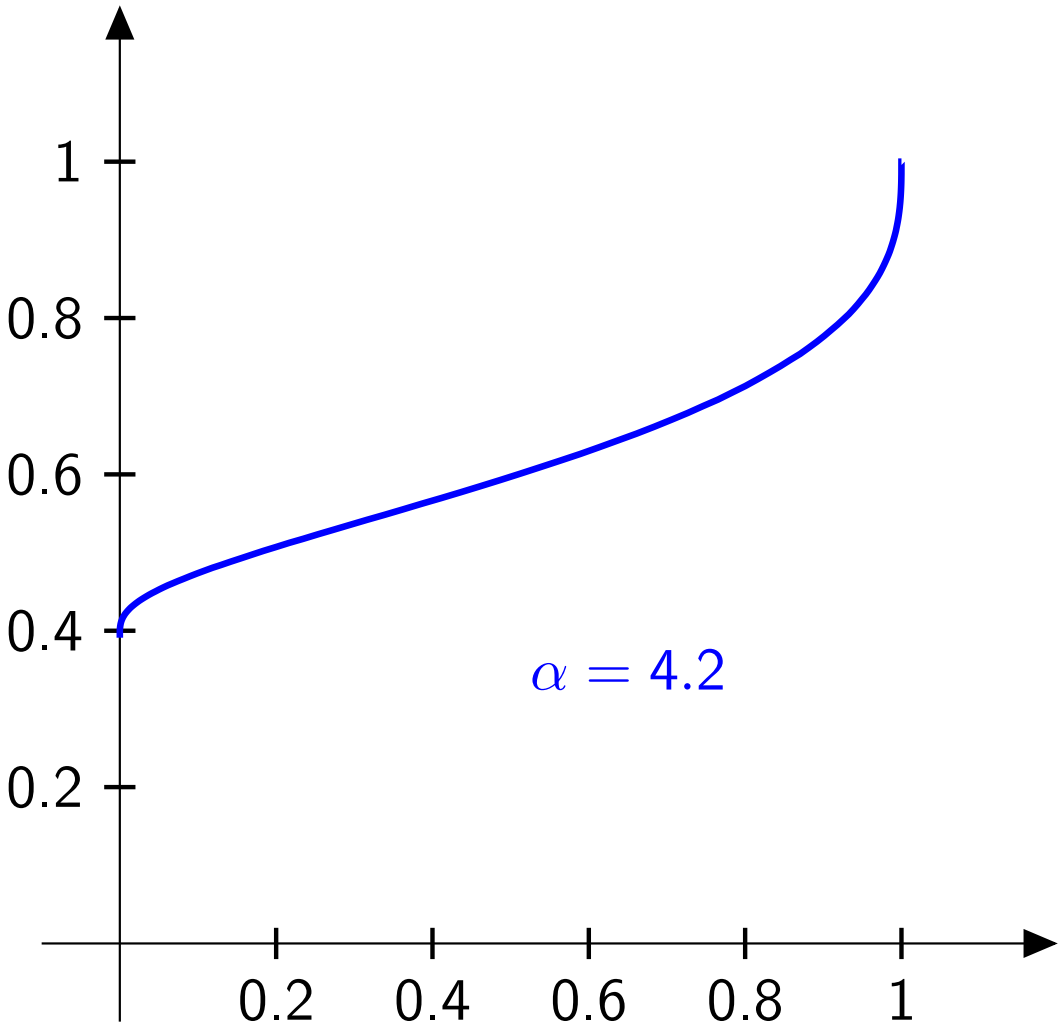

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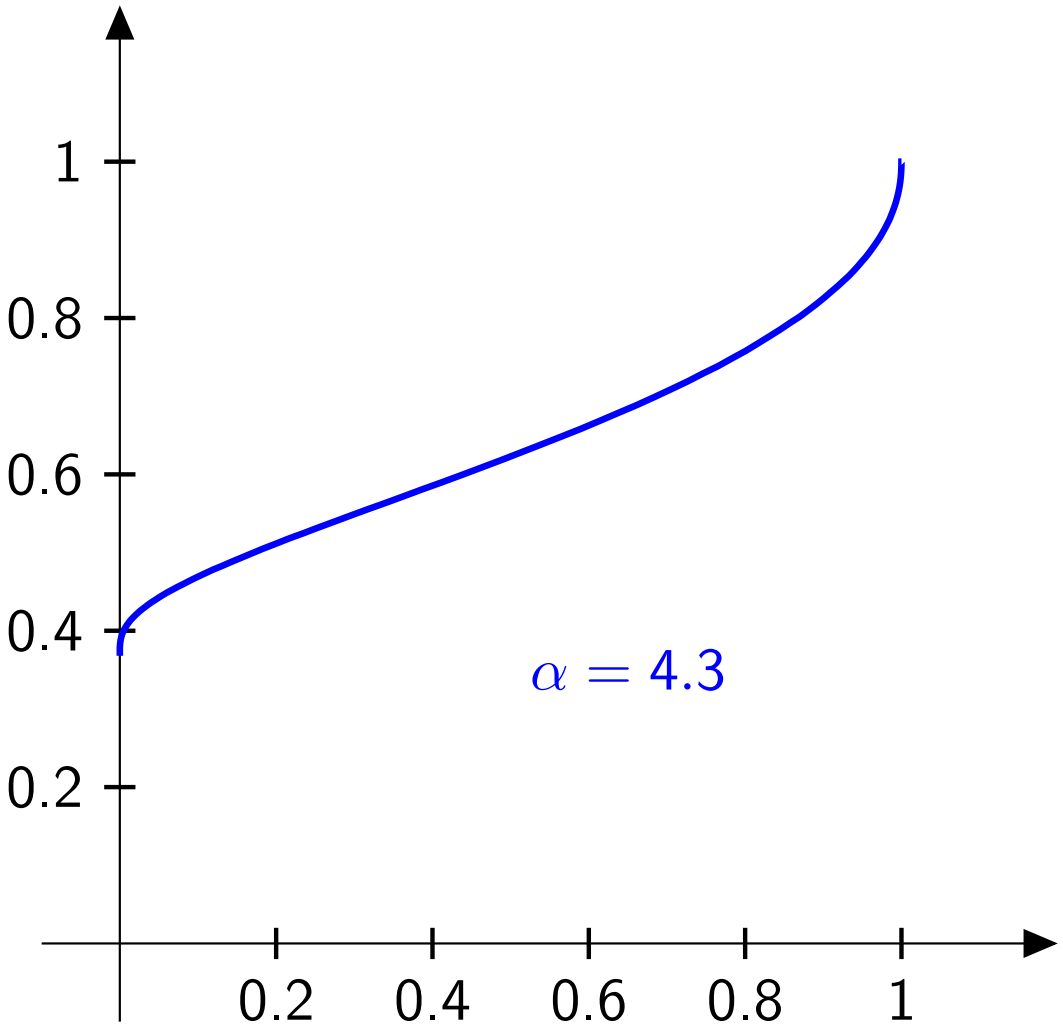

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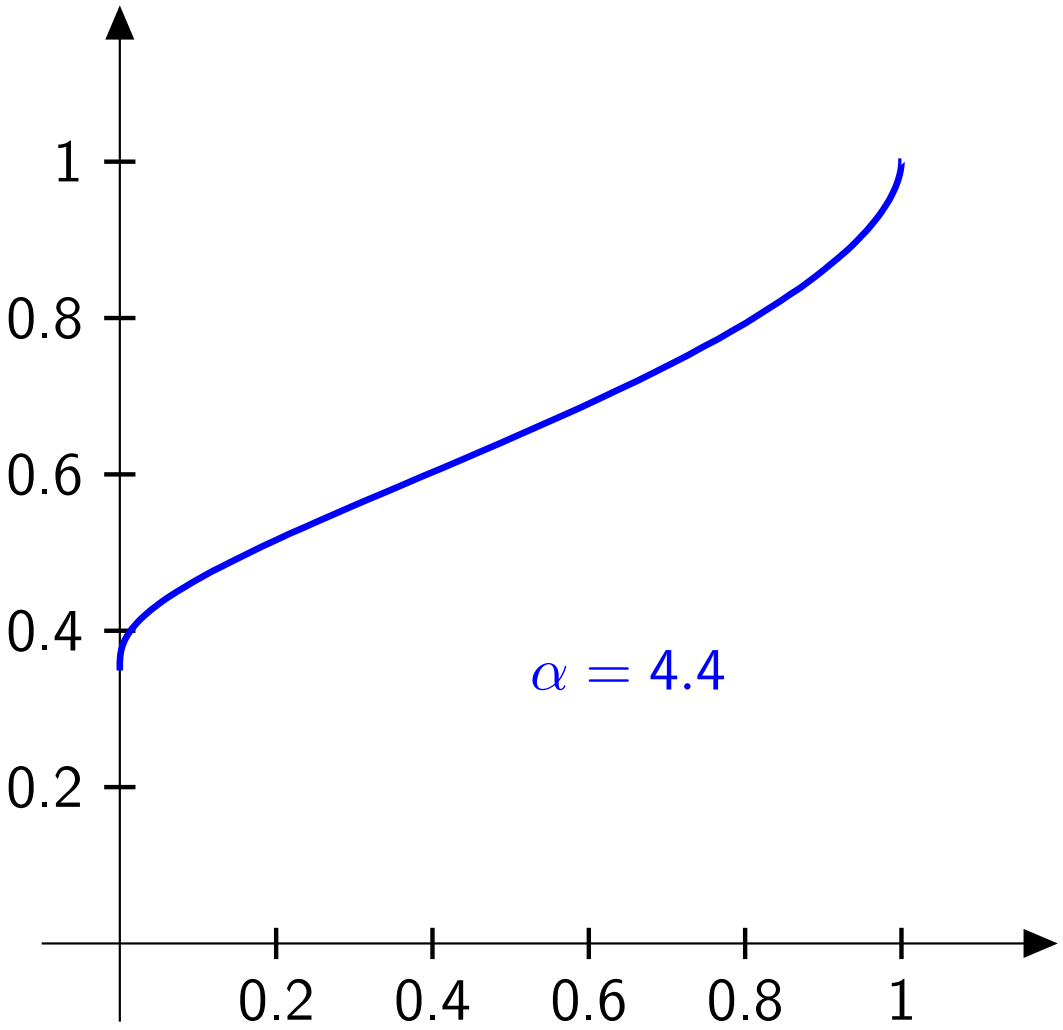

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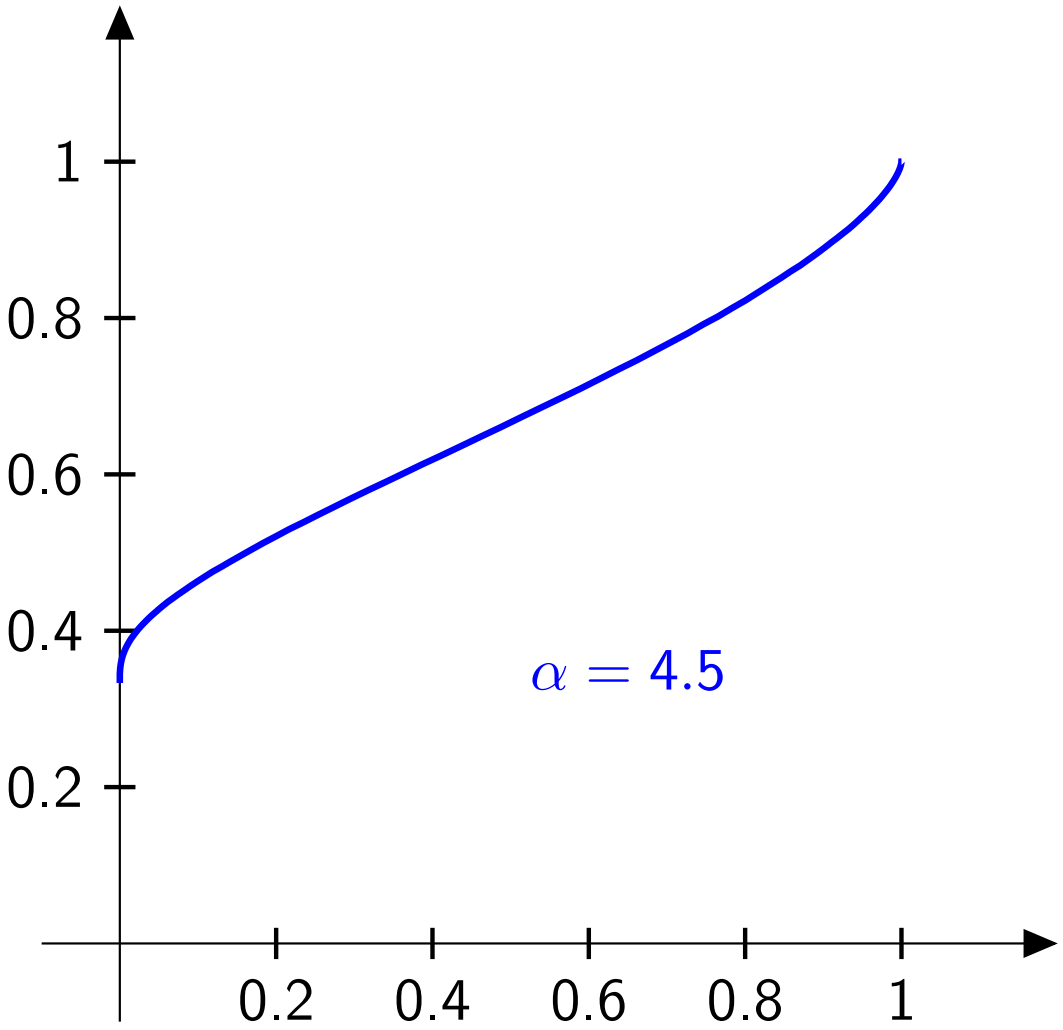

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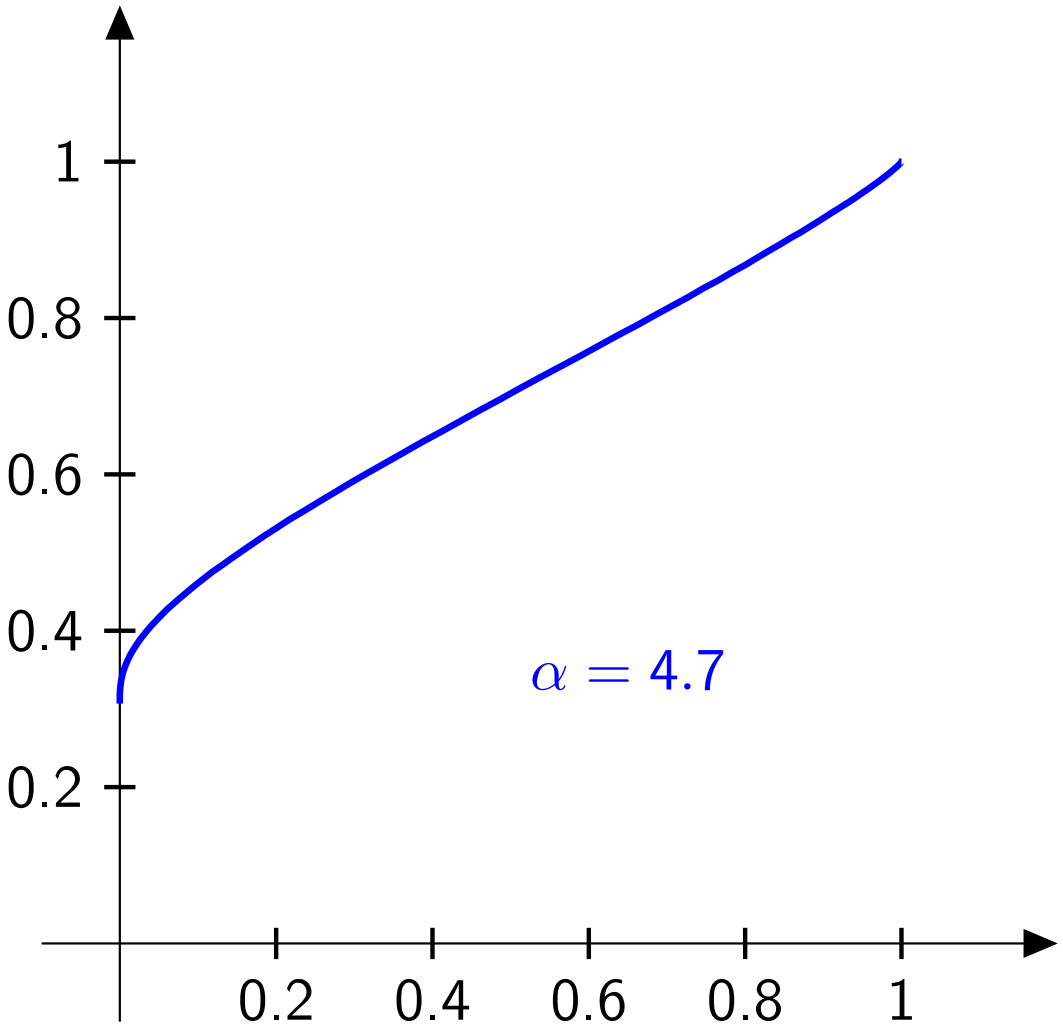

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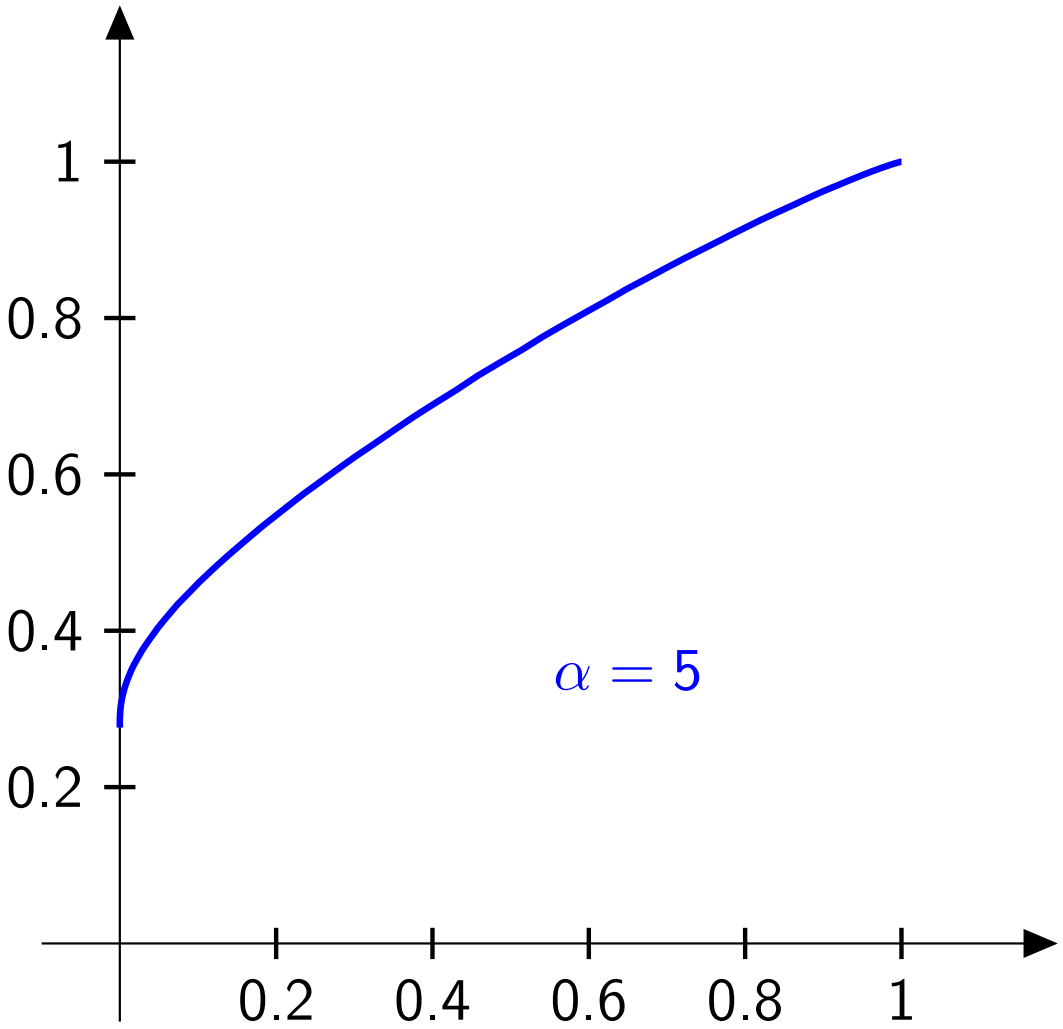

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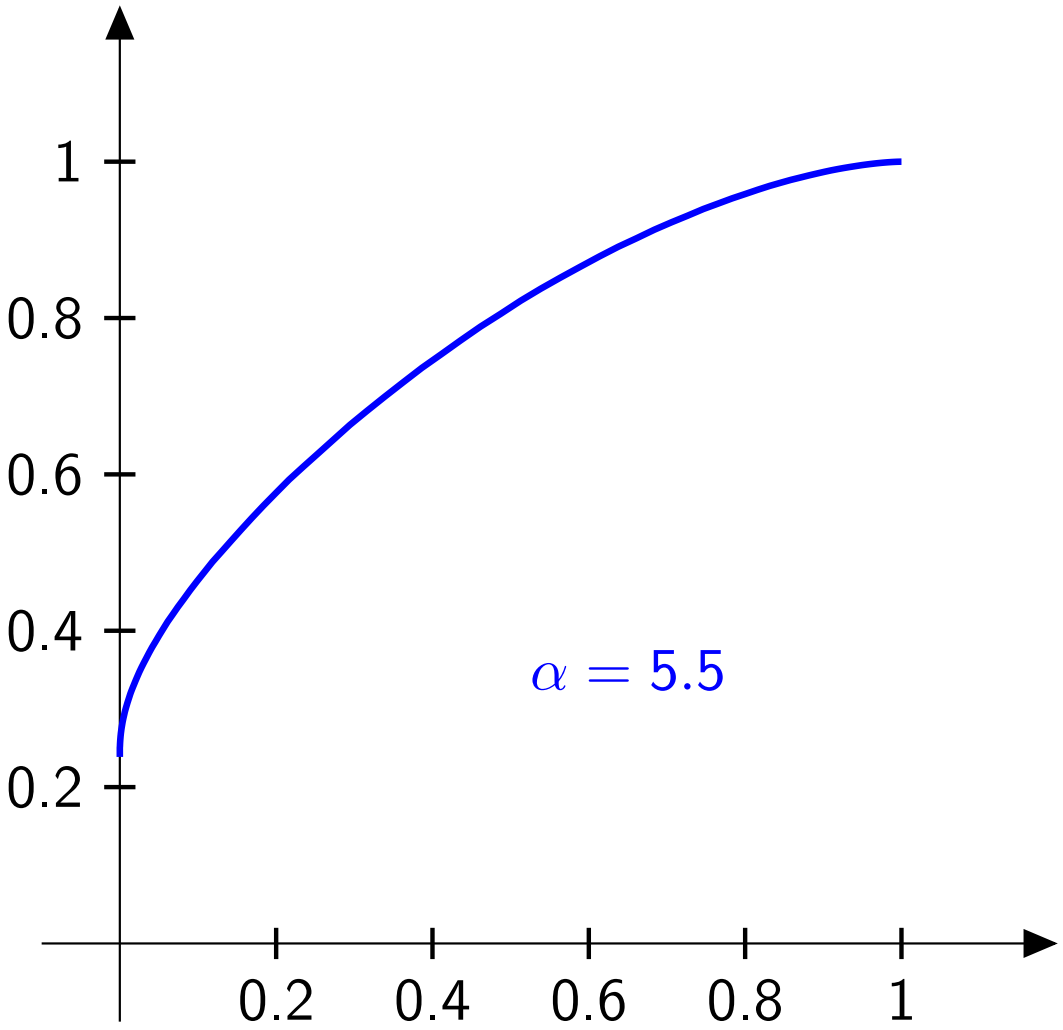

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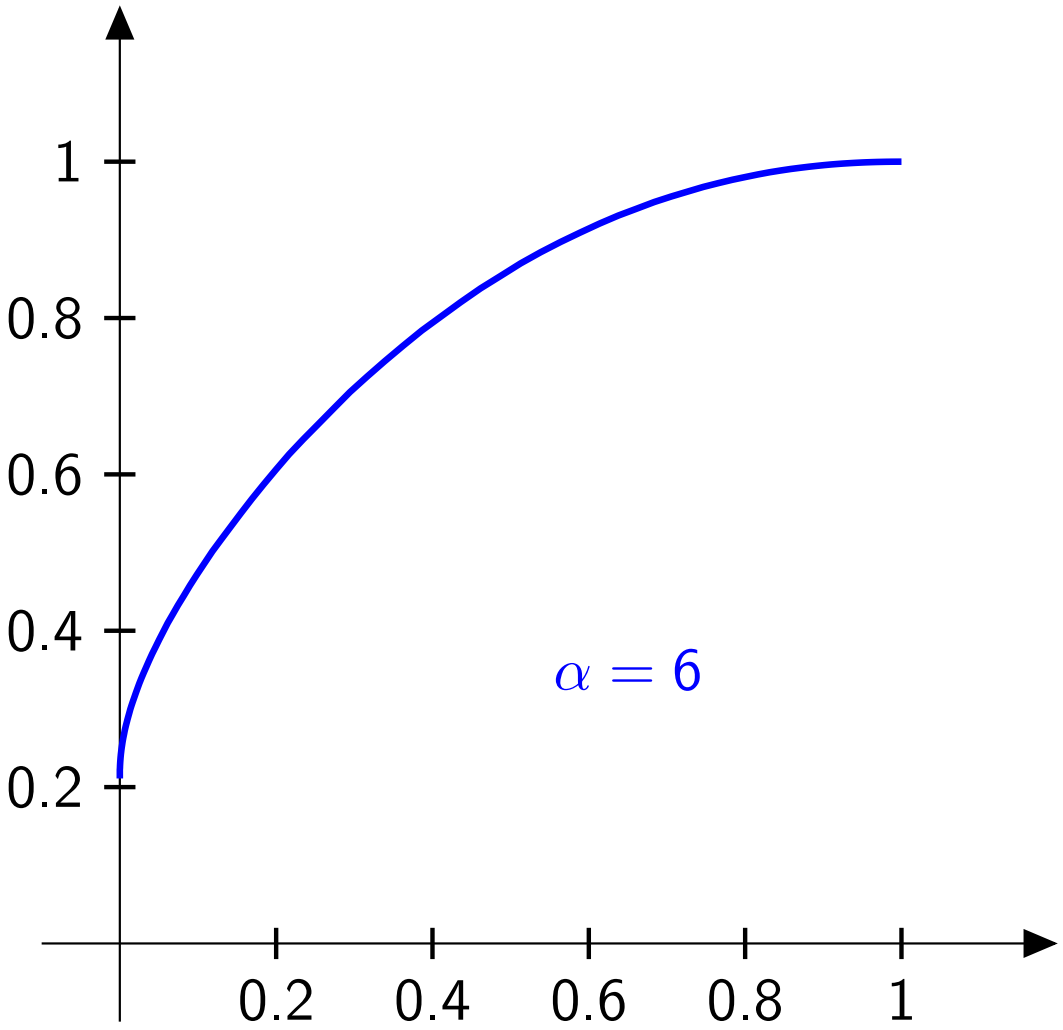

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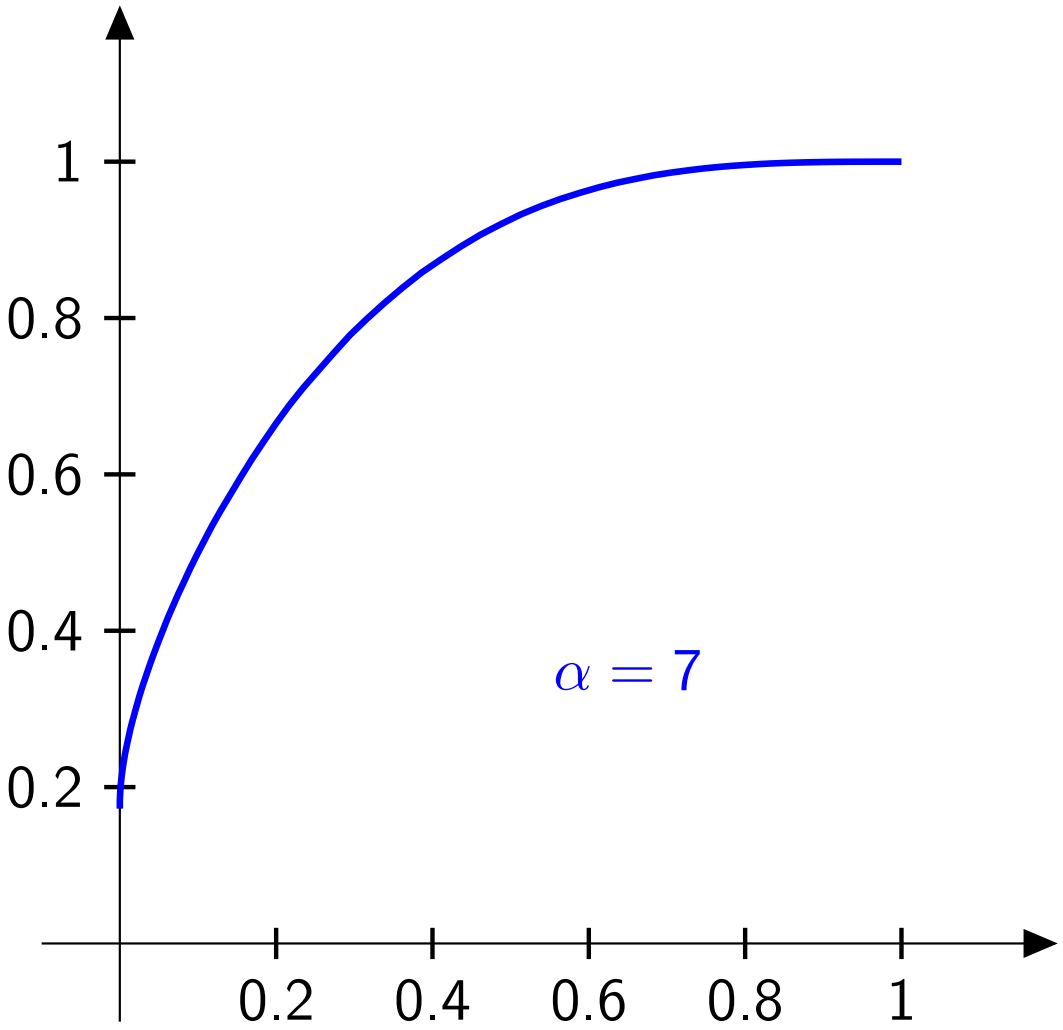

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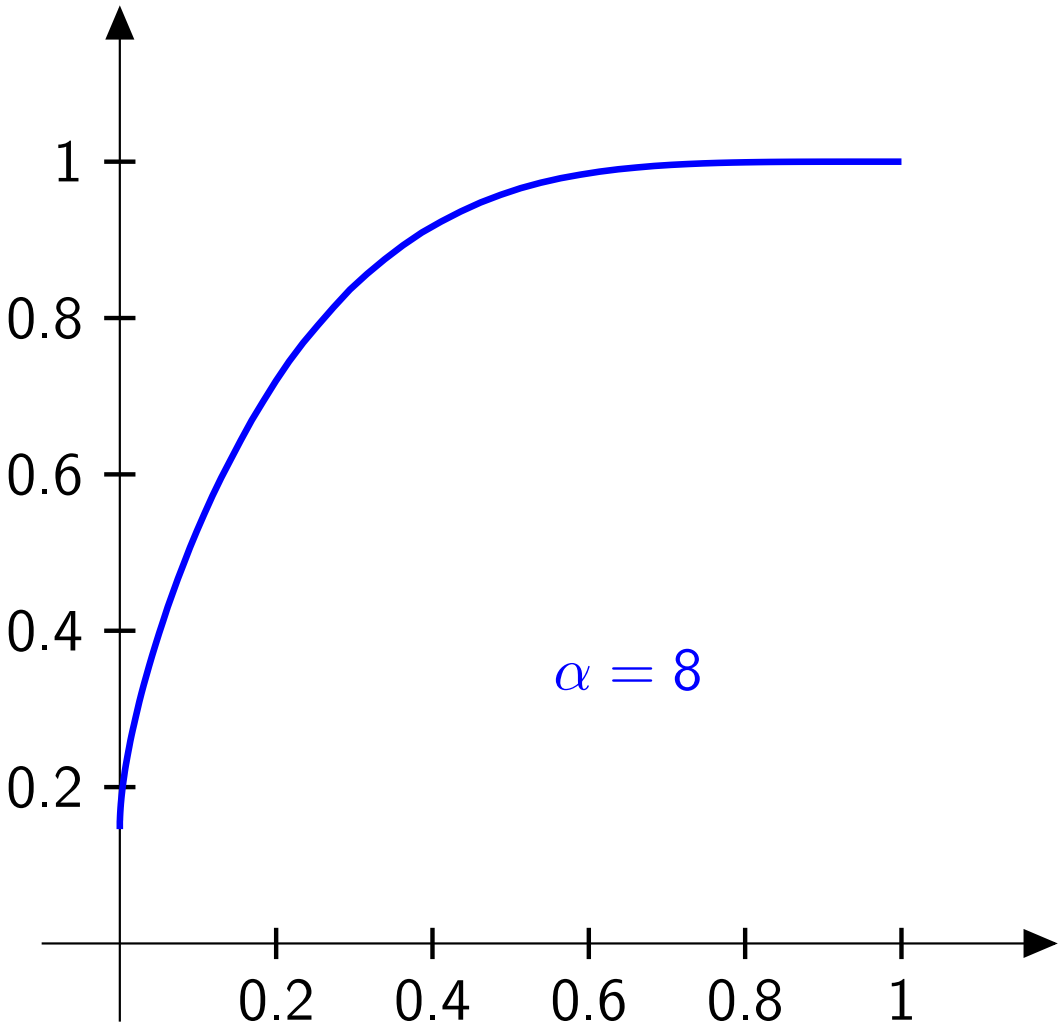

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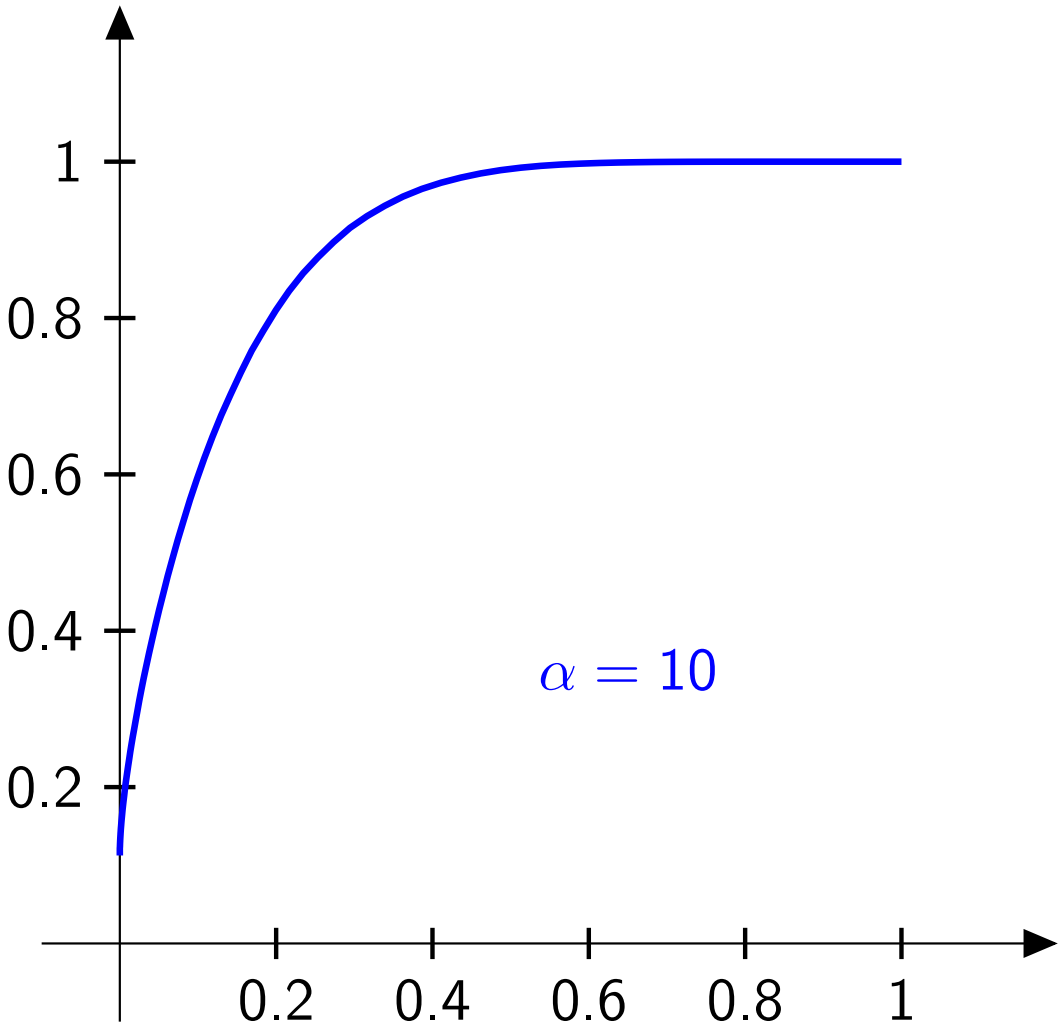

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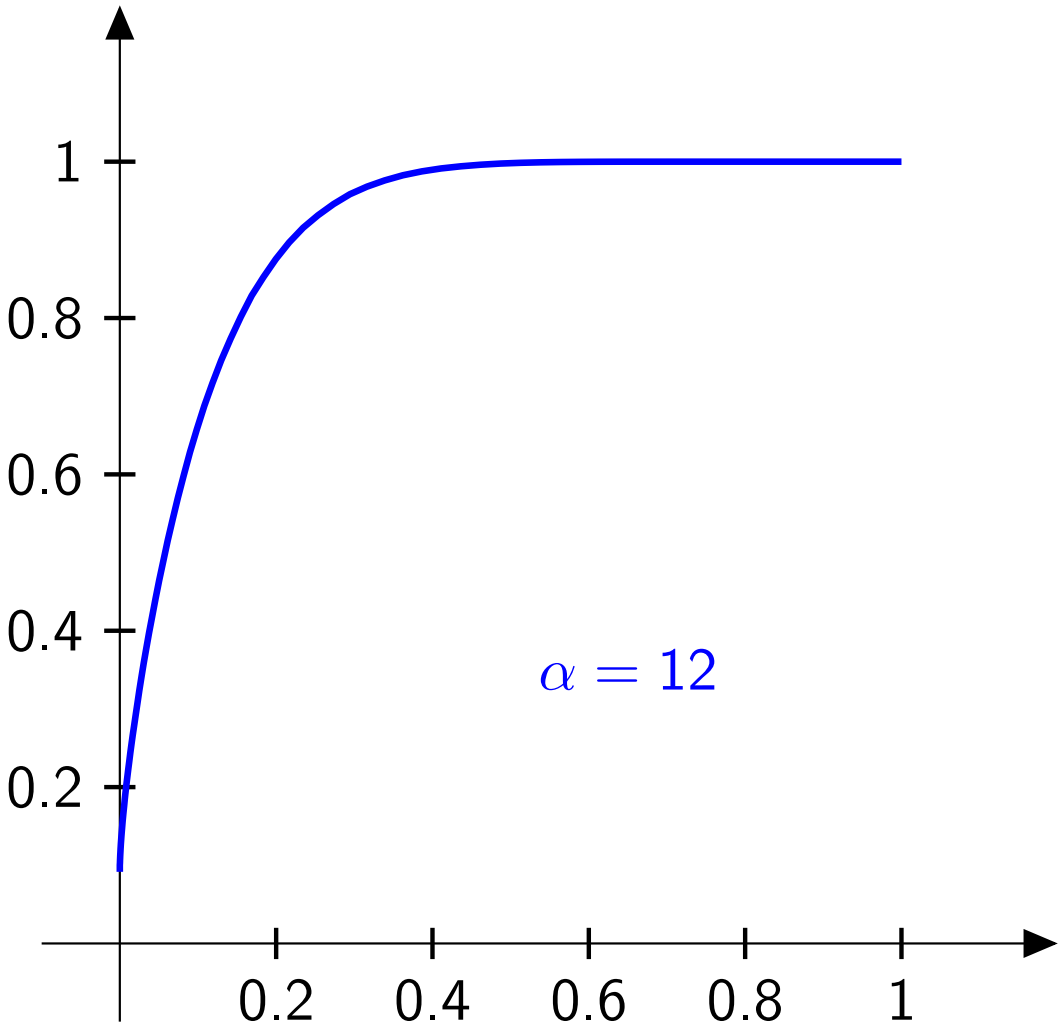

Numerical results