

biased annihilating branching process (BAB)

Chapter 1

$(X_t)_{t \geq 0}$ Markov process state space $\{0,1\}^{\mathbb{Z}}$ $\Rightarrow x = (x(i))_{i \in \mathbb{Z}}$
 $p \in [0,1]$ parameter

- site $i \in \mathbb{Z}$ updated after iid $\text{Exp}(1)$ waiting times
- during update, choose random neighbor $j \in \mathbb{Z}$ $|i-j|=1$
- if $X_t(j)=1$, then ~~update~~ $X_t(i)$ new value 1 probab. p
 0 " " $1-p$

• Similar to 1-facilitated Fredrickson-Andersen model

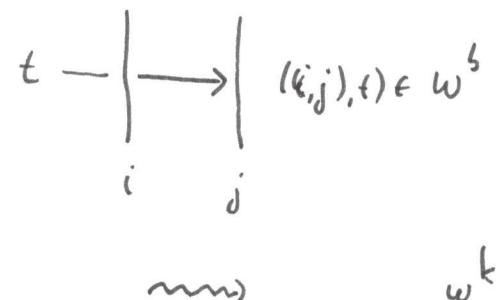
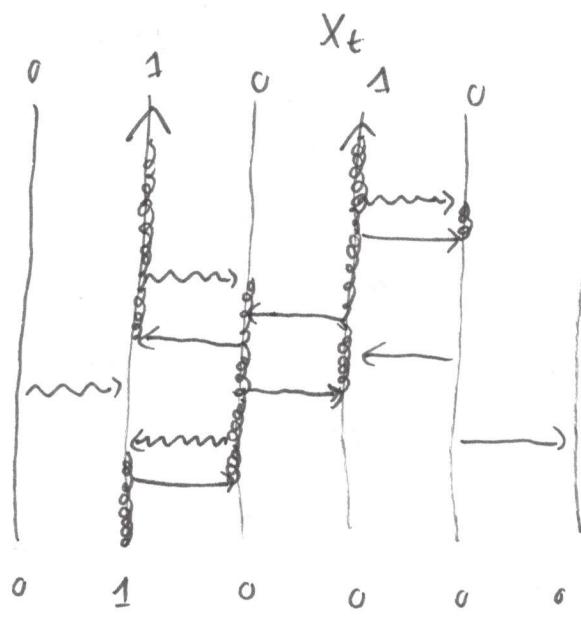
Construction $\vec{E} := \{(i,j) \in \mathbb{Z}^2 : |i-j|=1\}$

$$\text{bra}_{ij}(x)(k) := \begin{cases} 1 & \text{if } k=j, x(i)=1, \\ & x(k) \text{ otherwise.} \end{cases} \quad \text{kill}_{ij}(x)(h) := \begin{cases} 0 & \text{if } k=j, x(i)=1, \\ & x(h) \text{ otherwise.} \end{cases}$$

μ counting measure on \vec{E} ℓ Lebesgue measure on \mathbb{R}

w^b Poisson point set on $\vec{E} \times \mathbb{R}$ intensity $\frac{1}{2}p \mu \otimes \ell$
 w^k " " " " " " " " " " $\frac{1}{2}(1-p) \mu \otimes \ell$

$$X_t = \begin{cases} \text{bra}_{ij}(X_{t-}) & \text{if } ((i,j), t) \in w^b \\ \text{kill}_{ij}(X_{t-}) & \text{if } ((i,j), t) \in w^k \\ X_{t-} & \text{otherwise} \end{cases}$$



X_0

$$G_{BAB} f(n) = \frac{1}{2} p \sum_{(i,j) \in E} \{ f(bra_{ij}(n)) - f(n) \} + \frac{1}{2} (1-p) \sum_{(i,j) \in E} \{ f(bal_{ij}(n)) - f(n) \}$$

[Neinhausen & Sudbury 1993]

- product measures π_0, π_p reversible invariant laws

- all invariant laws convex combi. of π_0, π_p

- $\mathbb{P}^{\delta_0} [X_t \in \cdot] \xrightarrow[t \rightarrow \infty]{} \pi_p \quad \text{if } p \geq \frac{1}{4}$

improved to $p \geq 0.0335$

$R_t := \sup \{i : X_t(i) = 1\}$ in [Sudbury 1999]

Conjecture: true for all $p > 0$.

edge speed $\lim_{t \rightarrow \infty} \frac{1}{t} R_t$ exists? positive?

Conjecture $\sim \frac{1}{4} p^2$ as $p \rightarrow 0$

Aim understand limit $p \rightarrow 0$.

Intuition p small $\Rightarrow BAB \approx$ branching and coalescing random walks
(braco RW)



jump rate $\approx \frac{1}{2} p$



branching rate $\approx \frac{1}{4} p^2$

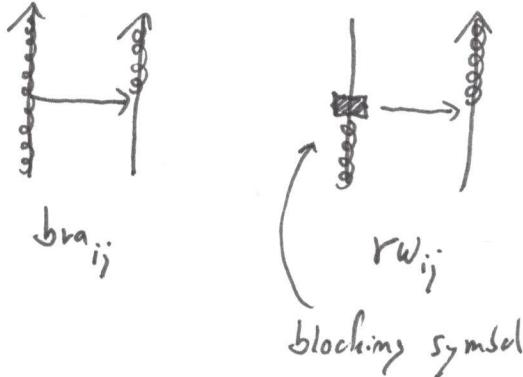
Plan • Study scaling limit of braco RW

- Then apply to BAB

braco RW

$$G_{\text{braco}} f(x) = \frac{1}{2} p \sum_{(i,j) \in \vec{\Sigma}} \{f(bra_{ij}(x)) - f(x)\} + \frac{1}{2} (1-p) \sum_{(i,j) \in \vec{\Sigma}} \{f(rw_{ij}(x)) - f(x)\}$$

$$rw_{ij}(x)(k) := \begin{cases} 0 & \text{if } k = i, \\ 1 & \text{if } k = j, x(i) = 1, \\ \geq 1 & \text{otherwise.} \end{cases}$$

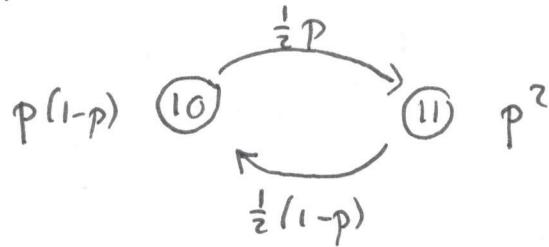


Def $\gamma: [s, u] \rightarrow \mathbb{Z}$ open path

- (i) $\gamma(t) \neq \gamma(t-)$ if \exists \blacksquare at $(\gamma(t-), t)$
- (ii) if $\gamma(t) \neq \gamma(t-)$, then $\exists \gamma_{(t-)} \rightarrow \gamma(t)$

- product law π_0, π_p reversible

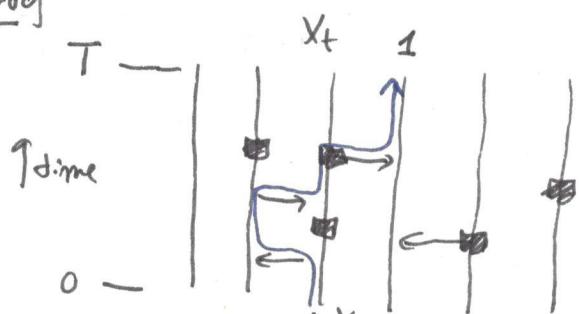
Proof detailed balance :



◻

- additive : $X_0 = X'_0 \vee X''_0 \Rightarrow X_t = X'_t \vee X''_t$ ($t \geq 0$)

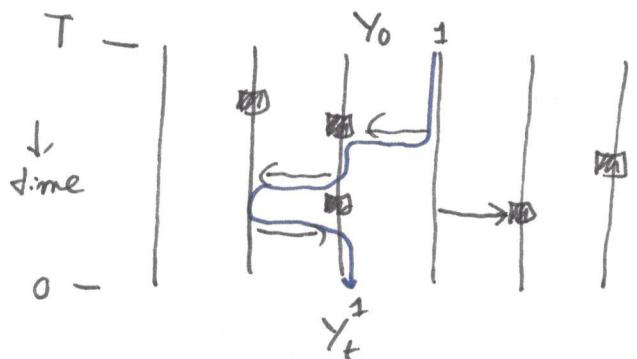
Proof



$$X_T(i) = 1 \Leftrightarrow$$

\exists open path $\gamma: [0, T] \rightarrow \mathbb{Z}$

$$X_0(\gamma(0)) = 1 \quad \gamma(T) = i$$

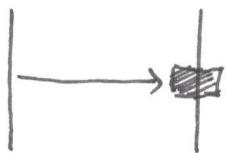


Dual process

$$Y_t(i) = 1 \Leftrightarrow$$

\exists open path $\gamma: [T-t, T] \rightarrow \mathbb{Z}$

$$\gamma(T-t) = i \quad Y_0(\gamma(T)) = 1$$



$$v_{0t} \delta_{ij}(y)(k) := \begin{cases} y^{(i)} & \text{if } k=j, \\ y^{(k)} & \text{otherwise} \end{cases}$$

biased voter model

$$G_{\text{rot}} f(y) := \frac{1}{2} p \sum_{(i,j) \in \bar{E}} \{ f(b_{\text{rot}}_{ij}(y)) - f(y) \} + \frac{1}{2} (1-p) \sum_{(i,j) \in \bar{E}} \{ f(v_{\text{rot}}_{ij}(y)) - f(y) \}.$$

• duality $\mathbb{P}[X_T \wedge Y_0 \neq 0] = \mathbb{P}[X_0 \wedge Y_T \neq 0] \quad (T \geq 0)$

Proof $\hookrightarrow \mathbb{P}[\exists \gamma: [0,T] \rightarrow \mathbb{Z}, X_0(\gamma(0))=1, Y_0(\gamma(T))=1] = \square$

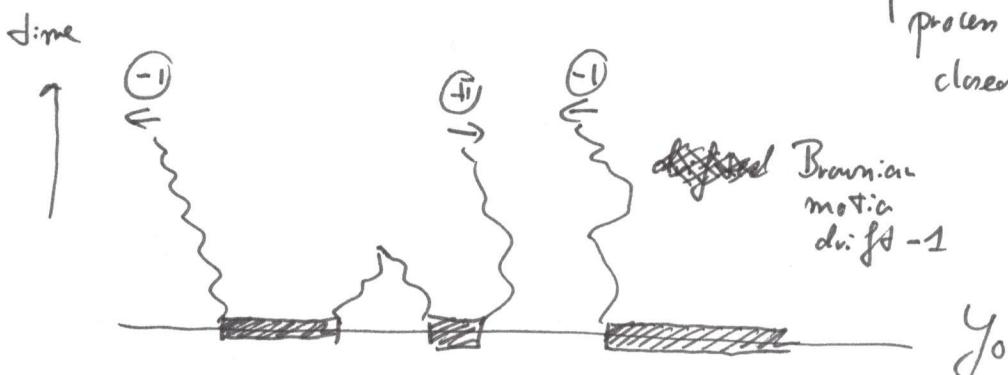
Def $Z_t(i) := Y_t(i) - Y_t(i+1) \quad \text{interface model}$

Y 0000111100111111	Z $0000-100010-100000$
random walk jump \leftarrow rate $\frac{1}{2}$ \rightarrow rate $\frac{1}{2}(1-p)$	random walk jump \rightarrow rate $\frac{1}{2}$ \leftarrow rate $\frac{1}{2}(1-p)$

• Scaling limit $\mathcal{Y}_{\left(\frac{p}{2}\right)t}^p := \left\{ \frac{p}{2} i : Y_t(i) = 1 \right\}$ Assume initial law converges

$$\mathbb{P}\left[\left(Y_t^p\right)_{t \geq 0} \in \cdot\right] \xrightarrow[p \rightarrow 0]{} \mathbb{P}\left[\left(Y_t\right)_{t \geq 0} \in \cdot\right]$$

↑ process taking values in closed sub sets of \mathbb{R} .



Theorem

$\exists!$ Markov process $(X_t)_{t \geq 0}$ taking values in closed subsets of \mathbb{R}

s.t.

$$\mathbb{P}[X_t \cap Y_0 \neq \emptyset] = \mathbb{P}[X_0 \cap Y_t \neq \emptyset] \quad (t \geq 0) \quad \forall Y$$

Theorem

$(X_t)_{t \geq 0}$ is the scaling limit of braco RW $(X_t^p)_{t \geq 0}$ as $p \rightarrow 0$.

Conjecture

For the BAB, let

$$X_{(\frac{p}{2})^3 t}^p := \left\{ \frac{p}{2} i : X_t^{(i)} = 1 \right\}$$

↑ 3rd power instead of 2 for voter, braco RW !

Then

$$\mathbb{P}\left[\left(X_t^p\right)_{t \geq 0} \in \cdot\right] \xrightarrow{p \rightarrow 0} \mathbb{P}\left[\left(X_t\right)_{t \geq 0} \in \cdot\right]$$

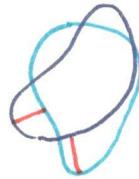
↑
universal scaling limit
branching-coalescing point set

Chapter 2Topology

(E, d) metric space $\mathcal{K}(E) := \{K \subset E : K \text{ compact}\}$ $\mathcal{K}_+(E) := \{K \in \mathcal{K}(E) : K \neq \emptyset\}$

$$d_H(K_1, K_2) := \sup_{x_1 \in K_1} d(x_1, K_2) \vee \sup_{x_2 \in K_2} d(x_2, K_1)$$

Hausdorff metric on $\mathcal{K}_+(E)$



- $d_H(K_n, K) \xrightarrow{n \rightarrow \infty} 0 \Leftrightarrow$
 - (i) $\exists C \text{ compact s.t. } K_n \subset C \forall n$
 - (ii) $K = \{x : \exists x_n \in K_n, x_n \rightarrow x\}$
 $= \{x : \exists x_n \in K_n, x \text{ cluster point of } (x_n)\}$

■ Topology generated by d_H
 does not depend on choice of d .

- (E, d) separable $\Rightarrow (\mathcal{K}_+(E), d_H)$ separable
- " complete " " " complete
- $A \subset \mathcal{K}_+(E)$ precompact $\Leftrightarrow \exists C \text{ compact s.t. } K \subset C \forall K \in A$.

- K_n r.v.: values in $\mathcal{K}_+(E)$
 laws of large numbers $\Leftrightarrow \forall \varepsilon > 0 \exists C \text{ compact s.t. } \mathbb{P}[K_n \subset C] \geq 1 - \varepsilon \quad \forall n$.

Def squeezed space $R(E) := (E \times \bar{\mathbb{R}}) \cup \{(*, -\infty), (*, +\infty)\}$

↑ extra point

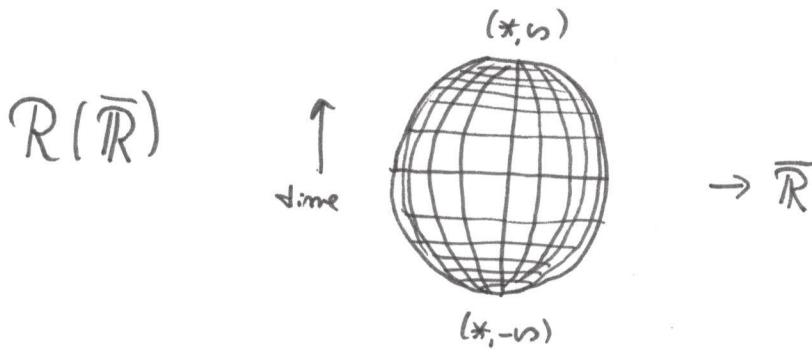
Let $d_{\bar{\mathbb{R}}}$ metric on $\bar{\mathbb{R}} := [-\infty, \infty]$

$\varphi : \bar{\mathbb{R}} \rightarrow [0, \infty)$ contin $\varphi(\pm \infty) = 0 \quad \varphi(t) > 0 \quad \forall t \in \bar{\mathbb{R}}$

$$\rho((x, s), (y, t)) := (\varphi(s) \wedge \varphi(t)) (d(x, y)) + |\varphi(s) - \varphi(t)| + d_{\bar{\mathbb{R}}}(s, t)$$

- ↑ metric on $R(E)$ $\rho((x_n, t_n), (y, t)) \rightarrow 0 \Leftrightarrow$
 - (i) $t_n \rightarrow t$
 - (ii) $\int t \in \bar{\mathbb{R}}, \text{ then } x_n \rightarrow x$.

■ Topology on $R(E)$ does not depend on choice of d .



- (E, d) separable $\Rightarrow (R(E), \rho)$ separable
- " complete " " complete"
- $A \subset R(E)$ precompact $\Leftrightarrow \forall T < \infty \exists C \subset E$ compact s.t.

$$A \cap (E \times [-T, T]) \subset C \times [-T, T]$$

Def path in E = compact $\pi \subset R(E)$ s.t.

(i) $\forall t \in \bar{R} \exists$ at most one $x \in E$ s.t. $(x, t) \in \pi$

(ii) $\bar{I}_\pi := \{t \in \bar{R} : (x, t) \in \pi \text{ for some } x \in E \cup \{\star\}\}$

is a compact interval
nonempty $\bar{I}_\pi \subset \bar{R}$

Def $\bar{I}_\pi =: [\sigma_\pi, \tau_\pi]$ $I_\pi := \bar{I}_\pi \cap \bar{R}$ domain of π

initial time \uparrow
final time

Def $\pi(t) := x$ with $(x, t) \in \pi$ ($t \in I_\pi$)

- $\pi : I_\pi \rightarrow E$ continuous

Proof $t_n, t \in I_\pi$, $t_n \rightarrow t \Rightarrow (\pi(t_n), t_n)$ precompact

unique cluster point is $(\pi(t), t)$ \square

- each continuum

$\pi : I \rightarrow E$ correspond
to a path.



Def $\Pi(E)$ space of paths equipped with Hausdorff metric

- $\Pi_m \rightarrow \Pi \Leftrightarrow C_{\Pi_m} \rightarrow C_\Pi, \tau_{\Pi_m} \rightarrow \tau_\Pi$

$\Pi_m \rightarrow \Pi$ locally uniformly.

↳ thanks to squeezed space!

Recall E Polish \Leftrightarrow separable

\exists complete metric generating the topology

! not every metric generating the topology is complete!

(unless E compact).

- E Polish $\Rightarrow \Pi(E)$ Polish.

Proof

Def $m_{T,\delta}(\Pi) := \sup \{ d(\pi(s), \pi(t)) : s, t \in I_\Pi, -T \leq s \leq t \leq T, |t-s| \leq \delta \}$

$m_{T,\delta}(K) := \sup \{ d(x, y) : (x, s), (y, t) \in K, \dots \}$

$\bar{I}_k := \{ t \in \bar{\mathbb{R}} : (x, t) \in K \text{ for some } x \in E \cup \{*\} \}$

~~$\Pi(E) = \{ K \in \mathcal{K}_+(\mathbb{R}(E)) : \bar{I}_k \text{ is an interval and } \lim_{\delta \rightarrow 0} m_{T,\delta}(K) = 0 \forall T < 0 \}$~~

~~Def $A_{T,\varepsilon,\delta} := \{ K \in \mathcal{K}' : m_{T,\delta}(K) \geq \varepsilon \}$~~

Def $\mathcal{K}' := \{ K \in \mathcal{K}_+(\mathbb{R}(E)) : \bar{I}_K \text{ is an interval} \}$ closed subset of $\mathcal{K}_+(\mathbb{R}(E))$

- $\Pi(E) = \{ K \in \mathcal{K}' : \lim_{\delta \rightarrow 0} m_{T,\delta}(K) = 0 \forall T < 0 \}$

Def $A_{T,\varepsilon,\delta} := \{ K \in \mathcal{K}' : m_{T,\delta}(K) \geq \varepsilon \}$

$$\Pi(E) = \bigcap_{n,m} \bigcup_k A_{n,\frac{1}{m},\frac{1}{k}}^c$$

$\underbrace{\hspace{10em}}$ open

G_δ -set $\Rightarrow \Pi(E)$ Polish



(7)

Def $A \subset \Pi(E)$ equi-continuous if $\lim_{\delta \rightarrow 0} \sup_{\pi \in A} m_{T,\delta}(\pi) = 0 \quad \forall T < \infty$

Def A compactly contained if $\forall T < \infty \exists C \subset E$ compact s.t.

$$\pi(t) \in C \quad \forall \pi \in A, t \in I_\pi \cap [-T, T]$$

• Arzela-Ascoli:

$A \subset \Pi(E)$ precompact \Leftrightarrow

A equi-continuous and compactly contained.

Proof $A \subset \Pi(E)$ precompact \Leftrightarrow $\underbrace{\bar{A} \subset K_+(R(E))}$ compact and $\underbrace{\bar{A} \subset \Pi(E)}$

$\Leftrightarrow A$ compactly contained $\quad A$ equi-continuous

$A \ni \pi_m \rightarrow k$ then I_k interval

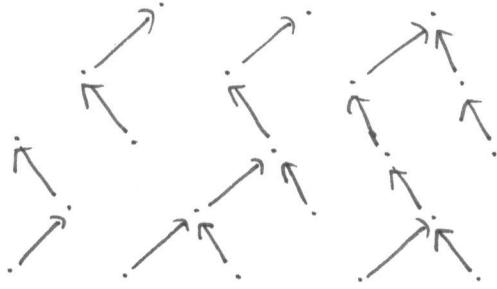
$\lim_{\delta \rightarrow 0} m_{T,\delta}(k) = 0 \quad \forall T < \infty$

□

Chapter 3The Brownian web

Brownian web = scaling limit of coalescing random walks

" net = " " " branching and coalescing " .. \rightarrow next chapter



$$\mathbb{Z}_{\text{even}}^2 := \{(x, t) \in \mathbb{Z}^2 : x+t \text{ is even}\},$$

$$w = (w_z)_{z \in \mathbb{Z}_{\text{even}}^2} \text{ iid unif } \{-1, 1\}.$$

Def $\Pi^\uparrow := \{\pi \in \Pi(\bar{\mathbb{R}}) : \tau_\pi = \infty\}$ $\quad z_\pi := (\pi(\sigma_\pi), \sigma_\pi)$ starting point

Def path $\pi \in \Pi^\uparrow$ in the arrow configuration w satisfies:

$$(i) \quad (\pi(t), t) \in \mathbb{Z}_{\text{even}}^2 \quad (t \in \mathbb{Z}, t \geq \sigma_\pi)$$

$$(ii) \quad \pi(t+1) = \pi(t) + w_{(\pi(t), t)} \quad (t \in \mathbb{Z}, t \geq \sigma_\pi)$$

$$(iii) \quad \pi(t+s) = (1-s)\pi(t) + s\pi(t+1) \quad (t \in \mathbb{Z}, t \geq \sigma_\pi, 0 \leq s \leq 1).$$

Def $\mathcal{U} = \mathcal{U}(w) = \{\pi \in \Pi^\uparrow : \pi \text{ is a path in } w\}$

• $\overline{\mathcal{U}}$ is a compact subset of Π^\uparrow



$\overline{\mathcal{U}} \setminus \mathcal{U}$ = set of trivial paths $\sigma_\pi \in \bar{\mathbb{Z}}, \pi(t) = -n \forall t \in \bar{\mathbb{Z}}$
or .. + n ..

Def $\theta_\varepsilon(x, t) := (\varepsilon x, \varepsilon^2 t)$ diffusive scaling map

extend continuously to $\mathcal{R}(\bar{\mathbb{R}})$.

$$\theta_\varepsilon(A) := \{\theta_\varepsilon(z) : z \in A\} \quad A \subset \mathcal{R}(\bar{\mathbb{R}})$$

$$\theta_\varepsilon(\mathcal{A}) := \{\theta_\varepsilon(\pi) : \pi \in \mathcal{A}\} \quad \mathcal{A} \subset \Pi^\uparrow$$

Theorem [Fontes, Isopi, Newman & Ravishankar 2004]

$$\mathbb{P}[\theta_\varepsilon(\bar{U}) \in \cdot] \xrightarrow{\varepsilon \rightarrow 0} \mathbb{P}[W \in \cdot]$$

weak convergence
on $\mathcal{K}(\mathbb{H}^1)$.

Step 1 Coalescing Brownian motions

Let $(z_i)_{i \in \mathbb{N}_+}$, $z_i = (x_i, \xi_i) \in \mathbb{R}^2$

$(B_t^i)_{t \geq s_i}$, BM started $B_{s_i}^i = x_i$.



$$\tau_i := \inf \left\{ t \geq s_i : (B_t^i, t) \in \bigcup_{j=1}^{i-1} A_j \right\} \quad \tau_i = \infty$$

$$A_i := \left\{ (B_t^i, t) : s_i \leq t < \tau_i \right\}$$

Def $k(i) < i$ (i22) by $(B_{\tau_i}^i, \tau_i) \in A_{k(i)}$

$$\tilde{B}_t^i := \begin{cases} B_t^i & s_i \leq t < \tau_i \\ \tilde{B}_{\tau_i}^{k(i)} & \tau_i \leq t \end{cases} \quad \text{coalescing BM}$$

Proposition

$$\varepsilon_k > 0, \varepsilon_k \rightarrow 0 \quad z_1^k, \dots, z_n^k \in \mathbb{Z}_{\text{even}}^2$$

$$\theta_{\varepsilon_k}(z_1^k, \dots, z_n^k) \xrightarrow{k \rightarrow \infty} z_1, \dots, z_n \in \mathbb{R}^2$$

π_1^k, \dots, π_n^k paths in $U(w)$ started z_1^k, \dots, z_n^k

Then:

$$\mathbb{P}[\theta_{\varepsilon_k}(\pi_1^k, \dots, \pi_n^k) \in \cdot] \xrightarrow{k \rightarrow \infty} \mathbb{P}[(\pi_1, \dots, \pi_n) \in \cdot]$$

Proof: Donsker's invariance principle + convergence of meeting times

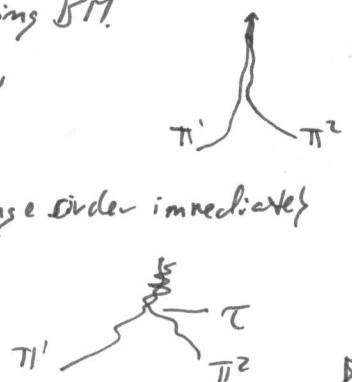
$$\pi_1^k \rightarrow \pi_1, \pi_2^k \rightarrow \pi_2$$

\uparrow
coalescing BM.

Assume π_1, π_2 change order immediately after τ .
Then $\tau_k \rightarrow \tau$

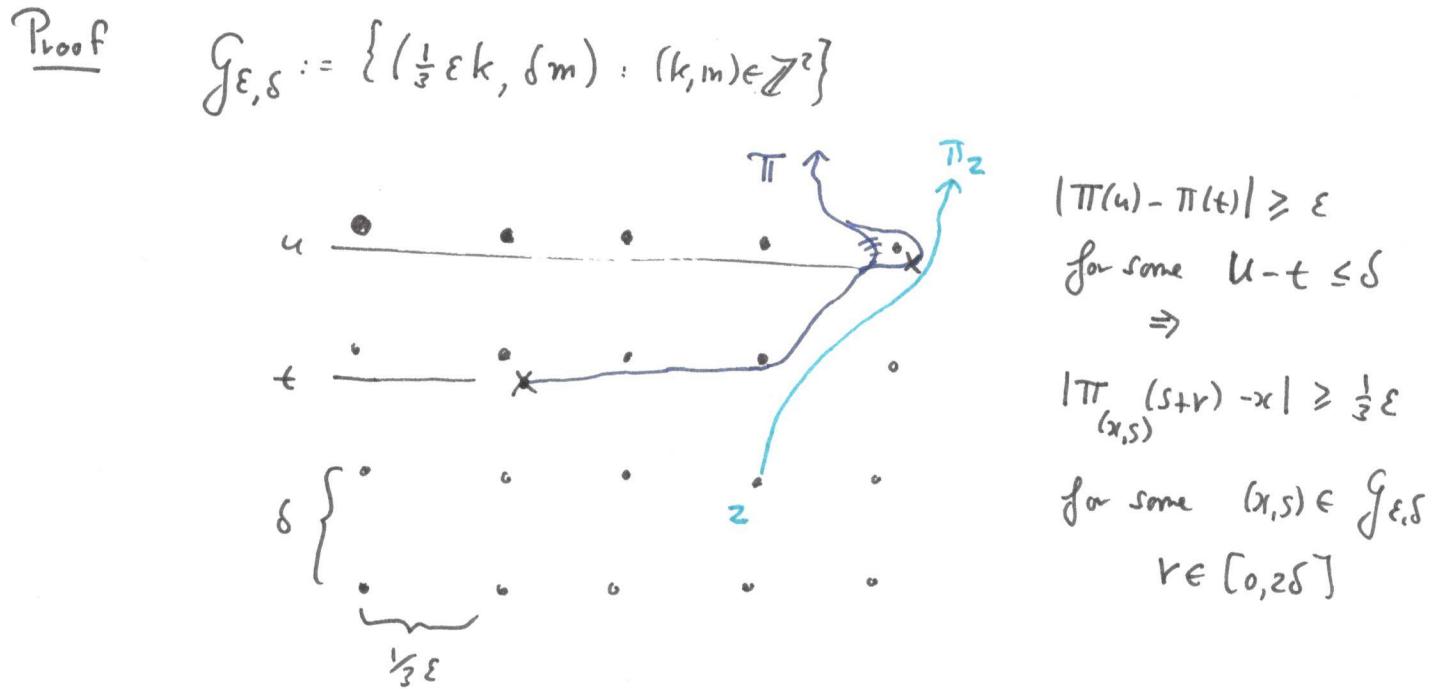
$$\tau_k := \inf \left\{ t \geq \tau_{\pi_1} \vee \tau_{\pi_2} : \pi_1(t) = \pi_2(t) \right\}$$

$$\tau := \pi_1 \quad \pi_2$$



□

- $(\pi_z)_{z \in D}$ coalescing BM started from $D \subset \mathbb{R}^2$ countable & dense
 $\Rightarrow \{\pi_z : z \in D\}$ precompact $\subset \Pi^\uparrow$



$$\mathbb{P} \left[\sup_{r \in [0, 2\delta]} |B_r| \geq \frac{1}{3}\varepsilon \right] \leq C e^{-c \frac{\varepsilon^2}{\delta}}$$

$$\mathbb{P} \left[m_{T, \delta}(\pi) \geq \varepsilon \text{ for some } \pi \in \mathcal{A} \right] \leq C_T \varepsilon^{-1} \delta^{-1} e^{-c \frac{\varepsilon^2}{\delta}} \xrightarrow[\delta \rightarrow 0]{} 0 \quad \forall \varepsilon > 0$$

↳ can be estimated in $\sup \left\{ |\pi(u) - \pi(t)| : |u-t| \leq \delta, (\pi(t), t), (\pi(u), u) \in [-T, T]^2 \right\}$

$\Rightarrow \mathcal{A}$ a.s. precompact. \square

Notation $\mathcal{A} \subset \Pi^\uparrow$, $D \in \mathcal{R}(\overline{\mathbb{R}})$: $\mathcal{A}(D) := \{\pi \in \mathcal{A} : z_\pi \in D\}$

$$\mathcal{A}(x) := \mathcal{A}(\{x\}).$$

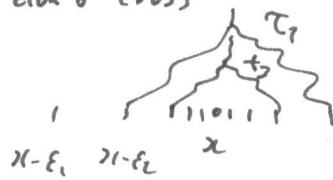
Theorem $\exists!$ (in law) r.v. value in $\mathcal{F}_+(\Pi^\uparrow)$ s.t.

- $\forall z \in \mathbb{R}^2$, a.s. $W(z) =: \{\pi_z\}$ singleton
- $\forall z_1, \dots, z_n$ $(\pi_{z_1}, \dots, \pi_{z_n})$ are coalescing BM
- $\forall D \subset \mathbb{R}^2$ countable & dense, a.s. $W = \overline{W(D)}$.

Proof Let $(\pi_z)_{z \in D}$ coalescing BM, set $W := \overline{\{\pi_z : z \in D\}}$.

- paths in W don't cross
- can add countably many pts to $D \rightarrow$ consistent

proof of (i)



$$\text{if } t_1 \geq t_2 \geq \dots \geq t_m \rightarrow 0 \text{ a.s.}$$

(ii) add $\{z_1, \dots, z_n\}$ to D

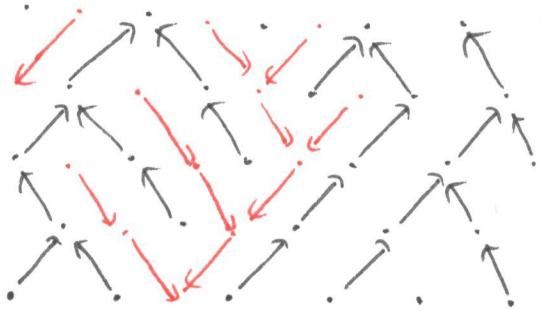
$$\text{(iii)} \quad W := \overline{\{\pi_z : z \in D\}} \quad W' := \overline{\{\pi_z : z \in D \cup D'\}}$$

$$z \in D \Rightarrow \pi_z \in W'(z) \supset W(z)$$

$$\text{by (i)} \nearrow \begin{matrix} \nearrow \\ \text{singletons} \end{matrix} \Rightarrow \pi_z \in W'$$

$$\{\pi_z : z \in D\} \subset W' \Rightarrow W \subset W' \quad \text{Also } W' \subset W \Rightarrow \text{(iii)} \quad \square$$

- $\pi \in W \Rightarrow \begin{cases} \pi(t) \in \mathbb{R} \quad \forall t \in I_\pi, \text{ or} \\ " = -\infty \quad " \quad ", \text{ or} \\ " = +\infty \quad " \end{cases}$



$$\hat{U} = \hat{U}(w) \subset \pi^\downarrow$$

dual paths

$$-\pi := \{(-x, -t) : (x, t) \in \pi\} \quad -A := \{-\pi : \pi \in A\}$$

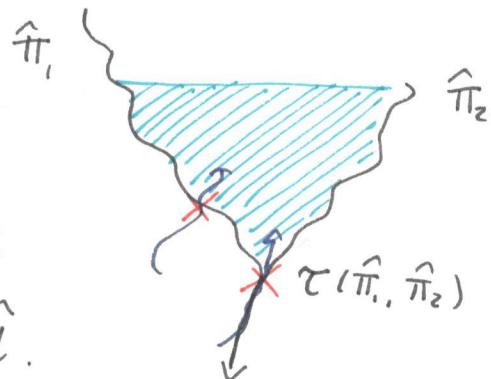
$$\hat{U} \stackrel{d}{=} -U.$$

Def wedge $W(\hat{\pi}_1, \hat{\pi}_2)$ open set

$$:= \{(x, t) : T < t < \zeta_{\hat{\pi}_1} \wedge \zeta_{\hat{\pi}_2}, \hat{\pi}_1(t) < x < \hat{\pi}_2(t)\}$$

- paths in U do not enter wedge of \hat{U} .

$$\nexists s < t \quad (\pi(s), s) \notin \bar{W} \quad (\pi(t), t) \in W$$



- $D, \hat{D} \subset \mathbb{R}^2$ countable \Rightarrow can construct $(\pi_z)_{z \in D}$ coalescing BM
 $(\hat{\pi}_z)_{z \in \hat{D}}$ dual ..
- s.t. π_z does not enter $W(\hat{\pi}_{z_1}, \hat{\pi}_{z_2})$ $\forall z, z_1, z_2$
 $\hat{\pi}_z$ etc.

consistent!

Proof Finite approximation, convergence of meeting times \square

- [Souriau, Tóth, Werner 2000] $\hat{\pi}_z$ reflected off π_z (Skorokhod reflection)



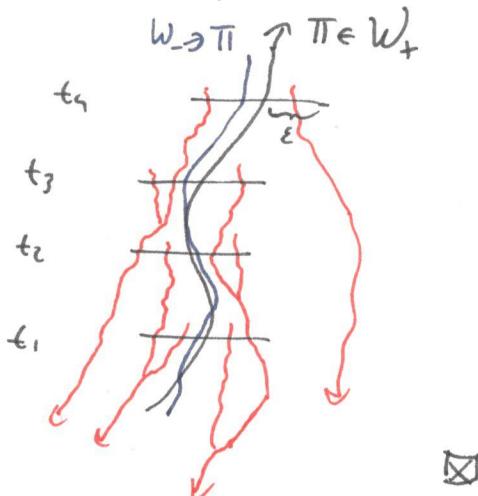
$$W_- := \overline{\{\pi_z : z \in D\}}$$

$$W_+ := \{\pi \in \Pi^\uparrow : \pi \text{ does not enter wedge of } (\hat{\pi}_z)_{z \in \hat{D}}\}$$

$$\Rightarrow W_- = W_+$$

Proof $W_- \subset W_+$ easy.

$$W_+ \subset W_-$$



\square

Theorem $\mathbb{P}[\theta_\varepsilon(\bar{u}, \bar{u}) \in \cdot] \xrightarrow[\varepsilon \rightarrow 0]{\text{B web}} \mathbb{P}[(w, \hat{w}) \in \cdot]$

Proof (Skorokhod representation thm \Rightarrow \exists coupling s.t.)

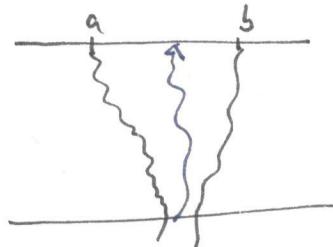
$\forall z \in D$ choose $\theta_{\varepsilon_n}(z^n) \rightarrow z$ $\pi_z^n := \theta_{\varepsilon_n}(\pi_{z^n})$
 $\pi_z^n \rightarrow \pi_z$ a.s. $\tau(\pi_{z_1}^n, \pi_{z_2}^n) \rightarrow \tau(\pi_{z_1}, \pi_{z_2})$ a.s.
 $\hat{\pi}_z^n \rightarrow \hat{\pi}_z$ a.s. ...

tighten \Rightarrow subseq. limit .. couple $\theta_{\varepsilon_n}(\bar{u}, \bar{v}) \rightarrow (v, \hat{v})$

$\vdash W_- \subset V \subset W_+$ $\Rightarrow V = W$
 $\pi_z \in V \forall z \in D$ $\pi \in V$ does not enter wedge $\text{also } \hat{V} = \bar{W}$ \square

Def $\xi_t^A := \{\pi(t) : \pi \in W(A \times \{0\})\}$ $(\xi_t^A)_{t \geq 0}$ ~~branching~~ ~~coalescing~~ point set.

- $\mathbb{E}[|\xi_t^R \cap [a, b]|] = \frac{b-a}{\sqrt{\pi t}}$

Proof 

$$\xi_t^R \cap [a, b] \neq \emptyset \Leftrightarrow \exists \hat{\pi}_a \in \hat{W}(a), \hat{\pi}_b \in \hat{W}(b)$$

s.t. $\tau(\hat{\pi}_a, \hat{\pi}_b) \leq 0$.

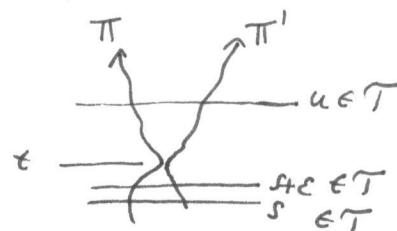
$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \mathbb{P}[\tau(\hat{\pi}_a, \hat{\pi}_{a+\varepsilon}) \geq t] = \frac{1}{\sqrt{\pi t}}$$

\square

- $\pi, \pi' \in W \quad \pi(t) = \pi'(t) \text{ for some } t > \sigma_\pi \vee \sigma_{\pi'} \Rightarrow \pi(u) = \pi'(u) \text{ for } u \geq t$.

Proof Let $T \subset \mathbb{R}$ countable & dense

at dist $s + \varepsilon$



$s + \varepsilon$ $\xrightarrow{\text{independent of } W \text{ above } s + \varepsilon}$
 \nwarrow countable set of points

sufficient to prove for deterministic starting points
 $\Rightarrow \text{OK}$

- $\pi \in W$ does not enter wedge of \hat{W} .

\square

Special points

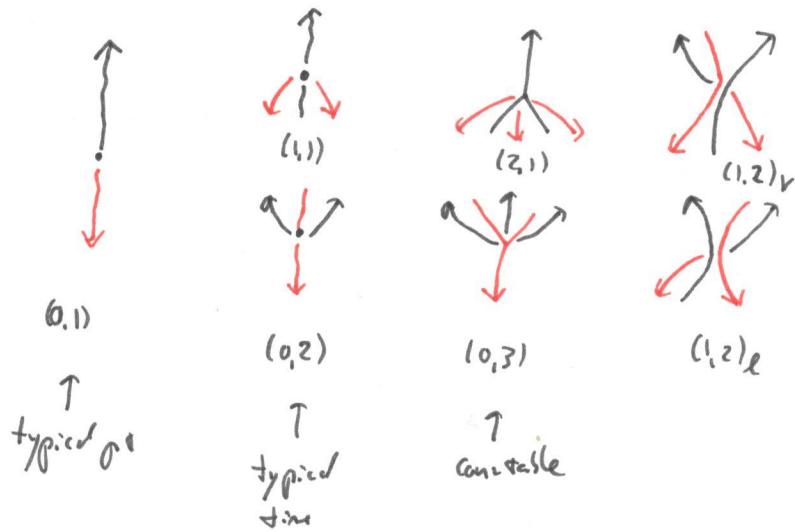
Def Π enters $(x, t) \Leftrightarrow \sigma_\Pi < t \quad \pi(t) = x$

Def $\Pi \sim \Pi' : \exists \sigma_\Pi \vee \sigma_{\Pi'} \leq s < t \text{ s.t. } \Pi = \Pi' \text{ on } [s, \infty)$

Def $m_{in}(z) := \# \text{ equiv classes of } \Pi \in \omega \text{ entering } z$

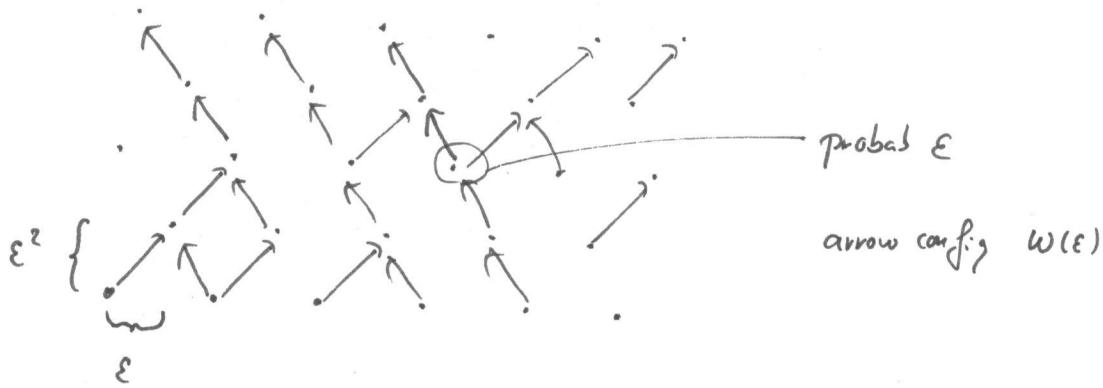
$m_{out}(z) := |\omega(z)|$

$(m_{in}(z), m_{out}(z)) = \underline{\text{type}}$ of $z \in \mathbb{R}^2$.



Chapter 9

The Brownian net



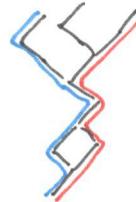
Theorem [Sun, S. 2008] $\epsilon_n > 0$ $\epsilon_n \rightarrow 0$ \mathcal{U}_n paths in $w(\epsilon_n)$

$$\mathbb{P}[\Theta_{\epsilon_n}(\bar{\mathcal{U}}_n) \in \cdot] \xrightarrow[n \rightarrow \infty]{\rightharpoonup} \mathbb{P}[N \in \cdot]$$

↑ Brownian net

Step 1 \mathcal{U}_n^ℓ left paths

\mathcal{U}_n^r right "



- $\mathbb{P}[\Theta_{\epsilon_n}(\bar{\mathcal{U}}_n^\ell) \in \cdot] \xrightarrow[n \rightarrow \infty]{\rightharpoonup} \mathbb{P}[w^\ell \in \cdot]$ left B web

" $\bar{\mathcal{U}}_n^r$ " " w^r " right B web

left-right SDE

$$\begin{cases} dL_t = 1_{\{L_t \neq R_t\}} dB_t^\ell + 1_{\{L_t = R_t\}} dB_t^r - dt \\ dR_t = 1_{\{L_t \neq R_t\}} dB_t^r + 1_{\{L_t = R_t\}} dB_t^\ell + dt \end{cases}$$

- $\exists!$ unique weak solution such that $L_t \leq R_t \quad \forall t \geq \tau := \inf\{s \geq 0 : L_s = R_s\}$.

Proof time change \Rightarrow (i) $dL_t = d\tilde{B}_{T_t}^\ell + d\tilde{B}_{S_t}^r - dt$

(ii) $dR_t = d\tilde{B}_{T_t}^r + d\tilde{B}_{S_t}^\ell + dt$

(iii) $T_t + S_t = t$

(iv) $\int_0^t 1_{\{L_s < R_s\}} dS_s = 0$

$$\left. \begin{array}{l} T_t = \int_0^t 1_{\{L_s < R_s\}} ds \\ S_t = \int_0^t 1_{\{L_s = R_s\}} ds \end{array} \right\}$$

Has pathwise unique solution

$$\bullet \quad I := \{t \geq 0 : L_t = R_t\} \quad \mu_I(A) := \int_A 1_I(t) dt$$

$\Rightarrow I$ nowhere dense $I = \text{supp } (\mu_I)$

(W^ℓ, W^r) left-right Bweb

- (i) paths evolve independently in disjoint parts of space
- (ii) left path stays left of right path after meeting & follow left-right SDE
- (iii) left + left coalesce, right + right coalesce

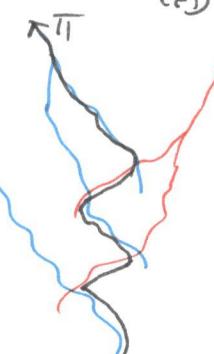
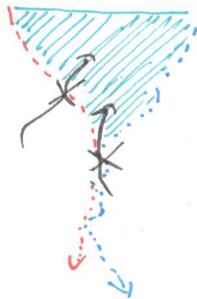
$$\{L(R)(L)(L(R)(L(R)(R)(R))\}$$

$$\bullet \quad \mathbb{P}[\Theta_{\epsilon_n}(\bar{U}_n^\ell, \bar{U}_n^r, \hat{U}_n^\ell, \hat{U}_n^r) \in \cdot] \xrightarrow{n \rightarrow \infty} \mathbb{P}[(W^\ell, W^r, \hat{W}^\ell, \hat{W}^r) \in \cdot]$$

Let $D \subset \mathbb{R}^2$ countable dense

Def $N_- := \overline{\{\pi : \pi \text{ obtained by hopping between } (\pi_z^\ell)_{z \in D} \text{ and } (\pi_z^r)_{z \in D}\}}$

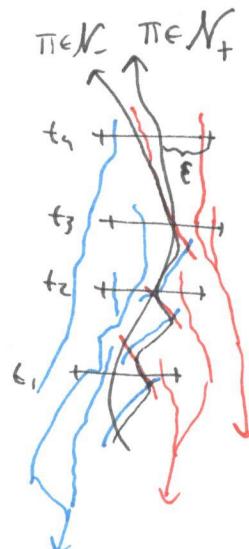
Def $N_+ := \{\pi : \pi \text{ does not enter wedges } W(\hat{\pi}_{z_1}^r, \hat{\pi}_{z_2}^r) \cup z_1, z_2 \in D\}$



$$\bullet \quad N_- = N_+$$

Proof \subset easy

Proof of $\mathbb{P}[\Theta_{\epsilon_n}(\bar{U}_n) \in \cdot] \xrightarrow{n \rightarrow \infty} \mathbb{P}[N \in \cdot]$
as far as.



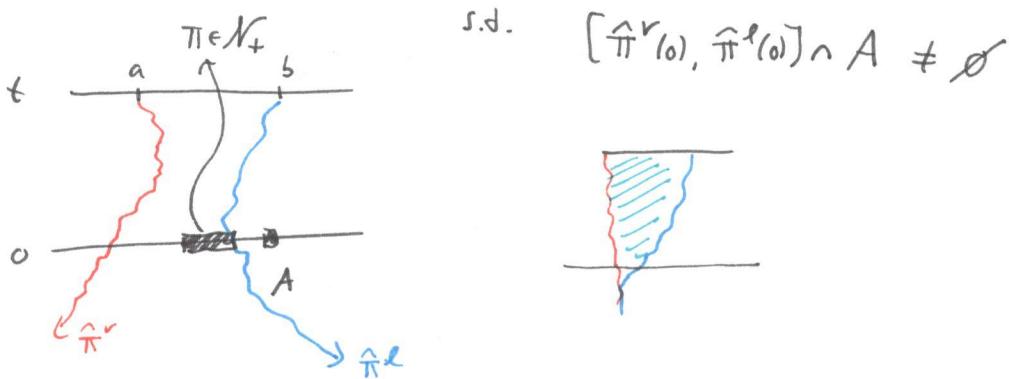
& equicontinuity

Def $\xi_t^A := \{\pi(t) : \pi \in \mathcal{N}(A \times \{0\})\}$ branching-coalescing point set

- $E[|\xi_t^R \cap [a,b]|] = (b-a) \cdot \left(\frac{e^{-t}}{\sqrt{\pi t}} + 2\Phi(\sqrt{2t}) \right)$

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}y^2} dy$$

- $\xi_t^A \cap [a,b] \neq \emptyset \Leftrightarrow \exists \hat{\pi}^r \in \mathcal{W}^r(a,t), \hat{\pi}^l \in \mathcal{W}^l(b,t)$



Chapter 5 The branching-coalescing point set

Fix N . $\forall s \leq t$, Def $X_{s,t} : \mathcal{K}(\bar{\mathbb{R}}) \rightarrow \mathcal{K}(\bar{\mathbb{R}})$

$$X_{s,t}(A) := \{\pi(t) : \pi \in N(A \times \{s\})\}$$

$$Y_{t,s}(A) := \{x \in \bar{\mathbb{R}} : \exists \pi \in N(x, s) \text{ s.t. } \pi(t) \in A\} = \mathcal{K}(\bar{\mathbb{R}})$$

$$\text{additive } X_{s,t}(\emptyset) = \emptyset \quad X_{s,t}(A \cup B) = X_{s,t}(A) \cup X_{s,t}(B)$$

- For deterministic $s \leq t \leq u$, a.s.

$$\begin{aligned} X_{t,t} &= \text{id} & X_{t,u} \circ X_{s,t} &= X_{s,u} \\ Y_{t,t} &= \text{id} & Y_{t,s} \circ Y_{u,t} &= Y_{u,s} \end{aligned} \quad \left. \begin{array}{l} \text{stochastic flow} \\ \text{concatenation} \end{array} \right\} \begin{array}{l} \overbrace{\pi}^{\pi'} \in N \\ \pi' \in N \end{array}$$

Proof At determ. t , if $\pi \in N$ incoming at (x, t) $\left. \begin{array}{l} \text{at determ. } t, \text{ if } \pi \in N \text{ incoming at } (x, t) \\ \pi' \in N(x, +) \end{array} \right\} \Rightarrow \text{concatenation}$

not true at all t $\boxed{\gamma_{(1,2)}^w}$

$$\begin{aligned} t_1 < \dots < t_n &\Rightarrow X_{t_1, t_2}, \dots, X_{t_{n-1}, t_n} \text{ indep} & \left. \begin{array}{l} \text{indep increments} \\ \text{indep} \end{array} \right\} \text{indep increments} \\ &Y_{t_n, t_{n-1}}, \dots, Y_{t_2, t_1} \text{ indep} \end{aligned}$$

Proof disjoint parts of N indep. $\boxed{\gamma}$

$$\forall s \in \mathbb{R}, A \in \mathcal{K}(\bar{\mathbb{R}}) \quad \text{def } P_t(A, \cdot) := \mathbb{P}[X_{s,s+t}(A) \in \cdot]$$

$$\begin{aligned} X_t &:= X_{s,s+t}(A) \\ Y_t &:= Y_{s,s-t}(A) \end{aligned} \quad \left. \begin{array}{l} \text{define Markov process} \\ \text{branch point set} \end{array} \right\} \begin{array}{l} (X_t)_{t \geq 0} \\ (Y_t)_{t \geq 0} \end{array} \quad \begin{array}{l} \uparrow \\ \text{expanding interval process} \end{array}$$

Proof $\mathbb{P}[X_{t_1} \in A_1, \dots, X_{t_n} \in A_n]$

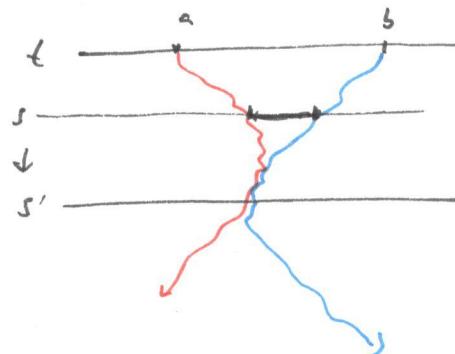
$$= \int_{A_1 \times \dots \times A_n} \mathbb{P}[X_{t_0, t_1}(A_0) \in dA_1, \dots, X_{t_{n-1}, t_n}(A_{n-1}) \in dA_n]$$

$$= \int_{A_1} P_{t_1}(A_0, dA_1) \cdot \dots \cdot \int_{A_n} P_{t_n}(A_{n-1}, dA_n) \quad \boxed{\gamma}$$

- $\mathbb{Y}_{t,s}([a,b]) = \begin{cases} [\hat{\pi}_{(a,t)}^r(s), \hat{\pi}_{(b,t)}^r(s)] & \text{if } s \geq \tau(\hat{\pi}_{(a,t)}^r, \hat{\pi}_{(b,t)}^r) \\ \emptyset & \text{otherwise} \end{cases}$

$$\mathbb{Y}_{t,s}(A \cup B) = \mathbb{Y}_{t,s}(A) \cup \mathbb{Y}_{t,s}(B)$$

$(Y_t)_{t \geq 0}$ = expanding interval process

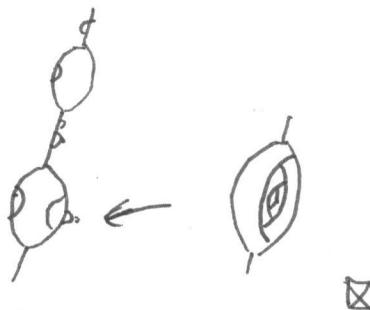


- $\mathbb{P}[X_t \cap Y_0 \neq \emptyset] = \mathbb{P}[X_0 \cap Y_t \neq \emptyset] \quad (t \geq 0)$

Proof $X_{0,t}(X_0) \neq \emptyset \Leftrightarrow \exists \pi \in N(X_0 \times \{0\}) \text{ s.t. } \pi(t) \in Y_0$
 $\Leftrightarrow \cancel{\exists \pi \in N(X_0 \times \{0\})} \quad X_0 \cap \mathbb{Y}_{t,0}(Y_0) \neq \emptyset \quad \square$

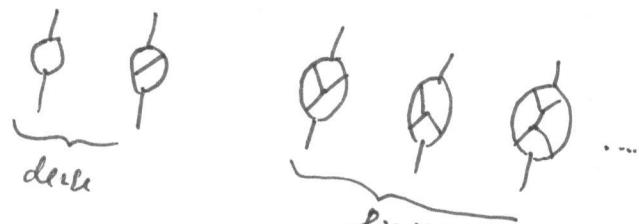
- $X_0 \subset \mathbb{R}$ compact \Rightarrow X_t finite a.s. $\forall t \geq 0$
 nonempty ! a.s. $\exists T \subset (0, \infty)$ dense s.t. $|X_t| = \infty \quad \forall t \in T$

Proof for $X_0 = \{x\}$. $I := \{t \geq 0 : \pi_{(x,t)}^\ell(t) = \pi_{(x,t)}^r(t)\}$ nowhere dense



Open Problem Describe excursion measure

Bubble conjecture:



Def $\bar{X}_t := \{\pi(t) : \pi \in N(-\infty, x)\}$
 \uparrow backbone

- $(\bar{X}_t)_{t \in \mathbb{R}}$ stationary & reversible, $\# \bar{X}_t = \text{Poisson}(2)$.

Proof Finite approximation $\mathbb{P}[\theta_{\epsilon_n}(U_n(-\infty, *))] \Rightarrow \dots \Rightarrow \bar{X}_t \geq \text{Poisson}(2)$

Density formula $\Rightarrow \mathbb{E}[(\bar{X}_t \cap [a,b])] \leq 2$

$$\bullet \quad X_0 \neq \emptyset \Rightarrow \mathbb{P}[X_t \in \cdot] \underset{t \rightarrow 0}{\rightarrow} \text{Pois}(z) \Leftarrow \mathbb{P}[\bar{X}_s \in \cdot] \quad (s \in \mathbb{R})$$

Proof w.l.o.g. $X_0 = \{x\}$.

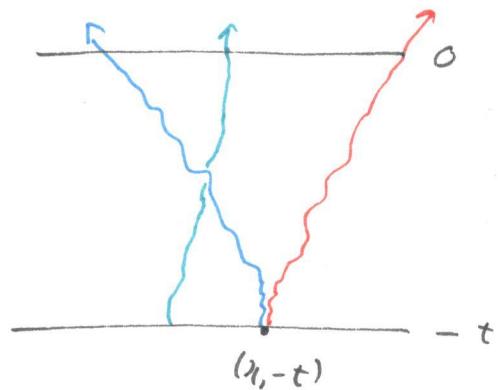
$$\mathbb{X}_{-t, 0}(\{x\}) = \mathbb{X}_{-t, 0}(\bar{\mathbb{R}}) \cap [\pi_{(x, -t)}^{\ell}(t_0), \pi_{(x, t)}^{\nu}(t_0)]$$

$$\downarrow$$

$$\downarrow$$

$$\downarrow$$

☒



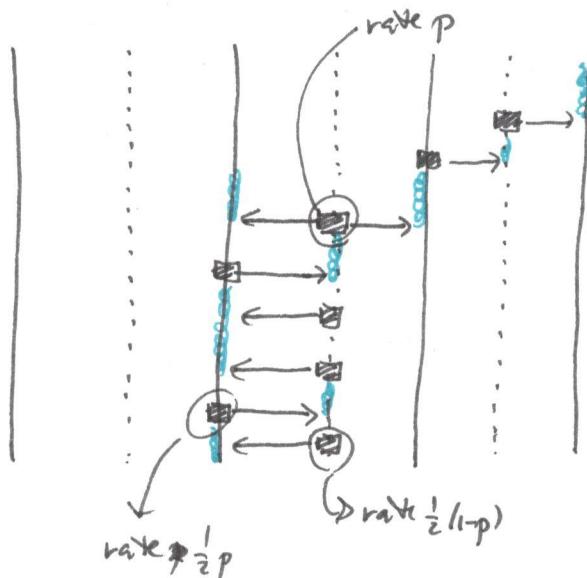
$$G_{BAD} f(x) = \frac{1}{2} p \sum_{(i,j) \in \tilde{\mathbb{F}}} \{f(bra_{ij}(x)) - f(x)\} + \frac{1}{2}(1-p) \sum_{(i,j) \in \tilde{\mathbb{F}}} \{f(k_{ij}(x)) - f(x)\}$$

$$\text{Def } \tilde{\mathbb{F}} := \{(i, j) \in (\frac{1}{2}\mathbb{Z})^2 : |i-j| = \frac{1}{2}\}$$

Def $(Z_t)_{t \geq 0}$ covering branching-coalescent $\quad Z_t \in \{0, 1\}^{\frac{1}{2}\mathbb{Z}}$

$$G_{\text{cov}} f(z) := \frac{1}{2} p \sum_{\substack{(i,j) \in \tilde{\mathbb{F}} \\ i \in \mathbb{Z}}} \{f(rw_{ij}(z)) - f(z)\} + \frac{1}{2}(1-p) \sum_{\substack{(i,j) \in \tilde{\mathbb{F}} \\ i \in \mathbb{Z} + \frac{1}{2}}} \{f(rw_{ij}(z)) - f(z)\} + p \sum_{i \in \mathbb{Z} + \frac{1}{2}} \{f(spl_i T_i(z)) - f(z)\}$$

$$spl_i T_i(z)(k) := \begin{cases} 0 & \text{if } k=i \\ 1 & \text{if } k \in \{i - \frac{1}{2}, i + \frac{1}{2}\}, z(i)=1 \\ z(k) & \text{otherwise} \end{cases}$$



$$\text{Def } \Theta_p(x, t) := (px, \frac{1}{2}p^3 t)$$

Conjecture (easy)

$$\mathbb{P}[\Theta_p(\bar{U}^p) \in \cdot] \xrightarrow[p \rightarrow 0]{} \mathbb{P}[N \in \cdot]$$

Brownian net.

Motivation $R_t :=$ position of right-most path \Rightarrow

$$M_t := R_t - \frac{1}{2}p \int_0^t 1_{\{R_s \in \mathbb{Z}\}} ds \quad \text{martingale}$$

quadratic variation

$$\approx R_t - \frac{1}{2}p^2 t$$

$$\langle M \rangle_t = \frac{1}{4}p \int_0^t 1_{\{R_s \in \mathbb{Z}\}} ds + \frac{1}{4} \int_0^t 1_{\{R_s \in \mathbb{Z} + \frac{1}{2}\}} ds \approx \frac{1}{2}p t$$

$$\mathbb{E}[R_t] \approx \frac{1}{2}p^2 t \quad \text{Var}(R_t) \approx \frac{1}{2}p t$$

$$\mathbb{E}[p R_{\frac{2t}{p^3}}] = p \cdot \frac{1}{2}p^2 \cdot \frac{2t}{p^3} = 1 \quad \text{Var}(p R_{\frac{2t}{p^3}}) = p^2 \cdot \frac{1}{2}p \cdot \frac{2t}{p^3} = 1$$

◻

$$\text{Def } \tilde{\Theta}_p(x, t) := \left(\frac{p}{2}x, \left(\frac{p}{2}\right)^3 t \right) \quad X_{\left(\frac{p}{2}\right)^3 t}^p := \left\{ \frac{p}{2}i : i \in \mathbb{Z}, X_t^p(i) = 1 \right\}$$

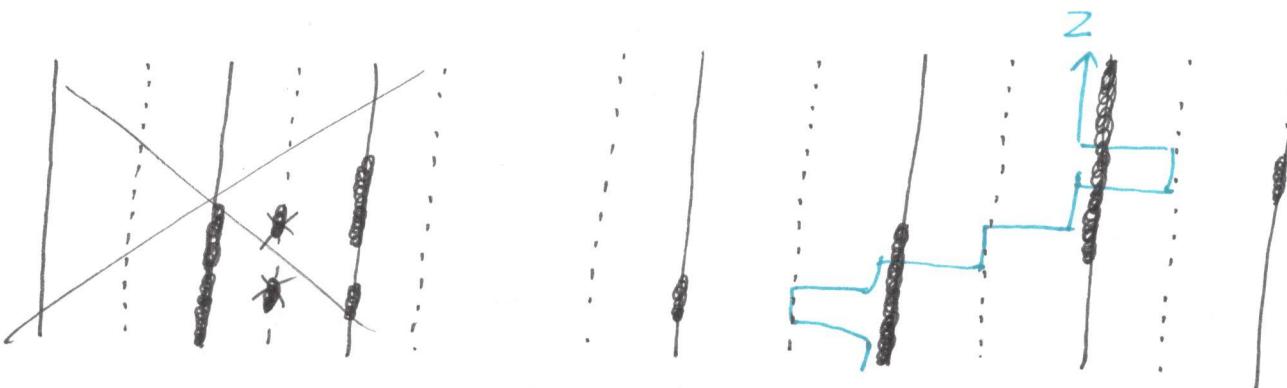
Conjecture (difficult)

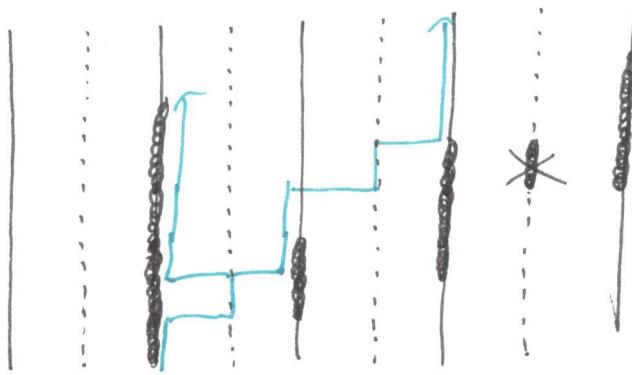
$$\mathbb{P}[(X_t^p)_{t \geq 0} \in \cdot] \xrightarrow[p \rightarrow 0]{} \mathbb{P}[(X_t)_{t \geq 0} \in \cdot]$$

↑ branco point set.

Motivation Couple B&B $(X_t)_{t \geq 0}$ to covering branching-coalescent $(Z_t)_{t \geq 0}$.

single particle



Branching

2 particles of covering branc Z at distance 1

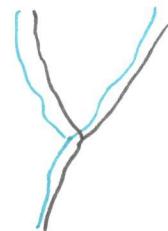
have probab $\frac{1}{L}$ to reach distance L from each other
before they coalesce.

$$\left. \begin{array}{l} \text{splitting rate} \\ p \cdot 1_{Z+\frac{1}{2}} \end{array} \right)$$

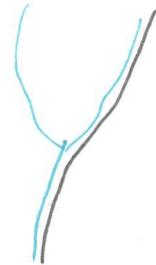
2 particles of $BAB \times$ at distance 2

have probab $\frac{1}{L-1}$ to reach distance L before coalescence.

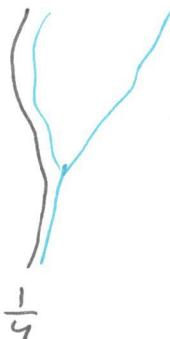
$$\left. \begin{array}{l} \text{effective branch rate} \\ \frac{1}{2} p \cdot 1_{Z+\frac{1}{2}} \end{array} \right)$$



$$\frac{1}{2}$$

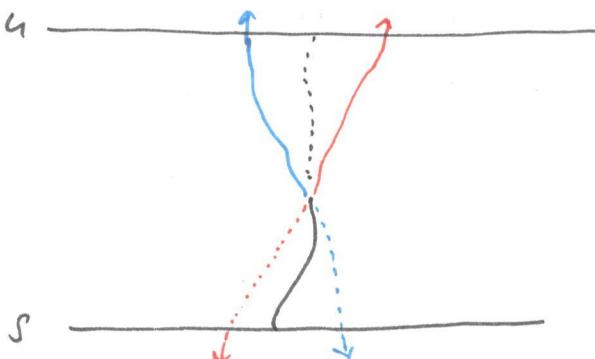


$$\frac{1}{3}$$



$$\frac{1}{4}$$

Def (S, u) -relevant separation point



- Set of (S, u) -relevant separation points is locally finite.