# Rank-based Markov chains, self-organized criticality, and order book dynamics

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joint with Marco Formentin, Jana Plačková

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# A model for email communication

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Priorities are i.i.d. with some atomless law. Without loss of generality we can take the uniform distribution on $[-\lambda_{\rm in}, 0]$.

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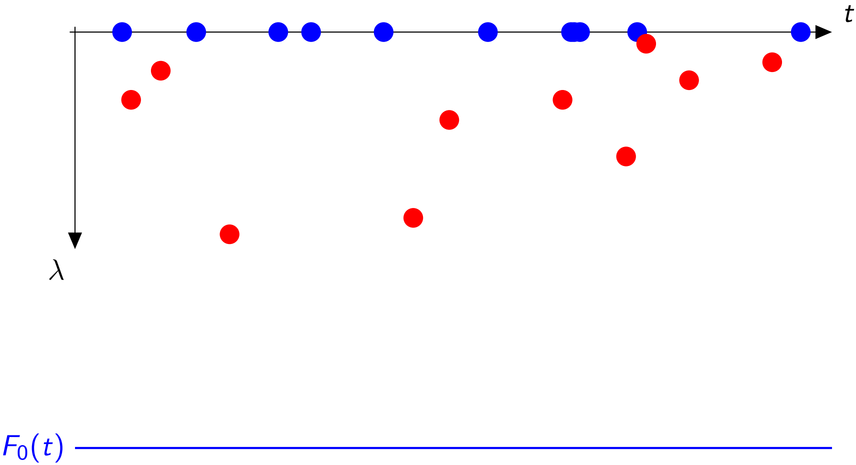
This random walk is positive recurrent for $\lambda < \lambda_{\rm out}$, null recurrent for $\lambda = \lambda_{\rm out}$, and transient for $\lambda > \lambda_{\rm out}$. ∎

# Poisson construction

Let $F_\lambda(t)$ denote the number of emails with priority in $[-\lambda, 0]$ that are in the inbox at time $t$.

We can read off $F_\lambda(t)$ from the Poisson processes describing the arrivals of new emails and answering times.

# Poisson construction



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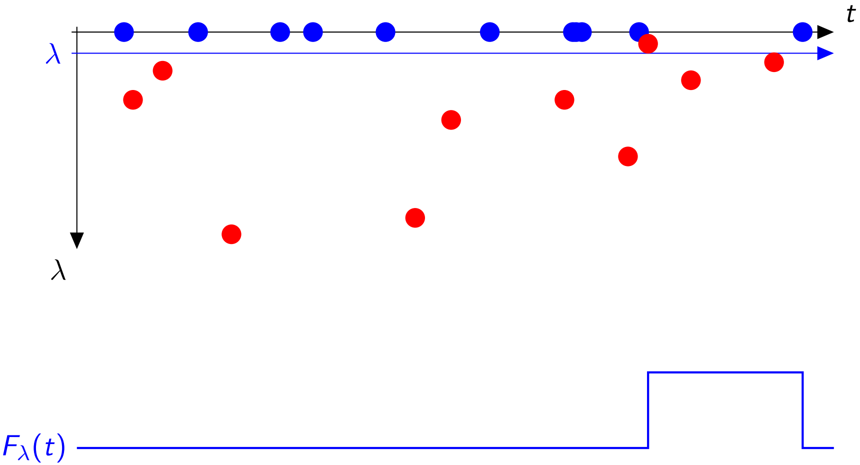

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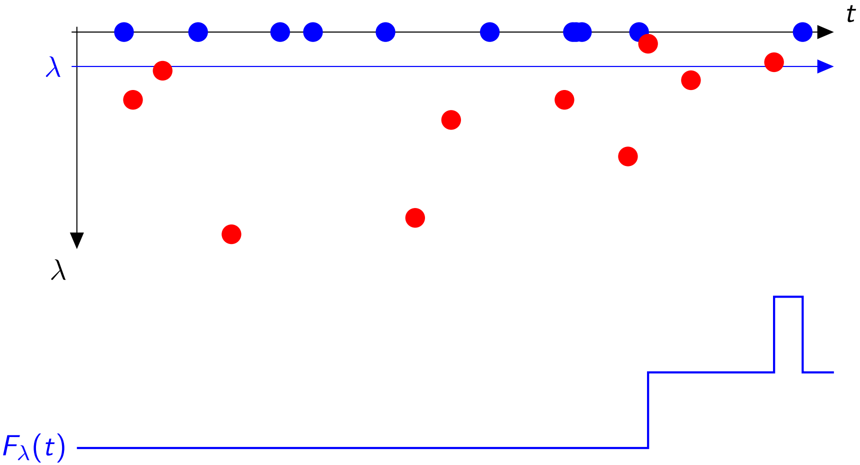

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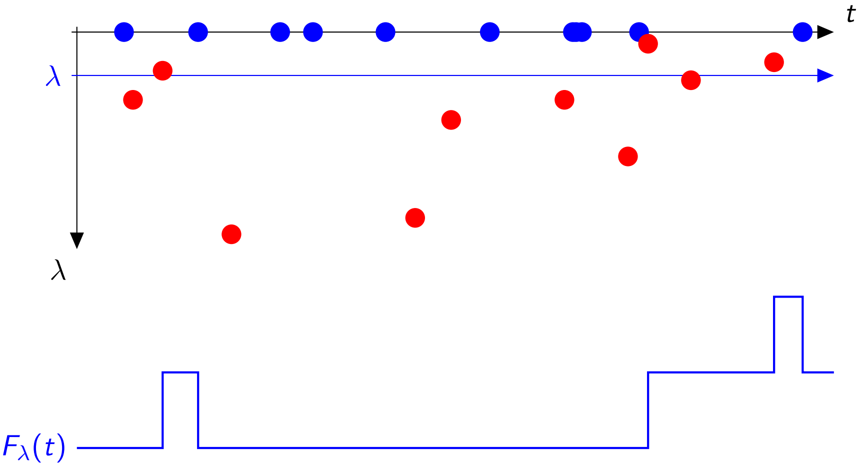

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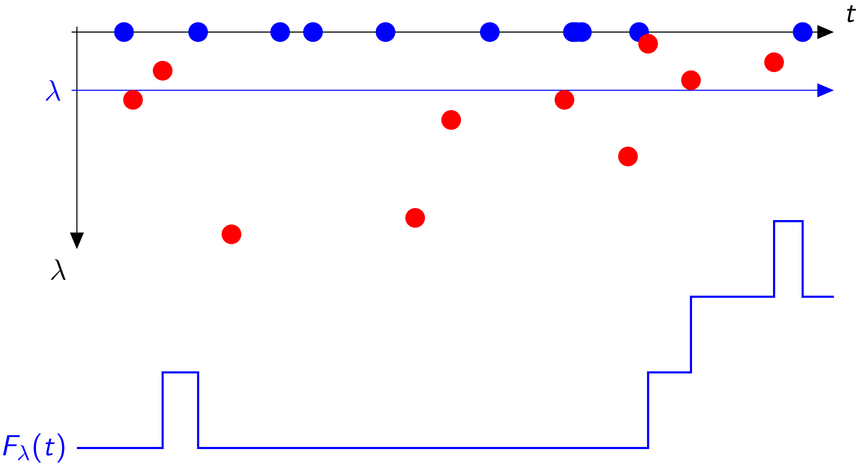

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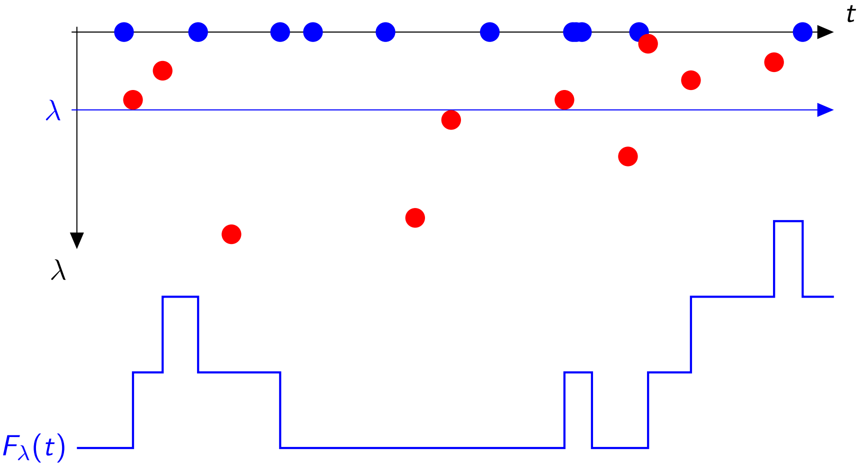

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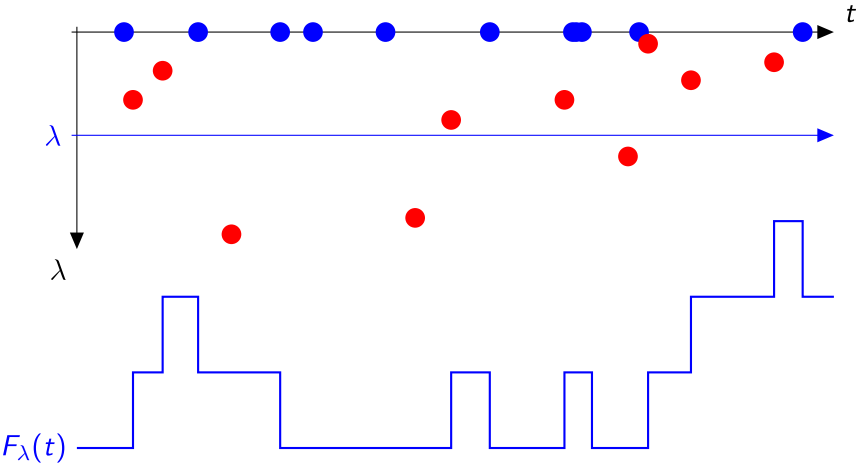
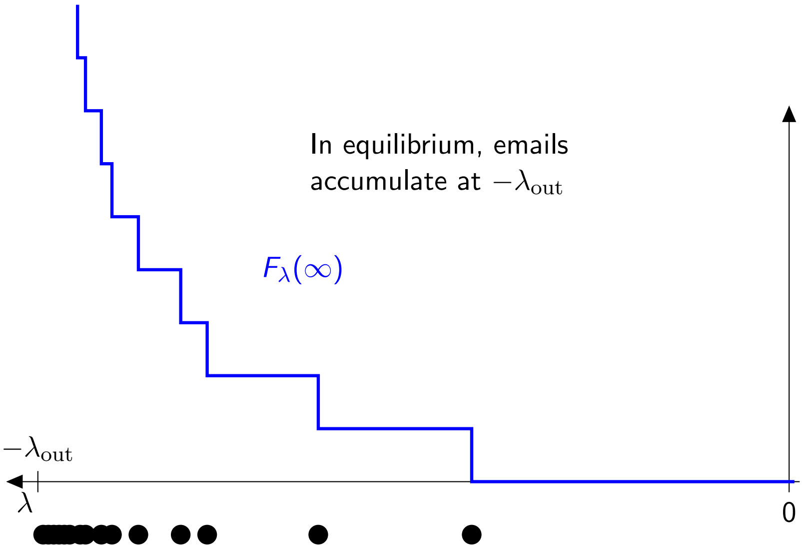
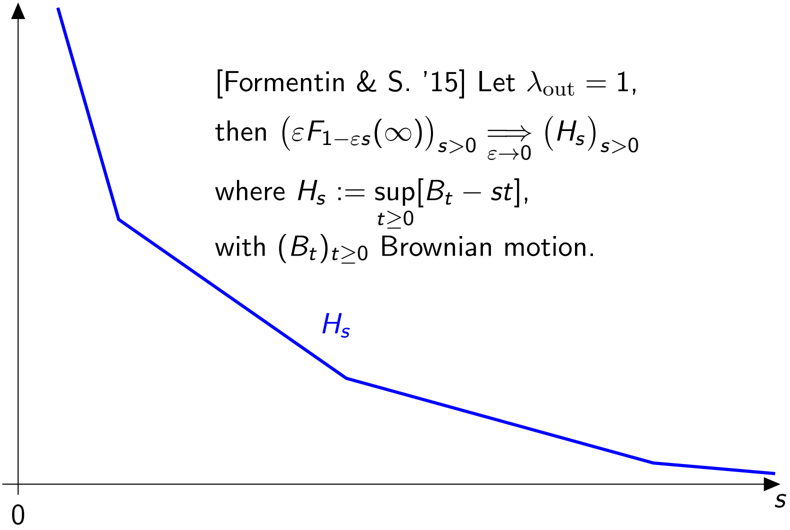


# The equilibrium distribution



# Critical behavior




[Formentin & S. '15] Let $\lambda_{\text{out}} = 1$,

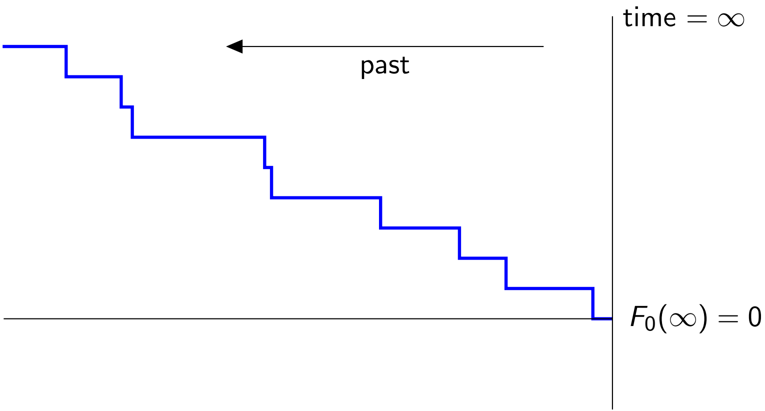
then $\big(\varepsilon F_{1-\varepsilon s}(\infty)\big)_{s>0} \underset{\varepsilon \to 0}{\Longrightarrow} \big(H_s\big)_{s>0}$

where $H_s := \sup_{t \geq 0} [B_t - st]$,

with $(B_t)_{t \geq 0}$ Brownian motion.

Groeneboom (1983): the concave majorant of Brownian motion.

# Coupling from the past



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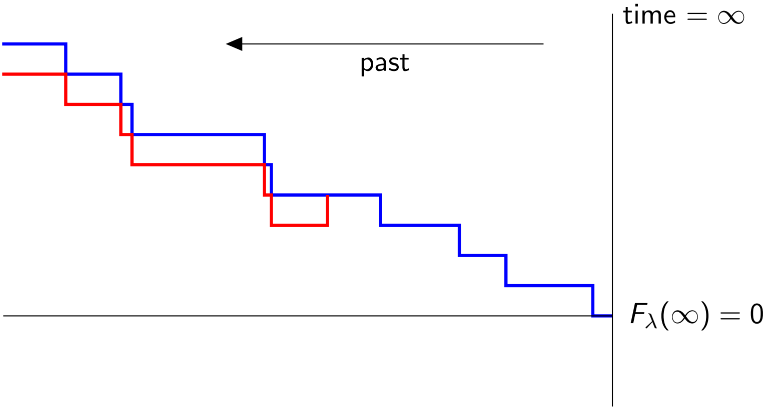

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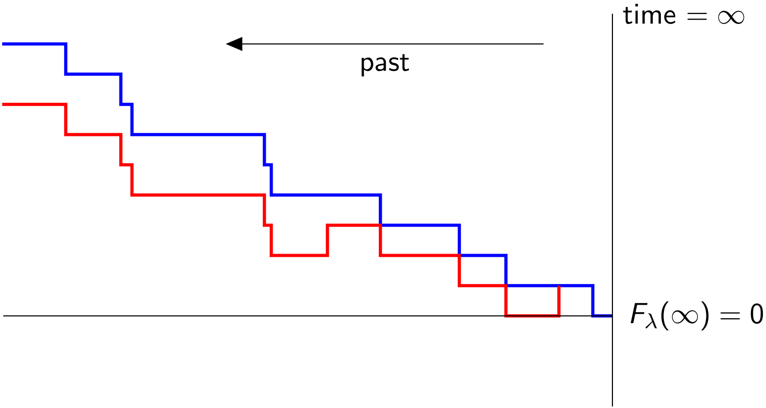

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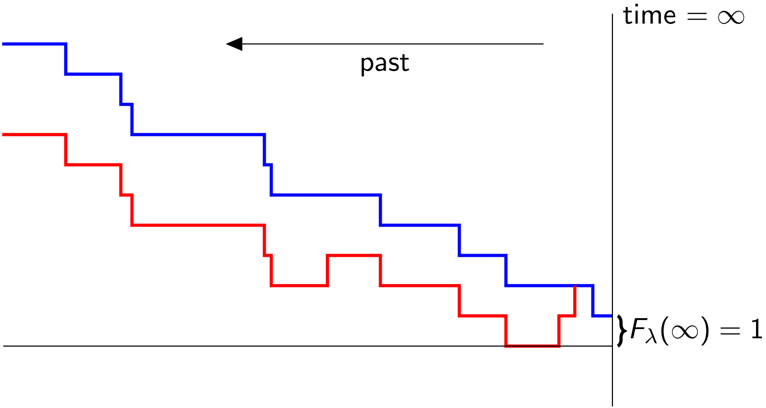

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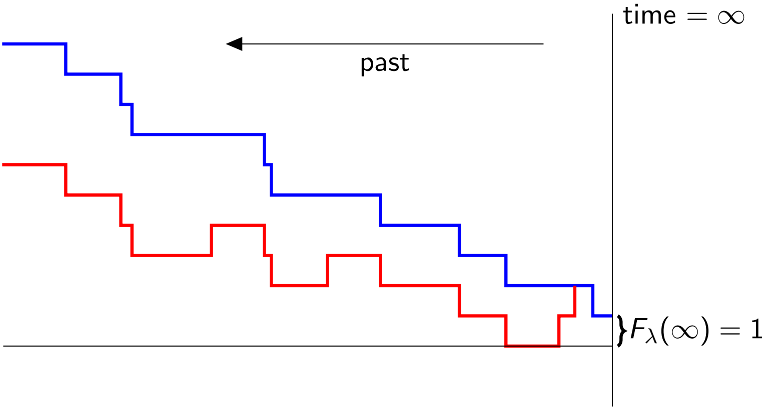

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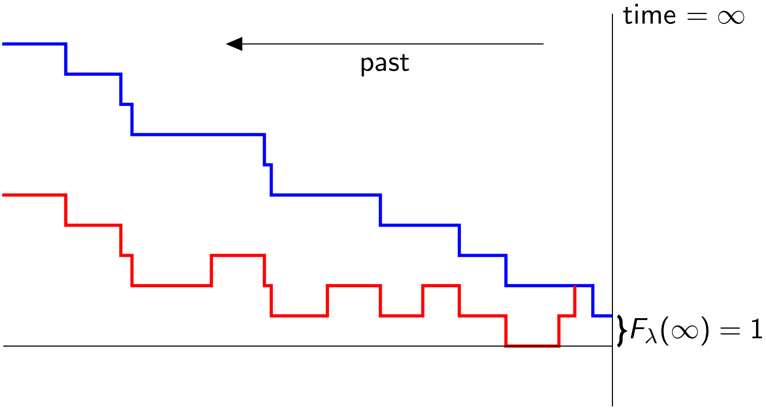

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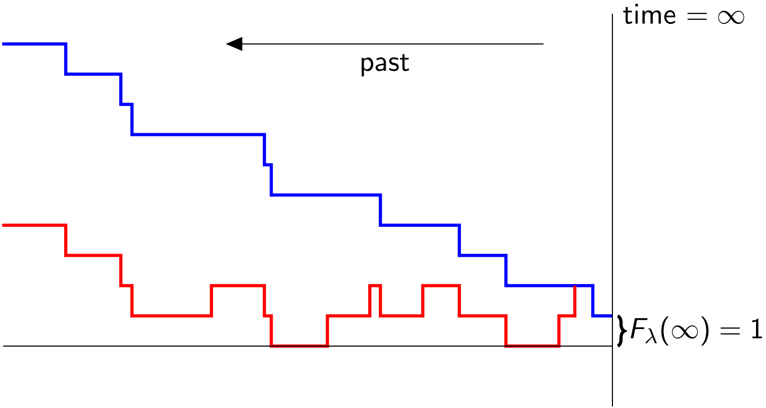

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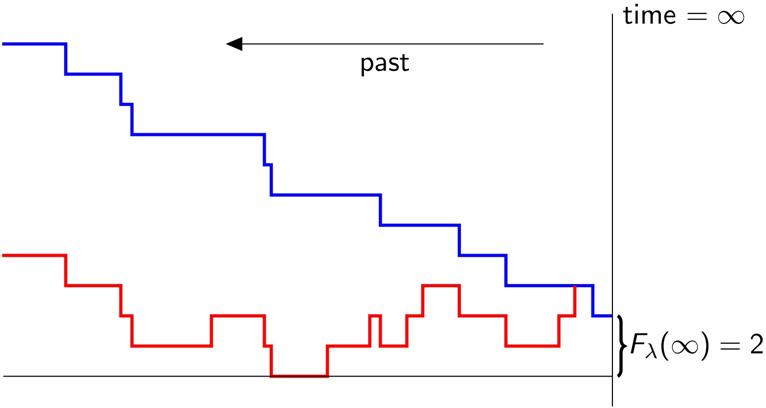

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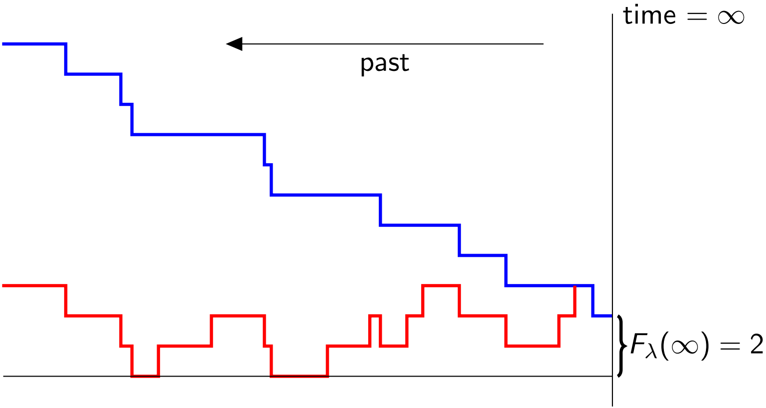

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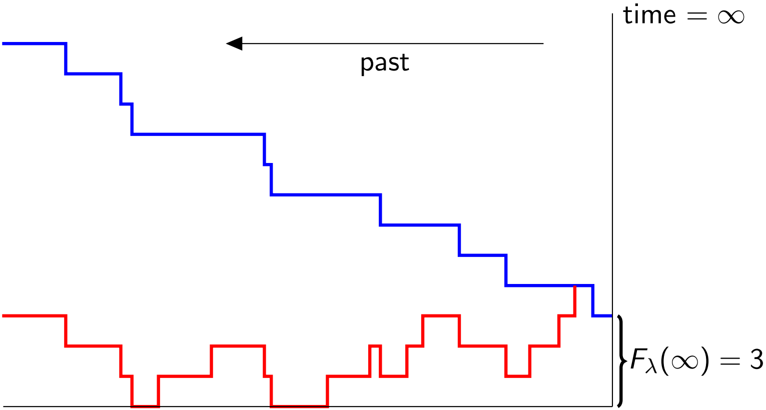


# Self-organized criticality

Physical systems with second order phase transitions exhibit *critical behavior* at the point of the phase transition, which is characterized by:

- Scale invariance.
- Power law decay of quantities.
- Critical exponents.

Usually, critical behavior is only observed when the parameter(s) of the system, such as the temperature, have just the right value so that we are at the point of the phase transition, also called (in this context) the *critical point*.

Some physical systems show critical behavior even without the necessity to tune a parameter to exactly the right value.

In particular, this happens for systems whose dynamics find the critical point themselves. Such systems are said to exhibit *self-organized criticality*.

A classical example are sandpiles, which automatically find the maximal slope that is still stable. Adding a single grain to such a sandpile causes an avalanche whose size has a power-law distribution.

The Bak Sneppen model is another classical example of self-organized criticality and a cornerstone of Bak's (1996) book.

# Self-organized criticality

In the email model, the distribution of serving times (of answered emails) has a power-law tail. Indeed, it seems that in equilibrium, at any time, the probability that the last email we have answered had spent a time $\geq t$ in our inbox decays as $t^{-1/2}$.

This is quite different from the usual exponential tails in queueing theory.

This sort of power law decay, with the exponent $1/2$, has even been observed in real data, provided time is measured in units proportional to the activity of the owner of the inbox (as judged from the number of emails sent). [Formentin, Lovison, Maritan, Zanzotto, J. Stat. Mech. 2015].

# The Bak Sneppen model

Introduced by Bak & Sneppen (1993).

Consider an ecosystem with $N$ species. Each species has a fitness in $[0, 1]$.

In each step, the species $i \in \{1, \ldots, N\}$ with the lowest fitness dies out, together with its neighbors $i - 1$ and $i + 1$ (with periodic b.c.), and all three are replaced by species with new, i.i.d. uniformly distributed fitnesses.

There is a critical fitness $f_\mathrm{c} \approx 0.6672(2)$ such that when $N$ is large, after sufficiently many steps, the fitnesses are approximately uniformly distributed on $(f_\mathrm{c}, 1]$ with only a few smaller fitnesses. Moreover, for each $\varepsilon > 0$, the lowest fitness spends a positive fraction of time above $f_\mathrm{c} - \varepsilon$, uniformly as $N \to \infty$.

# The modified Bak Sneppen model

Introduced by Meester & Sarkar (2012).

Instead of the neighbors of the least fit species, choose one arbitrary other species from the population that dies together with the least fit species.

Critical point exactly $f_{\rm c} = 1/2$.

Critical behavior at $f_{\rm c}$: intervals between times when all individuals have a fitness $> f_{\rm c}$ have a power-law distribution with $\mathbb{P}[\tau \geq k] \sim k^{-1/2}$.

Proof based on coupling to a branching process.

# Rank-based models

The email model and (modified) Bak Sneppen model share some common features:

- Only the relative order of the priorities/fitnesses matter. As a result, replacing the uniform distribution with any other atomless law basically yields the same model (up to a transformation of space).
- Both models use some version of the rule "kill the minimal element".
- Both models exhibit *self-organized criticality*.

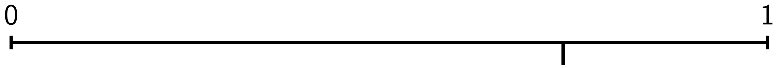
In what follows, we will look at some other models that fall into the same class. In particular:

- Two toy models for canyon formation.
- The Stigler-Luckock model for the evolution of an order book.

# A model for canyon formation


We start with a flat rock profile.

# A model for canyon formation




The river cuts into the rock at a uniformly chosen point.

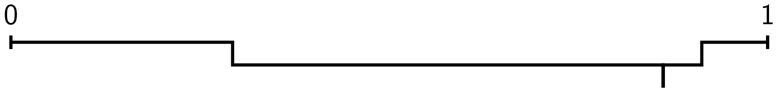
# A model for canyon formation


Rock between a next point and the river is eroded one step down.

# A model for canyon formation


We continue in this way.

# A model for canyon formation




Either the river cuts deeper in the rock.

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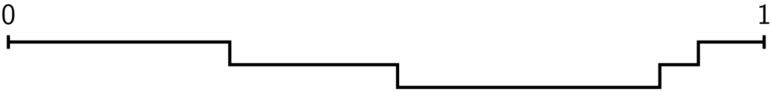


Or one side of the river is eroded down.

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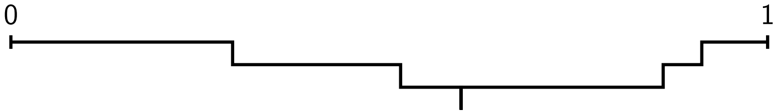


We are interested in the limit profile.

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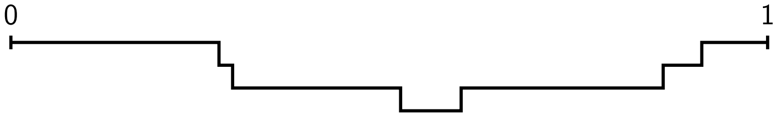


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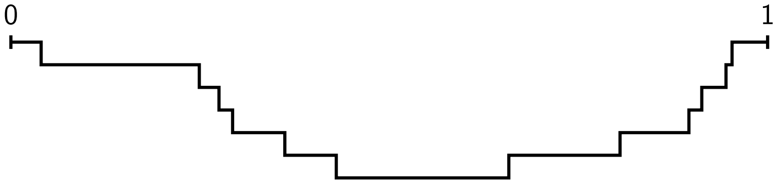


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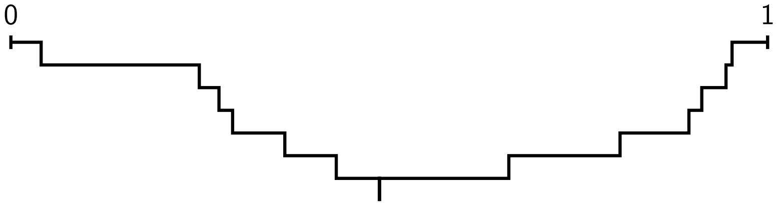


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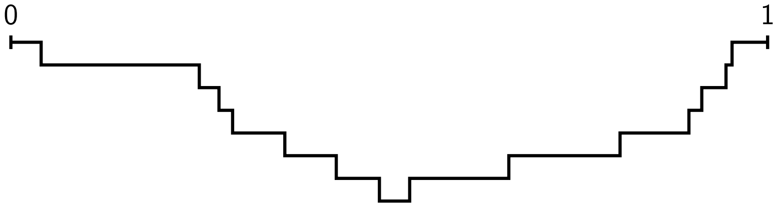


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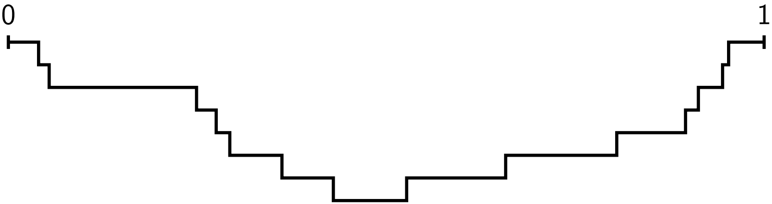


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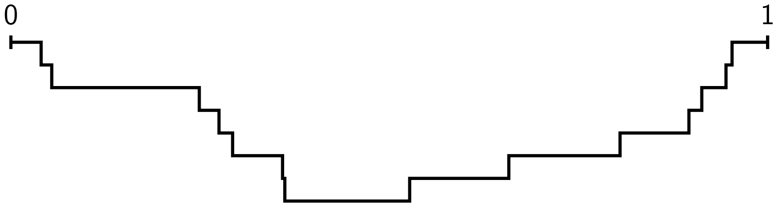


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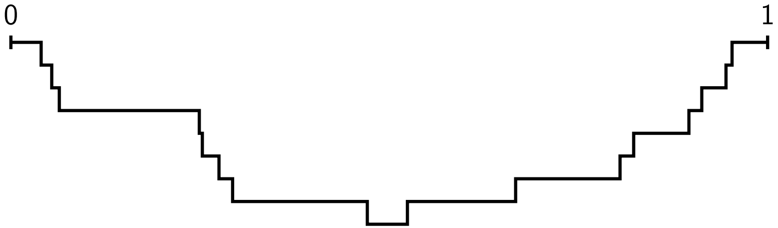


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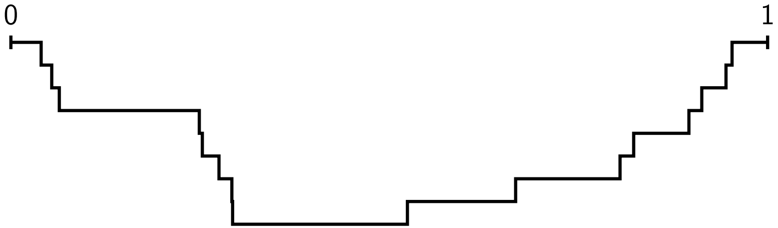


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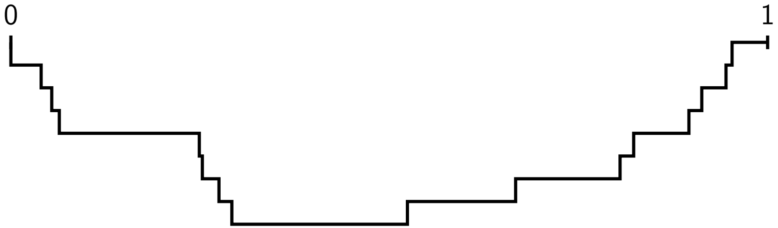


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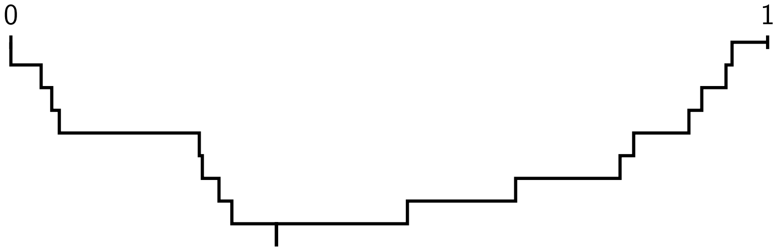


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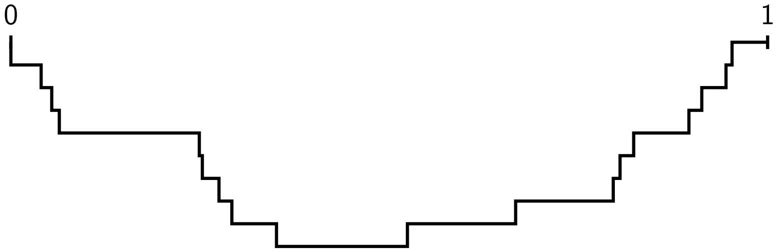


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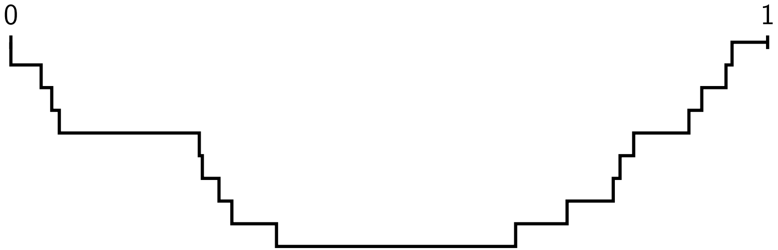


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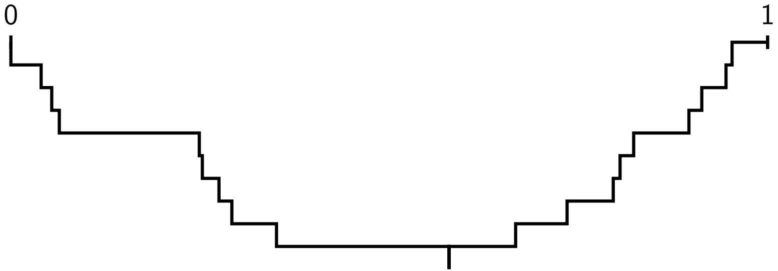


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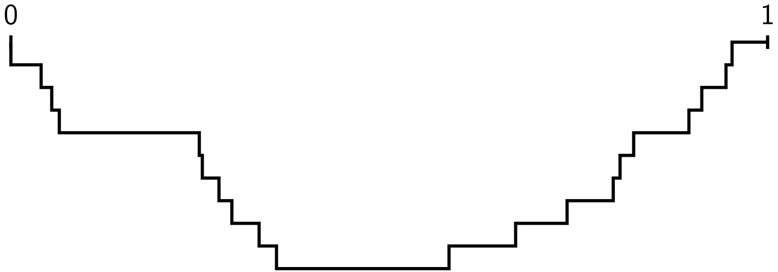


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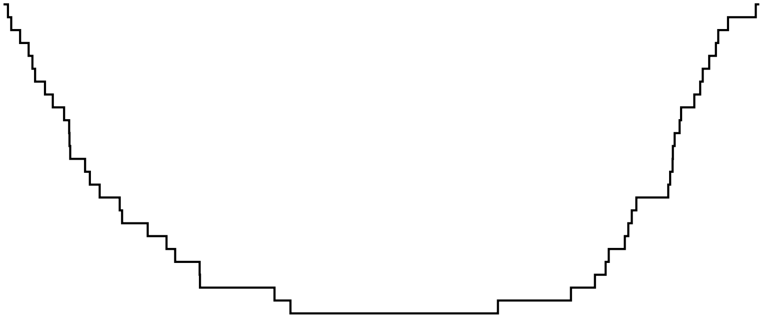
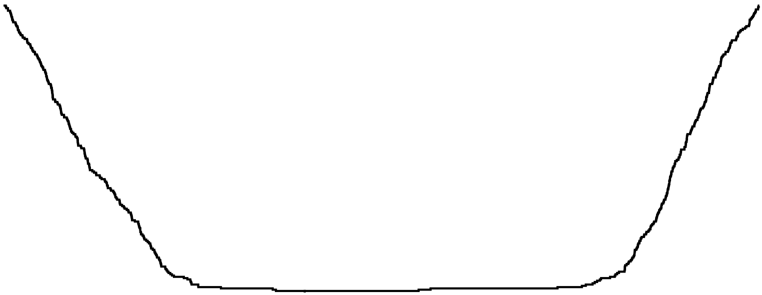


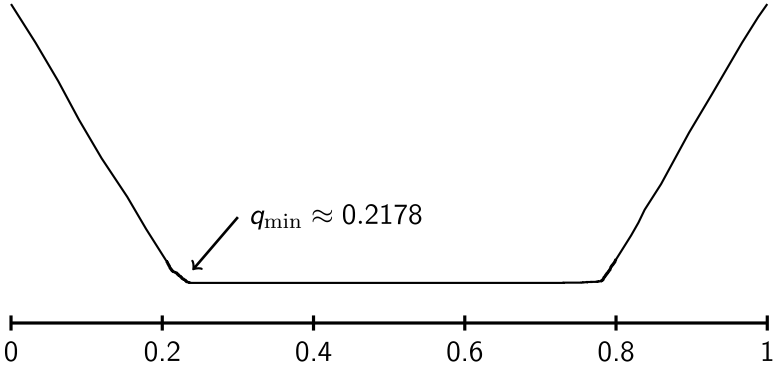
The profile after 100 steps.

# A model for canyon formation



The profile after 1000 steps.

# A model for canyon formation




The profile after 10,000 steps.

# A model for canyon formation

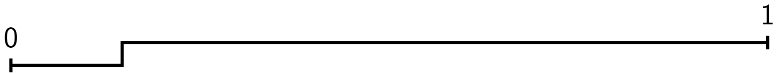
In simulations, the canyon model is very similar to the email model, but it is more difficult to treat mathematically.

- Can we prove the existence of the critical point and explain its value $q_{\min} \approx 0.2178$?
- In equilibrium, scaling in on the critical point, do we again find the convex hull of Brownian motion?

# A one-sided canyon model

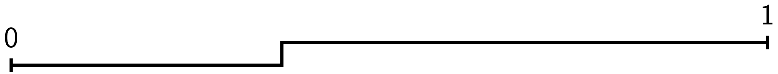

A river flows on the left.

# A one-sided canyon model




The river either cuts deeped into the rock.

# A one-sided canyon model




Or the shore is eroded down, starting from a random point.

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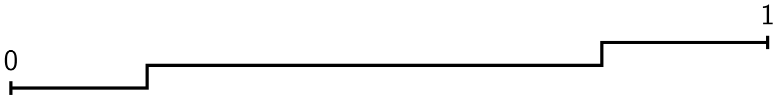
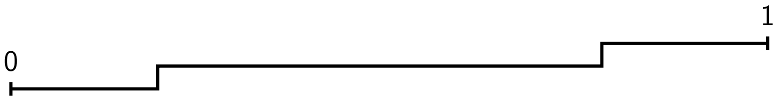



We either make the river deeper. . .

# A one-sided canyon model

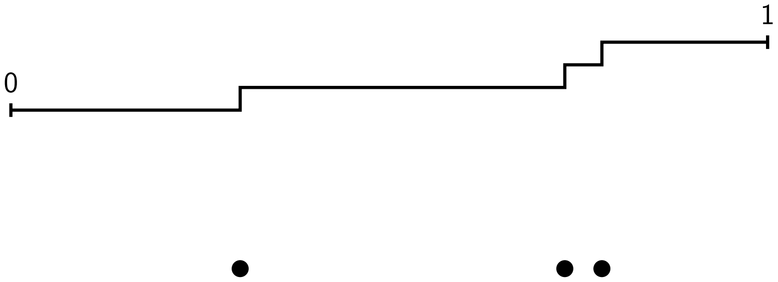



. . . or we erode the shore,

# A one-sided canyon model

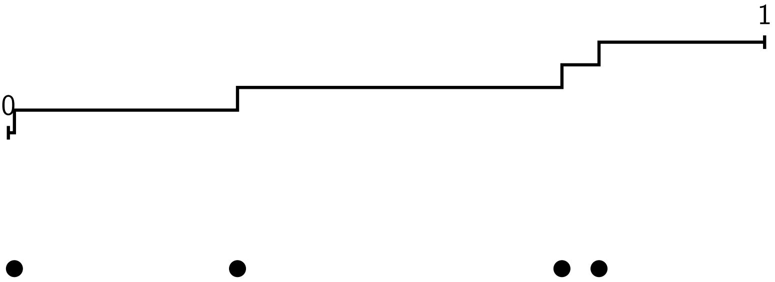

. . . depending on where the new point falls.

# A one-sided canyon model




Points on the left of all others are simply added.

# A one-sided canyon model

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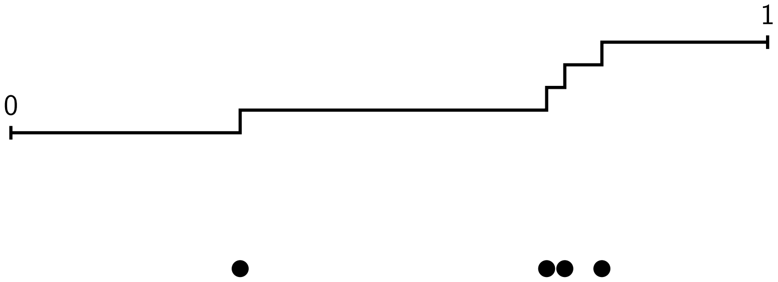
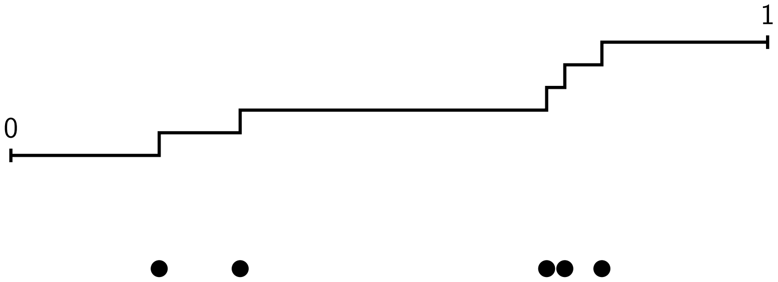



Otherwise, we remove the left-most point.

# A one-sided canyon model

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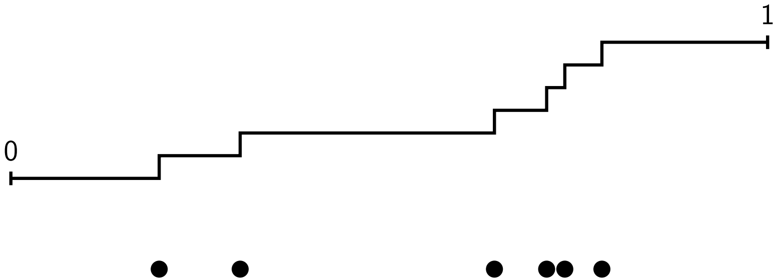


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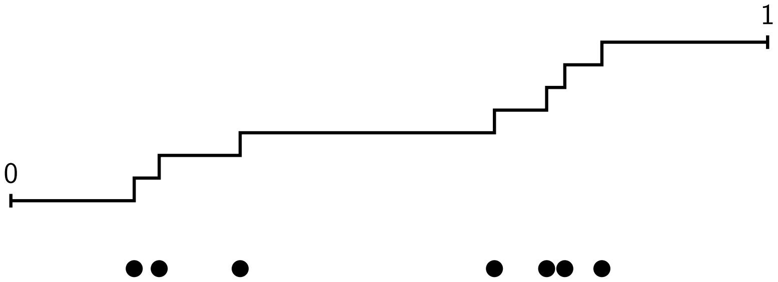


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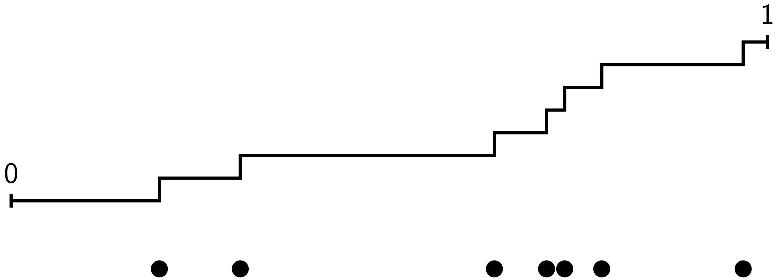


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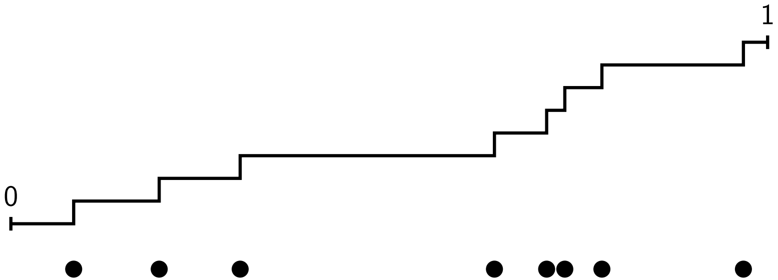


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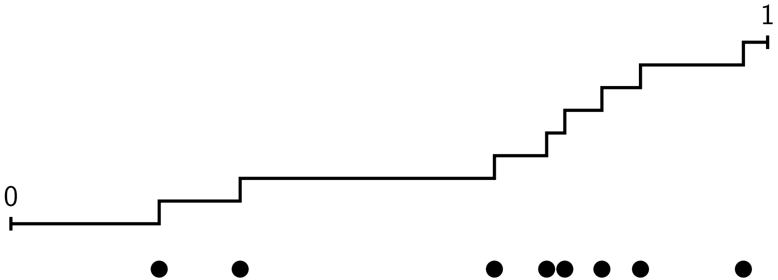


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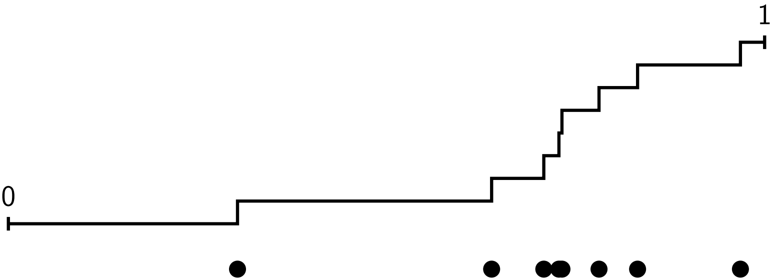


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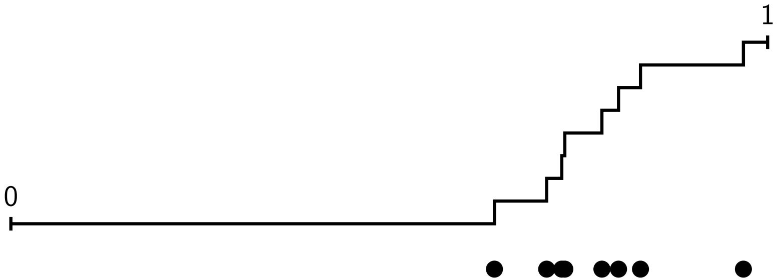


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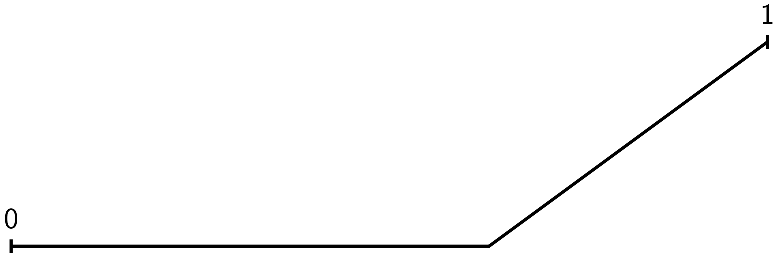



In this model, the critical point is $p_c = 1 - e^{-1} \approx 0.63212$.

# The one-sided canyon model

The process just described defines a Markov chain $(X_k)_{k\geq 0}$ where $X_k \subset [0,1]$ is a finite set.

**Consistency:** For each $0 < q < 1$, we observe that the *restricted process*

$$\big(X_k \cap [0,q]\big)_{k\geq 0}$$

is a Markov chain.

**Theorem 1** The restricted process is positively recurrent for $q < 1 - e^{-1}$ and transient for $q > 1 - e^{-1}$.

**Theorem 2** The restricted process is null recurrent at $q = 1 - e^{-1}$.

**Proof of Theorem 1** Since only the relative order of the points matters, transforming space we may assume that the $(U_k)_{k\geq 1}$ are i.i.d. *exponentially* distributed with mean one and $X_k \subset [0,\infty]$.

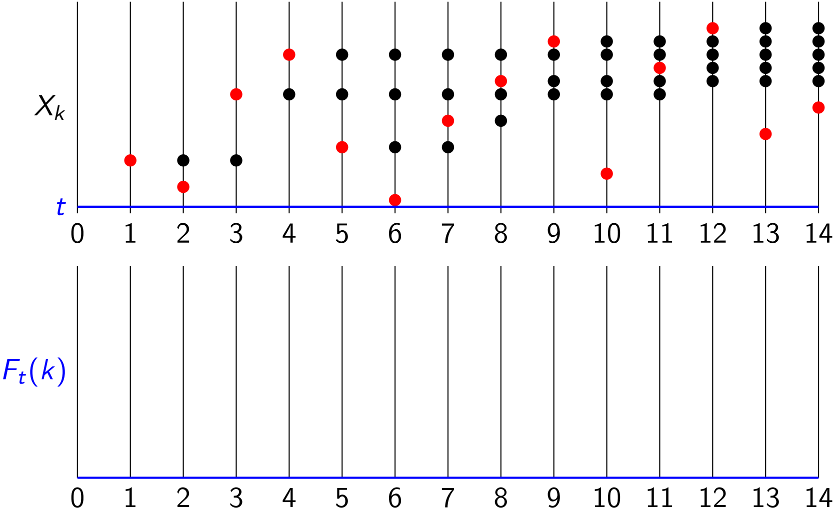
For the modified model, we must prove that $p_{\rm c} = 1$.

Start with $X_0 = \emptyset$ and define

$$F_t(k) := \big|X_k \cap [0,t]\big| \qquad (k \geq 0,\ t \geq 0).$$

**Claim** $(F_t)_{t\geq 0}$ is a continuous-time Markov process taking values in the functions $F : \mathbb{N} \to \mathbb{N}$.

# The point-counting function



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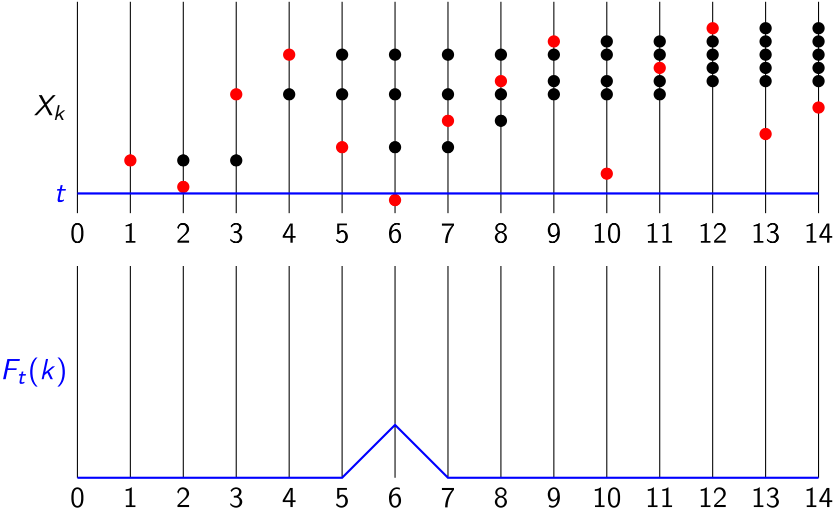

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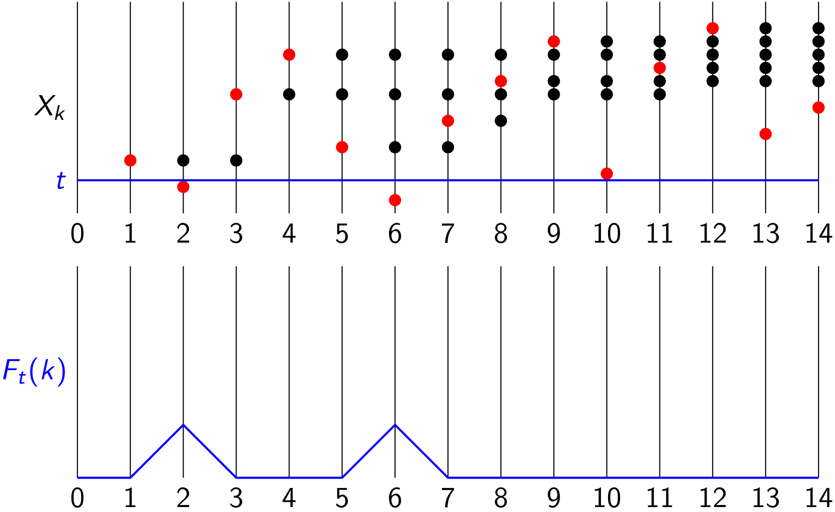

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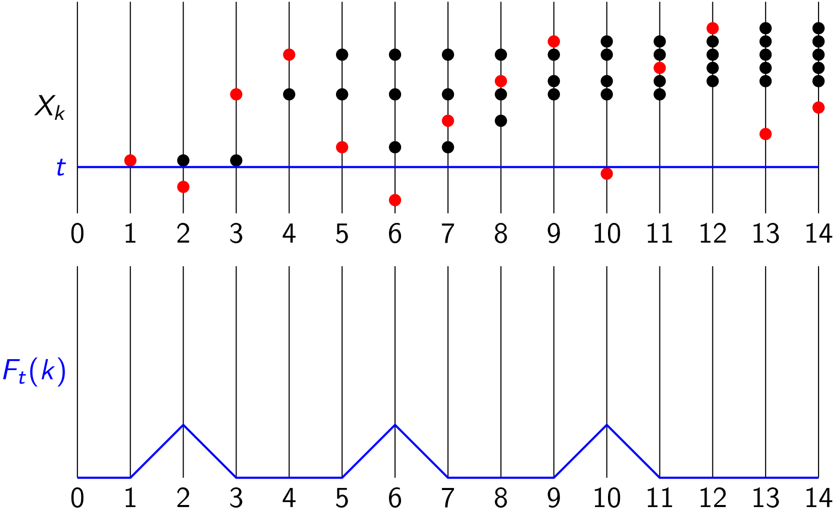

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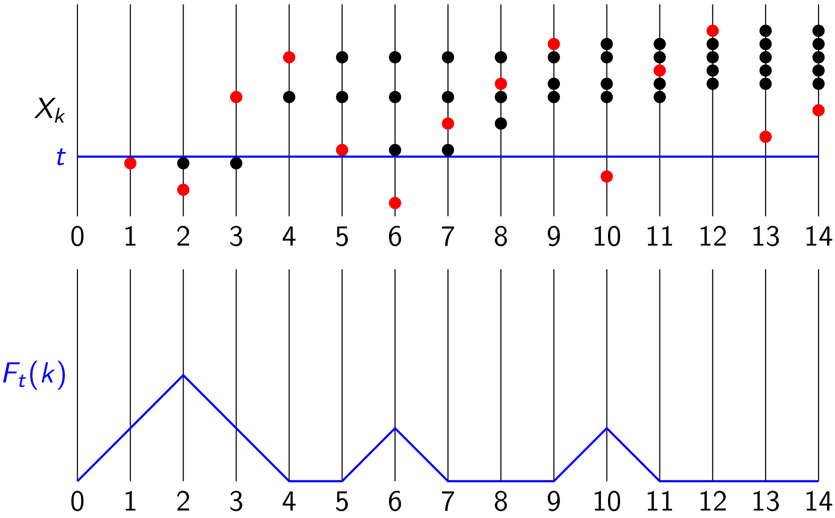

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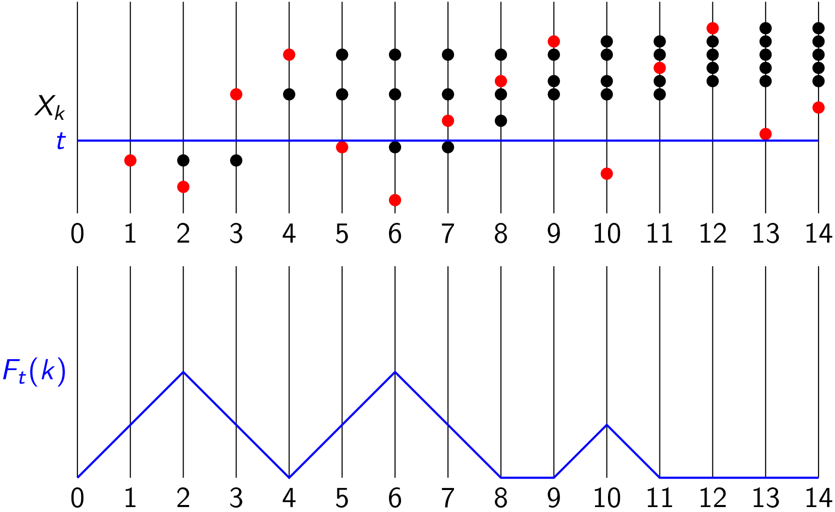

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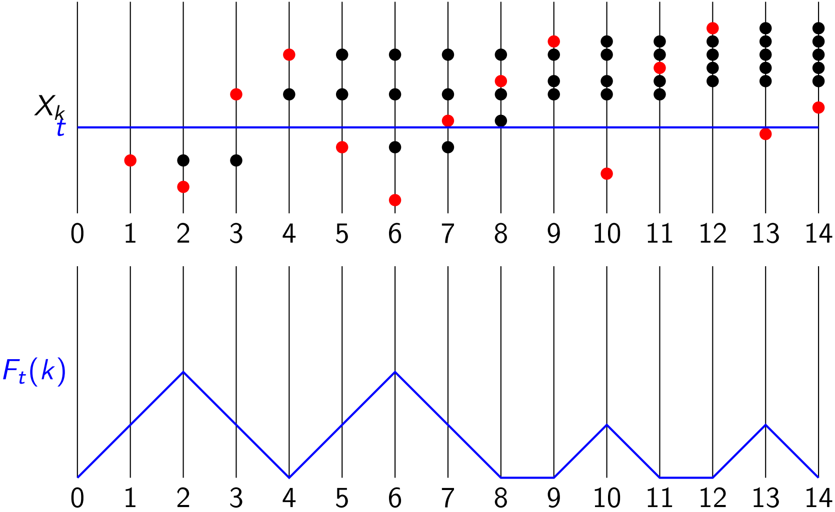

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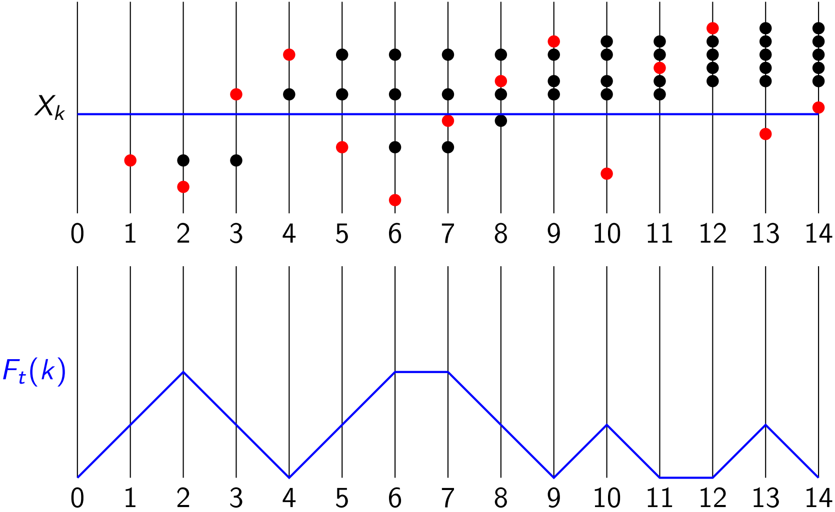

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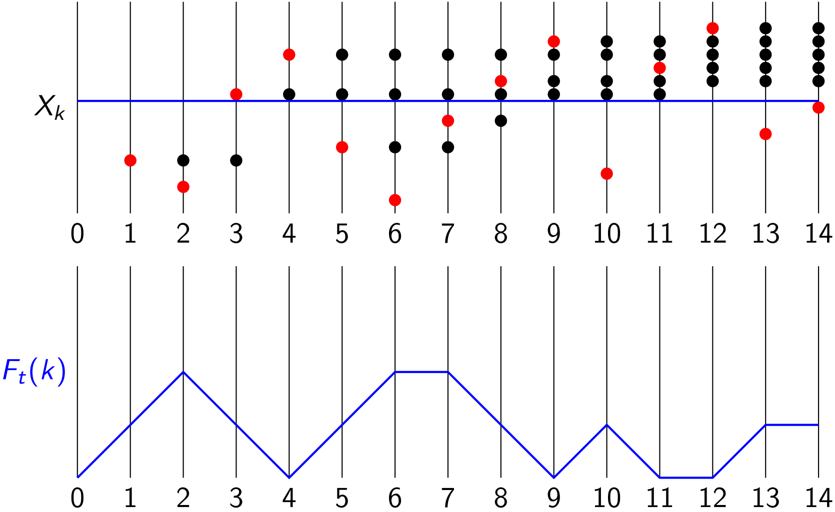

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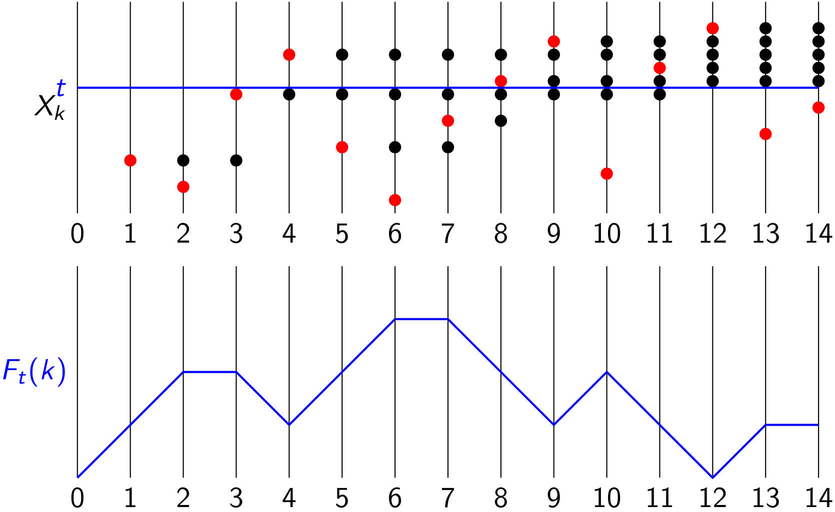

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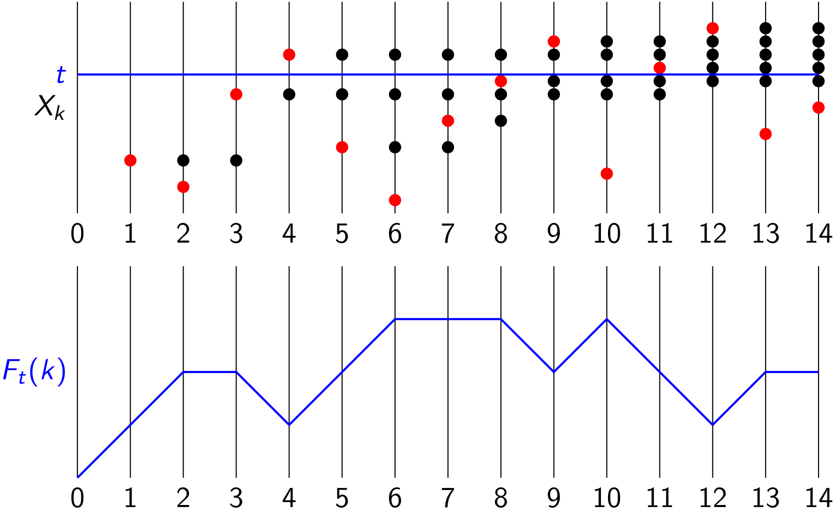

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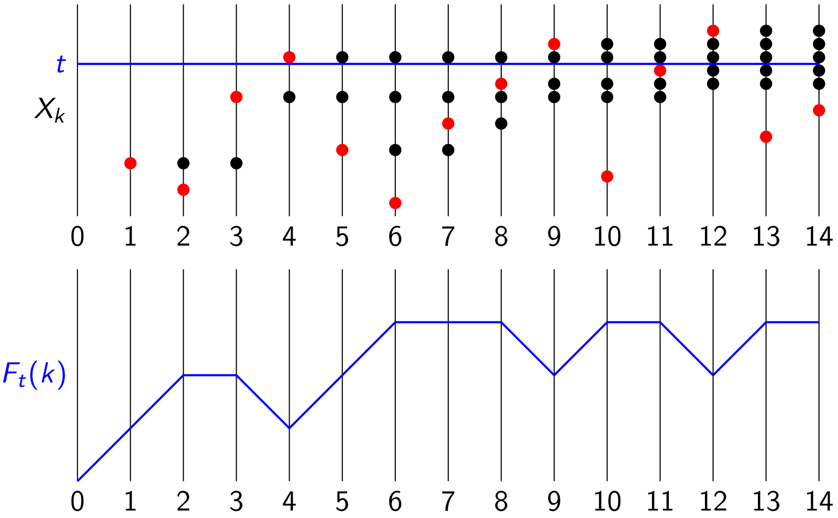

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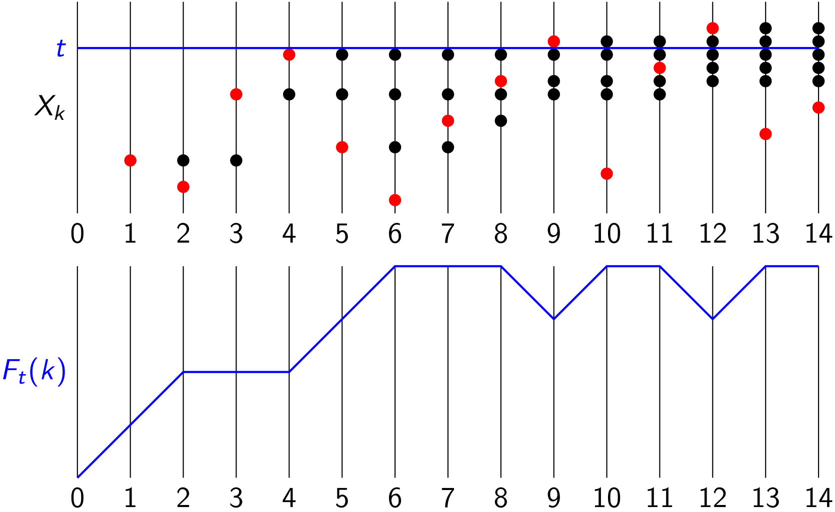

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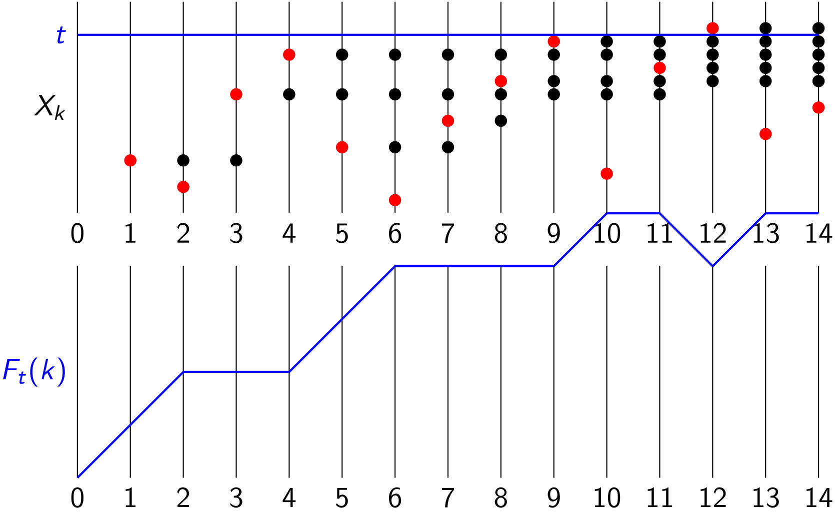

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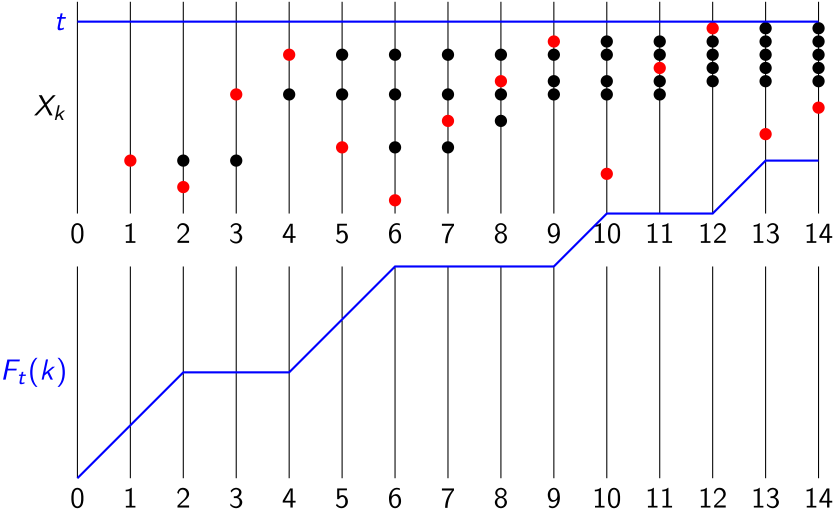

# Increments

Define

$$\Delta F_t(k) := \begin{cases} \underline{0} & \text{if } F_t(k) = F_t(k-1) = 0, \\ \overline{0} & \text{if } F_t(k) = F_t(k-1) > 0, \\ -1 & \text{if } F_t(k) = F_t(k-1) - 1, \\ +1 & \text{if } F_t(k) = F_t(k-1) + 1. \end{cases}$$

At the exponentially distributed time $t = U_k$, the increment $\Delta F_t(k)$ changes from $\underline{0}$ to $+1$ or from $-1$ to $\overline{0}$.

At the same time, the next $\underline{0}$ to the right of $k$, if there is one, is changed into a $-1$.

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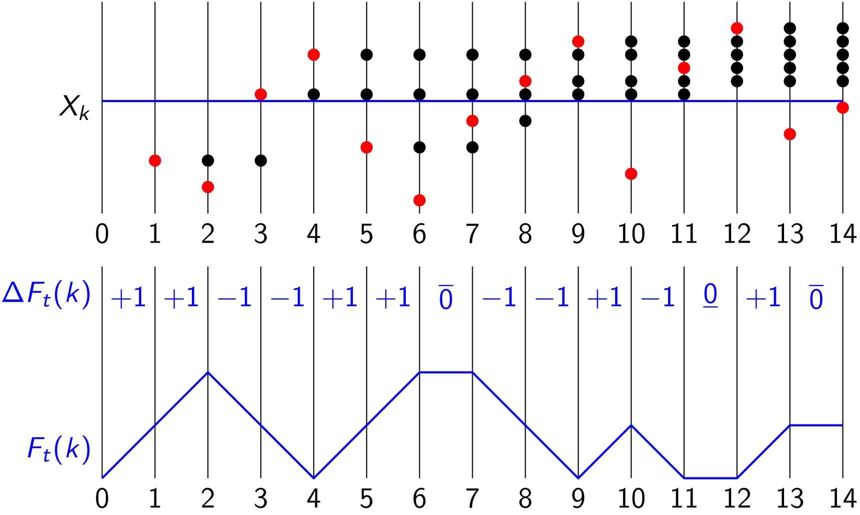

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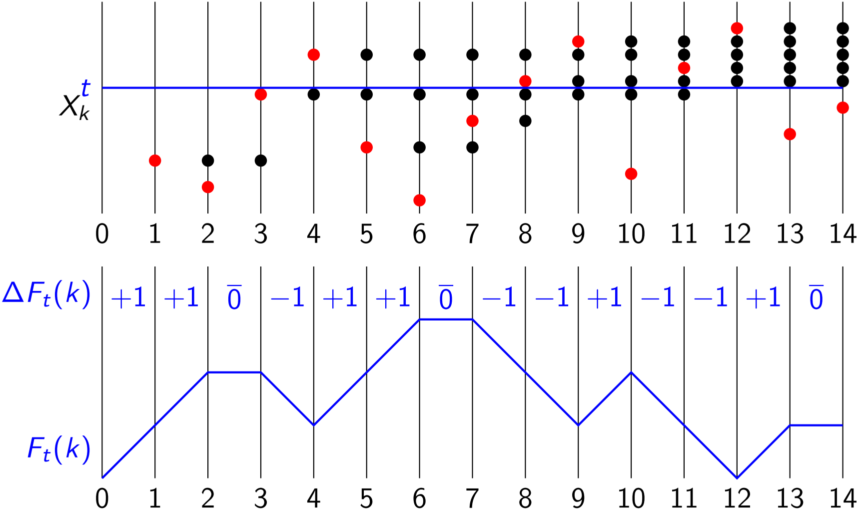


# A stationary increment process

We can define the Markov process $(\Delta F_t)_{t \geq 0}$ also on $\mathbb{Z}$ instead of on $\mathbb{N}_+$.

As long as the density of $\underline{0}$'s is nonzero, the process started in $\Delta F_0(k) = \underline{0}$ $(k \in \mathbb{Z})$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t}\mathbb{P}[\Delta F_t(k) = \underline{0}] &= -2\mathbb{P}[\Delta F_t(k) = \underline{0}] - \mathbb{P}[\Delta F_t(k) = -1], \\ \frac{\partial}{\partial t}\mathbb{P}[\Delta F_t(k) = -1] &= \mathbb{P}[\Delta F_t(k) = \underline{0}], \end{aligned}$$

from which we derive that the $\underline{0}$'s run out at $t_{\mathrm{c}} = 1$ and

$$\mathbb{P}[\Delta F_t(1) = \underline{0}] = (1-t)e^{-t} \quad \text{and} \quad \mathbb{P}[\Delta F_t(1) = -1] = te^{-t}$$

$(0 \leq t \leq 1)$.

# The increment process

The process $(F_t)_{t\geq 0}$, both on $\mathbb{N}_+$ and $\mathbb{Z}$, makes i.i.d. excursions away from 0.

For the process started in $X_0 = \emptyset$, define the return time

$$\tau_t^{\emptyset} := \inf\big\{k \geq 1 : X_k \cap [0,t] = \emptyset\big\}.$$

From the density of $\underline{0}$'s for the process $(F_t)_{t\geq 0}$ on $\mathbb{Z}$ we deduce that

$$\mathbb{E}[\tau_t^{\emptyset}] = (1-t)^{-1} \qquad (0 \leq t < 1).$$

This proves positive recurrence $\Leftrightarrow$ $t < 1$.

It is not hard to derive from this that the restricted process $\big(X_k \cap [0,t]\big)_{k\geq 0}$ is transient for $t > 1$. ∎

# A weight function

*New progress:*

**Proof of Theorem 2 and alternative proof of Theorem 1**
For $t > 0$, consider the weighted sum over points in $X_k$

$$W_k^{(t)} := \sum_{x \in X_k} e^x 1_{[0,t]}(x).$$

Then

$$\mathbb{E}\big[W_{k+1}^{(t)} - W_k^{(t)} \,\big|\, \min(X_k) = m\big] = t - 1_{[0,t]}(m).$$

In particular, the process $W^{(t)}$ stopped at the first time that $\min(X_k) > t$ is

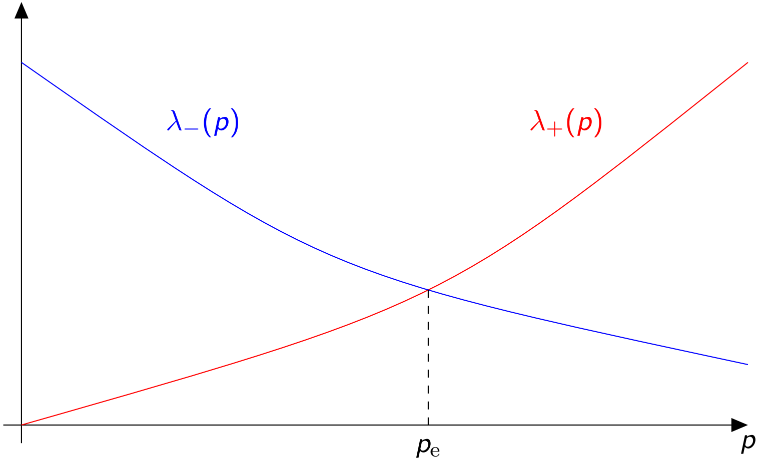
- A supermartingale for $t < 1$,
- A martingale for $t = 1$,
- A submartingale for $t > 1$.

In classical economic theory (Walras,[1] 1874), the *price* of a commodity is determined by *demand* and *supply*.

Let $\lambda_-(p)$ (resp. $\lambda_+(p)$) be the total *demand* (resp. *supply*) for a commodity at price level $p$, i.e., the total amount that people are willing to buy (resp. sell), per unit of time, for a price of at most (resp. at least) $p$ per unit.

[1] Walras developed the theory of equilibrium markets in his book *Éléments d' économie politique pure.*

# Some classical ecomomic theory




**Postulate** In an equilibrium market, the commodity is traded at the *equilibrium prize* $p_\mathrm{e}$.

# Stock & Commodity Exchanges & the Order Book

On stock & commodity exchanges, goods are traded using an *order book*.

The order book for a given asset contains a list of offers to buy or sell a given amount for a given price. Traders arriving at the market have two options.

- Place a **market order**, i.e., either *buy* (*buy market order*) or *sell* (*sell market order*) $n$ units of the asset at the best price available in the order book.

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Market orders are matched to existing limit orders according to a mechanism that depends on the trading system.

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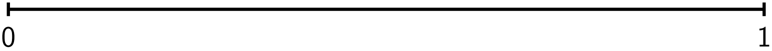
# Numerical simulation



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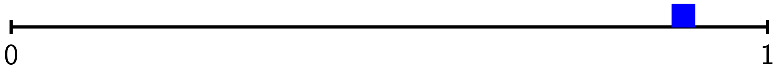

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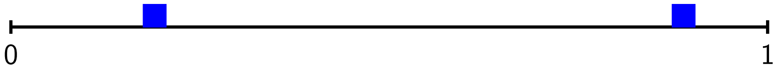

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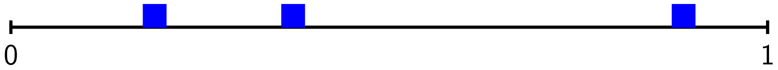

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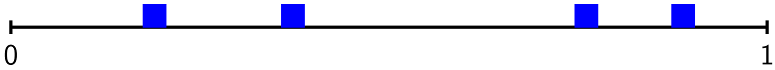

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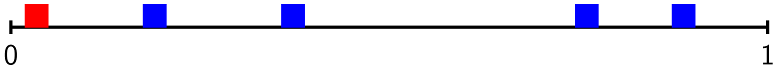

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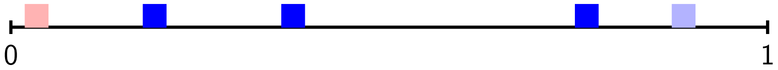

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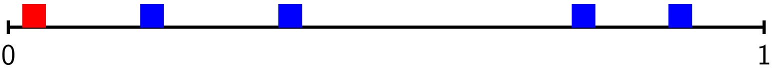

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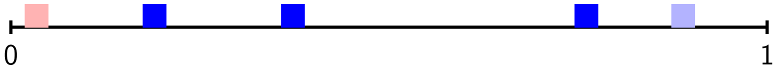

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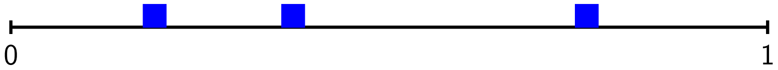

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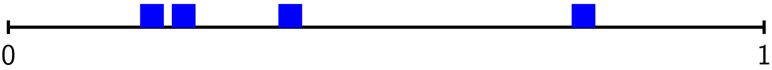

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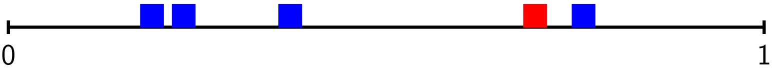

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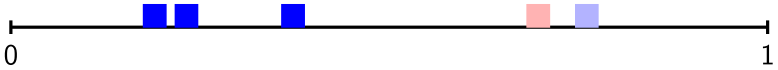

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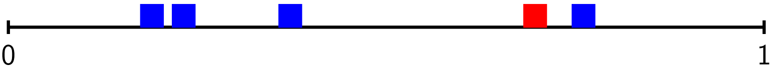

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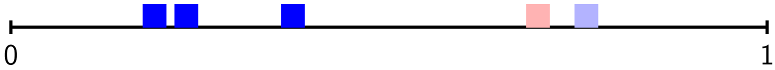

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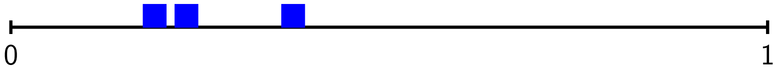

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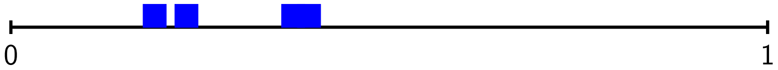

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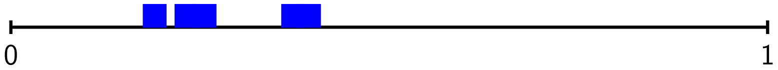

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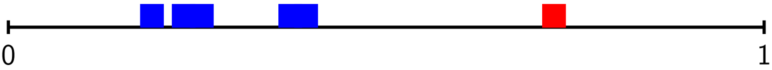

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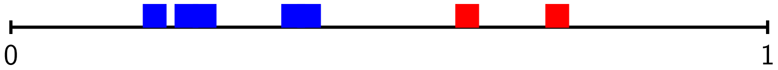

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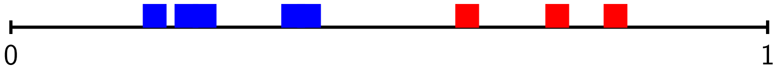

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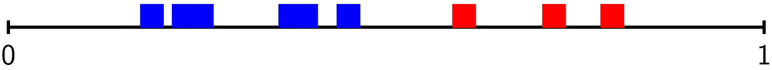

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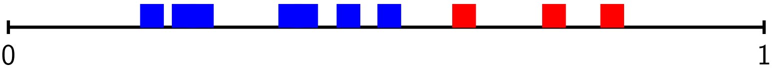

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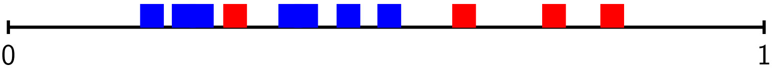

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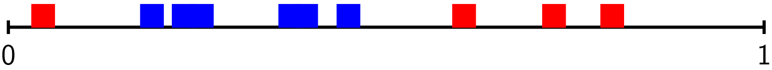

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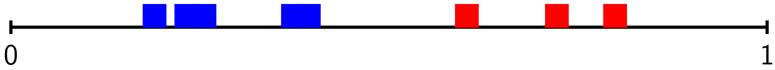

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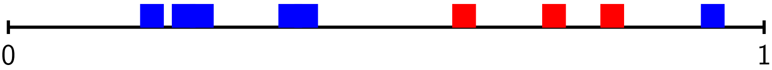

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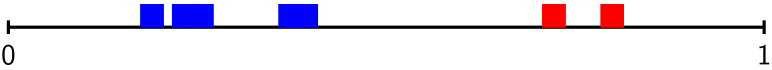

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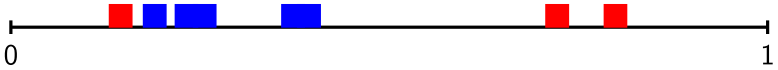

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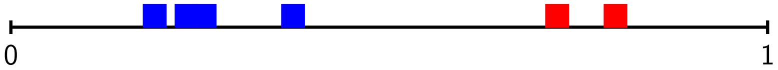

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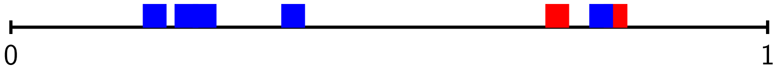

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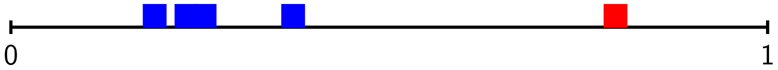

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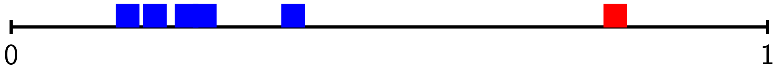

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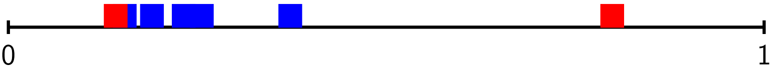

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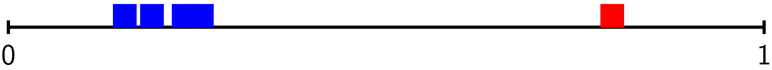

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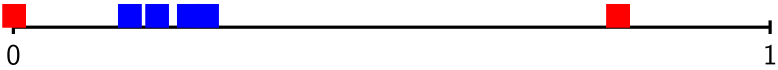

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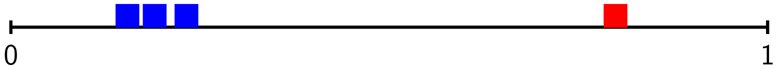

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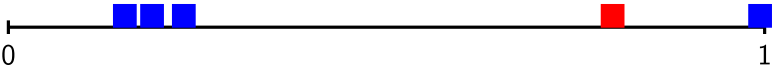

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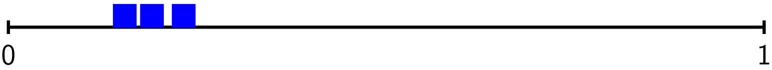

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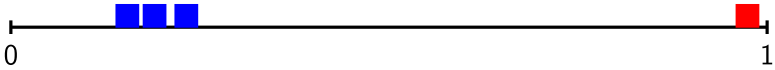

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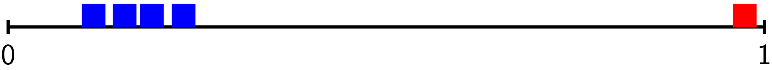

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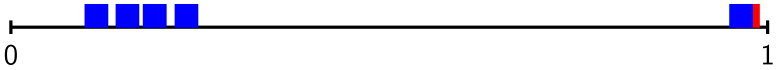

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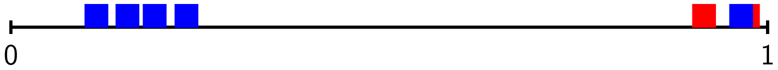

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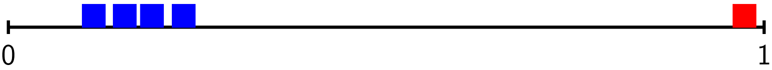

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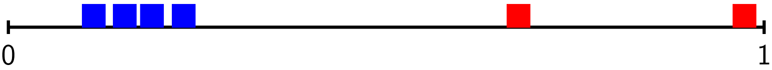

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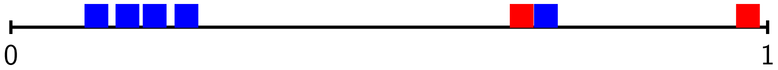

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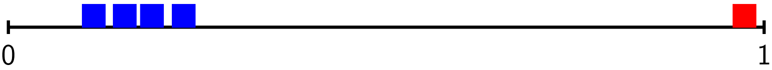

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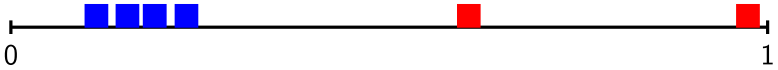

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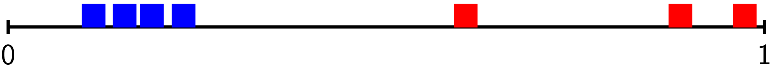

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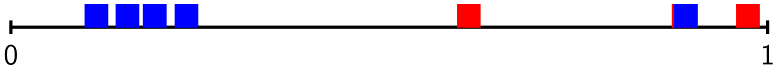

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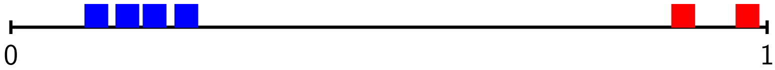

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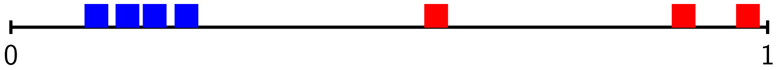

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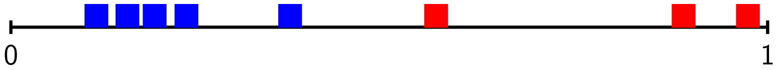

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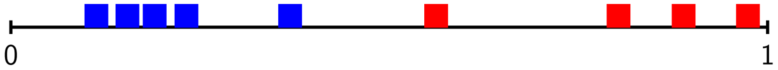

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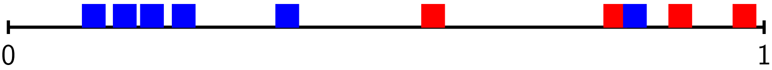

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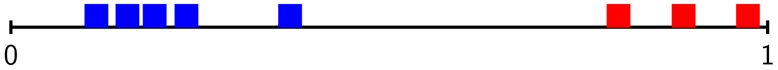

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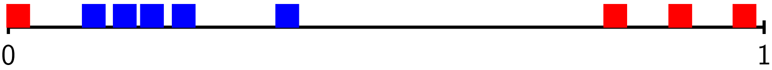

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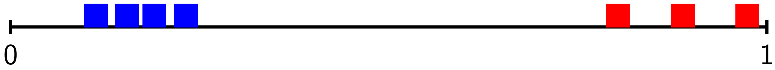

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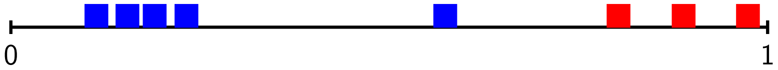
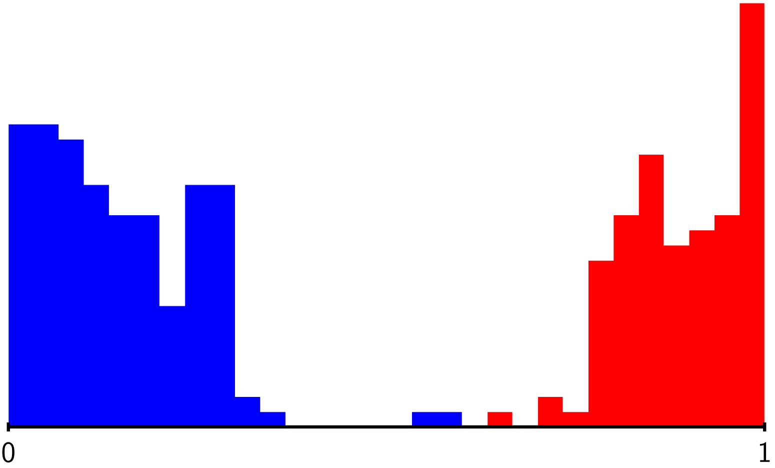


# Numerical simulation

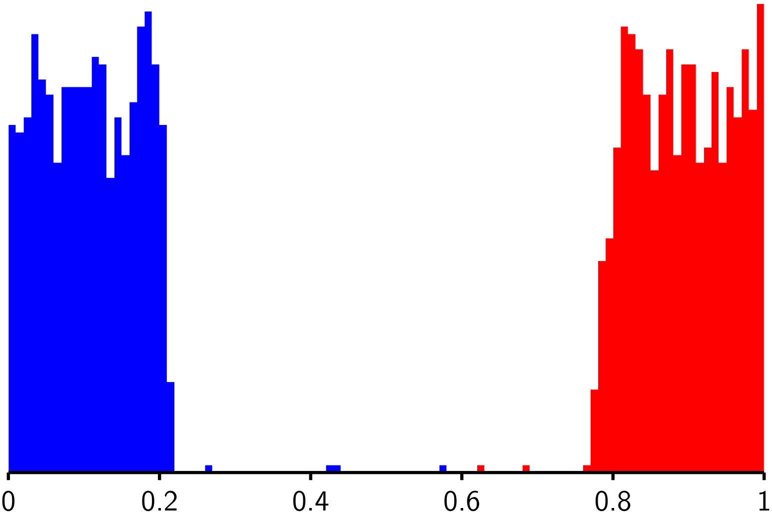

The order book after the arrival of 100 traders.

# Numerical simulation




The order book after the arrival of 1000 traders.

# Numerical simulation




The order book after the arrival of 10,000 traders.

# Stigler's model

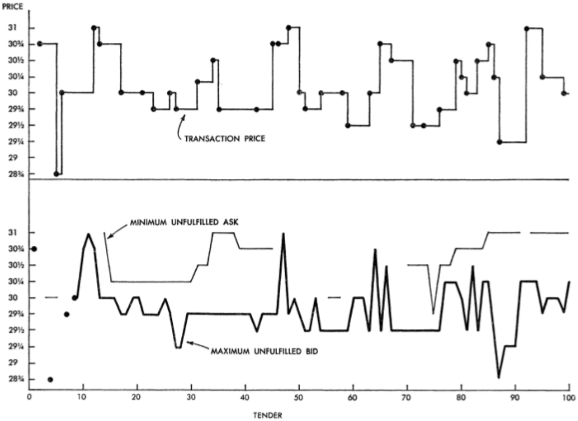
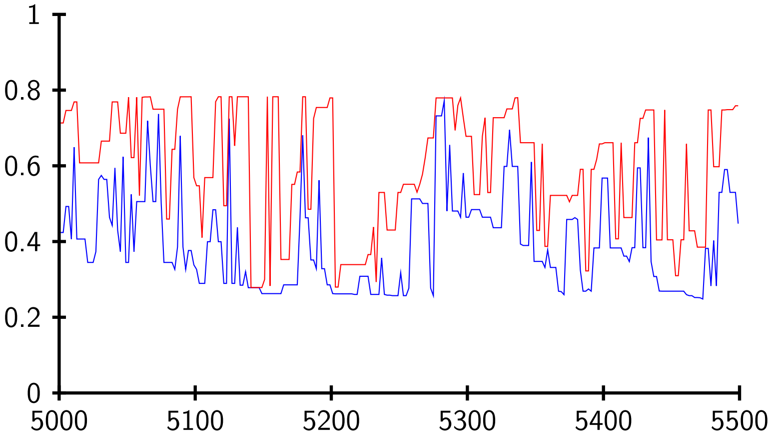



Fig. 1.—Hypothetical sequence of transaction prices, generated by sequence of random numbers, and maximum unfulfilled bid and minimum unfulfilled ask prices (equilibrium price of 29¾ or 30).

Stigler (1964) already simulated the same model with $\mu_\pm$ the uniform distributions on a set of 10 possible prices.

# Numerical simulation




Evolution of the highest bid and lowest ask prices between the arrivals of the 5000th and 5500th trader.

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- The spread is huge, most of the time.

*In fact, the model is very similar to the two-sided canyon model and even seems to have the same critical point.*

# The critical point

Luckock has a **formula** for $q_{\min}$ and $q_{\max}$.

In particular, for the model on $[0, 1]$ with $\lambda_-(x) = 1 - x$ and $\lambda_+(x) = x$, Luckcock claims: $q_{\min} := 1 + 1/z$ with $z$ the unique solution of the equation $1 + z + e^z = 0$.

Numerically, $q_{\min} \approx 0.21781170571980$.

Luckock proves his claim based on the following assumptions:

- The model is stationary.
- There exist $0 < q_{\min} < q_{\max} < 1$ such that buy (sell) limit orders below $q_{\min}$ (above $q_{\max}$) are never matched.
- All buy (sell) limit orders above $q_{\min}$ (below $q_{\max}$) are eventually matched.

# Adding market orders

Let $\bar{I} = [I_-, I_+]$ be the interval of possible prices.
We assume that $\lambda_\pm : \bar{I} \to [0, \infty)$ are continuous, $\lambda_-$ is nonincreasing, and $\lambda_+$ nondecreasing.

We drop the assumption that $\lambda_-(I_+) = 0 = \lambda_+(I_-)$.

Instead, with rate $\lambda_-(I_+)$ (resp. $\lambda_+(I_-)$), a trader arrives that places a buy market order (resp. sell market order) if the order book contains at least one sell limit order (resp. buy limit order), and does nothing else.

The advantage of allowing $\lambda_-(I_+), \lambda_+(I_-) > 0$ is that the process can be positive recurrent.

# Luckock's equation

**[Luckock '03]** Let $M^{\pm}$ denote the price of the best buy/sell offer. Assume that the process is in equilibrium. Then

$$f_-(x) := \mathbb{P}[M^- < x] \quad \text{and} \quad f_+(x) := \mathbb{P}[M^+ > x]$$

solve the differential equation

$$\begin{aligned} \text{(i)} \quad & f_- \mathrm{d}\lambda_+ = -\lambda_- \mathrm{d}f_+, \\ \text{(ii)} \quad & f_+ \mathrm{d}\lambda_- = -\lambda_+ \mathrm{d}f_- \\ \text{(iii)} \quad & f_+(I_-) = 1 = f_-(I_+). \end{aligned}$$

**Proof:** Since buy orders are added to $A \subset (q_{\min}, q_{\max})$ at the same rate as they are removed

$$\int_A \mathbb{P}[M^- < x]\, \mathrm{d}\lambda_+(\mathrm{d}x) = \int_A \lambda_-(x)\, \mathbb{P}[M^+ \in \mathrm{d}x].$$

# Luckock's equation

**Theorem** Assume $\lambda_-(I_+), \lambda_+(I_-) > 0$. Then Luckock's equation has a unique solution.

**Conjecture** A Stigler-Luckock model is positive recurrent if and only if the unique solution to Luckock's equation satisfies $f_-(I_+) > 0$ and $f_+(I_-) > 0$.

I have a proof under the "asymmetry" assumption that $\lambda_-(I_+) \neq \lambda_+(I_-)$.

With new methods, I am hopeful to prove the full conjecture soon.

# Weight functions

Let $\mathcal{X}_t^{\pm}$ denote the sets of buy and sell limit orders in the order book at time $t$ and consider a weighted sum over the limit orders of the form

$$W_t := \sum_{x \in \mathcal{X}_t^-} w_-(x) + \sum_{x \in \mathcal{X}_t^+} w_+(x),$$

where $w_\pm : \overline{I} \to \mathbb{R}$ are "weight" functions.

**Lemma** One has

$$\frac{\partial}{\partial t}\mathbb{E}[W_t] = q_-(M_t^-) + q_+(M_t^+),$$

where $q_-: [I_-, I_+) \to \mathbb{R}$ and $q_+ : (I_+, I_-] \to \mathbb{R}$ are given by

$$\begin{aligned} q_-(x) &:= \int_x^{I_+} w_+ \mathrm{d}\lambda_+ - w_-(x)\lambda_+(x) \qquad \big(x \in [I_-, I_+)\big), \\ q_+(x) &:= -\int_{I_-}^{x} w_- \mathrm{d}\lambda_- - w_+(x)\lambda_-(x) \qquad \big(x \in (I_-, I_+]\big). \end{aligned}$$

# Weight functions

**Theorem** For each $z \in \overline{I}$, there exist a unique pair of weight functions $(w_-, w_+)$ such that

$$\frac{\partial}{\partial t}\mathbb{E}[W_t] = 1_{\{M_t^- \leq z\}} - f_-(z),$$

where $(f_-, f_+)$ is the unique solution to Luckock's equation. Likewise, there exist a unique pair of weight functions $(w_-, w_+)$ such that

$$\frac{\partial}{\partial t}\mathbb{E}[W_t] = 1_{\{M_t^+ \geq z\}} - f_+(z).$$

*This gives an interpretation to Luckock's equation even when its solutions take negative values, Moreover, the theorem is useful even in non-stationary settings.*