Lecture 1
Interacting particle systems and the backward process

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Outline

- Poisson construction of finite state space Markov processes
- Countable state spaces
- Interacting particle systems
- The backward process

Probability kernels

Let $\mathbf{S}$ be a finite set.
A *probability kernel* on $\mathbf{S}$ is a function $K : \mathbf{S} \times \mathbf{S} \to [0,1]$ such that

$$\sum_{y \in \mathbf{S}} K(x,y) = 1 \qquad (x \in \mathbf{S}).$$

We can multiply probability kernels as matrices:

$$(KL)(x,z) := \sum_{y \in \mathbf{S}} K(x,y)L(y,z) \qquad (x,z \in \mathbf{S}).$$

We can also view kernels as matrices representing linear operators that act on functions $f : \mathbf{S} \to \mathbb{R}$ as

$$Kf(x) := \sum_{y \in \mathbf{S}} K(x,y)f(y).$$

Continuous-time Markov chains

Let $\mathbf{S}$ be a finite set. A *Markov semigroup* is a collection of probability kernels $(P_t)_{t\geq 0}$ on $\mathbf{S}$ such that

$$\lim_{t\downarrow 0} P_t = P_0 = 1 \quad \text{and} \quad P_s P_t = P_{s+t} \qquad (s,t \geq 0).$$

Each such Markov semigroup is of the form

$$P_t = e^{tG} := \sum_{n=0}^{\infty} \frac{1}{n!}(tG)^n,$$

where the *generator* G is a matrix of the form

$$G(x,y) \geq 0 \quad (x \neq y) \quad \text{and} \quad \sum_{y\in\mathbf{S}} G(x,y) = 0.$$

One has $P_t(x,y) = 1_{\{x=y\}} + tG(x,y) + O(t^2)$ as $t \to 0$. We call $G(x,y)$ the *rate* of jumps from x to y $(x \neq y)$.

Random mapping representations

We view generators as linear operators that act on functions $f : \mathbf{S} \to \mathbb{R}$ as

$$Gf(x) := \sum_{y \in \mathbf{S}} G(x,y) f(y) \qquad (x \in \mathbf{S}).$$

Then $P_t f = f + tGf + O(t^2)$ as $t \to 0$.
Each generator G can be written in the form

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\},$$

where $\mathcal{G}$ is a finite set whose elements are functions $m : \mathbf{S} \to \mathbf{S}$ and $(r_m)_{m \in \mathcal{G}}$ are nonnegative rates.

We call this a *random mapping representation* of G.

Random mapping representation are usually far from unique.

Poisson construction of Markov processes

Each random mapping representation of G corresponds to a Poisson construction of the Markov process.

Let ρ be the measure on $\mathcal{G} \times \mathbb{R}$ defined by

$$\rho\big(\{m\} \times [s,t]\big) := r_m(t-s) \qquad (m \in \mathcal{G},\ s \leq t).$$

Let ω be a Poisson point set on $\mathcal{G} \times \mathbb{R}$ with intensity ρ and let

$$\omega_{s,u} := \big\{(m,t) \in \omega : s < t \leq u\big\} \qquad (s \leq u).$$

Define random maps $(\mathbb{X}_{s,u})_{s \leq u}$ by

$$\begin{aligned} \mathbb{X}_{s,u} &:= m_n \circ \cdots \circ m_1 \\ &\text{with} \quad \omega_{s,u} := \big\{(m_1,t_1), \ldots, (m_n,t_n)\big\} \\ &\text{and} \quad t_1 < \cdots < t_n. \end{aligned}$$

Poisson construction of Markov processes

The random maps $(\mathbb{X}_{s,u})_{s\leq u}$ form a *stochastic flow:*

$$\mathbb{X}_{s,s} = 1 \quad \text{and} \quad \mathbb{X}_{t,u} \circ \mathbb{X}_{s,t} = \mathbb{X}_{s,u} \qquad (s \leq t \leq u).$$

This stochastic flow has *independent increments* in the sense that

$$\mathbb{X}_{t_0,t_1}, \ldots, \mathbb{X}_{t_{n-1},t_n} \quad \text{are independent} \quad \forall\ t_0 < \cdots < t_n.$$

Let X_0 be an **S**-valued random variable, independent of ω, and let $s \in \mathbb{R}$. Then

$$X_t := \mathbb{X}_{s,s+t}(X_0) \qquad (t \geq 0)$$

defines a Markov process $(X_t)_{t\geq 0}$ with generator G.

Poisson construction of Markov processes

Proof (sketch) Define

$$P_t(x,y) := \mathbb{P}\big[\mathbb{X}_{s,s+t}(x) = y\big] \qquad (t \geq 0,\ x,y \in \mathbf{S}).$$

Let $(\mathcal{F}^X_t)_{t\geq 0}$ be the filtration generated by $(X_t)_{t\geq 0}$ and let $\mathcal{F}_t := \sigma(\{\omega_{s,s+t}, X_0\})$. Then $\mathcal{F}^X_t \subset \mathcal{F}_t$, and for all $f : \mathbf{S} \to \mathbb{R}$ and $0 \leq t \leq u$, one has

$$\mathbb{E}\big[f(X_u)\,\big|\,\mathcal{F}_t\big] = \mathbb{E}\big[f(\mathbb{X}_{s+t,s+u} \circ \mathbb{X}_{s,s+t}(X_0))\,\big|\,\mathcal{F}_t\big] = P_{u-t}f(X_t).$$

It is easy to see $(P_t)_{t\geq 0}$ is a continuous semigroup so it suffices to identify its generator. Since

$$\mathbb{P}\big[\omega_{s,s+t} = \{(m,t)\}\big] = r_m t + O(t^2) \text{ and } \mathbb{P}\big[|\omega_{s,s+t}| \geq 2\big] = O(t^2),$$

it follows that

$$P_t f(x) = f(x) + t\sum_{m\in\mathcal{G}} r_m\big\{f\big(m(x)\big) - f\big(x\big)\big\} + O(t^2).$$

∎

The contact process

Let Λ be a finite set and let $\mathbf{S} := \{0,1\}^\Lambda$ be the set of all functions $x : \Lambda \to \{0,1\}$.
For $i,j \in \Lambda$, let $\mathtt{bra}_{ji} : \mathbf{S} \to \mathbf{S}$ be the *branching map*

$$\mathtt{bra}_{ji}(x)(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = i, \\ x(k) & \text{otherwise,} \end{cases}$$

and for $i \in \Lambda$ let $\mathtt{dth}_i : \mathbf{S} \to \mathbf{S}$ be the *death map*

$$\mathtt{dth}_i(x)(k) := \begin{cases} 0 & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Then the generator of the *contact process* with *infection rates* $\big(\lambda(j,i)\big)_{i,j\in\Lambda}$ has the random mapping representation

$$\begin{aligned} Gf(x) = \sum_{i,j\in\Lambda} \lambda(j,i)\big\{f\big(\mathtt{bra}_{ji}(x)\big) - f\big(x\big)\big\} \\ + \sum_{i\in\Lambda} \big\{f\big(\mathtt{dth}_i(x)\big) - f\big(x\big)\big\} \qquad (x \in \{0,1\}^\Lambda). \end{aligned}$$

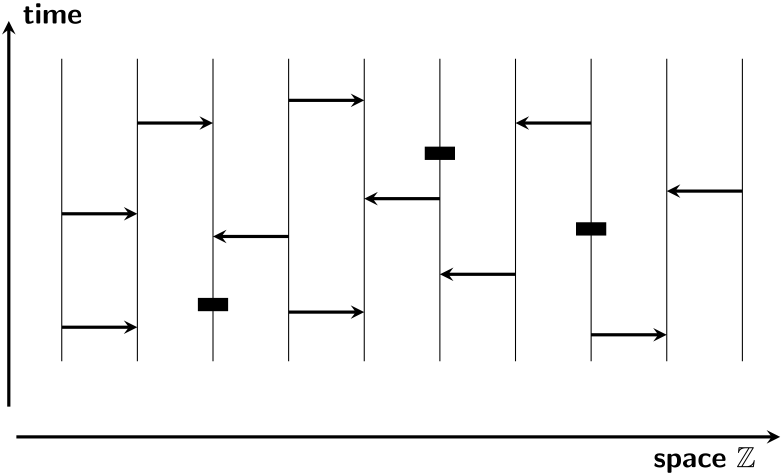
The graphical representation

We construct the contact process from
a graphical representation ω.

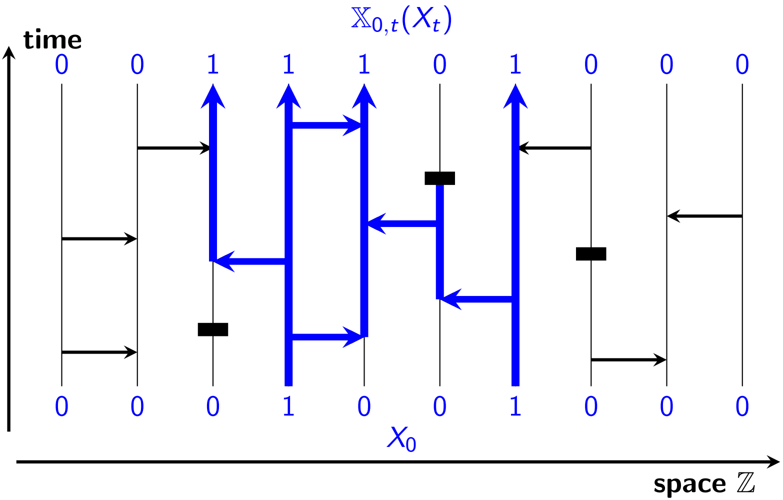
We visualise the Poisson point set ω by drawing
space Λ horizontally and time $\mathbb{R}$ vertically.

For each $(\texttt{bra}_{ji}, t) \in \omega$ we draw an arrow from (j, t) to (i, t).
For each $(\texttt{dth}_i, t) \in \omega$ we draw a blocking symbol ▬ at (i, t).

The graphical representation



The graphical representation



The threshold voter model

Let Λ be a finite graph and

$$\mathcal{N}_i := \{j \in \Lambda : j \text{ is adjacent to } i\} \quad \text{and} \quad \overline{\mathcal{N}}_i := \{i\} \cup \mathcal{N}_i.$$

For each $i \in \Lambda$, we define maps $\mathtt{min}_i$ and $\mathtt{max}_i$ by

$$\mathtt{min}_i(x)(k) := \begin{cases} \bigwedge_{j \in \mathcal{N}_i} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise,} \end{cases}$$

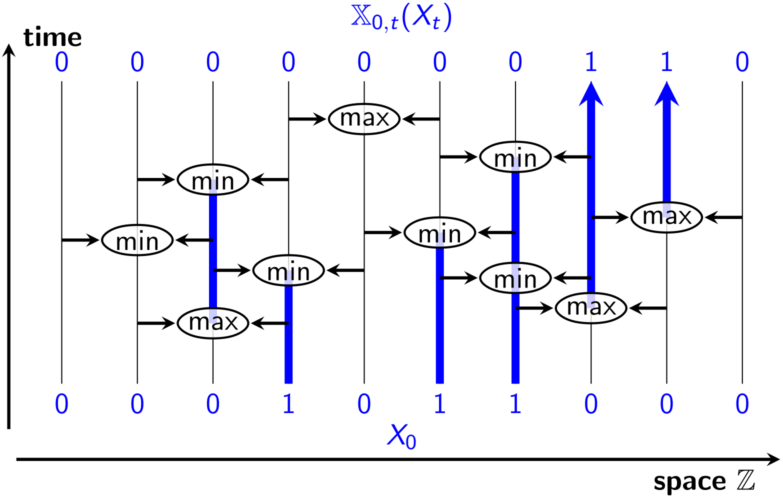
and

$$\mathtt{max}_i(x)(k) := \begin{cases} \bigvee_{j \in \mathcal{N}_i} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Then the generator of the *threshold voter model* has the random mapping representation

$$\begin{aligned} Gf(x) = & \sum_{i \in \Lambda} \big\{ f\big(\mathtt{min}_i(x)\big) - f\big(x\big) \big\} \\ & + \sum_{i \in \Lambda} \big\{ f\big(\mathtt{max}_i(x)\big) - f\big(x\big) \big\} \qquad \big(x \in \{0,1\}^{\Lambda}\big). \end{aligned}$$

The threshold voter model



The threshold voter model

The threshold voter model has a second graphical representation.

Let $\oplus$ denote addition modulo 2.

For each $i \in \Lambda$ and $\Delta \subset \overline{\mathcal{N}}_i$, we define a map $\mathtt{flip}_{i,\Delta}$ by

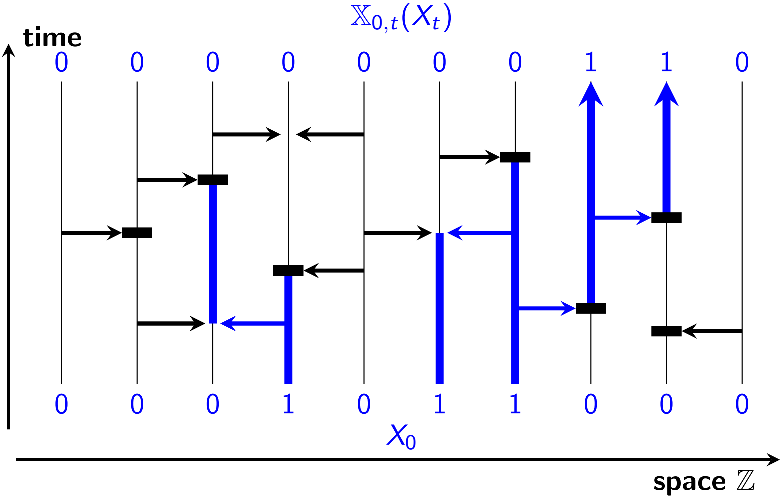
$$\mathtt{flip}_{i,\Delta}(x)(k) := \begin{cases} \bigoplus_{j\in\Delta} x(j) & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Then the generator of the threshold voter model has the alternative random mapping representation

$$Gf(x) = \sum_{i\in\Lambda} 2^{-|\overline{\mathcal{N}}_i|} \sum_{\substack{\Delta\subset\overline{\mathcal{N}}_i \\ |\Delta| \text{ is odd}}} \big\{ f\big(\mathtt{flip}_{i,\Delta}(x)\big) - f\big(x\big) \big\}$$

[Cox & R. Durrett, 1991].

The threshold voter model



Second random mapping representation

Note there are $2^{|\overline{\mathcal{N}}_i|-1}$ odd subsets of $\overline{\mathcal{N}}_i$.

We claim the threshold voter mode has the following description:

- Each site i is activated with Poisson rate 2.
- If i is activated, we uniformly chose an odd subset $\Delta \subset \overline{\mathcal{N}}_i$ and apply $\texttt{flip}_{i,\Delta}$.

If $x(j) = 0$ for all $j \in \overline{\mathcal{N}}_i$ then $\bigoplus_{j\in\Delta} x(j) = 0$ for all odd $\Delta \subset \overline{\mathcal{N}}_i$ so $\texttt{flip}_{i,\Delta}$ does nothing.

If $x(j) = 1$ for all $j \in \overline{\mathcal{N}}_i$ then $\bigoplus_{j\in\Delta} x(j) = 1$ for all odd $\Delta \subset \overline{\mathcal{N}}_i$ so $\texttt{flip}_{i,\Delta}$ does nothing.

In all other cases, $\Delta \cap \{j \in \overline{\mathcal{N}}_j : x(j) = 1\}$ is uniformly chosen from all subsets of $\{j \in \overline{\mathcal{N}}_j : x(j) = 1\}$ so $\bigoplus_{j\in\Delta} x(j)$ is uniformly distributed on $\{0, 1\}$ and there is a probability $1/2$ that $\texttt{flip}_{i,\Delta}$ changes the local state at i.

Poisson construction on countable spaces

Let $\mathbf{S}$ be a countably infinite set and let $\overline{\mathbf{S}} := \mathbf{S} \cup \{\infty\}$ be its one-point compactification, i.e., $\mathbf{S} \ni x_n \to \infty$ iff for all finite $\mathbf{S}' \subset \mathbf{S}$ there is an $N < \infty$ such that $x_n \notin \mathbf{S}'$ for all $n > N$.

Let $\mathcal{G}$ be a countable set whose elements are functions $m : \mathbf{S} \to \mathbf{S}$. Let $(r_m)_{m \in \mathcal{G}}$ be nonnegative rates.

We assume that

$$\sum_{\substack{m \in \mathcal{G} \\ m(x) \neq x}} r_m < \infty \qquad (x \in \mathbf{S}).$$

Poisson construction of Markov processes

As before let ρ be the measure on $\mathcal{G} \times \mathbb{R}$ defined by

$$\rho\big(\{m\} \times [s,t]\big) := r_m(t-s) \qquad (m \in \mathcal{G},\ s \leq t).$$

Let ω be a Poisson point set on $\mathcal{G} \times \mathbb{R}$ with intensity ρ.

Recall that a function f defined on a real interval is *cadlag* if it is right-continuous and the left-limit $f_{t-} := \lim_{s \uparrow t} f_s$ exists for all t.

Lemma For all $s \in \mathbb{R}$ and $x \in \mathbf{S}$, there exists a unique cadlag function $[s,\infty) \ni t \mapsto X_t \in \overline{\mathbf{S}}$ and time $0 < \tau \leq \infty$ such that

(i) $X_s = x$,

(ii) $X_t = m(X_{t-})$ if $(m,t) \in \omega$ for some necessarily unique $m \in \mathcal{G}$ and $X_t = X_{t-}$ otherwise $\big(t \in [0,\tau)\big)$.

(iii) If $\tau < \infty$, then $\lim_{t \uparrow \tau} X_t = \infty$ and $X_t = \infty$ for all $t \geq \tau$.

Explosion

If $\tau < \infty$, then we say that the Markov process *explodes*.
To prove that the process is nonexplosive, it suffices to find a Lyapunov function.

Lemma Assume that there exists a function $L : \mathbf{S} \to [0, \infty)$ such that $L(x) \to \infty$ as $x \to \infty$, and a constant $K < \infty$ such that

$$GL(x) \leq KL(x) \qquad (x \in \mathbf{S}).$$

Then $\tau = \infty$ a.s. and the process started in $X_s = x$ satisfies

$$\mathbb{E}^x \big[L(X_t) \big] \leq e^{K(t-s)} L(x) \qquad (s \leq t, \ x \in \mathbf{S}).$$

We define random maps $(\mathbb{X}_{s,u})_{s \leq u}$ by

$$\mathbb{X}_{s,u}(x) := X_u \quad \text{where} \quad (X_t)_{t \geq s} \text{ satisfies (i)–(iii).}$$

Infinite lattices

Let Λ be a countable set, called the *lattice*.
Let S be a finite set, called the *local state space*.
Let S^Λ denote the space of all functions $x : \Lambda \to S$.
We equip S^Λ with the product topology, under which it is compact.

For $x \in S^\Lambda$ and $\Delta \subset \Lambda$, let

$$x_\Delta = \big(x(i)\big)_{i \in \Delta}$$

denote the restriction of x to Δ.

Lemma Let T be a finite set. Then a function $f : S^\Lambda \to T$ is continuous if and only if it depends on finitely many coordinates, i.e., there exists a finite set $\Delta \subset \Lambda$ and a function $f' : S^\Delta \to T$ such that $f(x) = f'(x_\Delta)$ $(x \in S^\Lambda)$.

Relevant sites

For any function $f : S^\Lambda \to T$, we call

$$\mathcal{R}(f) := \big\{i \in \Lambda : \exists x, y \in S^\Lambda \text{ s.t. } f(x) \neq f(y) \text{ and } x_{\Lambda\backslash\{i\}} = y_{\Lambda\backslash\{i\}}\big\}.$$

the set of f*-relevant sites*.

If f is continuous, then $\mathcal{R}(f)$ is the smallest possible finite set $\Delta \subset \Lambda$ such that there exists a function $f' : S^\Delta \to T$ with $f(x) = f'(x_\Delta)$ $(x \in S^\Lambda)$.

If f is not continuous, then strange things can happen:

Example Set $S = T := \{0, 1\}$ and $f(x) := 1$ iff $\{i \in \Lambda : x(i) = 1\}$ is finite. Then $\mathcal{R}(f) = \emptyset$, but f is not constant.

Example Set $S = T := \{0, 1\}$ and $f(x) := 1$ iff $\{i \in \Lambda : x(i) = 1\}$ is finite and even. Then $\mathcal{R}(f) = \Lambda$.

Local maps

For any map $m : S^\Lambda \to S^\Lambda$ and $i \in \Lambda$, we define $m[i] : S^\Lambda \to S$ by

$$m[i](x) := m(x)(i) \qquad (x \in S^\Lambda).$$

Then m is continuous iff $m[i]$ is continuous for all $i \in \Lambda$.

By definition, a map $m : S^\Lambda \to S^\Lambda$ is *local* iff

(i) m is continuous,

(ii) $\mathcal{D}(m) := \big\{ i \in \Lambda : \exists x \in S^\Lambda \text{ s.t. } m(x)(i) \neq x(i) \big\}$ is finite.

We will be interested in interacting particle systems with generator of the form

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\},$$

where $\mathcal{G}$ is a countable set whose elements are local maps $m : S^\Lambda \to S^\Lambda$ and $(r_m)_{m \in \mathcal{G}}$ are nonnegative rates.

Finite configurations

Let $\mathbf{S} := S^\Lambda$. For each local map $m : \mathbf{S} \to \mathbf{S}$, set

$$\mathcal{R}_i^\uparrow(m) := \big\{ j \in \Lambda : i \in \mathcal{R}(m[j]),\ j \in \mathcal{D}(m) \big\},$$
$$\mathcal{R}_i^\downarrow(m) := \big\{ j \in \Lambda : j \in \mathcal{R}(m[i]),\ i \in \mathcal{D}(m) \big\}.$$

Fix $0 \in S$, let $\underline{0} \in \mathbf{S}$ denote the all zero configuration, and let

$$\mathbf{S}_{\rm fin} := \big\{ x \in \mathbf{S} : |x| < \infty \big\} \quad \text{with} \quad |x| := \big|\{ i \in \Lambda : x(i) \neq 0 \}\big|.$$

Theorem Assume that $m(\underline{0}) = \underline{0}$ for all $m \in \mathcal{G}$ and that

$$\sum_{\substack{m \in \mathcal{G} \\ \mathcal{D}(m) \ni i}} r_m < \infty \quad (i \in \Lambda) \quad \text{and} \quad \sup_{i \in \Lambda} \sum_{m \in \mathcal{G}} r_m \big|\mathcal{R}_i^\uparrow(m)\big| < \infty.$$

Then the Markov process with countable (!) state space $\mathbf{S}_{\rm fin}$ is nonexplosive.

Finite configurations

Proof idea Since $m(\underline{0})=\underline{0}$ for all $m\in\mathcal{G}$ and

$$\sum_{\substack{m\in\mathcal{G}\\ \mathcal{D}(m)\ni i}} r_m<\infty\quad (i\in\Lambda),$$

we have that

$$\sum_{\substack{m\in\mathcal{G}\\ m(x)\neq x}} r_m<\infty\qquad (x\in\mathbf{S}_{\rm fin}),$$

while the condition involving $\mathcal{R}^{\uparrow}_i(m)$ implies that $x\mapsto|x|$ is a Lyapunov function. ∎

The theorem allows us to construct the contact process, the threshold voter model, and many more interacting particle systems on infinite lattices, but only for finite initial states.

Theorem Assume that

$$\sum_{\substack{m\in\mathcal{G}\\ \mathcal{D}(m)\ni i}} r_m<\infty\quad(i\in\Lambda)\quad\text{and}\quad\sup_{i\in\Lambda}\sum_{m\in\mathcal{G}} r_m\big|\mathcal{R}_i^{\downarrow}(m)\big|<\infty.$$

Then almost surely, for each $s\in\mathbb{R}$ and $x\in S^\Lambda$, there exists a unique cadlag function $(X_t)_{t\geq s}$ such that

(i) $X_s=x$,

(ii) $X_t=m(X_{t-})$ if $(m,t)\in\omega$ for some necessarily unique $m\in\mathcal{G}$ and $X_t=X_{t-}$ otherwise.

We define random maps $(\mathbb{X}_{s,u})_{s\leq u}$ by

$$\mathbb{X}_{s,u}(x):=X_u\quad\text{where}\quad(X_t)_{t\geq s}\text{ satisfies (i)–(ii).}$$

If X_0 is independent of ω and $s\in\mathbb{R}$, then

$$X_t:=\mathbb{X}_{s,s+t}(X_0)\qquad(t\geq 0)$$

is a Markov process $(X_t)_{t\geq 0}$ with generator G.

The backtracking process

Idea of the proof Fix a finite "target" set T.

Then the set $\mathcal{C}(S^\Lambda, T)$ of continuous functions $f : S^\Lambda \to T$ is countable.

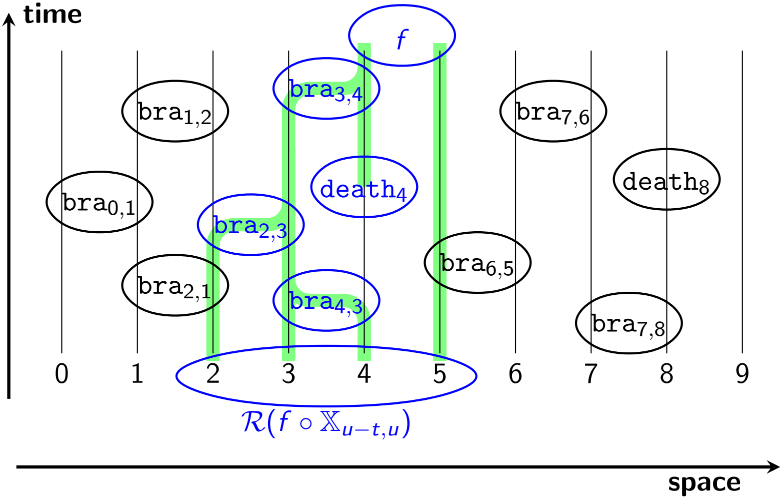
For fixed $f \in \mathcal{C}(S^\Lambda, T)$ and $u \in \mathbb{R}$, we want the *backtracking process*

$$(F_t)_{t\geq 0} := \big(f \circ \mathbb{X}_{u-t,t}\big)_{t\geq 0}$$

to be a well-defined Markov process with countable state space $\mathcal{C}(S^\Lambda, T)$ and generator

$$H\mathcal{F}(f) := \sum_{m\in\mathcal{G}} r_m \big\{\mathcal{F}\big(f \circ m\big) - \mathcal{F}\big(f\big)\big\}.$$

The backtracking process



The backtracking process

The condition

$$\sum_{\substack{m\in\mathcal{G}\\ \mathcal{D}(m)\ni i}} r_m < \infty \quad (i\in\Lambda)$$

guarantees that

$$\sum_{\substack{m\in\mathcal{G}\\ f\circ m\neq f}} r_m < \infty \qquad \big(f\in\mathcal{C}(S^\Lambda,T)\big),$$

and the Lyapunov function

$$L(f) := |\mathcal{R}(f)| \qquad \big(f\in\mathcal{C}(S^\Lambda,T)\big),$$

satisfies $HL(f)\leq KL(f)$ with

$$K := \sup_{i\in\Lambda}\sum_{m\in\mathcal{G}} r_m\big|\mathcal{R}^{\downarrow}_i(m)\big|.$$

Examples

The contact process is well-defined for finite initial states provided that

$$\sup_{i\in\Lambda}\sum_{j\in\Lambda}\lambda(i,j)<\infty.$$

and for arbitrary initial states provided that

$$\sup_{i\in\Lambda}\sum_{j\in\Lambda}\lambda(j,i)<\infty.$$

Our theorems imply that the threshold voter model is well-defined for finite or infinite initial states provided the graph Λ is of uniformly bounded degree.
(This condition can be relaxed by a more clever choice of the Lyapunov function.)

The backward stochastic flow

We set

$$\mathbb{F}_{u,s}(f) := f \circ \mathbb{X}_{s,u} \qquad \big(u \geq s,\ f \in \mathcal{C}(S^\Lambda, T)\big).$$

Then $(\mathbb{F}_{u,s})_{u \geq s}$ is a *backward stochastic flow:*

$$\mathbb{F}_{s,s} = 1 \quad \text{and} \quad \mathbb{F}_{t,s} \circ \mathbb{F}_{u,t} = \mathbb{X}_{s,u} \qquad (u \geq t \geq s).$$

If F_0 is a random variable with values in $\mathcal{C}(S^\Lambda, T)$, independent of ω, and $u \in \mathbb{R}$, then

$$F_t := \mathbb{F}_{u,u-t}(F_0) \qquad (t \geq 0)$$

defines a Markov process $(F_t)_{t \geq 0}$ with generator H.
It is a consequence of our construction that this *backtracking process* has *caglad* sample paths, i.e., $t \mapsto F_t$ left-continuous and the right limit $F_{t+} := \lim_{s \downarrow t} F_s$ exists for all $t \geq 0$.