Lecture 2
Monotone duality

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Outline

- Monotonicity and monotone representability
- Positive correlations
- The lower and upper invariant laws
- Survival and stability
- Monotone duality
- A cooperative contact process

Probability kernels

For finite sets S, T, let $\mathcal{F}(S,T)$ denote the set of all functions $f : S \to T$.

A *random mapping representation* of a probability kernel K from S to T is an $\mathcal{F}(S,T)$-valued random variable M such that

$$K(x,y) = \mathbb{P}[M(x) = y] \qquad (x \in S,\ y \in T).$$

We say that K is *representable* in $\mathcal{G} \subset \mathcal{F}(S,T)$ if M can be chosen so that it takes values in $\mathcal{G}$. Recall that

$$Kf(x) := \sum_{y \in T} K(x,y) f(y) = \mathbb{E}\big[f\big(M(x)\big)\big] \qquad \big(x \in S,\ f \in \mathcal{F}(T,\mathbb{R})\big).$$

Monotone probability kernels

For partially ordered sets S, T, let $\mathcal{F}_{\rm mon}(S, T)$ be the set of all monotone maps $m : S \to T$, i.e., those for which $x \leq x'$ implies $m(x) \leq m(x')$.

A probability kernel K is called *monotone* if

$$Kf \in \mathcal{F}_{\rm mon}(S,\mathbb{R}) \quad \forall f \in \mathcal{F}_{\rm mon}(T,\mathbb{R}),$$

and *monotonically representable* if K is representable in $\mathcal{F}_{\rm mon}(S, T)$.

Monotonical representability implies monotonicity:

$$f \in \mathcal{F}_{\rm mon}(T,\mathbb{R}) \quad \text{and} \quad x \leq x' \quad \Rightarrow$$
$$Kf(x) = \mathbb{E}\big[f\big(M(x)\big)\big] \leq \mathbb{E}\big[f\big(M(x')\big)\big] = Kf(x').$$

Monotone probability kernels

J.A. Fill & M. Machida (AOP 2001) (and also D.A. Ross (unpublished)) discovered that the converse does not hold. There are counterexamples with $S = T = \{0,1\}^2$.

On the positive side, Kamae, Krengel & O'Brien (1977) and Fill & Machida (2001) have shown that:

(Sufficient conditions for monotone representability) *Let S, T be finite partially ordered sets and assume that at least one of the following conditions is satisfied:*

(i) *S is totally ordered.*

(ii) *T is totally ordered.*

Then any monotone probability kernel from S to T is monotonically representable.

In particular, setting $S = \{1,2\}$, this proves that if μ_1, μ_2 are probability laws on T such that

$$\mu_1 f \leq \mu_2 f \quad \forall f \in \mathcal{F}_{\rm mon}(T, \mathbb{R}),$$

then it is possible to couple random variables M_1, M_2 with laws μ_1, μ_2 such that $M_1 \leq M_2$.

The statement remains true if T is replaced by a set of the form $\mathbf{T} = T^\Lambda$, equipped with the product order and $\mathcal{F}_{\rm mon}(T, \mathbb{R})$ is replaced by the space $\mathcal{C}_{\rm mon}(\mathbf{T}, \mathbb{R})$ of continuous monotone functions.

Monotone interacting particle systems

An interacting particle system $(X_t)_{t\geq 0}$ on a lattice Λ with a partially ordered local state space S and semigroup $(P_t)_{t\geq 0}$ is called *monotone* if P_t is a monotone probability kernel for all $t \geq 0$.

Lemma If the generator G has a random mapping representation

$$Gf(x) = \sum_{m\in\mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\}$$

such that each local map $m \in \mathcal{G}$ is monotone, then P_t is monotonically representable for all $t \geq 0$.

Proof Immediate, since

$$P_t(x,\,\cdot\,) = \mathbb{P}\big[\mathbb{X}_{0,t}(x) \in \,\cdot\,\big] \qquad (x \in S^\Lambda,\ t \geq 0)$$

and $\mathbb{X}_{0,t}[i]$ is a concatenation of finitely many monotone maps for each $i \in \Lambda$ and $t \geq 0$. ∎

Positive correlations

A probability measure μ on $\mathbf{S} = S^\Lambda$ has *positive correlations* if

$$\mathrm{Cov}_\mu(f,g) := \int (fg)\,\mathrm{d}\mu - \Big(\int f\,\mathrm{d}\mu\Big)\Big(\int g\,\mathrm{d}\mu\Big) \geq 0$$

for all $f, g \in \mathcal{C}_{\mathrm{mon}}(\mathbf{S}, \mathbb{R})$.

Theorem Assume that each $m \in \mathcal{G}$ is monotone and that

- $\forall x \in \mathbf{S}$ and $m \in \mathcal{G}$ either $m(x) \geq x$ or $m(x) \leq x$.

Then, if $\mathbb{P}[X_0 \in \cdot\,]$ has positive correlations,
so has $\mathbb{P}[X_t \in \cdot\,]$ for all $t \geq 0$.

Positive correlations

Proof sketch For any measure μ and bounded measurable function f write $\mu f := \int f \, \mathrm{d}\mu$. Define μP_t by $(\mu P_t)f := \mu(P_t f)$ so that $\mu P_t = \mathbb{P}[X_t \in \cdot\,]$.

The claim now comes from the covariance formula

$$\mathrm{Cov}_{\mu P_t}(f,g) = \mathrm{Cov}_\mu(P_t f, P_t g) + \int_0^t \mathrm{d}s \, \mu P_{t-s} \Gamma(P_s f, P_s g),$$

where

$$\begin{aligned}\Gamma(f,g)(x) &:= G(fg)(x) - Gf(x)g(x) - f(x)Gg(x) \\ &= \sum_{m \in \mathcal{G}} r_m \big\{ f\big(m(x)\big) - f\big(x\big) \big\} \big\{ g\big(m(x)\big) - g\big(x\big) \big\}.\end{aligned}$$

∎

The lower and upper invariant laws

Lemma Assume that S is partially ordered with least element 0 and greatest element 1 and that each $m \in \mathcal{G}$ is monotone.
Then there exist invariant laws $\underline{\nu}$ and $\overline{\nu}$ such that

$$\mathbb{P}^{\underline{0}}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \underline{\nu} \quad \text{and} \quad \mathbb{P}^{\underline{1}}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}$$

and each other invariant law ν satisfies $\underline{\nu} \leq \nu \leq \overline{\nu}$
in the stochastic order.

We call $\underline{\nu}$ and $\overline{\nu}$ the *lower* and *upper* invariant laws.

They are extremal elements of the convex set of all invariant laws.

The lower and upper invariant laws

Proof sketch Since $\mathbb{X}_{t,u}(\underline{1}) \leq \underline{1}$ and $\mathbb{X}_{s,t}$ is monotone

$$\mathbb{X}_{s,u}(\underline{1}) = \mathbb{X}_{t,u}\big(\mathbb{X}_{s,t}(\underline{1})\big) \leq \mathbb{X}_{t,u}(\underline{1}) \quad (s \leq t \leq u)$$

which proves that the decreasing limit

$$\overline{X}_u := \lim_{t \to -\infty} \mathbb{X}_{t,u}(\underline{1})$$

exists. It is easy to check $(\overline{X}_u)_{u \in \mathbb{R}}$ is a stationary process that a.s. dominates any other stationary process adapted to the same graphical representation.

If $\overline{\nu} = p\nu_1 + (1-p)\nu_2$ with $0 < p < 1$ and ν_1, ν_2 stationary, then $\nu_i \leq \overline{\nu}$ $(i = 1, 2)$ implies $\nu_1 f = \nu_2 f = \nu f$ for all $f \in \mathcal{C}_{\rm mon}(\mathbf{S}, \mathbb{R})$.

In the absencse of monotone representability, replace a.s. arguments by arguments involving the semigroup and the stochastic order. ∎

Survival and stability

Assume that S is partially ordered with least element 0 and greatest element 1 and that each $m \in \mathcal{G}$ is monotone with $m(\underline{0}) = \underline{0}$.

We say that the process $(X_t)_{t\geq 0}$ *survives* if

$$\mathbb{P}^x\big[X_t \neq \underline{0}\ \forall t \geq 0\big] > 0 \quad \text{for some } x \in \mathbf{S}_{\rm fin}.$$

We say that the process $(X_t)_{t\geq 0}$ is *stable* if

$$\overline{\nu} \text{ is } \textit{nontrivial} \text{ in the sense that } \overline{\nu} \neq \delta_{\underline{0}}.$$

By the extremality of $\overline{\nu}$, nontriviality implies $\overline{\nu}(\{\underline{0}\}) = 0$.

A bit of order theory

Let S be a partially ordered set and $A \subset S$.
Define the *upset* $A^{\uparrow}$ and *downset* $A^{\downarrow}$ of A as

$$A^{\uparrow} := \big\{y \in S : \exists x \in A \text{ s.t. } x \leq y\big\},$$
$$A^{\downarrow} := \big\{y \in S : \exists x \in A \text{ s.t. } x \geq y\big\}.$$

Then A is *increasing* if $A = A^{\uparrow}$ and *decreasing* if $A = A^{\downarrow}$.

S is a *lattice* if $\forall x, y \in S\ \exists! x \vee y, x \wedge y \in S$ s.t.

$$\{x\}^{\uparrow} \cap \{y\}^{\uparrow} = \{x \vee y\}^{\uparrow} \quad \text{and} \quad \{x\}^{\downarrow} \cap \{y\}^{\downarrow} = \{x \wedge y\}^{\downarrow}.$$

A finite lattice has a unique least element 0 and greatest element 1.
The set $\mathbf{S} := \{0,1\}^{\Lambda}$ is a lattice with
least element $\underline{0}$ and greatest element $\underline{1}$.

Monotone and additive maps

Let $\mathbf{S} := \{0,1\}^\Lambda$ and $\mathbf{T} := \{0,1\}^\Delta$.
Let $\mathcal{L}_+(\mathbf{S},\mathbf{T})$ denote the set of maps $m : \mathbf{S} \to \mathbf{T}$ such that

(i) m is lower semi-continuous,

(ii) $m(\underline{0}) = \underline{0}$,

(iii) $x \leq y \;\Rightarrow\; m(x) \leq m(x)$ $(x, y \in \mathbf{S})$.

Let $\mathcal{L}_{\rm add}(\mathbf{S},\mathbf{T})$ denote the set of maps $m \in \mathcal{L}_+(\mathbf{S},\mathbf{T})$ such that

(iii)′ $m(x \vee y) = m(x) \vee m(y)$ $(x, y \in \mathbf{S})$.

Let $\mathcal{C}_+(\mathbf{S},\mathbf{T})$ and $\mathcal{C}_{\rm add}(\mathbf{S},\mathbf{T})$ denote the sets of maps $m \in \mathcal{L}_+(\mathbf{S},\mathbf{T})$ and $m \in \mathcal{L}_{\rm add}(\mathbf{S},\mathbf{T})$, respectively, such that

(i)′ m is continuous.

Maps $m \in \mathcal{L}_{\rm add}(\mathbf{S},\mathbf{T})$ are called *additive*.
Note that (iii) implies $m(x \vee y) \geq m(x) \vee m(y)$ $(x, y \in \mathbf{S})$.

Monotone and additive systems

Let $(X_t)_{t\geq 0}$ be an interacting particle system on a lattice Λ with local state space $S=\{0,1\}$, semigroup $(P_t)_{t\geq 0}$, and generator

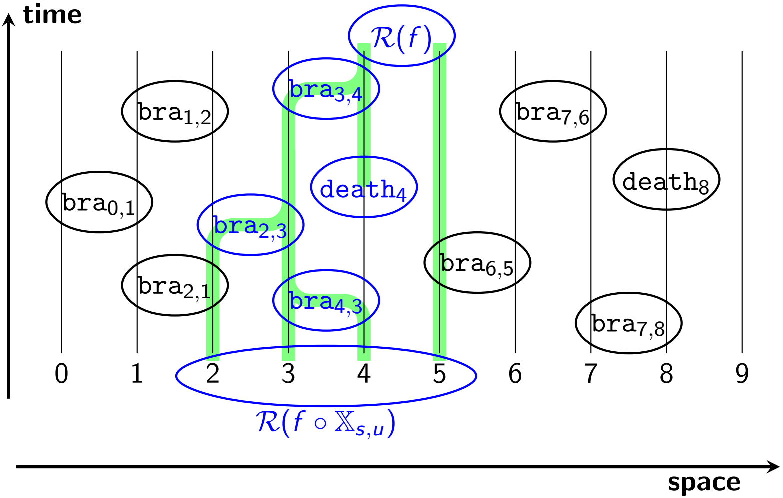
$$Gf(x)=\sum_{m\in\mathcal{G}} r_m\big\{f\big(m(x)\big)-f\big(x\big)\big\}.$$

Let $(\mathbb{X}_{s,u})_{s\leq u}$ be the associated stochastic flow and let $(\mathbb{F}_{u,s})_{u\geq s}$ be the backward stochastic flow defined as

$$\mathbb{F}_{u,s}(f):=f\circ\mathbb{X}_{s,u}\qquad\big(u\geq s,\ f\in\mathcal{C}(S^\Lambda,\{0,1\})\big).$$

- If $m\in\mathcal{C}_+(S^\Lambda,S^\Lambda)$ for all $m\in\mathcal{G}$, then $\mathbb{F}_{u,s}(f)\in\mathcal{L}_+(S^\Lambda,\{0,1\})$ for all $f\in\mathcal{L}_+(S^\Lambda,\{0,1\})$.
- If $m\in\mathcal{C}_{\rm add}(S^\Lambda,S^\Lambda)$ for all $m\in\mathcal{G}$, then $\mathbb{F}_{u,s}(f)\in\mathcal{L}_{\rm add}(S^\Lambda,\{0,1\})$ for all $f\in\mathcal{L}_{\rm add}(S^\Lambda,\{0,1\})$.

The backtracking process



Minimal one-states

Let $f \in \mathcal{L}_+(\mathbf{S}, \{0,1\})$. We can write $f(x) = 1_A(x)$ where

$$A := \big\{x \in \mathbf{S} : f(x) = 1\big\}$$

is the set of *one-states* of f. One has
$1_A \in \mathcal{L}_+(\mathbf{S}, \{0,1\}) \ \Leftrightarrow\ A$ is open and increasing.

We say $y \in A$ is *minimal* if $\nexists y' \neq y,\ y' \leq y,\ y' \in A$.
We set

$$A^\circ := \big\{y \in A : y \text{ is minimal}\big\},$$

$$\mathbf{S}^+_{\rm fin} := \big\{y \in \mathbf{S} : 0 < |y| < \infty\big\}, \quad \mathcal{H} := \big\{Y \subset \mathbf{S}^+_{\rm fin} : Y^\circ = Y\big\}.$$

Proposition Each $Y \in \mathcal{H}$ defines a function $f \in \mathcal{L}_+(\mathbf{S}, \{0,1\})$ via

$$f(x) := 1_{Y^\uparrow}(x) = 1_{\{\exists y \in Y \text{ s.t. } x \geq y\}} \qquad (x \in \mathbf{S}),$$

and $\mathcal{H} \ni Y \mapsto 1_{Y^\uparrow} \in \mathcal{L}_+(\mathbf{S}, \{0,1\})$ is a bijection.
Set $\mathcal{H}_{\rm fin} := \big\{Y \in \mathcal{H} : |Y| < \infty\big\}$. Then also
$\mathcal{H}_{\rm fin} \ni Y \mapsto 1_{Y^\uparrow} \in \mathcal{C}_+(\mathbf{S}, \{0,1\})$ is a bijection.

Duality

Define $\psi : \mathbf{S} \times \mathcal{H} \to T$ by

$$\psi(x, Y) := 1_{\{\exists y \in Y \text{ s.t. } x \geq y\}} \qquad (x \in \mathbf{S},\ \ Y \in \mathcal{H}).$$

There exists random maps $\mathbb{Y}_{u,s} : \mathcal{H} \to \mathcal{H}$ $(u \geq s)$ such that

$$\psi\big(\mathbb{X}_{s,u}(x), Y\big) = \psi\big(x, \mathbb{Y}_{u,s}(Y)\big) \qquad (x \in \mathbf{S},\ \ Y \in \mathcal{H},\ \ s \leq u).$$

These form a *dual stochastic flow*

$$\mathbb{Y}_{s,s} = 1 \quad \text{and} \quad \mathbb{Y}_{t,s} \circ \mathbb{Y}_{u,t} = \mathbb{Y}_{u,s} \qquad (u \geq t \geq s).$$

If Y_0 is independent of $(\mathbb{Y}_{u,s})_{u \geq s}$ and $u \in \mathbb{R}$, then

$$Y_t := \mathbb{Y}_{u-t,u}(Y_0) \qquad (t \geq 0)$$

defines a Markov process $(Y_t)_{t \geq 0}$ with state space $\mathcal{H}$.

Pathwise construction

Define *branching maps*, *cooperative branching maps*, and *death maps* by

$$\mathtt{bra}_{ij}(x)(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

$$\mathtt{cob}_{ii'j}(x)(k) := \begin{cases} \big(x(i) \wedge x(i')\big) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

$$\mathtt{dth}_{j}(x)(k) := \begin{cases} 0 & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

Note that $\underbrace{\mathtt{cob}_{123}(100 \vee 010)}_{\mathtt{cob}_{123}(110) = 111} > \underbrace{\mathtt{cob}_{123}(100)}_{100} \vee \underbrace{\mathtt{cob}_{123}(010)}_{010}$.

Pathwise construction

Assume that Λ is a graph.
Let $\mathcal{N}_j := \{i \in \Lambda : i \text{ is adjacent to } j\}$.
and $\mathcal{N}_j^2 := \{(i,i') : i,i' \in \mathcal{N}_j,\ i \neq i'\}$.

Consider the "cooperative contact process" with generator

$$\begin{aligned} Gf(x) := {} & (1-\alpha)\sum_{j\in\Lambda}\frac{1}{|\mathcal{N}_j|}\sum_{i\in\mathcal{N}_j}\big\{f\big(\mathtt{bra}_{ij}(x)\big) - f\big(x\big)\big\} \\ & +\alpha\sum_{j\in\Lambda}\frac{1}{|\mathcal{N}_j^2|}\sum_{(i,i')\in\mathcal{N}_j^2}\big\{f\big(\mathtt{cob}_{ii'j}(x)\big) - f\big(x\big)\big\} \\ & +\delta\sum_{j\in\Lambda}\big\{f\big(\mathtt{dth}_j(x)\big) - f\big(x\big)\big\}. \end{aligned}$$

For $\alpha = 0$ this is a contact process and
for $\alpha = 1$ there is only cooperative branching.

The graphical representation

We construct the cooperative contact process from a graphical representation ω.

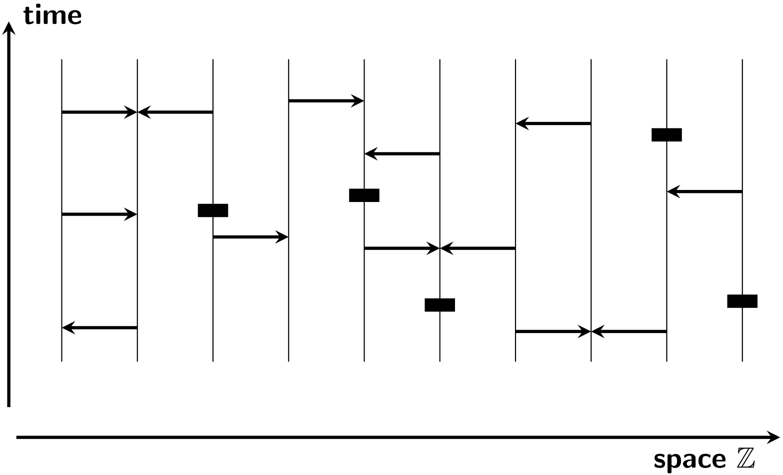
We visualise the Poisson point set ω by drawing space Λ horizontally and time $\mathbb{R}$ vertically.

For each $(\mathtt{bra}_{ij}, t) \in \omega$ we draw an arrow from (i, t) to (j, t).

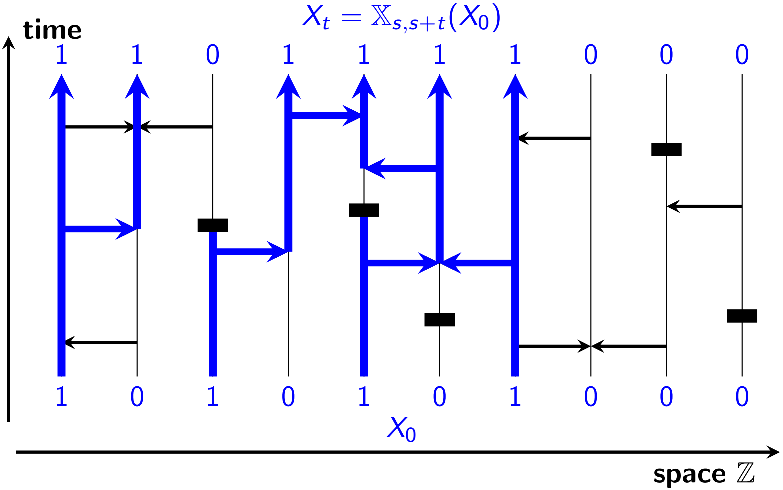
For each $(\mathtt{cob}_{ii'j}, t) \in \omega$ we draw two arrows, one from (i, t) to (j, t) and the other from (i', t) to (j, t).

For each $(\mathtt{dth}_j, t) \in \omega$ we draw a blocking symbol $\blacksquare$ at (j, t).

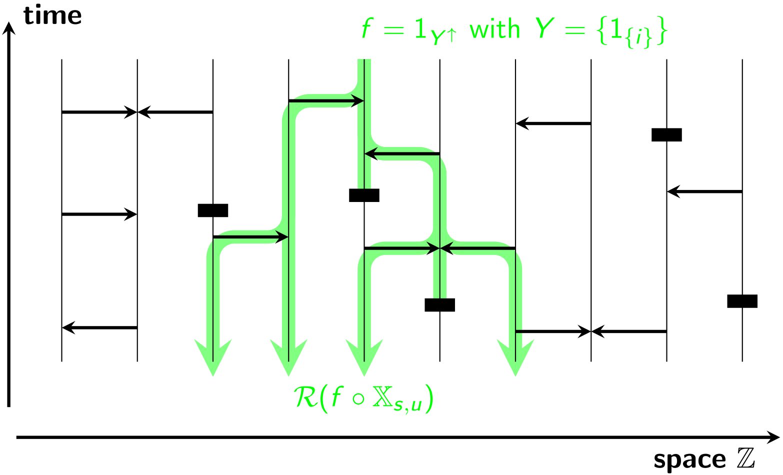
Pathwise construction



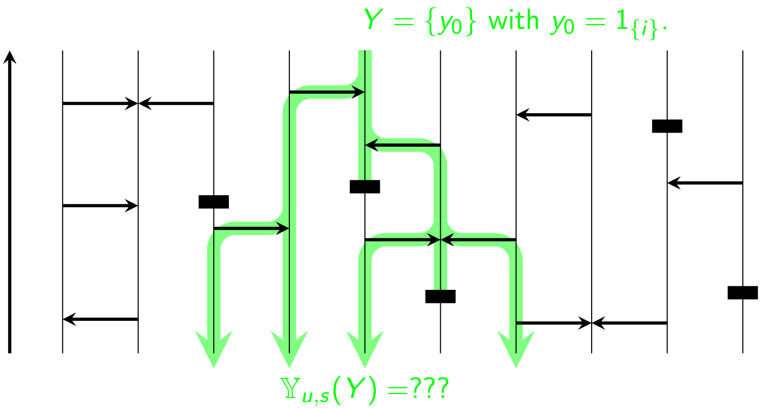
Pathwise construction



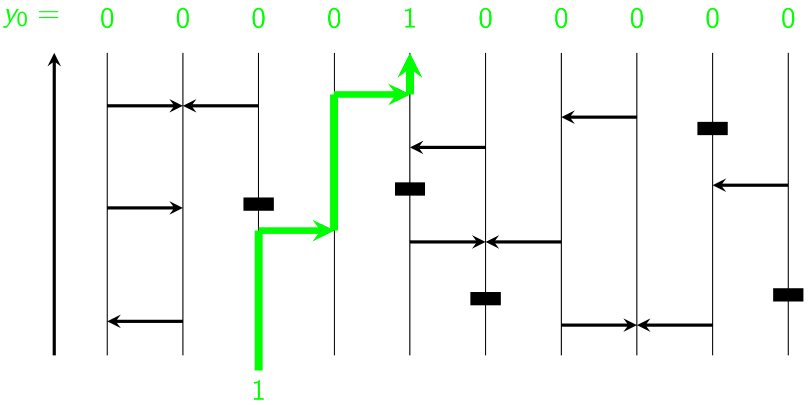
Pathwise construction



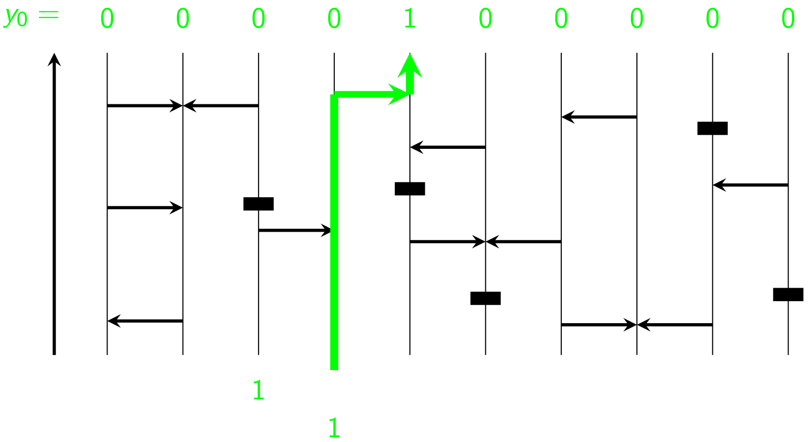
Percolation picture



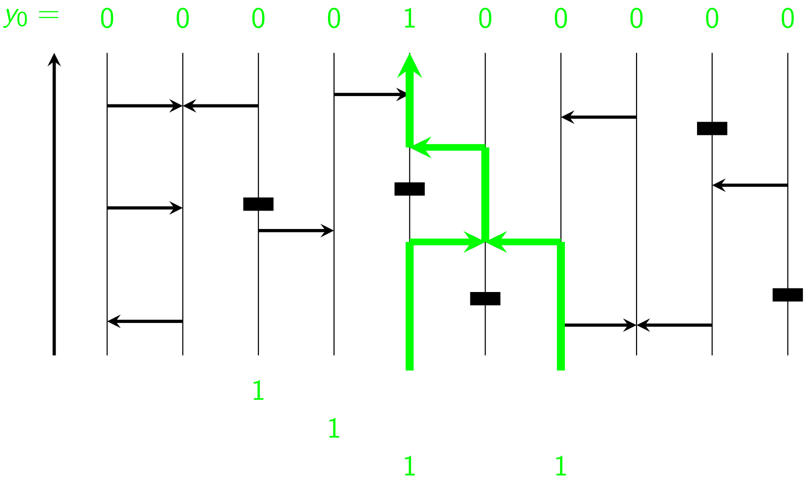
Percolation picture



Percolation picture

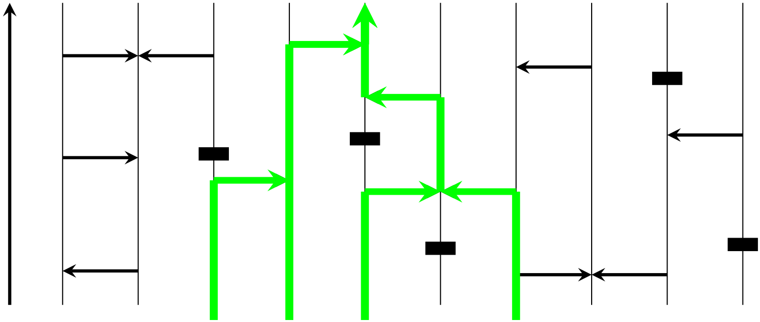


Percolation picture



Percolation picture

$y_0 = \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$



$y_1 = \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$

$y_2 = \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$

$y_3 = \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0$

$$\mathbb{Y}_{u,s}(\{y_0\}) = \{y_1, y_2, y_3\}$$

Duality

There exists a metrisable topology on $\mathcal{H}$ such that

$$Y_n \to Y \quad \Leftrightarrow \quad \psi(x, Y_n) \to \psi(x, Y) \quad \forall x \in \mathbf{S}_{\mathrm{fin}}.$$

The space $\mathcal{H}$ is compact under this topology.

We define a partial order on $\mathcal{H}$ by

$$Y_1 \leq Y_2 \quad \Leftrightarrow \quad \psi(x, Y_1) \leq \psi(x, Y_2).$$

The least element of $\mathcal{H}$ in this order is $\emptyset$ and the greatest element is

$$\top := \{1_{\{i\}} : i \in \Lambda\}.$$

One has

$$\mathbb{P}^{\top}\big[Y_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\mu},$$

where $\overline{\mu}$ is the *upper invariant law* of the dual process.

Survival and stability

- Either $\overline{\mu}$ is *trivial* in the sense that $\overline{\mu} = \delta_\emptyset$,
- or $\overline{\mu}$ is *nontrivial* in the sense that $\overline{\mu}(\{\emptyset\}) = 0$.

If $\overline{\mu}$ is nontrivial, then we say the dual process $(Y_t)_{t\geq 0}$ is *stable*.

We say that the dual process *survives* if

$$\mathbb{P}^Y\big[Y_t \neq \emptyset\ \forall t \geq 0\big] > 0 \quad \text{for some } Y \in \mathcal{H}_{\rm fin}.$$

Lemma [Gray '86, Latz & S. '23] One has

$$\begin{aligned} X \text{ is stable} \quad &\Leftrightarrow \quad Y \text{ survives}, \\ X \text{ survives} \quad &\Leftrightarrow \quad Y \text{ is stable} \end{aligned}$$

The main novelty of our work is the construction of $(Y_t)_{t\geq 0}$ for infinite initial states.

Proof

$$\mathbb{P}\big[\exists y \in Y \text{ s.t. } \mathbb{X}_{-t,0}(\underline{1}) \geq y\big] = \mathbb{P}\big[\exists y \in \mathbb{Y}_{0,-t}(Y) \text{ s.t. } \underline{1} \geq y\big]$$
$$= \mathbb{P}\big[\mathbb{Y}_{0,-t}(Y) \neq \emptyset\big] \underset{t\to\infty}{\longrightarrow} \mathbb{P}\big[\mathbb{Y}_{0,-s}(Y) \neq \emptyset \ \forall s \geq 0\big],$$

and the limit is > 0 for some $Y \in \mathcal{H}_{\rm fin}$ iff $\overline{\nu}$ is nontrivial. Similarly

$$\mathbb{P}\big[\exists y \in \mathbb{Y}_{t,0}(\top) \text{ s.t. } x \geq y\big] = \mathbb{P}\big[\exists y \in \top \text{ s.t. } \mathbb{X}_{0,t}(x) \geq y\big]$$
$$= \mathbb{P}\big[\mathbb{X}_{0,t}(x) \neq \underline{0}\big] \underset{t\to\infty}{\longrightarrow} \mathbb{P}\big[\mathbb{X}_{0,s}(x) \neq \underline{0} \ \forall s \geq 0\big],$$

and the limit is > 0 for some $x \in \mathbf{S}^+_{\rm fin}$ iff $\overline{\mu}$ is nontrivial. ∎

Additive duality

Let $\mathcal{H}_1 := \{Y \in \mathcal{H} : |y| = 1 \ \forall y \in Y\}$.
We can naturally identify $Y \in \mathcal{H}_1$ with $x \in \mathbf{S}$ defined as

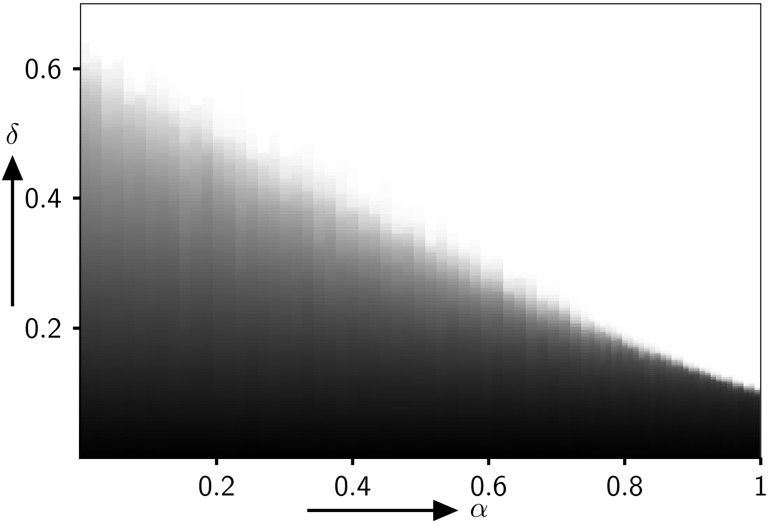
$$x(i) = 1 \quad \Leftrightarrow \quad 1_{\{i\}} \in Y.$$

If all maps $m \in \mathcal{G}$ are additive, then the dual stochastic flow $(\mathbb{Y}_{u,s})_{u \geq s}$ maps $\mathcal{H}_1$ into itself and the corresponding Markov process on $\mathcal{H}_1 \cong \mathbf{S}$ is itself an additive particle system.

For the contact process, this yields a well-known *self-duality*.
As a result:

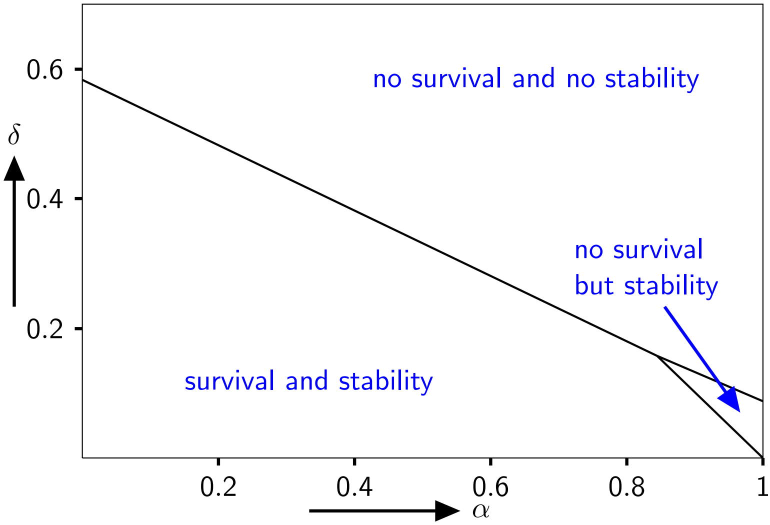
$$X \text{ is stable} \quad \Leftrightarrow \quad X \text{ survives.}$$

Numerical data



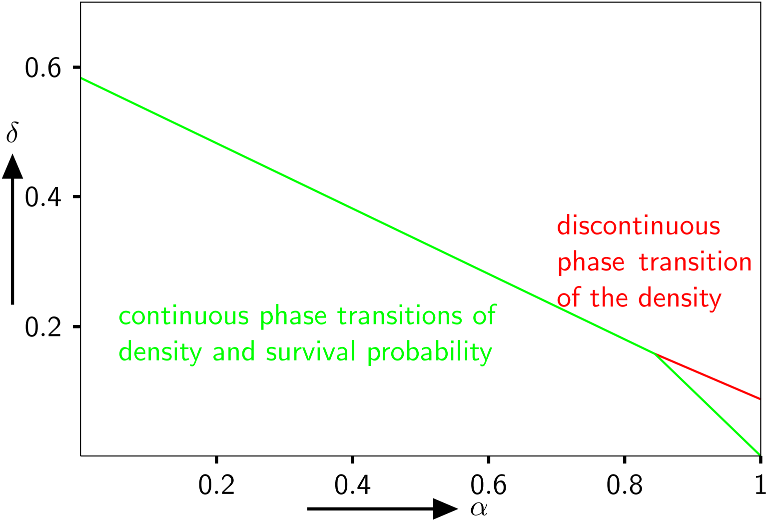
Density of the upper invariant law for the process on $\mathbb{Z}^2$.

Conjectured phase diagram



Conjectured phase diagram for the process on $\mathbb{Z}^2$.

Conjectured phase diagram



Conjectured phase diagram for the process on $\mathbb{Z}^2$.

Let $\theta(\alpha,\delta) := \lim_{t\to\infty} \mathbb{P}^{\underline{1}}\big[X_t(i)=1\big]$,
and $\theta'(\alpha,\delta) := \mathbb{P}^{1_{\{i\}}}\big[X_t \neq \underline{0}\ \forall t\geq 0\big]$.

- θ and θ' are nonincreasing in α and δ.
- $\forall\alpha\in[0,1]\ \exists 0\leq\delta_{\rm c}(\alpha)<\infty$ s.t. $\theta(\alpha,\delta)>0$ for $\delta<\delta_{\rm c}(\alpha)$ and $\theta(\alpha,\delta)=0$ for $\delta>\delta_{\rm c}(\alpha)$.
- $\forall\alpha\in[0,1]\ \exists 0\leq\delta'_{\rm c}(\alpha)<\infty$ s.t. $\theta'(\alpha,\delta)>0$ for $\delta<\delta'_{\rm c}(\alpha)$ and $\theta'(\alpha,\delta)=0$ for $\delta'>\delta_{\rm c}(\alpha)$.
- $\theta(0,\delta)=\theta'(0,\delta)$ and $\delta_{\rm c}(0)=\delta'_{\rm c}(0)$.
- $\delta_{\rm c}(0)>0$.
- $\delta\mapsto\theta(0,\delta)$ is continuous on $[0,\delta_{\rm c}(0))$.
- $\theta(0,\delta_{\rm c}(0))=0$.
- $\delta'_{\rm c}(1)=0$ and $\delta_{\rm c}(1)>0$.
- $\delta'_{\rm c}(\alpha)\leq\delta_{\rm c}(\alpha)\quad\forall\alpha\in[0,1]$.

Open problems

- $\exists 0 < \alpha_c < 1$ s.t. $\delta'_c(\alpha) = \delta_c(\alpha)$ for $\alpha \leq \alpha_c$ and $\delta'_c(\alpha) < \delta_c(\alpha)$ for $\alpha > \alpha_c$.
- Discontinuity of $\delta \mapsto \theta(\alpha, \delta)$ at $\delta_c(\alpha)$ for $\alpha > \alpha_c$.
- Continuity of θ and θ' everywhere else (partial results are known as will be discussed later).