Lecture 3
The mean-field limit

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Outline

- The mean-field ODE
- The mean-field phase diagram
- The mean-field backtracking process
- The mean-field dual process

The cooperative contact process

Let Λ be a graph.
Let $\mathcal{N}_j := \big\{i \in \Lambda : i \text{ is adjacent to } j\big\}$.
and $\mathcal{N}_j^2 := \big\{(i,i') : i,i' \in \mathcal{N}_j,\ i \neq i'\big\}$.

The *cooperative contact process* has generator

$$\begin{aligned}
Gf(x) := (1-\alpha) &\sum_{j\in\Lambda} \frac{1}{|\mathcal{N}_j|} \sum_{i\in\mathcal{N}_j} \big\{f\big(\mathtt{bra}_{ij}(x)\big) - f\big(x\big)\big\} \\
&+\alpha \sum_{j\in\Lambda} \frac{1}{|\mathcal{N}_j^2|} \sum_{(i,i')\in\mathcal{N}_j^2} \big\{f\big(\mathtt{cob}_{ii'j}(x)\big) - f\big(x\big)\big\} \\
&+\delta \sum_{j\in\Lambda} \big\{f\big(\mathtt{dth}_j(x)\big) - f\big(x\big)\big\}.
\end{aligned}$$

The *critical death rate for stability* is $\delta_{\rm c}(\alpha)$
and the *critical death rate for survival* is $\delta'_{\rm c}(\alpha)$.

The mean-field limit

Let Λ_N be the *complete graph* with N vertices.

If we start with a positive fraction of ones, then in the *mean-field limit* $N \to \infty$, the frequency of ones

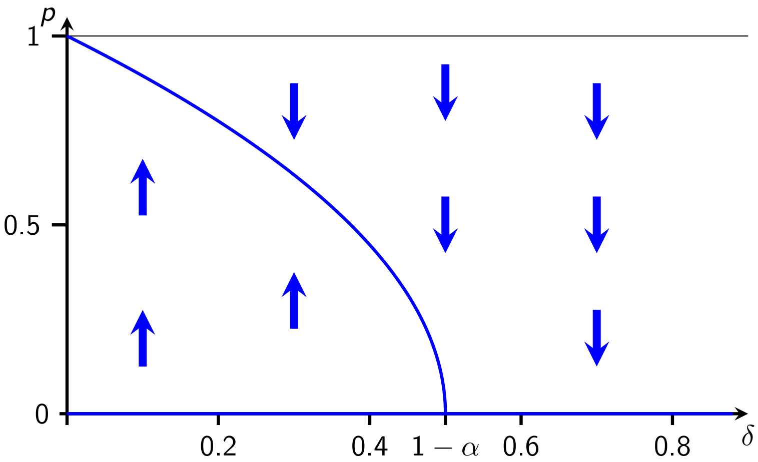
$$P_t := \frac{1}{N} \sum_{i \in V} X_t(i)$$

solves the *mean-field equation*

$$\tfrac{\partial}{\partial t} p_t = \alpha p_t^2 (1 - p_t) + (1 - \alpha) p_t (1 - p_t) - \delta p_t. \qquad (1)$$

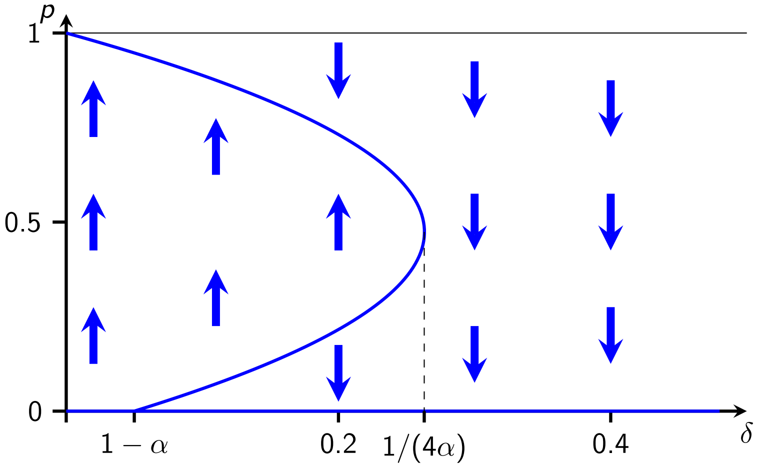
If we start with a single one, then in the limit the number of ones is a branching process with binary branching rate $1 - \alpha$ and death rate δ.

The mean-field limit



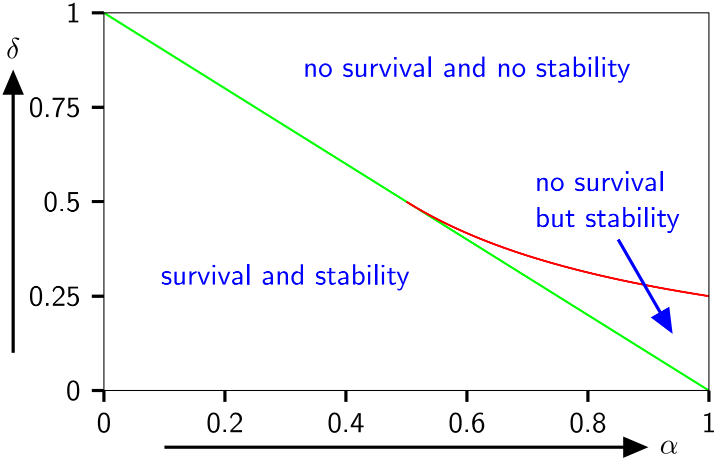
Fixed points of the mean-field equation for $\alpha = 0.5$ and their domains of attraction, as a function of δ.

The mean-field limit



Fixed points of the mean-field equation for $\alpha = 0.95$ and their domains of attraction, as a function of δ.

The mean-field phase diagram



$$\delta_c(\alpha) = \begin{cases} 1-\alpha & \text{if } \alpha \leq 1/2 \\ 1/(4\alpha) & \text{if } \alpha \geq 1/2 \end{cases} \qquad \text{and} \qquad \delta'_c(\alpha) = 1-\alpha.$$

The mean-field backward process

Recall that the backward stochastic flow $(\mathbb{F}_{u,s})_{u \geq s}$, defined as

$$\mathbb{F}_{u,s}(f) := f \circ \mathbb{X}_{s,u} \qquad \big(u \geq s,\ f \in \mathcal{C}(S^\Lambda, \{0,1\})\big),$$

has the property that

$$f \in \mathcal{C}_+(S^\Lambda, \{0,1\}) \quad \Rightarrow \quad \mathbb{F}_{u,s}(f) \in \mathcal{C}_+(S^\Lambda, \{0,1\}).$$

We are interested in the case that $f(x) = x(i)$ for some $i \in \Lambda_N$.

In the mean-field limit, the backtracking process

$$F_t := \mathbb{F}_{u,u-t}(f) \qquad (t \geq 0)$$

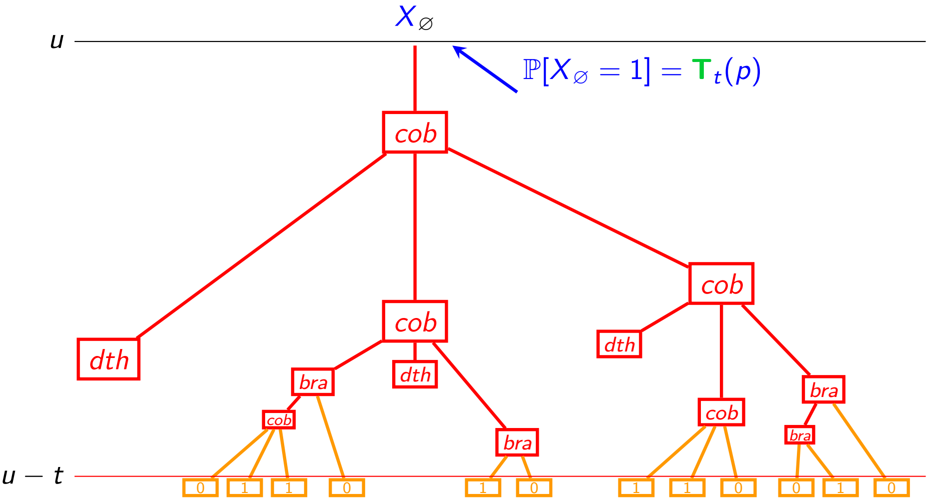
is the concatenation of local maps that are attached to the nodes of the ancestral tree of a branching process.

The mean-field backward process

Define $cob : \{0,1\}^3 \to \{0,1\}$, $bra : \{0,1\}^2 \to \{0,1\}$, and $dth : \{0,1\}^0 \to \{0,1\}$ by

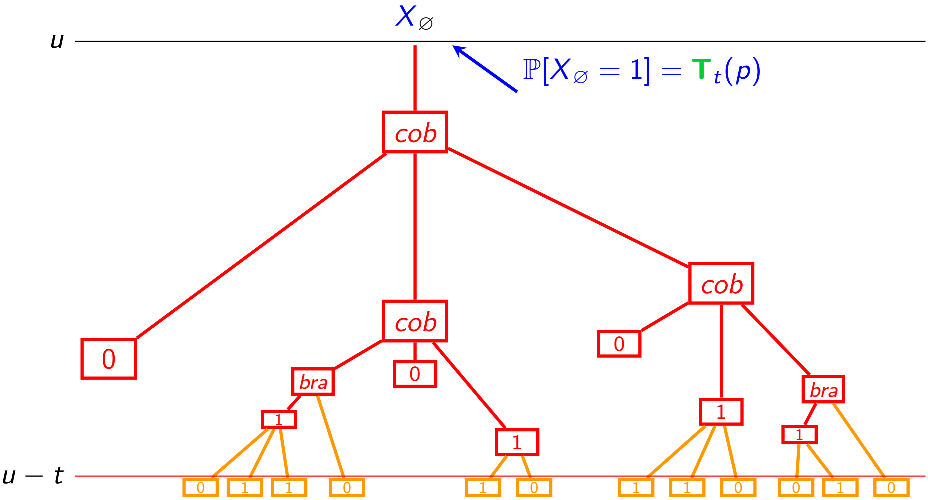
$$\begin{aligned} bra(x_1, x_2) &:= x_1 \vee x_2, \\ cob(x_1, x_2, x_3) &:= x_1 \vee (x_2 \wedge x_3), \\ dth(\varnothing) &:= 0. \end{aligned}$$

The mean-field backward process



$(X_{\mathbf{i}})_{\mathbf{i} \in \nabla \mathbb{S}_t}$ i.i.d. intensity p

The mean-field backward process



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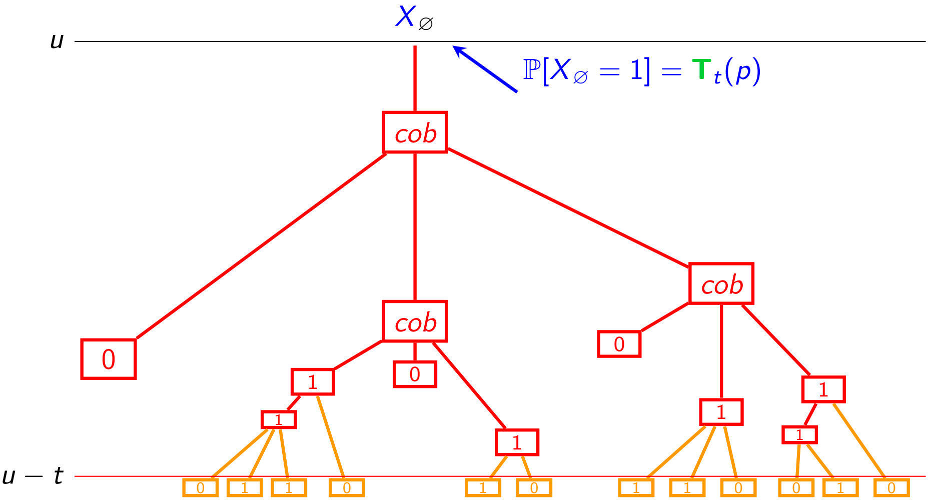

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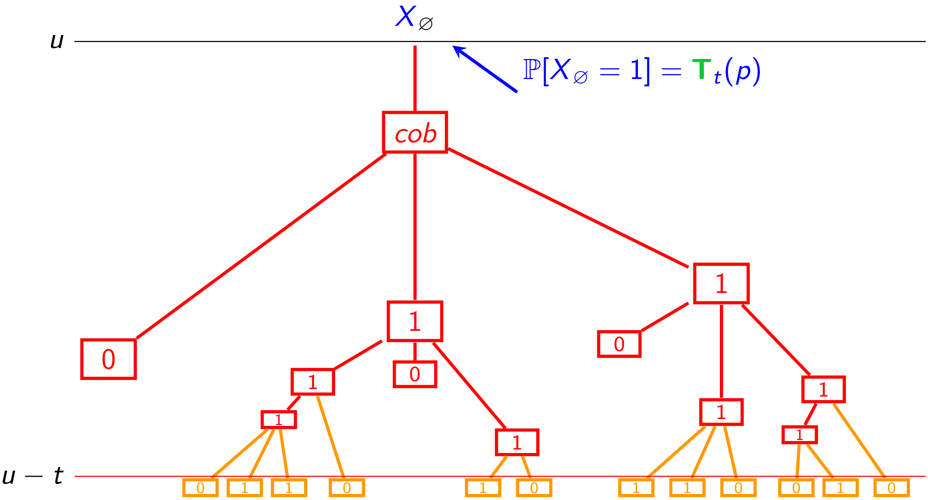

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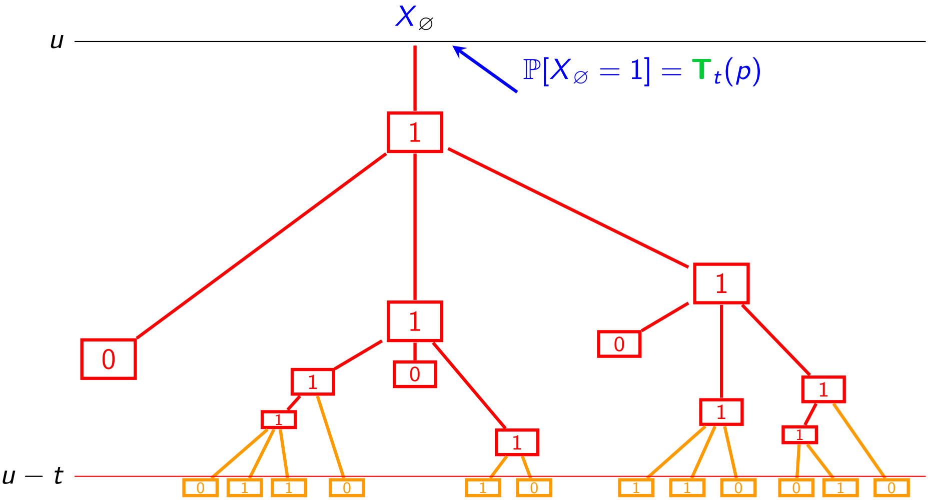

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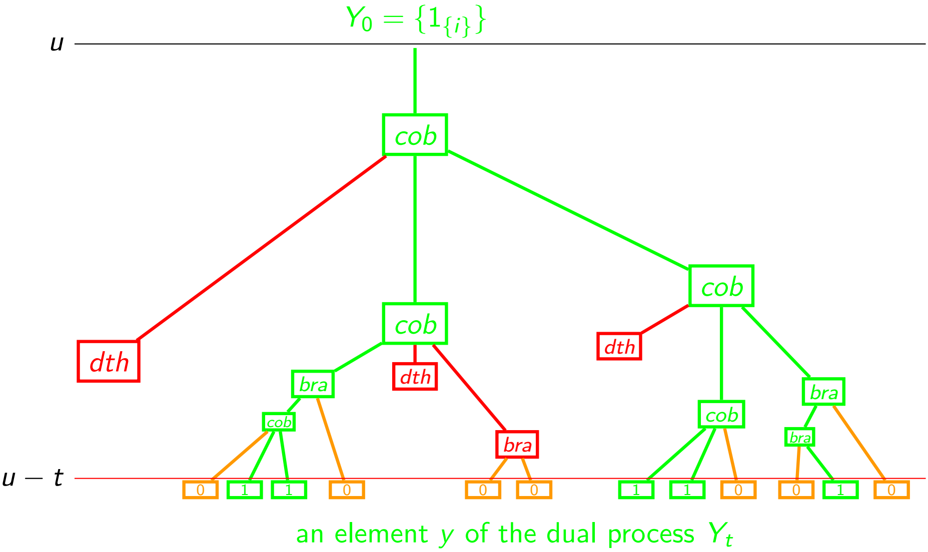

The mean-field dual process

In the regime where the process X is stable but does not survive, the dual process Y survives, but

$$\inf\big\{|y| : y \in Y_t\big\} \underset{t\to\infty}{\longrightarrow} \infty \quad \text{a.s.}$$

This means that the minimal one-states at time $u-t$ required for $X_u(i)=1$ are all very large when t is large.