Lecture 4
The cooperative contact process

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Outline

- Monotone coupling
- Extinction for large death rates.
- Equality of the critical death rates of the contact process.
- Survival for small death rates.
- Continuity of the density of the upper invariant law.
- Extinction in the purely cooperative case.
- Survival implies stability.

The cooperative contact process

Let $\Lambda = \mathbb{Z}^d$ with nearest-neighbour edges.
Let $\mathcal{N}_j := \big\{ i \in \Lambda : i \text{ is adjacent to } j \big\}$.
and $\mathcal{N}_j^2 := \big\{ (i,i') : i,i' \in \mathcal{N}_j,\ i \neq i' \big\}$.

The *cooperative contact process* has generator

$$
\begin{aligned}
Gf(x) := {} & (1-\alpha) \sum_{j\in\Lambda} \frac{1}{|\mathcal{N}_j|} \sum_{i\in\mathcal{N}_j} \big\{ f\big(\mathtt{bra}_{ij}(x)\big) - f\big(x\big) \big\} \\
& + \alpha \sum_{j\in\Lambda} \frac{1}{|\mathcal{N}_j^2|} \sum_{(i,i')\in\mathcal{N}_j^2} \big\{ f\big(\mathtt{cob}_{ii'j}(x)\big) - f\big(x\big) \big\} \\
& + \delta \sum_{j\in\Lambda} \big\{ f\big(\mathtt{dth}_j(x)\big) - f\big(x\big) \big\}.
\end{aligned}
$$

The *density of the upper inv. law* is $\theta(\alpha,\delta) := \lim_{t\to\infty} \mathbb{P}^{\underline{1}}\big[X_t(i) = 1\big]$,
and the *survival probability* is $\theta'(\alpha,\delta) := \mathbb{P}^{1_{\{i\}}}\big[X_t \neq \underline{0}\ \forall t \geq 0\big]$.

The cooperative contact process

Lemma θ and θ' are nonincreasing in α and δ.

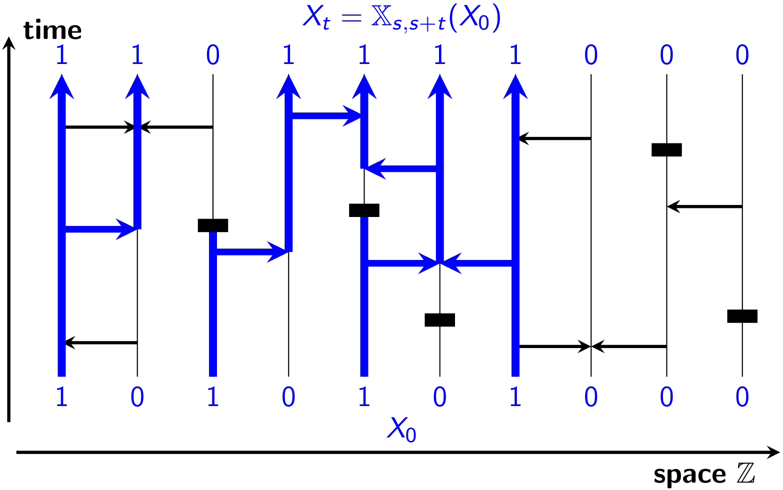
Proof Let X and $\tilde{X}$ have parameters $\alpha \leq \tilde{\alpha}$ and $\delta \leq \tilde{\delta}$.

We can couple their graphical representations ω and $\tilde{\omega}$
such that $\tilde{\omega}$ has more death maps
and some of the maps $\texttt{coop}_{ii'j}$ in $\tilde{\omega}$ are replaced by $\texttt{bra}_{ij}$ in ω.

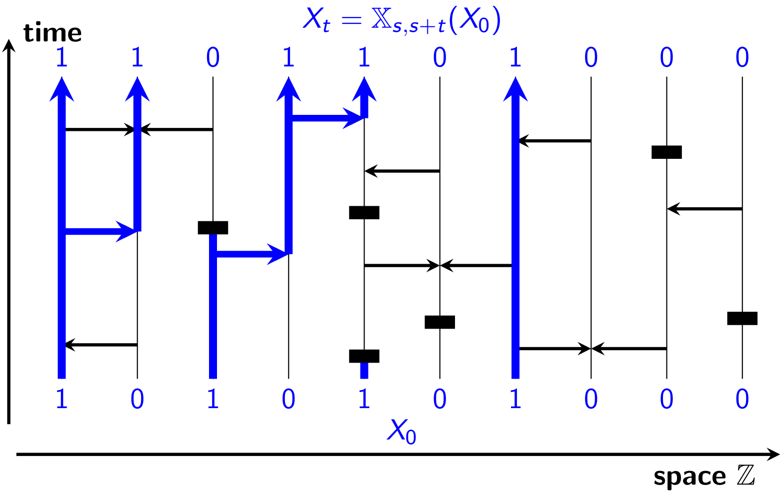
Then it is easy to see that $X_0 \geq \tilde{X}_0$ implies $X_t \geq \tilde{X}_t$ a.s. ∎

Note One of the difficulties of non-monotone interacting particle systems is that this sort of coupling arguments fail and many "obvious" monotonicities lack a rigorous proof.

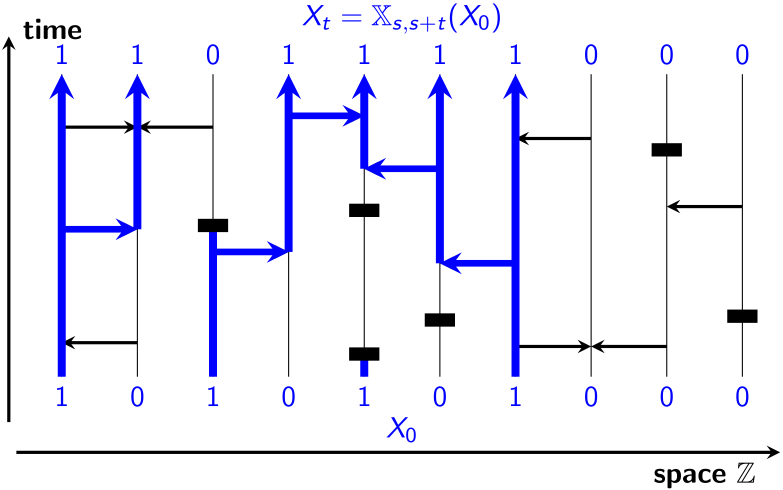
Coupling argument



Coupling argument



Coupling argument



The critical death rates

Lemma For all $\alpha \in [0,1]$, there exist $0 \leq \delta_{\rm c}(\alpha), \delta'_{\rm c}(\alpha) < \infty$ such that

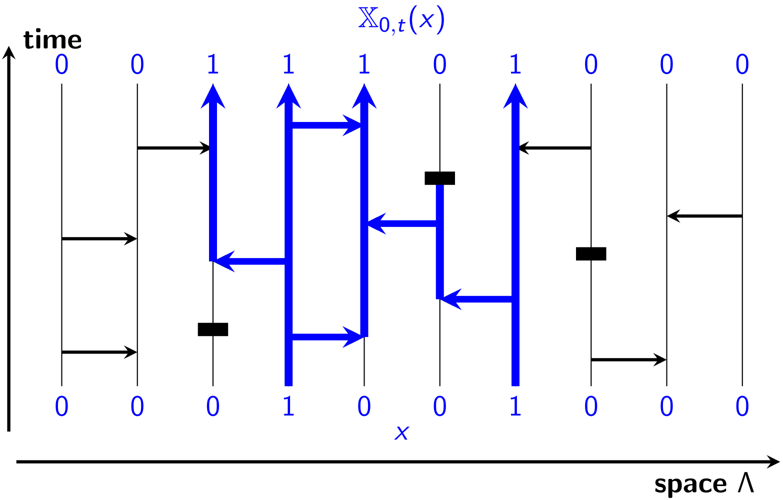
- $\theta(\alpha,\delta) > 0$ for $\delta < \delta_{\rm c}(\alpha)$ and $\theta(\alpha,\delta) = 0$ for $\delta > \delta_{\rm c}(\alpha)$,
- $\theta'(\alpha,\delta) > 0$ for $\delta < \delta'_{\rm c}(\alpha)$ and $\theta'(\alpha,\delta) = 0$ for $\delta' > \delta_{\rm c}(\alpha)$.

Proof We claim that $\theta(\alpha,\delta) = \theta'(\alpha,\delta) = 0$ for all $\delta > 1$. Indeed, each particle dies with rate δ and produces new particles with rate ≤ 1, so

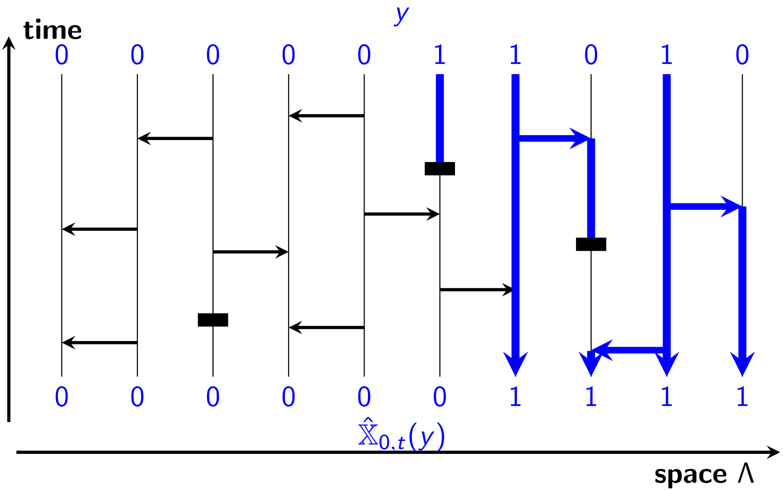
$$\mathbb{E}^x\big[|X_t|\big] \leq e^{(1-\delta)t}|x| \quad \text{and} \quad \mathbb{E}^{\underline{1}}\big[X_t(i) = 1\big] \leq e^{(1-\delta)t}.$$

The claims now follow from monotonicity in the parameters. ∎

The contact process



The dual contact process



Self-duality of the contact process

$$
\begin{aligned}
&\mathbb{X}_{0,t}(x) \wedge y \neq \underline{0} \\
&\quad \Leftrightarrow \quad \text{there is an open path from } x \text{ to } y \\
&\quad \Leftrightarrow \quad x \wedge \hat{\mathbb{X}}_{t,0}(y) \neq \underline{0}.
\end{aligned}
$$

A consequence of self-duality

Lemma $\theta(0,\delta) = \theta'(0,\delta)$ and $\delta_{\rm c}(0) = \delta'_{\rm c}(0)$.

Proof Self-duality of the contact process implies

$$\begin{aligned}\mathbb{P}^{\underline{1}}\big[X_t(i) = 1\big] &= \mathbb{P}^{\underline{1}}\big[X_t \wedge 1_{\{i\}} \neq \underline{0}\big] \\ &\stackrel{!}{=} \mathbb{P}^{1_{\{i\}}}\big[\underline{1} \wedge X_t \neq \underline{0}\big] = \mathbb{P}^{1_{\{i\}}}\big[X_t \neq \underline{0}\big].\end{aligned}$$

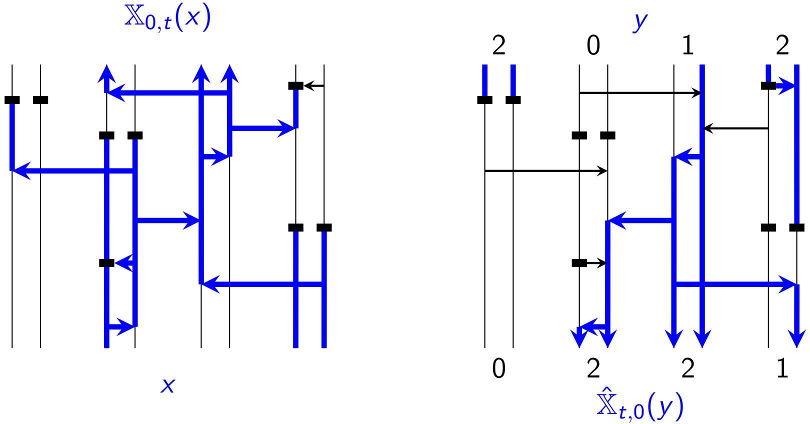
Letting $t \to \infty$ gives $\theta(0,\delta) = \theta'(0,\delta)$. ∎

Remark There are many interesting additive particle systems that are not self-dual. For such systems,

$$\begin{aligned}\theta(\delta) &= \mathbb{P}^{1_{\{i\}}}\big[\hat{X}_t \neq \underline{0}\ \forall t \geq 0\big], \\ \theta'(\delta) &= \mathbb{P}^{1_{\{i\}}}\big[X_t \neq \underline{0}\ \forall t \geq 0\big],\end{aligned}$$

where $(\hat{X}_t)_{t\geq 0}$ is the additive dual of $(X_t)_{t\geq 0}$, and $\theta(\delta) \neq \theta'(\delta)$.

The two-stage contact process



An example is Krone's two-stage contact process.

The two-stage contact process

Also for Krone's two-stage contact process we expect $\delta_{\rm c} = \delta'_{\rm c}$.

Proof idea Use the percolation representation on $\mathbb{Z}^d \times \{1, 2\}$. Let

$$C_t(i,j) := \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \mathbb{P}\big[(i,\alpha,0) \rightsquigarrow (j,\beta,t)\big] \qquad (i,j \in \mathbb{Z}^d),$$

and let $C_t := \sum_i C_t(i,j) = \sum_j C_t(i,j)$. Then it suffices to prove *sharpness of the phase transition*

$$\delta_{\rm c} = \inf\big\{\delta \geq 0 : \lim_{t\to\infty} t^{-1} \log C_t < 0\big\} = \delta'_{\rm c}.$$

∎

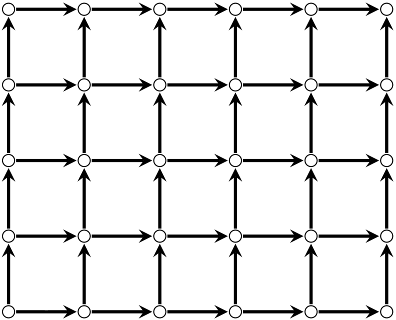
Survival of the contact process

Theorem For the coooperative contact process, $\delta_{\rm c}(0) > 0$.

Proof By a simple coupling argument, it suffices to prove the statement in one dimension.

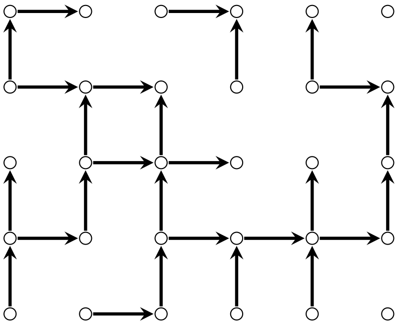
We will use *comparison with oriented percolation*.

Oriented Percolation



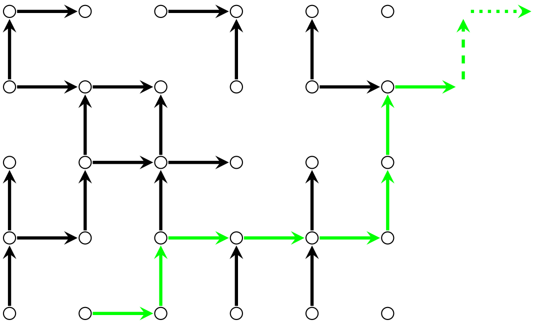
Equip $\mathbb{Z}^2$ with the structure of an oriented graph by drawing at each (i_1, i_2) two arrows, pointing to $(i_1 + 1, i_2)$ and $(i_1, i_2 + 1)$.

Oriented Percolation



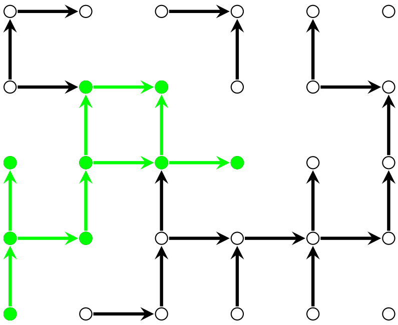
Thin the collection of arrows by independently keeping each arrow with probability p.

Oriented Percolation



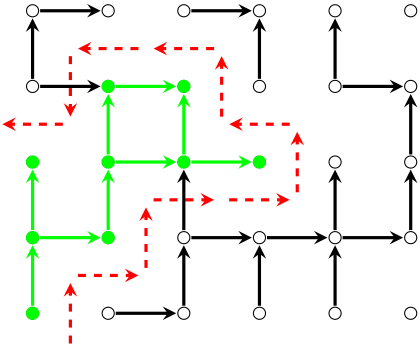
We want to prove that for p large enough *percolation* occurs, i.e., there are infinite oriented paths.

Oriented Percolation



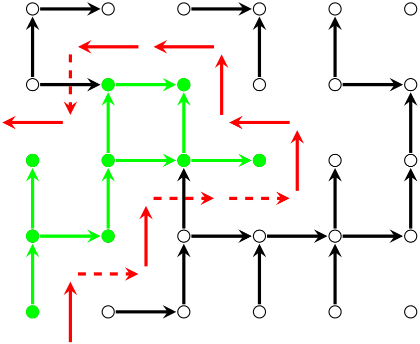
If the set C of points that can be reached by an open path starting at the origin is finite...

Oriented Percolation



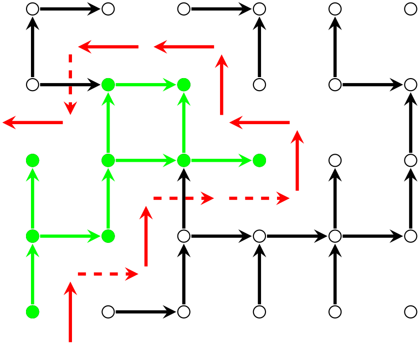
. . . then there is an oriented path separating this set from the infinite component of $\mathbb{N}^2 \backslash C$.

Oriented Percolation



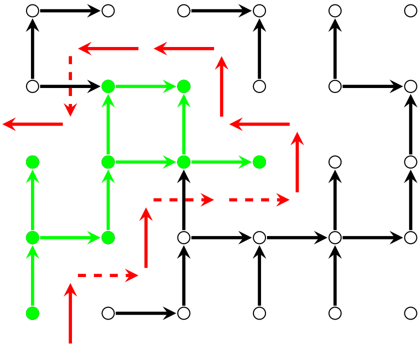
The *up* and *left* steps of this path cannot cross black arrows.

Oriented Percolation

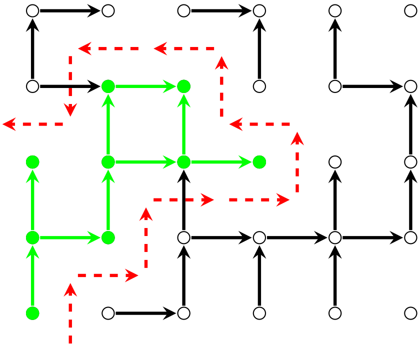


There are more up steps than down steps,
and more left steps than right steps.

Oriented Percolation

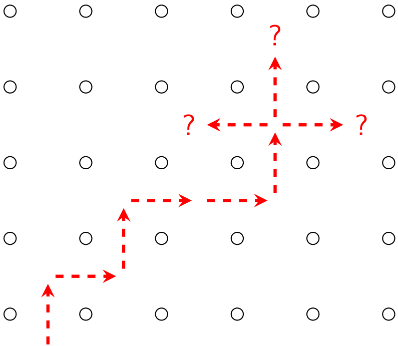


The probability that for a *given* path of L steps, no up or left steps cross a black arrow is $\leq (1-p)^{L/2}$.



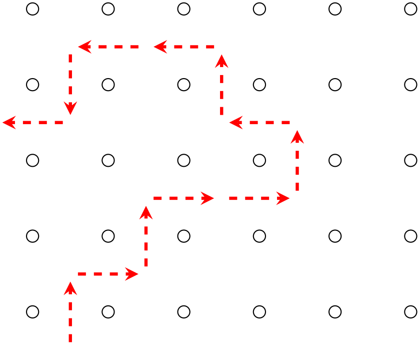
A path of length L must start somewhere between $(0,0)$ and $(L,0)$.

Oriented Percolation



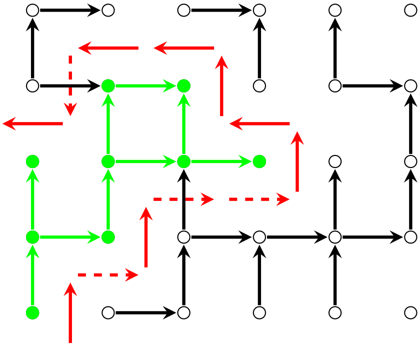
In each point, there are at most three directions in which the path can continue.

Oriented Percolation



It follows that the total number of red paths of length L is $\leq L3^L$.

Oriented Percolation



And the *expected number* of paths with the property that no up or left step crosses a black arrow is $\leq \sum_{L=2}^{\infty} L3^L(1-p)^{L/2}$.

Oriented Percolation

If $p > 8/9$, then

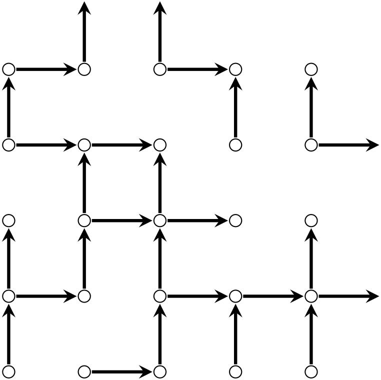
$$\begin{aligned}
&\mathbb{P}\big[\text{there is } \textit{no} \text{ infinite green path starting at } (0,0)\big] \\
&\quad = \mathbb{P}\big[\text{there is a red path blocking } (0,0)\big] \\
&\quad \leq \mathbb{E}\big[\#\text{ red paths blocking } (0,0)\big] \\
&\quad \leq \sum_{L=2}^{\infty} L 3^L (1-p)^{L/2} < \infty.
\end{aligned}$$

By choosing p very close to 1, we can make this sum as small as we wish. In particular, choosing p such that the sum is less than 1, we have proved that:

$$\mathbb{P}\big[\text{there is an infinite green path starting at } (0,0)\big] > 0.$$

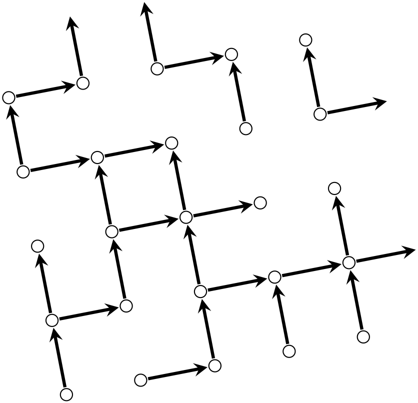
This is a *Peierls argument*.

Comparison with Oriented Percolation



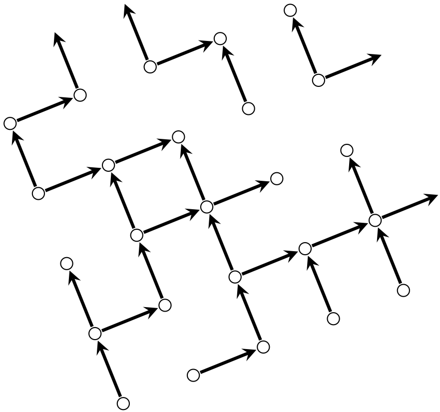
We take our percolation picture. . .

Comparison with Oriented Percolation



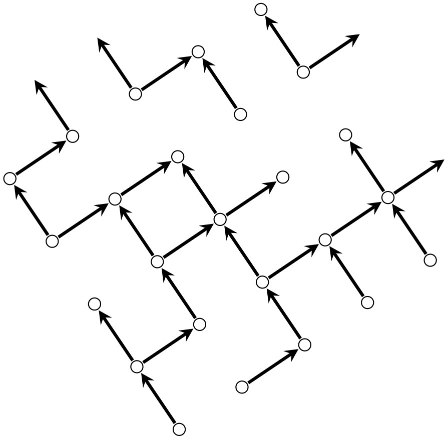
. . . and rotate it over 45°.

Comparison with Oriented Percolation



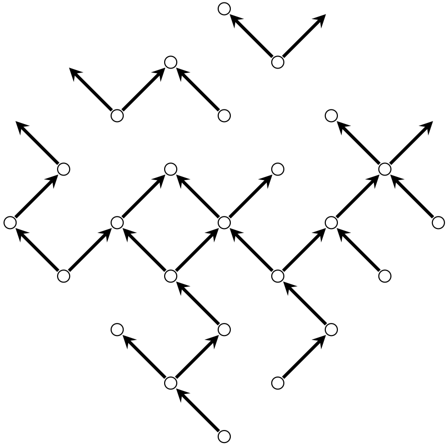
. . . and rotate it over 45°.

Comparison with Oriented Percolation



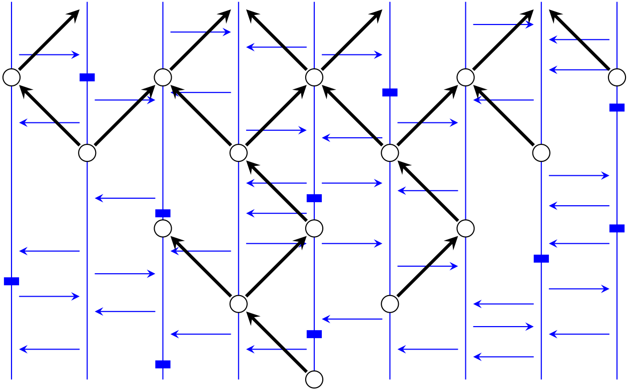
. . . and rotate it over 45°.

Comparison with Oriented Percolation



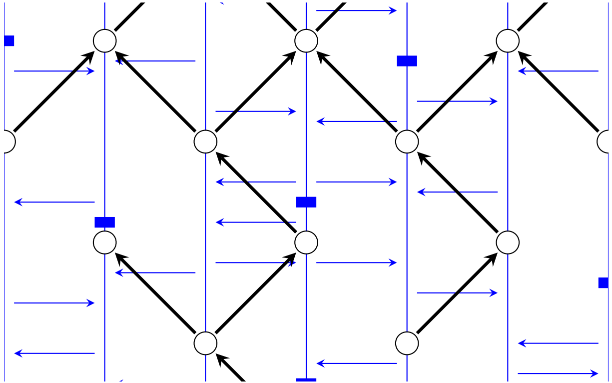
. . . and rotate it over 45°.

Comparison with Oriented Percolation



We overlay this with the graphical representation for the contact process.

Comparison with Oriented Percolation



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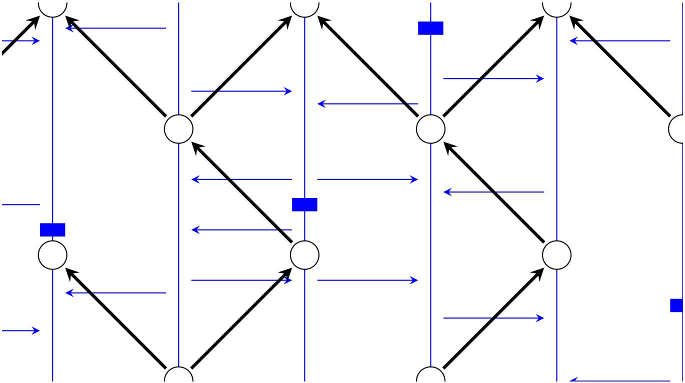
Comparison with Oriented Percolation



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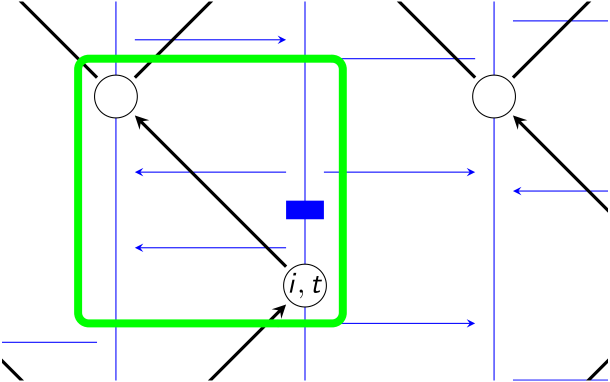
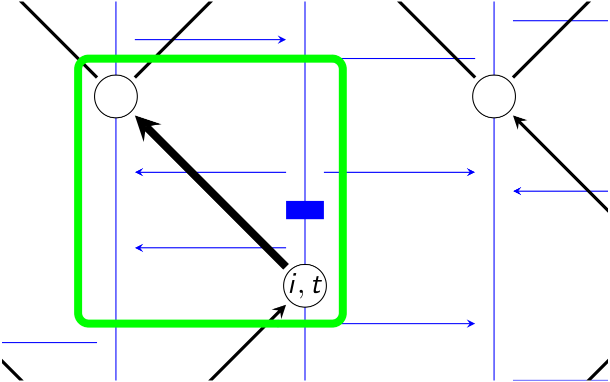
Comparison with Oriented Percolation



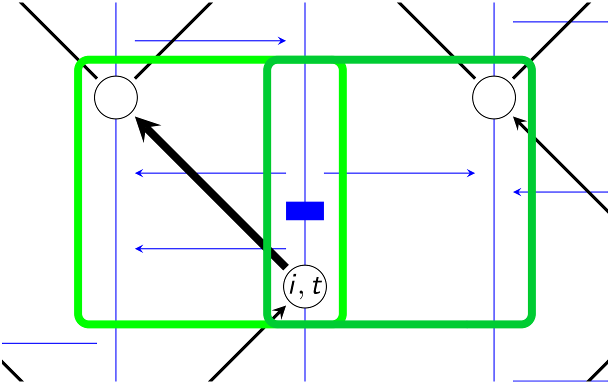
We draw a black arrow from (i, t) to $(i \pm 1, t + 1)$ if within the green square, there is an open path connecting these points.

Comparison with Oriented Percolation



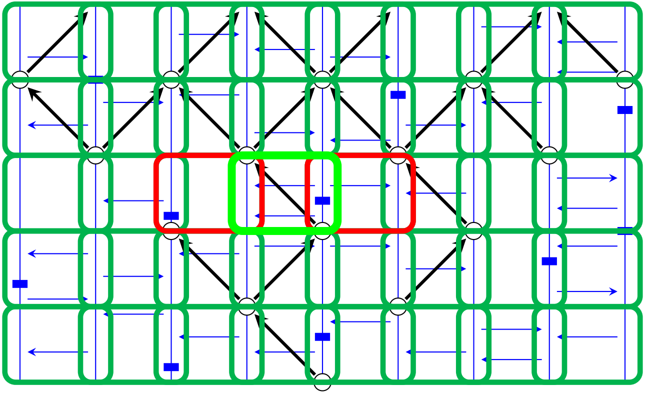
By choosing the vertical dimension of the box large enough and the death rate δ small enough, we can make the probability p of a black arrow as close to one as we wish.

Comparison with Oriented Percolation



The only problem is that, since the green squares overlap, these probabilities are not independent.

Comparison with Oriented Percolation



They are, however, *almost* independent. In fact, the bright green square is independent of all other squares, except the red ones.

K-dependence

Now apply "K-dependence" [Liggett, Schonmann & Stacey, 1997].

Theorem 1 *Let Λ be a countable set and let p, K be constants. Let $(\chi_i)_{i\in\Lambda}$ be Bernoulli random variables such that for each $i \in \Lambda$: 1° $P[\chi_i = 1] \geq p$, and 2° there exists $i \in \Delta_i \subset \Lambda$ with $|\Delta_i| \leq K$, such that*

$$\chi_i \text{ is independent of } (\chi_j)_{j\in\Lambda\backslash\Delta_i}.$$

Assume also that

$$\tilde{p} := \big(1 - (1-p)^{1/K}\big)^2 \geq 1/4.$$

Then it is possible to couple $(\chi_i)_{i\in\Lambda}$ to a collection of independent Bernoulli random variables $(\tilde{\chi}_i)_{i\in\Lambda}$ with $P[\tilde{\chi}_i = 1] = \tilde{p}$ in such a way that $\tilde{\chi}_i \leq \chi_i$ for all $i \in \Lambda$.

Left-continuity

Lemma For each $\alpha \in [0,1]$, the function $\delta \mapsto \theta(\alpha,\delta)$ is left-continuous.

Proof Let $\overline{\nu}_\delta$ denote the upper invariant law for a given δ.
By monotone coupling, $\delta \mapsto \overline{\nu}_\delta$ is nonincreasing in the stochastic order.
By the compactness of $\{0,1\}^{\mathbb{Z}^d}$, the space of probability measures on $\{0,1\}^{\mathbb{Z}^d}$ is compact in the topology of weak convergence.
Let $\delta_n \uparrow \delta$. Then each cluster point ν^* of $\overline{\nu}_{\delta_n}$ as $n \to \infty$ is an invariant law such that $\nu^* \geq \overline{\nu}_\delta$.
But then $\nu^* = \overline{\nu}_\delta$, since the upper invariant law is the largest one.
It follows that $\delta \mapsto \overline{\nu}_\delta$ is left-continuous and hence the same is true for its density $\delta \mapsto \theta(\alpha,\delta)$. ∎

Right-continuity

We say that a probability μ on $\{0,1\}^{\mathbb{Z}^d}$ is *nontrivial* if $\mu(\{\underline{0}\}) = 0$ and *homogeneous* if it is translation invariant.

Theorem The function $\delta \mapsto \theta(0,\delta)$ is right-continuous on $[0,\delta_{\rm c}(0)]$.

Proof Let $\delta_{\rm c} \geq \delta_n \downarrow \delta$. Then each cluster point of $\overline{\nu}_{\delta_n}$ as $n \to \infty$ is a nontrivial homogeneous invariant law. By the same argument as before, it suffices to prove that $\overline{\nu}$ is the only nontrivial homogeneous invariant law.

Note For $\alpha > 1/2$, the mean-field model suggests the existence of an "intermediate" homogeneous invariant law that lies between $\delta_{\underline{0}}$ and $\overline{\nu}$. Hard to rule out.

Ergodicity of the contact process

Theorem Assume that $\mathbb{P}[X_0 \in \cdot\,]$ is nontrivial and homogeneous. Then

$$\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}.$$

Proof idea Need to show

$$\mathbb{P}\big[X_t \wedge y \neq 0\big] \underset{t\to\infty}{\longrightarrow} \mathbb{P}^y\big[Y_t \neq \underline{0}\ \forall t \geq 0\big] =: \rho(y) \quad (y \in \mathbf{S}_{\rm fin}).$$

Fix $N > 0$. Then

$$\mathbb{P}\big[X_t \wedge y \neq 0\big] = \mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \hat{\mathbb{X}}_{t,1}(y) \neq \underline{0}\big]$$

$$= \mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \hat{\mathbb{X}}_{t,1}(y) \neq \underline{0}\,\big|\,|\hat{\mathbb{X}}_{t,1}(y)| = 0\big]\,\mathbb{P}\big[|\hat{\mathbb{X}}_{t,1}(y)| = 0\big]$$
$$+\mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \hat{\mathbb{X}}_{t,1}(y) \neq \underline{0}\,\big|\,0 < |\hat{\mathbb{X}}_{t,1}(y)| < N\big]\,\mathbb{P}\big[0 < |\hat{\mathbb{X}}_{t,1}(y)| < N\big]$$
$$+\mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \hat{\mathbb{X}}_{t,1}(y) \neq \underline{0}\,\big|\,N \leq |\hat{\mathbb{X}}_{t,1}(y)|\big]\,\mathbb{P}\big[N \leq |\hat{\mathbb{X}}_{t,1}(y)|\big].$$

Ergodicity of the contact process

Almost surely

$$\exists t < \infty \text{ s.t. } \hat{\mathbb{X}}_{t,1}(y) \neq \underline{0} \quad \text{or} \quad |\hat{\mathbb{X}}_{t,1}(y)| \underset{t\to\infty}{\longrightarrow} \infty.$$

As a consequence

$$\underbrace{\mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \hat{\mathbb{X}}_{t,1}(y) \neq \underline{0} \,\big|\, |\hat{\mathbb{X}}_{t,1}(y)| = 0\big]}_{=0} \mathbb{P}\big[|\hat{\mathbb{X}}_{t,1}(y)| = 0\big]$$

$$+\mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \hat{\mathbb{X}}_{t,1}(y) \neq \underline{0} \,\big|\, 0 < |\hat{\mathbb{X}}_{t,1}(y)| < N\big] \underbrace{\mathbb{P}\big[0 < |\hat{\mathbb{X}}_{t,1}(y)| < N\big]}_{\underset{t\to\infty}{\longrightarrow} 0}$$

$$+\underbrace{\mathbb{P}\big[\mathbb{X}_{0,1}(X_0) \wedge \hat{\mathbb{X}}_{t,1}(y) \neq \underline{0} \,\big|\, N \leq |\hat{\mathbb{X}}_{t,1}(y)|\big]}_{\approx 1} \underbrace{\mathbb{P}\big[N \leq |\hat{\mathbb{X}}_{t,1}(y)|\big]}_{\underset{t\to\infty}{\longrightarrow} \rho(y)}.$$

∎

Ergodicity of the cooperative contact process

Conjecture Consider the cooperative contact process started in product measure with intensity $q \in [0,1]$. Then in the regime where the process is stable but does not survive, there exists a critical value $0 < q_{\rm c} < 1$ such that

- $\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \delta_{\underline{0}}$ if $q < q_{\rm c}$,
- $\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}$ if $q \geq q_{\rm c}$.

If the process is stable, then $\overline{\nu}$ is the only nontrivial homogeneous invariant law.

Question Assume that the initial law $\mathbb{P}[Y_0 \in \cdot\,]$ of the *dual* of the cooperative contact process is nontrivial and homogeneous. Is it then always true that

$$\mathbb{P}\big[Y_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\mu}.$$

where $\overline{\mu}$ is the upper invariant law of the dual process?

The critical contact process dies out

Theorem $\theta'\big(\alpha, \delta'_{\rm c}(\alpha)\big) = 0$ for all $\alpha \in [0,1]$.

Proof Bezuidenhout & Grimmett (1990) have shown that $\theta'\big(0, \delta'_{\rm c}(0)\big) = 0$.
Bezuidenhout & Gray (1994) generalised this to a class of monotone interacting particle systems with positive correlations that includes the cooperative contact process. $\blacksquare$

Lemma $\delta_{\rm c}'(1) = 0$.

Proof

The process cannot escape from rectangles

$$
\begin{array}{cc|ccc|cc}
\multicolumn{2}{c}{0\ 0} & \multicolumn{3}{c}{0\ 0\ 0} & \multicolumn{2}{c}{0\ 0} \\
\multicolumn{2}{c}{0\ 0} & \multicolumn{3}{c}{0\ 0\ 0} & \multicolumn{2}{c}{0\ 0} \\
\multicolumn{2}{c}{0\ 0} & \multicolumn{3}{c}{0\ 0\ 0} & \multicolumn{2}{c}{0\ 0} \\
\cline{3-5}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\cline{3-5}
\multicolumn{2}{c}{0\ 0} & \multicolumn{3}{c}{0\ 0\ 0} & \multicolumn{2}{c}{0\ 0} \\
\multicolumn{2}{c}{0\ 0} & \multicolumn{3}{c}{0\ 0\ 0} & \multicolumn{2}{c}{0\ 0} \\
\multicolumn{2}{c}{0\ 0} & \multicolumn{3}{c}{0\ 0\ 0} & \multicolumn{2}{c}{0\ 0}
\end{array}
$$

∎

Survival implies stability

Theorem $\delta'_{\rm c}(\alpha) \leq \delta_{\rm c}(\alpha) \quad \forall \alpha \in [0,1]$.

Proof idea Assuming that the contact process survives, Bezuidenhout & Grimmett (1990) have shown that on sufficiently large, carefully chosen space-time scales, the process can be compared with highly supercritical oriented percolation. Bezuidenhout & Gray (1994) generalised this to a class of monotone interacting particle systems with positive correlations. As a consequence, survival of a cooperative contact process implies nontriviality of its upper invariant law. ∎