Lecture 5
Toom contours

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Stability without survival

Let $\Lambda = \mathbb{Z}^2$ with nearest-neighbour edges.
Let $\mathcal{N}_j := \{i \in \Lambda : i \text{ is adjacent to } j\}$.
and $\mathcal{N}_j^2 := \{(i,i') : i,i' \in \mathcal{N}_j,\ i \neq i'\}$.

We are interested in the *cooperative process* with generator

$$Gf(x) := \sum_{j\in\Lambda} \frac{1}{|\mathcal{N}_j^2|} \sum_{(i,i')\in\mathcal{N}_j^2} \big\{ f\big(\mathtt{cob}_{ii'j}(x)\big) - f\big(x\big) \big\} + \delta \sum_{j\in\Lambda} \big\{ f\big(\mathtt{dth}_j(x)\big) - f\big(x\big) \big\}.$$

The *survival probability* $\theta'(\delta)$ is 1 for $\delta = 0$ and 0 for $\delta > 0$.
The *density of the upper inv. law* is $\theta(\delta) := \lim_{t\to\infty} \mathbb{P}^{\underline{1}}\big[X_t(i) = 1\big]$.
We define $\delta_c := \inf\{\delta \geq 0 : \theta(\delta) = 0\}$.

Stability without survival

Aim Prove that $\delta_c > 0$.

Recall that for the contact process, we proved $\delta_c > 0$ in two steps:

1. We proved survival for a discrete-time process, oriented percolation, by means of a Peierls argument.
2. We proved survival for the contact process by comparison with oriented percolation.

We will take a similar approach here. We will focus on step 1. We will prove stability of some discrete-time cellular automata by means of a Peierls argument that goes back to Toom (1980).

Monotone cellular automata

Let $\mathbf{S} = \{0,1\}^{\mathbb{Z}^d}$ be equipped with the product topology.

Recall that $\mathcal{C}_+(\mathbf{S},\{0,1\})$ is the set of continuous monotone functions $f : \mathbf{S} \to \{0,1\}$ with $f(\underline{0}) = 0$.
Functions in $\mathcal{C}_+(\mathbf{S},\{0,1\})$ depend on finitely many coordinates.

Let $\phi^0(x) := 0$ $(x \in \mathbf{S})$ denote the "death" map.
Let $\phi_1,\ldots,\phi_m \in \mathcal{C}_+(\mathbf{S},\{0,1\})$, $\phi_k \neq \phi^0$ $\forall k = 1,\ldots,m$.
Let $\Phi = \big(\Phi_{(i,t)}\big)_{(i,t)\in\mathbb{Z}^d\times\mathbb{Z}}$ be i.i.d. with values in $\{\phi^0,\phi_1,\ldots,\phi_m\}$ and

$$\mathbb{P}\big[\Phi_{(i,t)} = \phi^0\big] = p \quad \text{and} \quad \mathbb{P}\big[\Phi_{(i,t)} = \phi_k\big] = (1-p)r_k.$$

For each $x \in \mathbf{S}$ and $s \in \mathbb{Z}$, there exists a unique $\mathbf{S}$-valued Markov chain $(X_t)_{t\geq s}$ such that $X_s = x$ and

$$X_t(i) = \Phi_{(i,t)}\big((X_{t-1}(i+j))_{j\in\mathbb{Z}^d}\big) \qquad \forall i \in \mathbb{Z}^d,\ t > s.$$

We set $\mathbb{X}_{s,u}(x) := X_u$ $(s \leq u)$.

Monotone cellular automata

Lemma The decreasing limit $\overline{X}_t := \lim_{s \to -\infty} \mathbb{X}_{-s,t}(\underline{1})$ exists a.s. and defines a stationary Markov chain $(\overline{X}_t)_{t \in \mathbb{Z}}$.

We let $\theta(p, r_1, \ldots, r_m) := \mathbb{P}\big[\overline{X}_t(i) = 1\big] \quad \big((i,t) \in \mathbb{Z}^{d+1}\big)$.
Clearly $\theta(0, r_1, \ldots, r_m) = 1$.

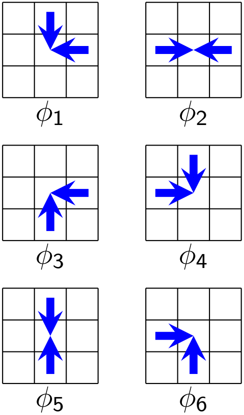
Fix a probability law $r_1, \ldots, r_m$. We say that the *unperturbed* cellular automaton with $p = 0$ is *stable* if

$$\lim_{p \to 0} \theta(p, r_1, \ldots, r_m) = 1.$$

A cooperative cellular automaton

For example, we would like to prove stability of the cellular automaton with $m = 6$, $r_1 = \cdots = r_6 = 1/6$, and

$$\phi_1(x) := x(0,0) \vee \big(x(1,0) \wedge x(0,1)\big),$$
$$\phi_2(x) := x(0,0) \vee \big(x(1,0) \wedge x(-1,0)\big),$$
$$\phi_3(x) := x(0,0) \vee \big(x(1,0) \wedge x(0,-1)\big),$$
$$\phi_4(x) := x(0,0) \vee \big(x(0,1) \wedge x(-1,0)\big),$$
$$\phi_5(x) := x(0,0) \vee \big(x(0,1) \wedge x(0,-1)\big),$$
$$\phi_6(x) := x(0,0) \vee \big(x(-1,0) \wedge x(0,-1)\big),$$



which corresponds to a discrete time version of the cooperative process.

Toom's rule

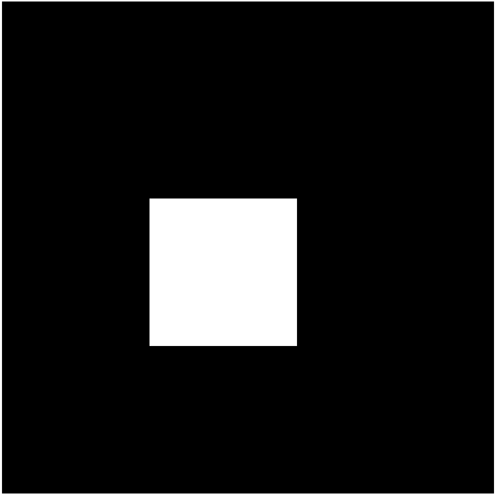
We first look at the case $m = 1$. Thus, we fix one $\phi \in \mathcal{C}_+(\mathbf{S}, \{0,1\})$ and let $\Phi = \big(\Phi_{(i,t)}\big)_{(i,t)\in\mathbb{Z}^d\times\mathbb{Z}}$ be i.i.d. with

$$\mathbb{P}\big[\Phi_{(i,t)} = \phi^0\big] = p \quad \text{and} \quad \mathbb{P}\big[\Phi_{(i,t)} = \phi\big] = 1 - p.$$

For example, *Toom's rule*, also known as the *North East Center majority rule* on $\mathbb{Z}^2$, is given by

$$\phi^{\mathrm{NEC}}(x) := \begin{cases} 1 & \text{if } x(0,0) + x(0,1) + x(1,0) \geq 2, \\ 0 & \text{if } x(0,0) + x(0,1) + x(1,0) \leq 1. \end{cases}$$

Toom's model



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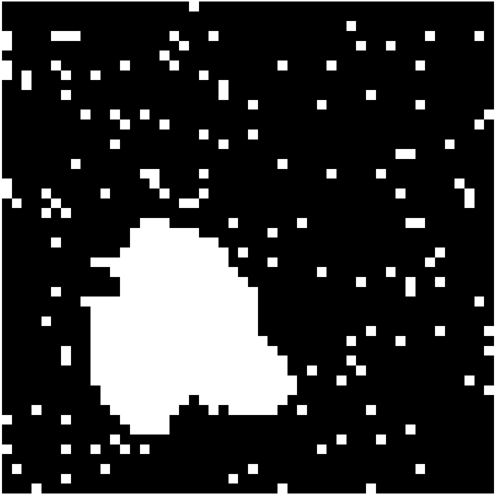

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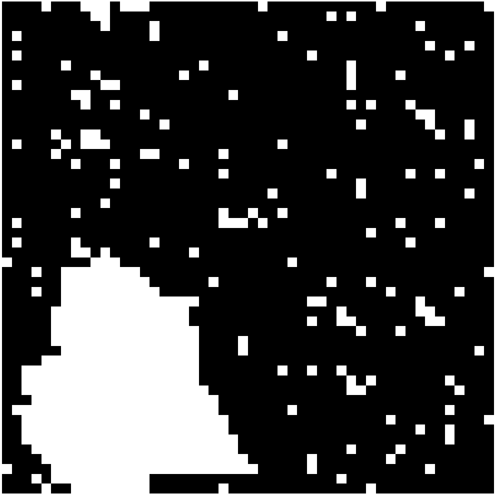

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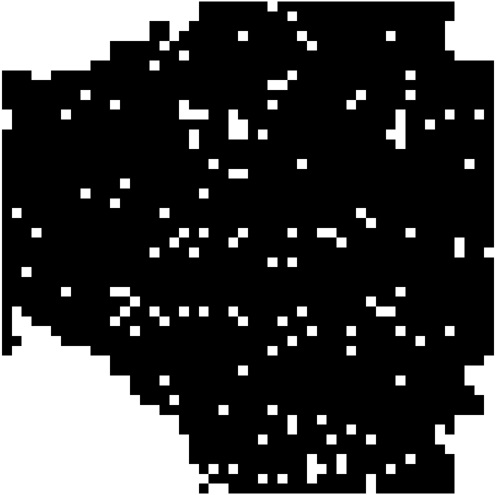

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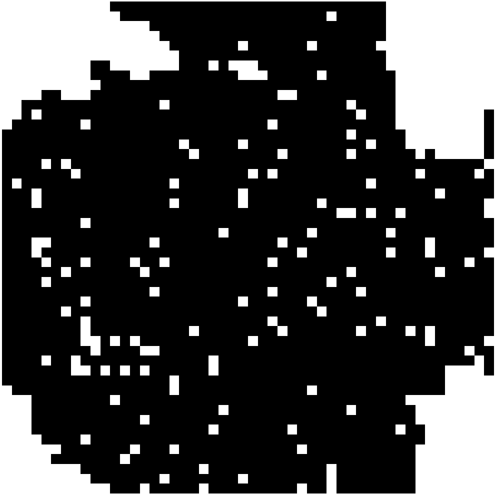

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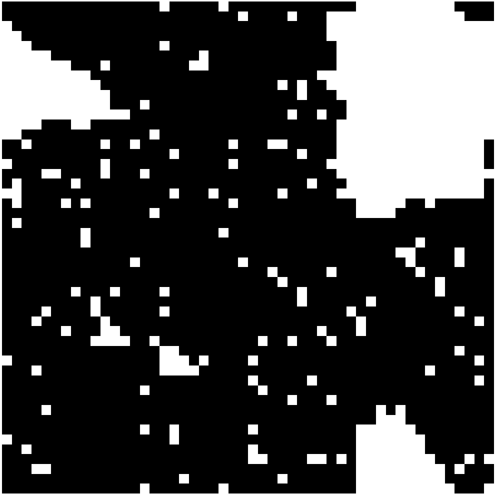

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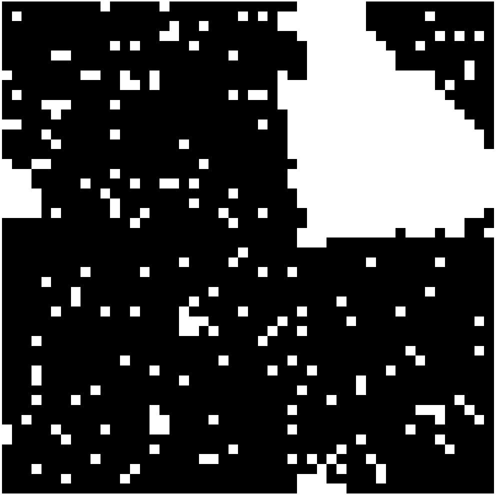

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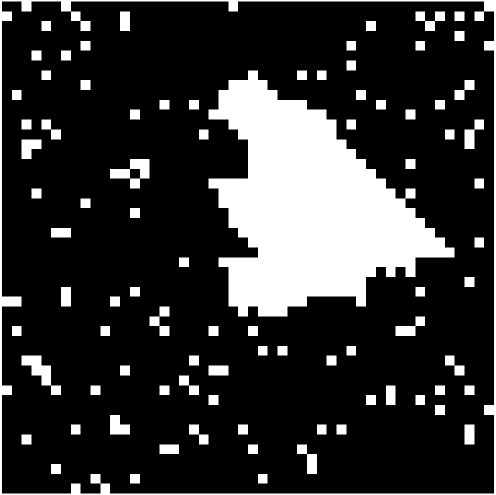

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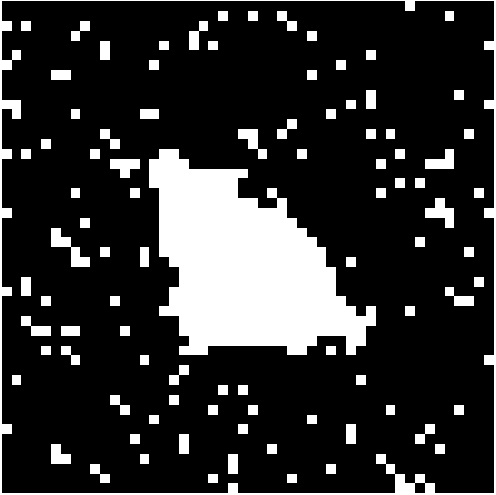

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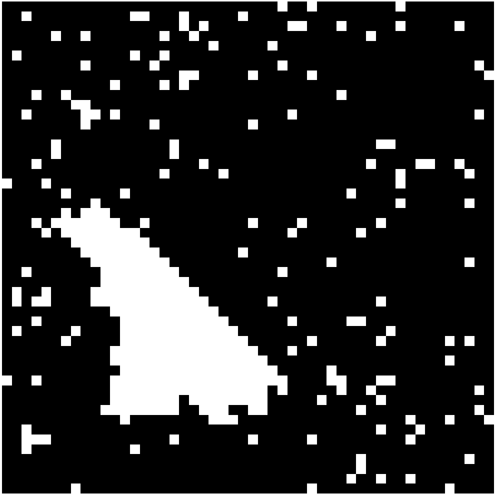

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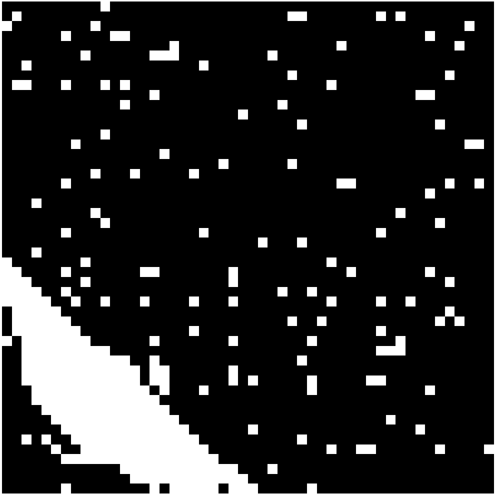

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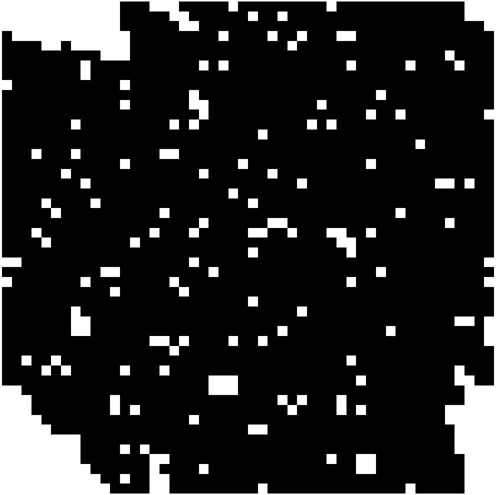

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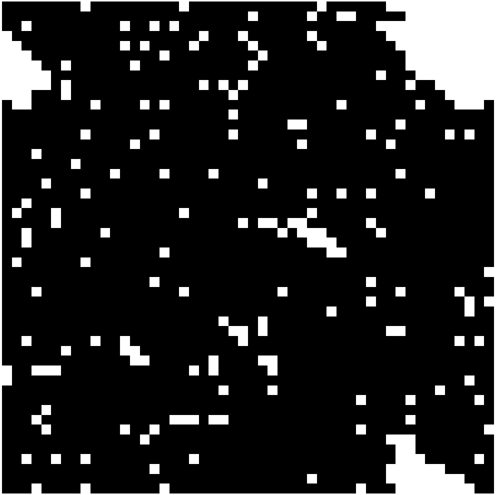

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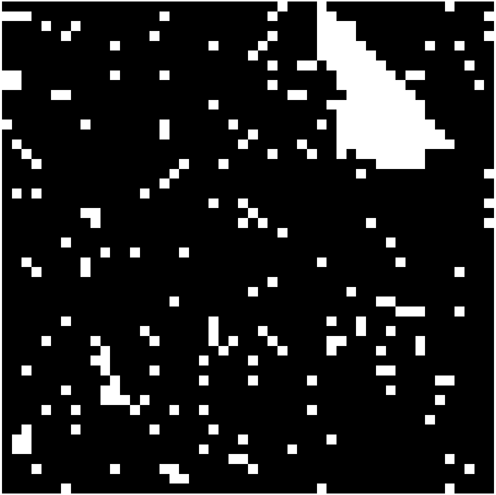

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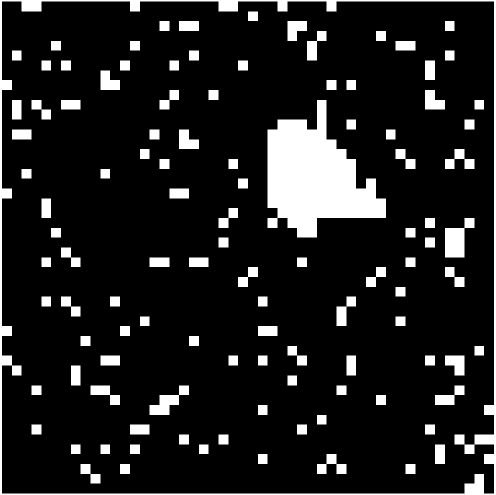

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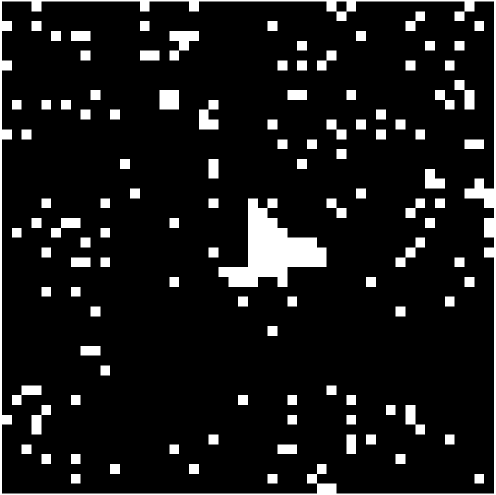

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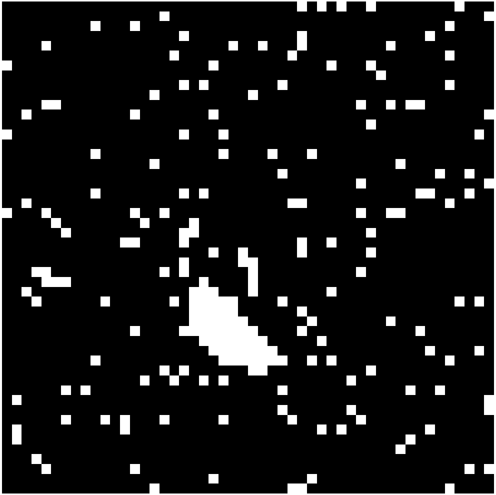

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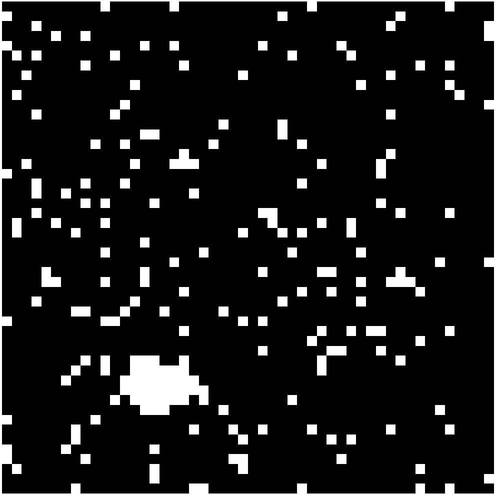

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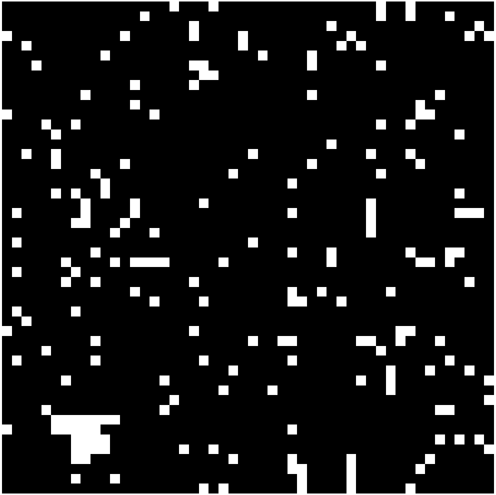

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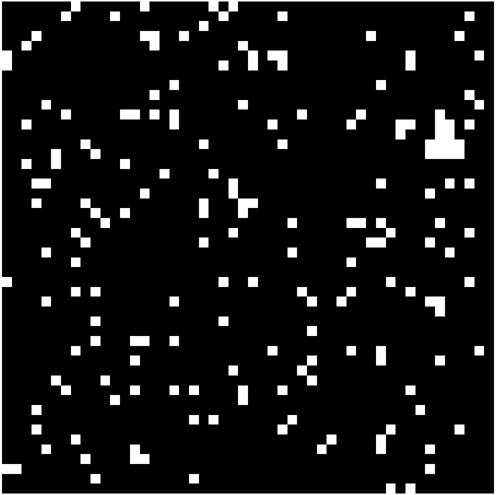

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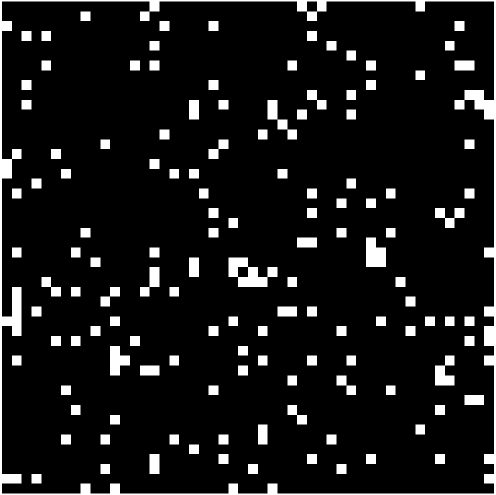


Toom's model

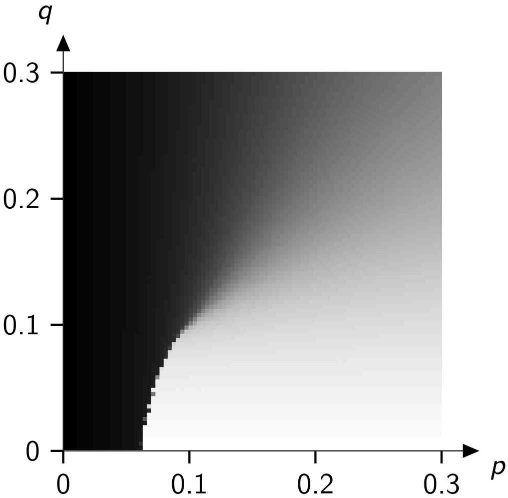
We can generalise a bit and let

$$\mathbb{P}\big[\Phi_{(i,t)} = \phi^0\big] = p, \quad \mathbb{P}\big[\Phi_{(i,t)} = \phi^1\big] = q$$
$$\text{and} \quad \mathbb{P}\big[\Phi_{(i,t)} = \phi\big] = 1 - p - q,$$

where $\phi^1(x) := 1$ $(x \in \mathbf{S})$ denotes the "spontaneous birth" map. Let $\theta(p, q)$ denote the density of the upper invariant law.

$$\textbf{Toom (1980)} \ \lim_{p \to 0} \theta(p, 0) = 1.$$

Toom's model



Density of the upper invariant law for Toom's model.

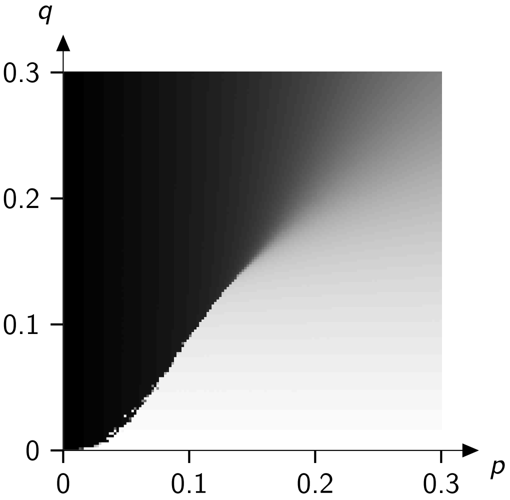
Nearest neighbour voting

The *Nearest Neighbor voting map* is defined as

$$\phi^{\mathrm{NN}}(x) := \begin{cases} 1 & \text{if} \quad x(0,0)+x(0,1)+x(1,0) \\ & \qquad +x(0,-1)+x(-1,0) \geq 3, \\ 0 & \text{if} \quad x(0,0)+x(0,1)+x(1,0) \\ & \qquad +x(0,-1)+x(-1,0) \leq 2. \end{cases}$$

Toom (1980) $\theta(p,0)=0$ for all $p>0$.

Nearest neighbour voting



Density of the upper invariant law for nearest neighbour voting.

Def ϕ is an *eroder* if for the unperturbed cellular automaton, any finite collection of zeros disappears in finite time.

Toom's stability theorem (1980)

If ϕ is an eroder, then $\theta(p) \to 1$ as $p \to 0$.

If ϕ is not an eroder, then $\theta(p) = 0$ for all $p > 0$.

The eroder property

Let $\mathcal{A}(\phi) := \{A \subset \mathbb{Z}^d : 1_A \text{ is a minimal one-state for } \phi\}$.
Then we can write

$$\phi(x) = \bigvee_{A \in \mathcal{A}(\phi)} \bigwedge_{i \in A} x(i).$$

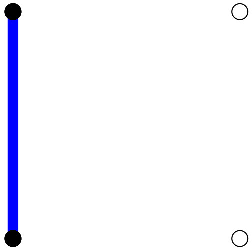
Theorem (Toom 1980, Ponselet 2013) ϕ is an eroder if and only if

$$\bigcap_{A \in \mathcal{A}(\phi)} \mathrm{Conv}(A) = \emptyset,$$

where $\mathrm{Conv}(A)$ is the convex hull of A.

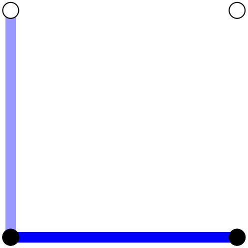
By *Helly's theorem* w.l.o.g. $|\mathcal{A}(\phi)| \leq d + 1$.

The eroder property



Toom's model ϕ^{NEC}

The eroder property



Toom's model ϕ^{NEC}

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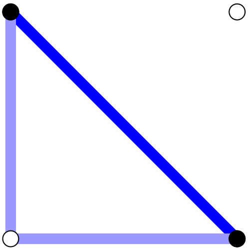

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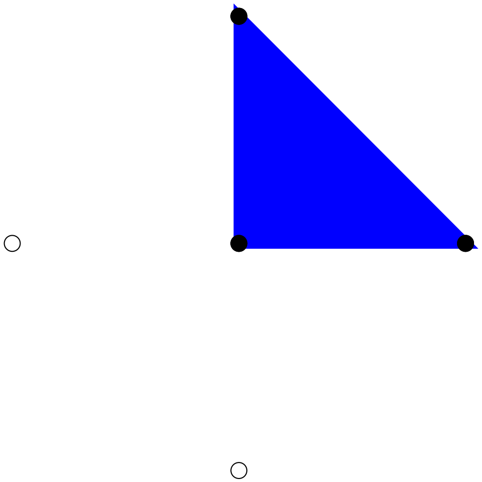
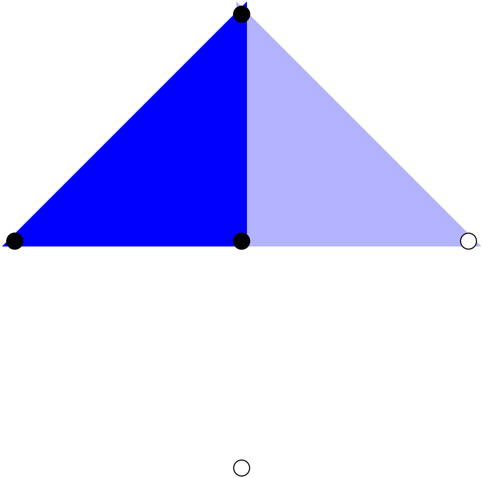


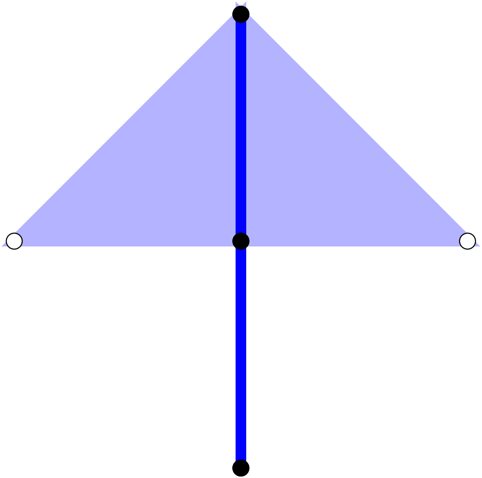
Nearest neighbour voting ϕ^{NN}.

The eroder property



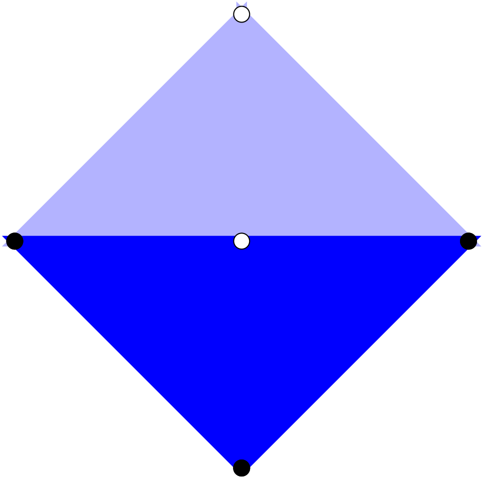
Nearest neighbour voting ϕ^{NN}.

The eroder property



Nearest neighbour voting ϕ^{NN}.

The eroder property



Nearest neighbour voting ϕ^{NN}.

The eroder property

Def A *linear polar function* is a linear function

$$\mathbb{R}^d \ni z \mapsto \big(L_1(z),\ldots,L_\sigma(z)\big) \in \mathbb{R}^\sigma$$

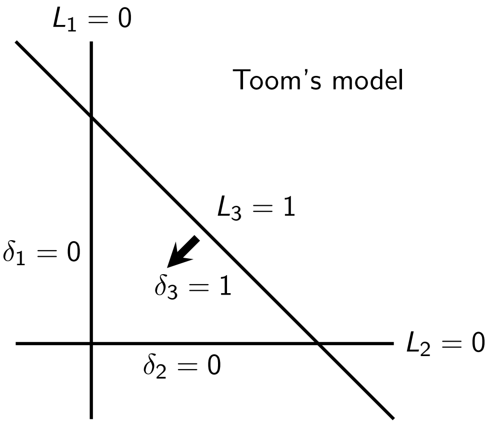
such that $\displaystyle\sum_{s=1}^{\sigma} L_s(z) = 0 \ (z \in \mathbb{R}^d)$.
For $x \in \{0,1\}^{\mathbb{Z}^d}$, let $\displaystyle\ell_s(x) := \sup_{i\in\mathbb{Z}^d:\, x(i)=0} L_s(i)$.

Then for the unperturbed cellular automaton:

$$\ell_s(X_n) \leq \ell_s(X_0) - \delta_s n \quad \text{with} \quad \delta_s := \sup_{A\in\mathcal{A}(\phi)} \inf_{i\in A} L_s(i).$$

The constants δ_s $(1 \leq s \leq \sigma)$ are *edge speeds*.

The eroder property



$$L_1(z) := -z_1, \quad L_2(z) := -z_2, \quad L_3(z) := z_1 + z_2.$$

The eroder property

Lemma (Toom 1980, Ponselet 2013) ϕ is an eroder if and only if there exists a linear polar function L such that

$$\delta := \sum_{s=1}^{\sigma} \delta_s > 0 \quad \text{with} \quad \delta_s := \sup_{A \in \mathcal{A}(\phi)} \inf_{i \in A} L_s(i).$$

Proof of sufficiency Define the *extent* of x by

$$\mathrm{ext}(x) := \sum_{s=1}^{\sigma} \ell_s(x) \quad \text{with} \quad \ell_s(x) := \sup_{i \in \mathbb{Z}^d : \, x(i)=0} L_s(i).$$

Then $\mathrm{ext}(x) \geq 0$ if there is at least one zero since $\sum_{s=1}^{\sigma} L_s(z) = 0$.

Moreover $\mathrm{ext}(X_n) \leq \mathrm{ext}(X_0) - \delta n$. ∎

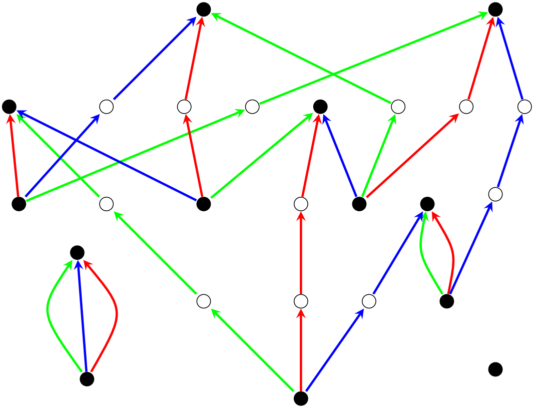
Toom contours

Def A *Toom graph* is a directed graph with edges of σ different *charges* and three types of vertices:

- At a *source*, σ directed edges emerge, one of each charge.
- At a *sink*, σ directed edges converge, one of each charge.
- At an *internal vertex*, there is one incoming edge and one outgoing edge, and they are of the same charge.

In addition, there can be *isolated vertices* which we can think of as a source and sink at the same time.

Toom contours



A Toom graph with three charges.

Peierls argument

Main idea A *Toom contour* is a connected Toom graph embedded in the plain, with one special source called the *root*.

Theorem (incomplete statement) If $\overline{X}_0(0) = 0$, then there exists a Toom contour T rooted at $(0,0)$ such that the sinks of T correspond to *defective* space-time points, where the trivial map ϕ^0 is applied. Consequently:

$$\mathbb{P}\big[\overline{X}_0(0) = 0\big] \leq \sum_T \mathbb{P}\big[T \text{ is present in } \Phi\big] \leq \sum_T p^{n_{\rm sink}(T)}.$$

This tends to zero as $p \to 0$ provided

$$N_n^{\rm sink} := \#\{T : n_{\rm sink}(T) = n\}$$

grows at most exponentially in n.

Peierls argument

It is not hard to show that there exists a $R < \infty$ such that

$$N_n^{\mathrm{edge}} \leq R^n \quad \text{with} \quad N_n^{\mathrm{edge}} := \#\{T : n_{\mathrm{edge}}(T) = n\}.$$

Need to show that $n_{\mathrm{sink}}(T) \geq c n_{\mathrm{edge}}(T)$ for some $c > 0$.

Idea: edges with charge s move in the direction where L_s increases, *except* for edges coming out of sources. As a result:

$$n_{\mathrm{sink}}(T) = n_{\mathrm{source}}(T) \geq c n_{\mathrm{edge}}(T)$$

for some $c > 0$.

Toom contours

Def An *embedding* of a Toom graph with vertex set V is a map

$$V \ni v \mapsto \big(\psi(v), -h(v)\big) \in \mathbb{Z}^d \times \mathbb{Z}$$

- The *height* (=negative time) h increases by 1 along each directed edge.
- Sinks do not overlap with any other vertices.
- Internal vertices of the same charge do not overlap.

A *Toom contour* is an embedded connected Toom graph with one special source, the *root*, whose height is minimal among all vertices.

Toom contours

Let ϕ be an eroder. For each $1 \leq s \leq \sigma$, choose $A_s(\phi) \in \mathcal{A}(\phi)$ such that

$$\delta_s := \sup_{A \in \mathcal{A}(\phi)} \inf_{i \in A} L_s(i) = \inf_{i \in A_s(\phi)} L_s(i).$$

Def A Toom contour is *present* in $\Phi = \big(\Phi_{(i,t)}\big)_{(i,t) \in \mathbb{Z}^d \times \mathbb{Z}}$ if:

- Sinks correspond to vertices where the trivial map ϕ^0 is applied.
- If (v, w) is a directed edge of charge s coming out of an internal vertex or the root, then $\psi(w) - \psi(v) \in A_s(\phi)$.
- For directed edges emerging at other sources $\psi(w) - \psi(v) \in \bigcup_{s=1}^{\sigma} A_s(\phi)$.

Theorem (complete statement) If $\overline{X}_0(0) = 0$, then there is a Toom contour rooted at $(0, 0)$ present in Φ.

Cooperative branching

Consider the cooperative branching map defined as

$$\phi^{\text{coop}}(x) := x(0,0) \vee \big(x(0,1) \wedge x(1,0)\big).$$

One has $\mathcal{A}(\phi^{\text{coop}}) = \{A_1, A_2\}$ with

$$A_1 := \{(0,1),(1,0)\} \quad \text{and} \quad A_2 := \{(0,0)\}.$$

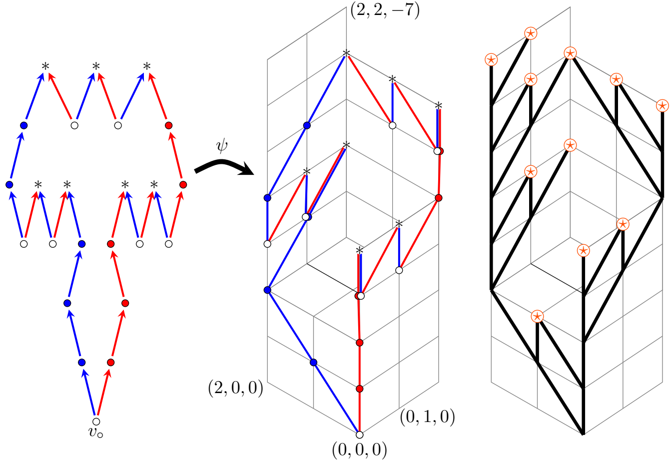
We choose the linear polar function

$$L_1(z) := z_1 + z_2, \quad L_2(z) := -z_1 - z_2.$$

The corresponding edge speeds are given by

$$\begin{aligned}
\delta_1 &= \sup_{A \in \mathcal{A}(\phi)} \inf_{i \in A} L_1(i) = \inf_{i \in A_1} L_1(i) = 1, \\
\delta_2 &= \sup_{A \in \mathcal{A}(\phi)} \inf_{i \in A} L_2(i) = \inf_{i \in A_2} L_2(i) = 0.
\end{aligned}$$

Toom contours



A Toom contour for the cooperative branching map.

Lemma There exists a $c>0$ such that $n_{\rm sink}\geq cn_{\rm edge}+1$.

Proof
$$\sum_{s=1}^{\sigma}\sum_{(v,w)\in E_s}\big(L_s(\psi(w))-L_s(\psi(v))\big)$$
$$=\sum_{v\in V}\sum_{s=1}^{\sigma}\Big\{\sum_{u:\,(u,v)\in E_s}L_s(\psi(v))-\sum_{w:\,(v,w)\in E_s}L_s(\psi(v))\Big\}=0.$$

Let E°_s denote the edges of charge s out of a source different from the root and E^*_s the other edges. Then

$$0=\sum_{s=1}^{\sigma}\sum_{(v,w)\in E^*_s}\underbrace{\big(L_s(\psi(w))-L_s(\psi(v))\big)}_{\geq\delta_s}$$
$$+\sum_{s=1}^{\sigma}\sum_{(v,w)\in E^\circ_s}\underbrace{\big(L_s(\psi(w))-L_s(\psi(v))\big)}_{\geq -K}.$$

∎

The Peierls argument

Lemma The number of Toom contours rooted at $(0,0)$ with N edges is bounded by R^n for some $R < \infty$.

Let $\mathcal{T}_0$ denote the set of all Toom contours rooted at $(0,0)$.
Let $n_{\rm sink}(T)$ denote the number of sinks of T.
Let $N_n^{\rm edge}$ denote the number of $T \in \mathcal{T}_0$ with n edges. Then

$$\mathbb{P}\big[\overline{X}_0(0) = 0\big] \leq \sum_{T \in \mathcal{T}_0} \mathbb{P}\big[T \text{ is present in } \Phi\big] \leq \sum_{T \in \mathcal{T}_0} p^{n_{\rm sink}(T)}$$

$$\leq p \sum_{T \in \mathcal{T}_0} p^{c n_{\rm edge}(T)} = p \sum_{n=0}^{\infty} N_n^{\rm edge} p^{cn} \leq p \sum_{n=0}^{\infty} R^n p^{cn}.$$

Intrinsic randomness

Recall that originally, we were interested in the case that there are multiple maps $\phi_1, \ldots, \phi_k$, and

$$\mathbb{P}\big[\Phi_{(i,t)} = \phi^0\big] = p \quad \text{and} \quad \mathbb{P}\big[\Phi_{(i,t)} = \phi_k\big] = (1-p)r_k.$$

Unfortunately, this is more difficult than the case of a single map.

Example Apply $\phi^{\rm NEC}, \phi^{\rm NWC}, \phi^{\rm SWC}, \phi^{\rm SEC}$ with equal probabilities. In spite of individually being eroders, this random cellular automaton is believed to be *unstable*.

Intuitively, the "edge speed" in each direction is zero.

Intrinsic randomness

Gray (1999) has given sufficient conditions for continuous-time Markov chains to be stable.

He uses a combination of Toom contours and a renormalisation argument of Bramson and Gray (1991).

Can we handle more general random cellular automata with intrinsic randomness?

Is the renormalisation argument really needed?

Work in progress with Réka Szabó and Cristina Toninelli. . .

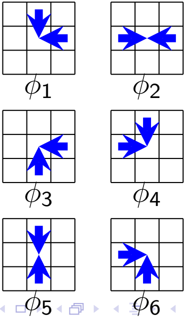
Continuous time

We were originally interested in the cooperative process with generator

$$Gf(x) := \sum_{j\in\Lambda} \frac{1}{|\mathcal{N}_j^2|} \sum_{(i,i')\in\mathcal{N}_j^2} \big\{ f\big(\mathtt{cob}_{ii'j}(x)\big) - f\big(x\big)\big\} + \delta \sum_{j\in\Lambda} \big\{ f\big(\mathtt{dth}_j(x)\big) - f\big(x\big)\big\}.$$

Recall that

$$\phi_1(x) := x(0,0) \vee \big(x(1,0) \wedge x(0,1)\big),$$
$$\phi_2(x) := x(0,0) \vee \big(x(1,0) \wedge x(-1,0)\big),$$
$$\phi_3(x) := x(0,0) \vee \big(x(1,0) \wedge x(0,-1)\big),$$
$$\phi_4(x) := x(0,0) \vee \big(x(0,1) \wedge x(-1,0)\big),$$
$$\phi_5(x) := x(0,0) \vee \big(x(0,1) \wedge x(0,-1)\big),$$
$$\phi_6(x) := x(0,0) \vee \big(x(-1,0) \wedge x(0,-1)\big),$$



Continuous time

We can think of the continuous-time model as the $\varepsilon \to 0$ limit of a discrete-time model that applies three maps:

$$\begin{array}{ll} \phi_1, \ldots, \phi_6 & \text{with probability } \varepsilon/6 \text{ each,} \\ \phi^0 & \text{with probability } \varepsilon\delta, \\ \phi^{\mathrm{id}} & \text{with the remaining probability,} \end{array}$$

where $\phi^{\mathrm{id}}(x) := x(0)$ is the *identity map*.

Gray (1999) has shown that combining the identity map with an eroder can spoil stability. Let:

$$\phi(x) := \begin{cases} 0 & \text{if } x(-2,0) = x(-1,0) = 0, \\ 1 & \text{if } x(-3,k) = x(-2,k) = 1 \ \forall |k| \leq n, \\ x(0,0) & \text{in all other cases.} \end{cases}$$

Continuous time