Some dualities for a class of one-dimensional cancellative systems

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Interface tightness

Let $X = (X_t)_{t \geq 0}$ be an interacting particle system, i.e., X is a Markov process taking values in the space $S = \{0,1\}^{\mathbb{Z}^d}$.

Assume that the constant configurations $\underline{0}$ and $\underline{1}$ are traps.

By definition, X exhibits *coexistence* if there exists an invariant law that gives zero probability to $\underline{0}$ and $\underline{1}$.

Intuition: mutual invadability should imply coexistence.

Durrett (2002) has proved a result of this type for systems with fast stirring.
Similar results can be found in Durrett & Neuhauser (1997), and for more specific systems in Neuhauser & Pacala (1999) and Cox & Perkins 2007.

Interface tightness

Restrict, from now on, to the one-dimensional case.

Assume that X is *type symmetric*, i.e., $(1 - X_t)_{t \geq 0}$ has the same dynamics as $(X_t)_{t \geq 0}$.

Intuition: mutual noninvadability should imply noncoexistence.

Let $\mathbb{Z} + \frac{1}{2} := \{k + \frac{1}{2} : k \in \mathbb{Z}\}$ and let $\mathbb{I} = \mathbb{Z}$ or $= \mathbb{Z} + \frac{1}{2}$.
Define a gradient operator $\nabla : \{0,1\}^{\mathbb{I}} \to \{0,1\}^{\mathbb{I}+\frac{1}{2}}$ by

$$\nabla x(i) := 1_{\{x(i-\frac{1}{2}) \neq x(i+\frac{1}{2})\}}.$$

By type symmetry, the *interface model* $(\nabla X_t)_{t \geq 0}$ is a Markov process.

X	0		1		1		1		0		0		1		0
∇X		1		0		0		1		0		1		1	

Interface tightness

Let $|y| := \sum_{i \in \mathbb{Z}} y(i)$ denote the number of 1's in y.

The interface process Y is *parity preserving* in the sense that if $|Y_0|$ is finite and odd, then a.s. $|Y_t|$ is finite and odd for all $t \geq 0$.

If $|Y_0|$ is finite and odd, then let $l_t := \inf\{i \in \mathbb{Z} + \frac{1}{2} : Y_t(i) = 1\}$ denote the left-most one and let

$$\hat{Y}_t(i) := Y(l_t + i) \qquad (t \geq 0,\ i \in \mathbb{N})$$

denote the process Y *viewed from the left-most one*.

Interface tightness

Assuming that X is symmetric w.r.t. translations, the process $\hat{Y}$ is a continuous-time Markov chain with countable state space

$$\{y \in \{0,1\}^{\mathbb{N}} : y(0) = 1,\ |y| \text{ is odd}\}.$$

Following Cox & Durrett (1995), we say that X exhibits *interface tightness* if $\hat{Y}$ started from δ_0 is positively recurrent.

Say *strong interface tightness* holds if $\mathbb{E}\big[|\hat{Y}_\infty|\big] < \infty$ in equilibrium.

Strong interface tightness will be our way of making the idea of "mutual noninvadability" rigorous.

> **[Swart '13]** *Assume that X is type symmetric and symmetric under translations. Assume moreover that its dynamics are cancellative and have a voter model component. Then strong interface tightness implies noncoexistence.*

Examples

We will define “cancellative” in a moment. We start with some examples.

Fix $R \geq 1$ and given x, let

$$f_\tau(i) := \frac{|\{j \in \mathcal{N}_i : x(j) = \tau\}|}{|\mathcal{N}_i|} \quad \mathcal{N}_i := \{j : 0 < |i - j| \leq R\}.$$

denote the *local frequency* of type $\tau = 0, 1$ near i.

In the **Neuhauser-Pacala model**, spin $x(i)$ flips:

$$0 \mapsto 1 \text{ with rate } f_1(f_0 + \alpha f_1),$$
$$1 \mapsto 0 \text{ with rate } f_0(f_1 + \alpha f_0).$$

For $\alpha = 1$ voter model, for $\alpha < 1$ advantage to rare types.

Examples

Known facts:

Case $R = 1$ *disagreement voter model*. Noncoexistence and interface tightness known for all $0 < \alpha \leq 1$. Strong interface tightness for $\alpha = \frac{1}{2}$ as a side-result of Andel, Liggett & Mountford (1992).

Case $R \geq 2$.
Coexistence and absence of interface tightness known for α close to zero, noncoexistence and strong interface tightness conjectured for α close to one.

Examples

In the **affine voter model**, spin $x(i)$ flips:

$$0 \mapsto 1 \text{ with rate } \alpha f_1 + (1-\alpha)1_{\{f_1>0\}},$$
$$1 \mapsto 0 \text{ with rate } \alpha f_0 + (1-\alpha)1_{\{f_0>0\}}.$$

Interpolates between the voter model ($\alpha = 1$) and *threshold voter model* ($\alpha = 0$).

Coexistence and absence of interface tightness known for $\alpha = 0$, conjectured for α close to zero, noncoexistence and interface tightness conjectured for α close to one.

Examples

In the **rebellious voter model**, spin $x(i)$ flips

$$0 \leftrightarrow 1 \text{ with rate } \tfrac{1}{2}\alpha\big(1_{\{x(i-1)\neq x(i)\}} + 1_{\{x(i)\neq x(i+1)\}}\big)$$
$$+\tfrac{1}{2}(1-\alpha)\big(1_{\{x(i-2)\neq x(i-1)\}} + 1_{\{x(i+1)\neq x(i+2)\}}\big).$$

For $\alpha = 1$ voter model,
for $\alpha < 1$ advantage to rare types.

Coexistence and absence of interface tightness known for α close to zero.

Numerically: $\alpha_{\rm c} \approx 0.510$ such that coexistence for $\alpha < \alpha_{\rm c}$,
(strong) interface tightness for $\alpha > \alpha_{\rm c}$.

Cancellative systems

Equip $\{0,1\}$ with the usual product and with addition modulo 2, denoted as $\oplus$. Then $\{0,1\}$ is a *finite field*. We may view $\{0,1\}^{\mathbb{Z}^d}$ (equipped with $\oplus$) as a *linear space* over $\{0,1\}$.

Let $(A(i,j))_{i,j\in\mathbb{Z}^d}$ be a matrix with $0,1$-valued entries, such that $A(i,j)=1$ for finitely many i,j and $A(i,j)=0$ otherwise. Then we define

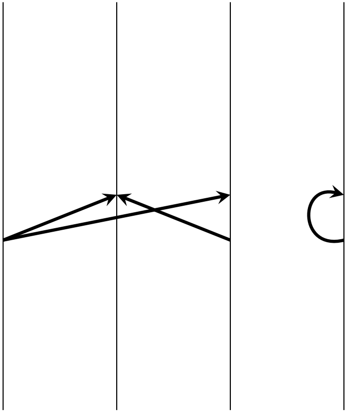
$$Ax(i) := \bigoplus_{j\in\mathbb{Z}^d} A(i,j)x(j).$$

A *cancellative system* $X=(X_t)_{t\geq 0}$ is a *linear system* w.r.t. to the finite field $\{0,1\}$. For certain A there is a nonnegative rate $r(A)$ such that the system makes the transition

$$x \mapsto x \oplus Ax$$

at Poisson times with rate $r(A)$.

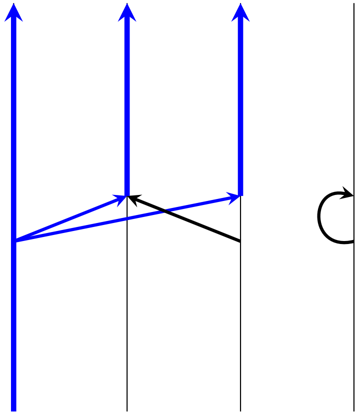
Graphical representation



Draw an arrow $i \to j$ whenever $A(j,i) = 1$.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Graphical representation



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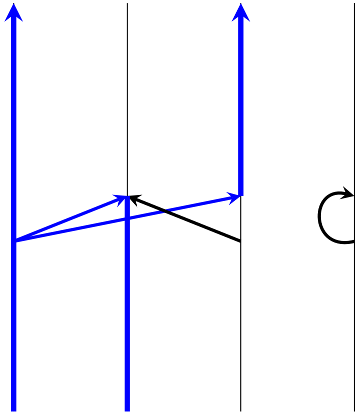

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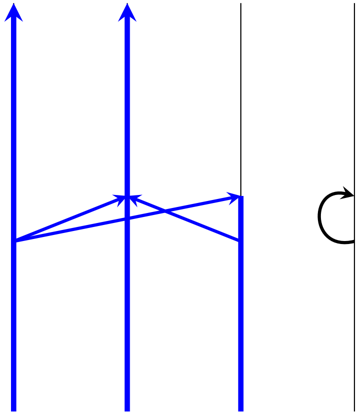

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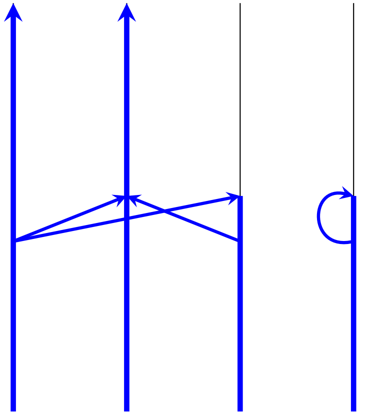

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$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Cancellative systems duality

For $x, y \in \{0,1\}^{\mathbb{Z}^d}$, define

$$|x| := \sum_i x(i) \quad \text{and} \quad \langle\langle x, y \rangle\rangle := \bigoplus_i x(i)y(i).$$

Then $|x \wedge y|$ is the number of sites i with $x(i) = 1 = y(i)$ and

$$\langle\langle x, y \rangle\rangle = 1_{\{|x \wedge y| \text{ is odd}\}}.$$

For any A,

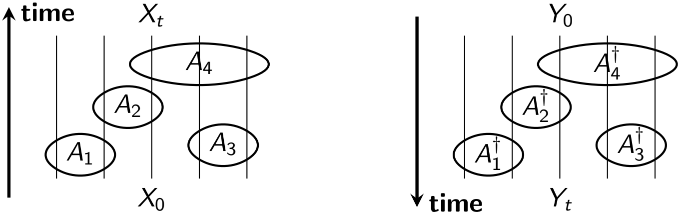
$$\langle\langle x, Ay \rangle\rangle = \langle\langle A^\dagger x, y \rangle\rangle,$$

where $A^\dagger(i,j) := A(j,i)$ is the *adjoint* of A.

Cancellative systems duality

Let X and Y be cancellative systems with rates satisfying

$$r_X(A) = r_Y(A^\dagger).$$



For each $t > 0$, we can *couple* such that for each $0 < u < t$, the processes $(X_s)_{0 \leq s \leq u}$ and $(Y_s)_{0 \leq s \leq t-u}$ are independent, and

$$\langle\langle X_t, Y_0 \rangle\rangle = \langle\langle X_u, Y_{t-u} \rangle\rangle = \langle\langle X_0, Y_t \rangle\rangle \qquad (0 \leq u \leq t).$$

Cancellative systems duality

Once again, if X and Y satisfy

$$r_X(A) = r_Y(A^\dagger).$$

Then X and Y are *pathwise dual* in the sense that for each $t > 0$ there exists a coupling such that

$$\langle\langle X_t, Y_0 \rangle\rangle = \langle\langle X_0, Y_t \rangle\rangle \qquad \text{a.s.}$$

In particular, they are dual in the sense that

$$\mathbb{P}\big[|X_t \wedge Y_0| \text{ is odd}\big] = \mathbb{P}\big[|X_0 \wedge Y_t| \text{ is odd}\big] \qquad (t \geq 0).$$

This formula holds also for random X_0 and Y_0 when we let X_t be independent of Y_0 and X_0 independent of Y_t.

Type symmetry and parity preservation

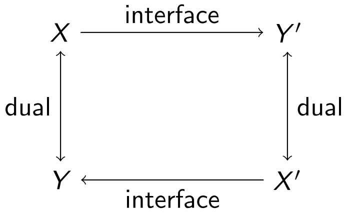
Lemma

- X type symmetric iff only the A's involved are such that *each row contains an even number of ones*. (Even number of incoming arrows at each site.)
- X parity preserving iff for all A's involved, *each column contains an even number of ones*. (Even number of outgoing arrows at each site.)

Consequence X type symmetric $\Leftrightarrow$ dual Y is parity preserving.

Interfaces and duality

[S. '13] The interface model of a type symmetric cancellative spin system is a parity preserving cancellative spin system. Conversely, every parity preserving cancellative spin system is the interface model of a unique type symmetric cancellative spin system. Moreover, the following commutative diagram holds:



Here X, X' are type symmetric and Y, Y' are parity preserving. X and X' are dual with the non-local duality function $\langle\langle X, \nabla X' \rangle\rangle$.

Interfaces and duality

Proof (sketch) The gradient operator $\nabla : \{0,1\}^{\mathbb{I}} \to \{0,1\}^{\mathbb{I}+\frac{1}{2}}$ is linear w.r.t. $\oplus$:

$$\nabla x(i) := x(i - \tfrac{1}{2}) \oplus x(i + \tfrac{1}{2}),$$

and self-adjoint in the sense that

$$\langle\!\langle x, \nabla y \rangle\!\rangle = \langle\!\langle \nabla x, y \rangle\!\rangle \qquad (x \in \{0,1\}^{\mathbb{I}},\ y \in \{0,1\}^{\mathbb{I}+\frac{1}{2}}).$$

If A is type symmetric, then $A^\dagger$ is the dual action and $\nabla A \nabla^{-1}$ is the corresponding action on interfaces. Now

$$(\nabla A \nabla^{-1})^\dagger = \nabla^{-1} A^\dagger \nabla$$

correspond to the dual of the interface model resp. the model whose interface model is the dual.

(Some care is needed to define ∇^{-1} but this is the basic idea.) ∎

Duality

$X = (X_t)_{t \geq 0}$ Markov with state space S, generator G, semigroup $(P_t)_{t \geq 0}$.
$Y = (Y_t)_{t \geq 0}$ Markov with state space R, generator H, semigroup $(Q_t)_{t \geq 0}$.

Def X and Y *dual* with *duality function* $\psi : S \times R \to \mathbb{R}$ iff

$$\mathbb{E}^x[\psi(X_t, y)] = \mathbb{E}^y[\psi(x, Y_t)] \qquad (t \geq 0).$$

Equivalent formulations:

- $\displaystyle\sum_{x'} P_t(x, x')\psi(x', y) = \sum_{y'} \psi(x, y')Q_t(y, y')$,
- $P_t\psi = \psi Q_t^\dagger$,
- $G\psi = \psi H^\dagger$.

Local duality functions

Let G, H be generators of particle systems, i.e., $S = R = \{0,1\}^\Lambda$, with Λ a locally finite graph.

Write the space of all functions $f : \{0,1\}^\Lambda \to \mathbb{R}$ as a tensor product

$$\mathbb{R}^S = \mathbb{R}^{\{0,1\}^\Lambda} \cong \bigotimes_{i\in\Lambda} \mathbb{R}^{\{0,1\}}.$$

Assume G has the form $G = \sum_{\{i,j\}} G_{ij}$ where we sum over all edges of the graph and G_{ij} acts only on the coordinates i and j, and similarly $H = \sum_{\{i,j\}} H_{ij}$.
By definition ψ is *local* if it is a commutative product $\psi = \prod_i \psi_i$ where ψ_i is an operator that acts only on coordinate i.
For $k \neq i, j$, ψ_k commutes with G_{ij}, so to check that $G\psi = \psi H^\dagger$, it suffices to check for each edge $\{i,j\}$

$$G_{ij}\psi_i\psi_j = \psi_i\psi_j H_{ij}^\dagger.$$

Lloyd-Sudbury duals

Lloyd and Sudbury ('95, '97, '00) have systematically classified local dualities for interacting particle systems with symmetric two-point interactions. In particular, they have found lots of examples of particle systems that are dual with the local duality function

$$\psi(x,y) = q^{|x \wedge y|},$$

where $q \in \mathbb{R}$.

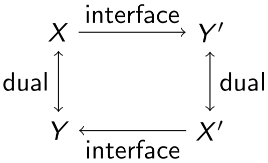
Example 1 $q = 0$ gives

$$0^{|x \wedge y|} = 1_{\{x \wedge y = \underline{0}\}} \qquad \text{additive duality.}$$

Example 2 $q = -1$ gives

$$(-1)^{|x \wedge y|} = 1 - 2\langle\!\langle x, y \rangle\!\rangle \qquad \text{cancellative duality.}$$

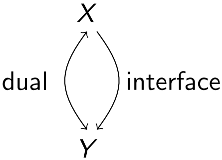
Interfaces and duality



In the commutative diagram, X and X' are dual with the *nonlocal* duality function $\langle\langle X, \nabla X' \rangle\rangle$.

Examples of nonlocal duality functions are pretty rare and it seems they always depend on a specific choice for the lattice (like $\mathbb{Z}$).

The rebellious voter model is *self-dual* w.r.t. this duality function:



The exclusion process

Recall that in the symmetric, nearest-neighbor exclusion process, pairs of neighboring 0's and 1's make the transitions $01 \leftrightarrow 10$ at rate one. This model is both type symmetric and parity preserving. It is part of a commutative diagram where:

$X =$ pure disagreement dynamics
$Y =$ exclusion process
$Z =$ double branching annihilating process

$$\begin{array}{ccccc} X & \xrightarrow{\text{interface}} & Y & \xrightarrow{\text{interface}} & Z \\ \text{dual} \updownarrow & & \text{dual} \updownarrow & & \text{dual} \updownarrow \\ Z & \xleftarrow[\text{interface}]{} & Y & \xleftarrow[\text{interface}]{} & X \end{array}$$

The symmetric Neuhauser-Pacala model

Claim The symmetric Neuhauser-Pacala model with $0 \leq \alpha \leq 1$ is cancellative.

Proof For each i:

- With rate α, choose uniform $j \in \mathcal{N}_i$ and jump $x(i) \mapsto x(i) \oplus x(i) \oplus x(j)$ (voter dynamics).
- With rate $1 - \alpha$, choose uniform, independent $j, k \in \mathcal{N}_i$ and jump $x(i) \mapsto x(i) \oplus x(j) \oplus x(k)$ (rebellious dynamics).

Check that this yields the desired flip rates.

Claim The affine and rebellious voter models are cancellative. (Similar.)

Dual models

The duals Y of all these models are *parity preserving* system of *branching* and *annihilating* random walks.

The parameter $1 - \alpha$ plays the role of the *branching rate*.
For $\alpha = 1$ we obtain annihilating random walks, which are dual to the voter model.

In the Neuhauser-Pacala model and rebellious voter model, in each branching event, two new particles are created.
In the dual of the affine voter model, particles can also give birth to $4, 6, \ldots$ new particles.

Classification of behavior

Let X be type symmetric.

Def X exhibits *coexistence* if there exists an invariant law that is concentrated on states other than $\underline{0}$ and $\underline{1}$.

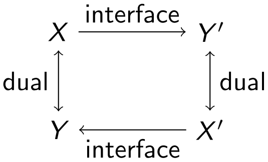
Def X *survives* if $\mathbb{P}^x[X_t \neq \underline{0}\ \forall t \geq 0] > 0$ for some *finite* initial state x.

Let Y be parity preserving.

Def Y *persists* if there exists an invariant law that is concentrated on states other than $\underline{0}$ (all zero).

Def Y *survives* if $\mathbb{P}^y[Y_t \neq \underline{0}\ \forall t \geq 0] > 0$ for some *even* initial state y.

Abstract results



Claim interface model Y' persists $\Leftrightarrow$ X coexists $\Leftrightarrow$ dual Y survives.

Proof of second claim

Start X in product measure with intensity 1/2. Then

$$\mathbb{P}\big[X_t(i) \neq X_t(j)\big] = \mathbb{P}\big[|X_t \wedge (\delta_i + \delta_j)| \text{ is odd}\big] =$$
$$\mathbb{P}^{\delta_i+\delta_j}\big[|X_0 \wedge Y_t| \text{ is odd}\big] = \tfrac{1}{2}\mathbb{P}^{\delta_i+\delta_j}\big[Y_t \neq \underline{0}\big]$$
$$\underset{t\to\infty}{\longrightarrow} \tfrac{1}{2}\mathbb{P}^{\delta_i+\delta_j}\big[Y_t \neq \underline{0}\ \forall t \geq 0\big].$$ *Odd upper invariant law.*

Claim X survives $\Leftrightarrow$ dual Y persists. (Similar.)

Thm [S. '13] Strong interface tightness implies noncoexistence.

Strong interface tightness implies noncoexistence

Lemma Assume that strong interface tightness holds for X. Let $\hat{Y}_\infty + i$ denote the configuration $\hat{Y}_\infty$ shifted by i. Then

$$h(x) := \sum_{i \in \mathbb{Z}+\frac{1}{2}} \mathbb{E}\big[\langle\langle x, \hat{Y}_\infty + i \rangle\rangle\big]$$

is a harmonic function for the process X' (dual of interface model of X). Moreover, there exist constants $0 < c \leq C < \infty$ s.t.

$$c|x| \leq h(x) \leq C|x|.$$

Proof of Thm (sketch) By martingale convergence, $h(X'_t)$ converges a.s., which implies that X' dies out a.s. The same holds for its interface model Y which is dual to X, so by duality X exhibits noncoexistence.