Rank-based Markov chains, self-organized criticality, and order book dynamics

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joint with Marco Formentin, Jana Plačková

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Some classical ecomomic theory

In classical economic theory (Walras,[1] 1874), the *price* of a commodity is determined by *demand* and *supply*.

Let $\lambda_-(p)$ (resp. $\lambda_+(p)$) be the total *demand* (resp. *supply*) for a commodity at price level p, i.e., the total amount that could be sold (resp. bought), per unit of time, for a price of at most (resp. at least) p per unit.

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[1] Walras developed the theory of equilibrium markets in his book *Éléments d' économie politique pure.*

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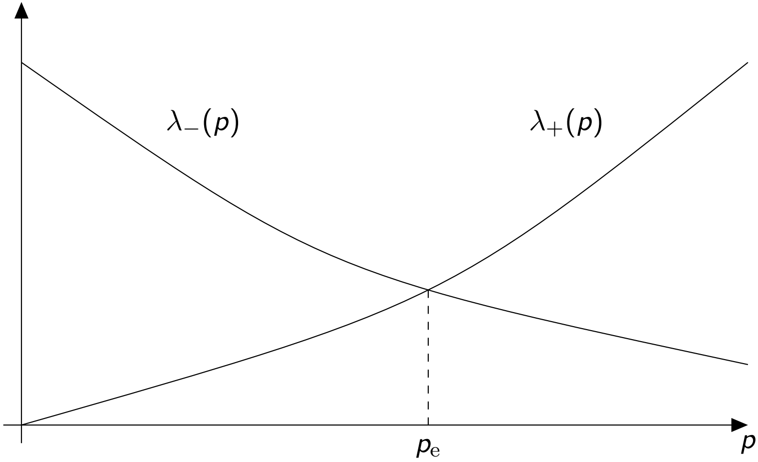

Postulate In an equilibrium market, the commodity is traded at the *equilibrium prize* p_e.

Stock & Commodity Exchanges & the Order Book

On stock & commodity exchanges, goods are traded using an *order book*.

The order book for a given asset contains a list of offers to buy or sell a given amount for a given price. Traders arriving at the market have two options.

- Place a **market order**, i.e., either *buy* (*buy market order*) or *sell* (*sell market order*) n units of the asset at the best price available in the order book.

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Market orders are matched to existing limit orders according to a mechanism that depends on the trading system.

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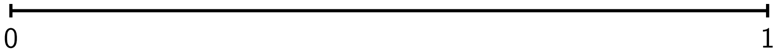
- If the order book contains a suitable offer, then the trader places a *market order*, i.e., sells to the highest bidder or buys from the cheapest seller.
- If the order book contains no suitable offer, then the trader places a *limit order* at his/her minimal sell or maximal buy price.

Numerical simulation



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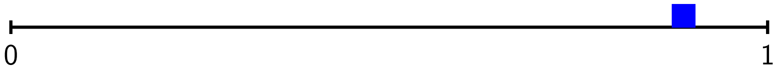

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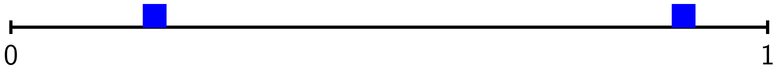

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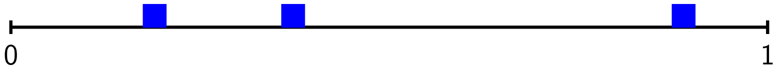

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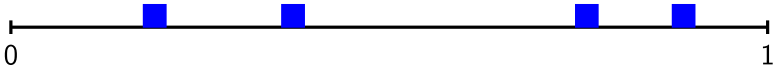

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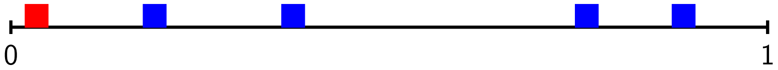

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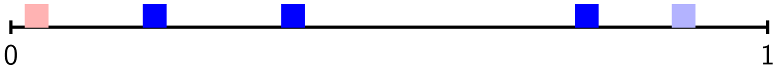

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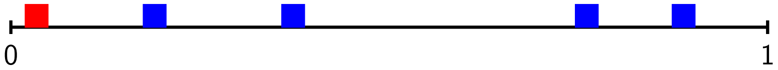

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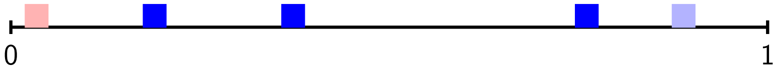

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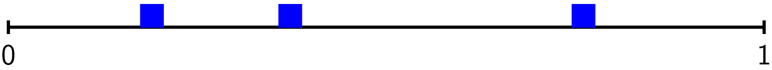

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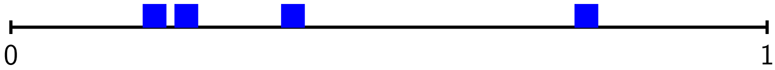

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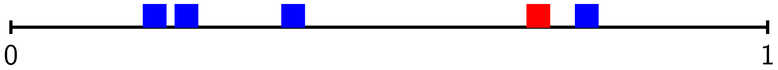

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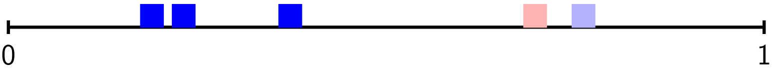

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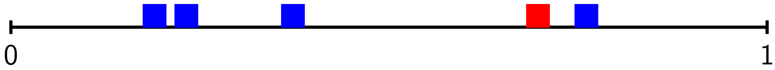

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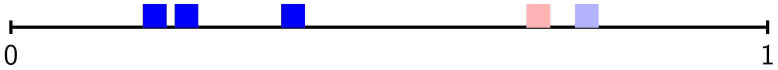

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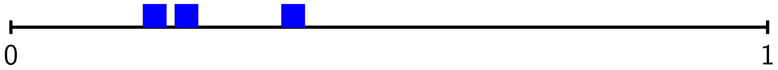

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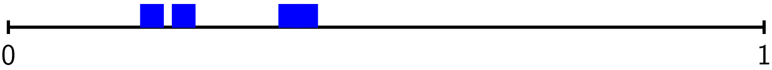

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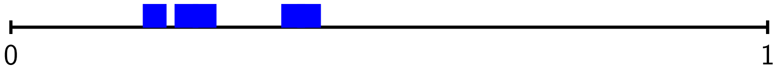

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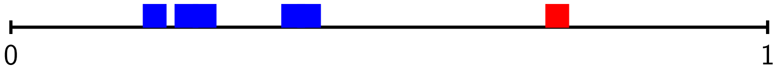

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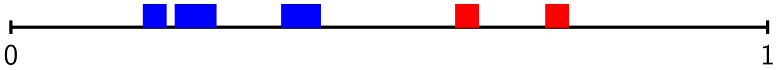

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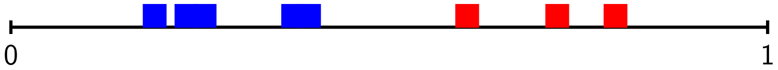

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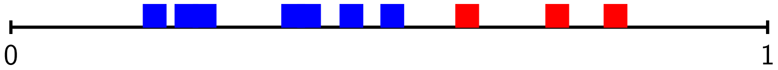

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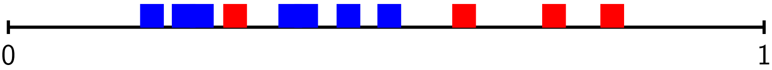

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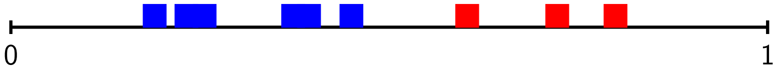

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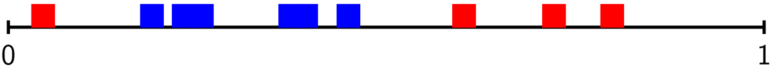

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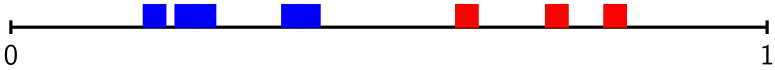

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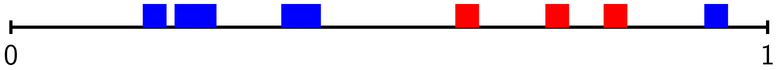

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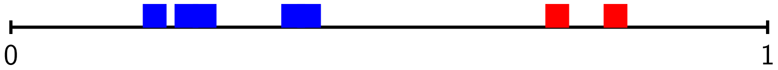

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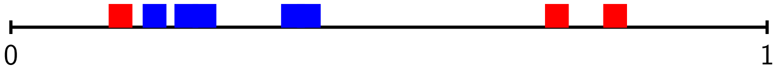

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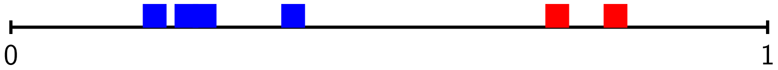

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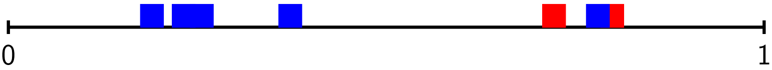

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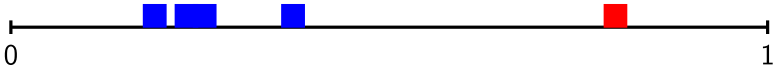

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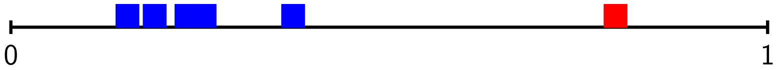

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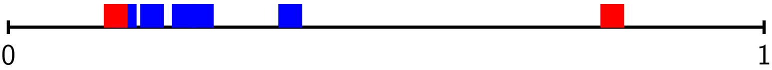

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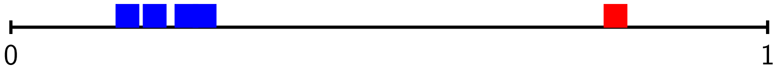

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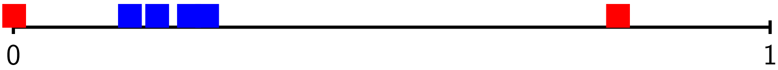

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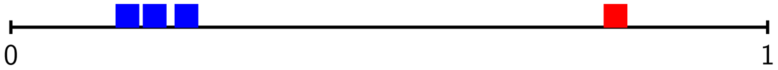

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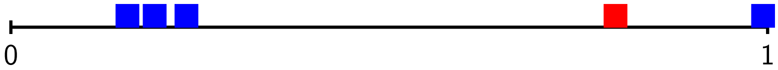

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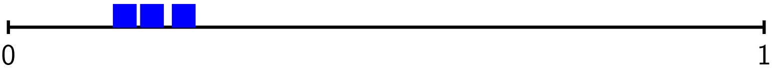

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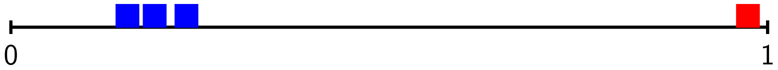

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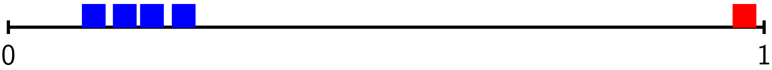

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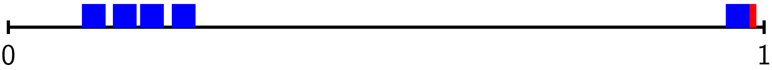

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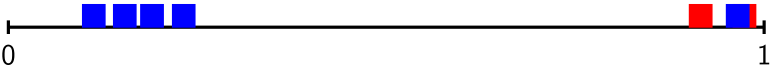

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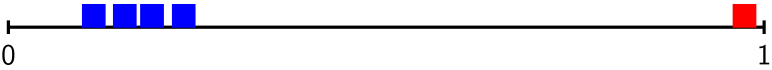

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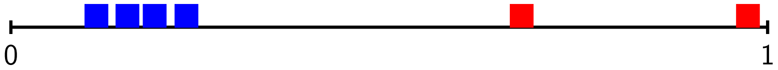

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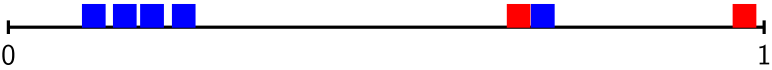

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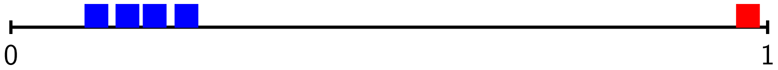

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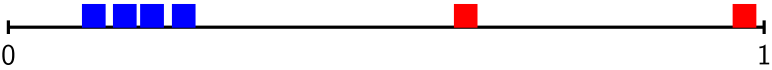

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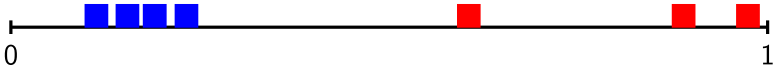

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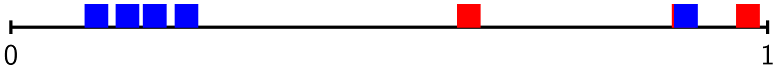

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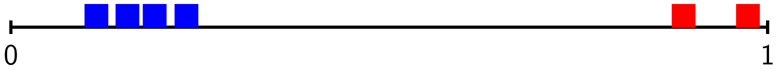

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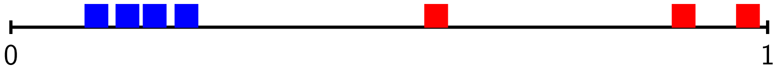

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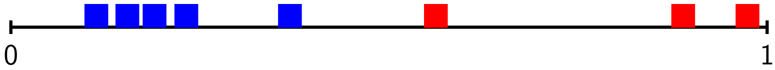

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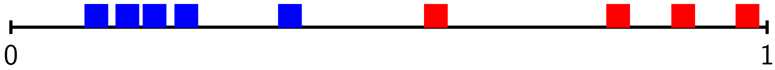

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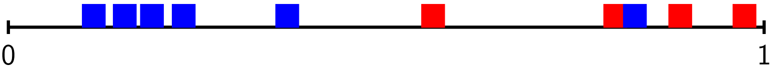

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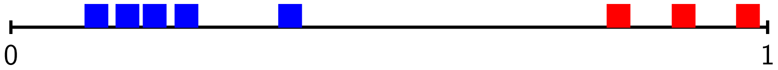

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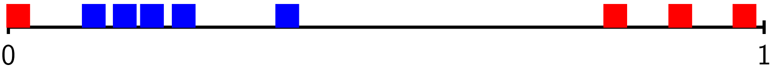

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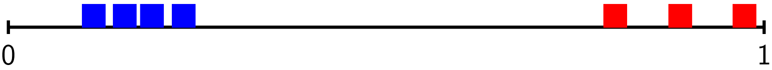

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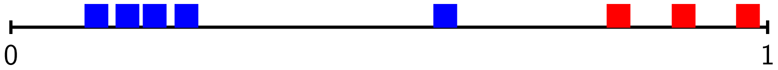

Numerical simulation

The order book after the arrival of 100 traders.

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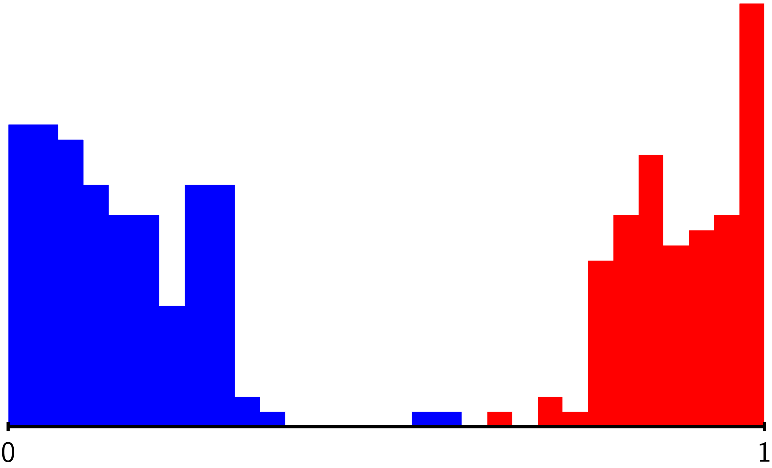

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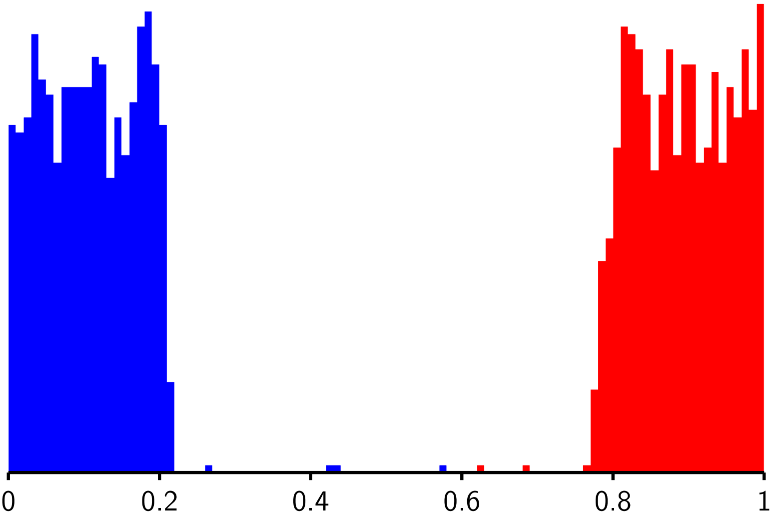


The order book after the arrival of 10,000 traders.

Stigler's model

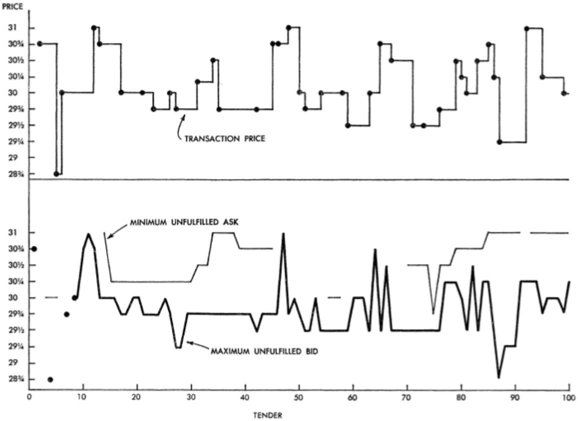
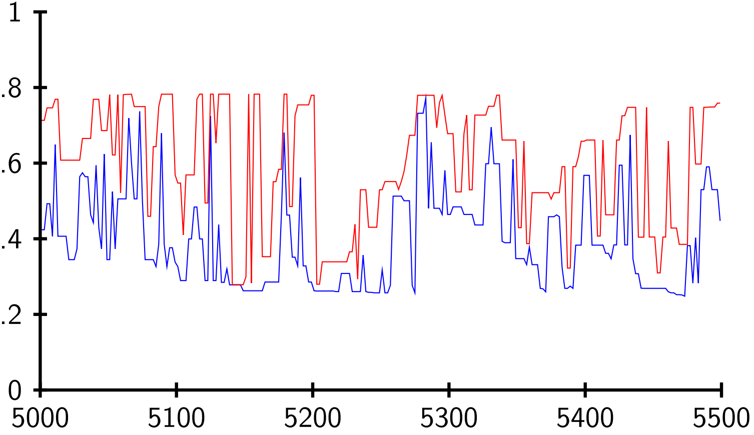


Fig. 1.—Hypothetical sequence of transaction prices, generated by sequence of random numbers, and maximum unfulfilled bid and minimum unfulfilled ask prices (equilibrium price of 29¾ or 30).

Stigler (1964) already simulated the same model with $\mu_\pm$ the uniform distributions on a set of 10 possible prices.

Numerical simulation



Evolution of the bid and ask prices between the arrivals of the 5000th and 5500th trader.

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- Buy limit orders at a price below $q_{\min}$ are never matched with a market order.
- Sell limit orders at a price above $q_{\max}$ are never matched.
- The bid and ask prices keep fluctuating between $q_{\min}$ and $q_{\max}$.
- The spread is huge, most of the time.

The critical point

Luckock has a **formula** for $q_{\min}$ and $q_{\max}$.

In particular, for the model on $[0,1]$ with $\lambda_-(x) = 1 - x$ and $\lambda_+(x) = x$, Luckcock claims: $q_{\min} := 1 + 1/z$ with z the unique solution of the equation $1 + z + e^z = 0$.

Numerically, $q_{\min} \approx 0.21781170571980$.

Luckock proves his claim based on the following assumptions:

- The model is stationary.
- There exist $0 < q_{\min} < q_{\max} < 1$ such that buy (sell) limit orders below $q_{\min}$ (above $q_{\max}$) are never matched.
- All buy (sell) limit orders above $q_{\min}$ (below $q_{\max}$) are eventually matched.

The critical point

Proof: Let $M^{\pm}$ denote the price of the best buy/sell offer. Since buy orders are added to $A \subset (q_{\rm min}, q_{\rm max})$ at the same rate as they are removed

$$\int_A \mathbb{P}[M^- < x]\,\mu_+(\mathrm{d}x) = \int_A \lambda_-(x)\,\mathbb{P}[M^+ \in \mathrm{d}x].$$

Write

$$f_-(x) := \mathbb{P}[M^- < x] \quad \text{and} \quad f_+(x) := \mathbb{P}[M^+ > x].$$

Then

$$\begin{aligned} &\text{(i)} \quad f_-\mathrm{d}\lambda_+ = -\lambda_-\mathrm{d}f_+,\\ &\text{(ii)} \quad f_+\mathrm{d}\lambda_- = -\lambda_+\mathrm{d}f_-, \end{aligned}$$

With the right boundary conditions, this can be solved for a unique $q_{\rm min}$ and $q_{\rm max}$.

The critical point

Open problems:

- Prove the existence of such a stationary state.
- Convergence to stationarity started from an empty order book.

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I am working on this.

Market orders

We extend the model by adding traders who *always place a market order*, i.e.,

- With rate m_- (resp. m_+), a trader arrives who takes the cheapest available offer (resp. sells to the highest bidder) in the order book, if there is one, and otherwise *does nothing*.

This is like a trader who wants to buy (resp. sell) at the highest (resp. lowest) possible price, except that if the order book contains no matching offer, no limit order is written down.

Advantage: is possible for the extended process to be *positive recurrent*.

Valid solutions

Assume that $m_\pm > 0$. Then Luckock's differential equation

$$\begin{aligned} \text{(i)} \quad & f_- \mathrm{d}\lambda_+ = -\lambda_- \mathrm{d}f_+, \\ \text{(ii)} \quad & f_+ \mathrm{d}\lambda_- = -\lambda_+ \mathrm{d}f_-, \end{aligned}$$

on an interval $[I_-, I_+]$ has a unique solution such that $f_-(I_+) = 1 = f_+(I_-)$. If the process is positive recurrent, then

$$f_-(x) = \mathbb{P}[M^- < x] \quad \text{and} \quad f_+(x) = \mathbb{P}[M^+ > x],$$

so positive recurrence implies $f_\pm(I_\mp) > 0$.

Proposition Assume $\lambda_-(I_-) \neq \lambda_+(I_+)$. Then $f_\pm(I_\mp) > 0$ implies positive recurrence.

Valid solutions

Set

$$g_-(x) := \frac{1}{\lambda_-(I_+)} - \frac{1}{\lambda_-(x)} \quad \text{and} \quad g_+(x) := \frac{1}{\lambda_+(I_-)} - \frac{1}{\lambda_+(x)}.$$

Then

$$c := \int_{I_-}^{I_+} g_- \, \mathrm{d}g_+ = -\int_{I_-}^{I_+} g_+ \, \mathrm{d}g_-$$

measures how well buy and sell limit orders match up. If $c = 0$ then buy and sell limit orders never match. If $c = \|g_-\| \cdot \|g_+\|$, then buy and sell limit orders always match.

If $|\mu_-| < |\mu_+| + m_+$ and $|\mu_+| < |\mu_-| + m_-$, but $|\mu_-| \geq m_+$ or $|\mu_+| \geq m_-$, then keeping $|\mu_\pm|$ and $m_\pm$ fixed but decreasing c, the system goes through a phase transition where it stops being positive recurrent.

Similar models

- Gabrielli and Caldarelli's (2007,2009) modification of Barabási's queueing model (2005).
- Two models for canyon formation.
- The modified Bak-Sneppen model (Meester & Sarkar, 2012).

All these models contain a rule "kill the largest (smallest) particle" and (seem to) exhibit *self-organized criticality*.

A model for email communication

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The recipient assigns a *priority* to each incoming email, and always answers the email with the highest priority in the inbox (or does nothing if the inbox is empty).

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The recipent assigns a *priority* to each incoming email, and always answers the email with the highest priority in the inbox (or does nothing if the inbox is empty).

Priorities are i.i.d. with some atomless law. Without loss of generality we can take the uniform distribution on $[-\lambda_{\rm in}, 0]$.

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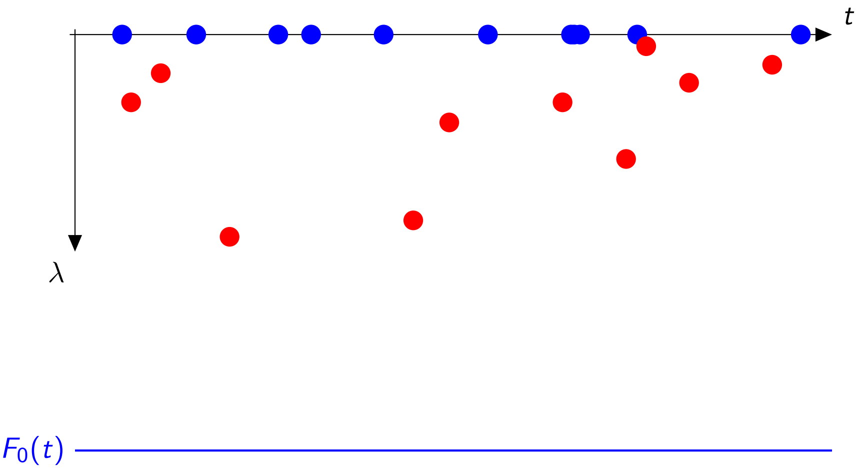
Critical behavior at $\lambda_{\rm out}$: intervals between times when all emails with priority above $\lambda_{\rm out}$ have been answered have a power-law distribution with $\mathbb{P}[\tau \geq k] \sim k^{-1/2}$.

Poisson construction

Let $F_\lambda(t)$ denote the number of emails with priority in $[-\lambda, 0]$ that are in the inbox at time t.

We can read off $F_\lambda(t)$ from the Poisson processes describing the arrivals of new emails and answering times.

Poisson construction



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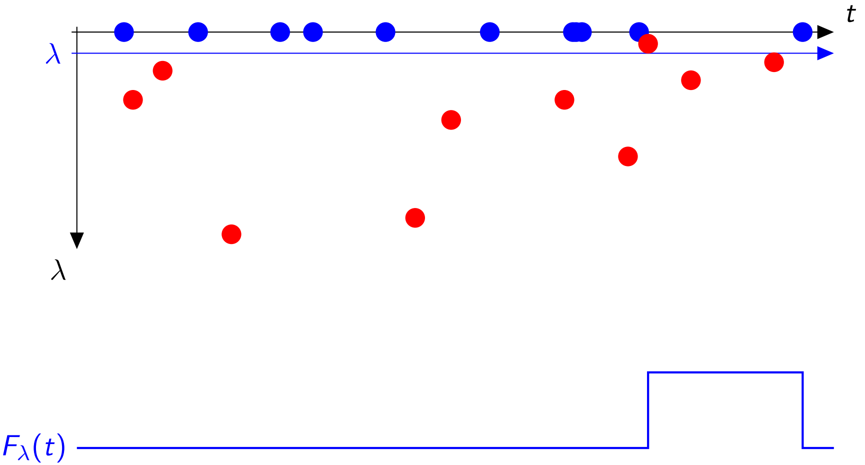

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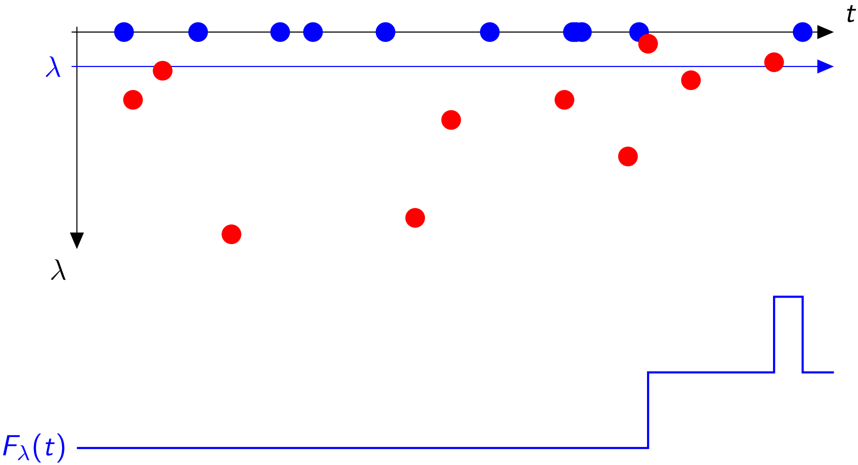

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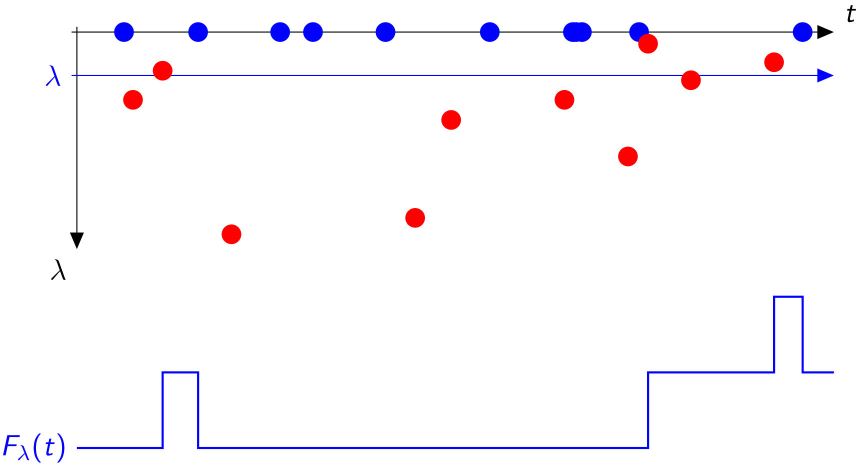

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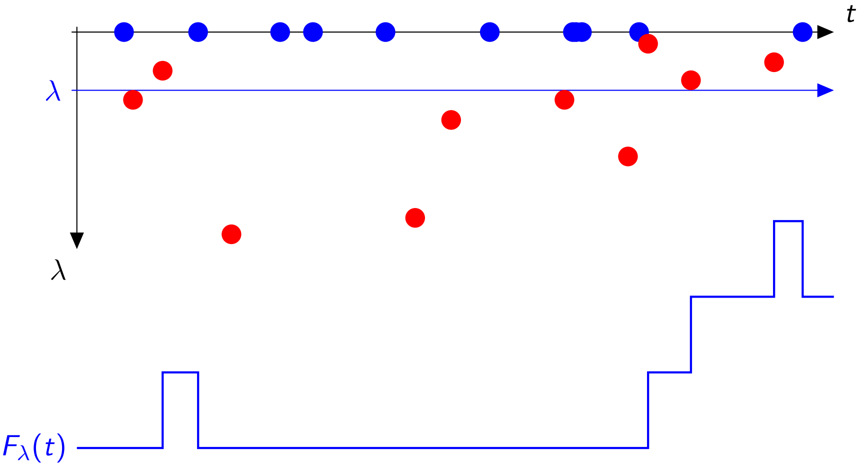

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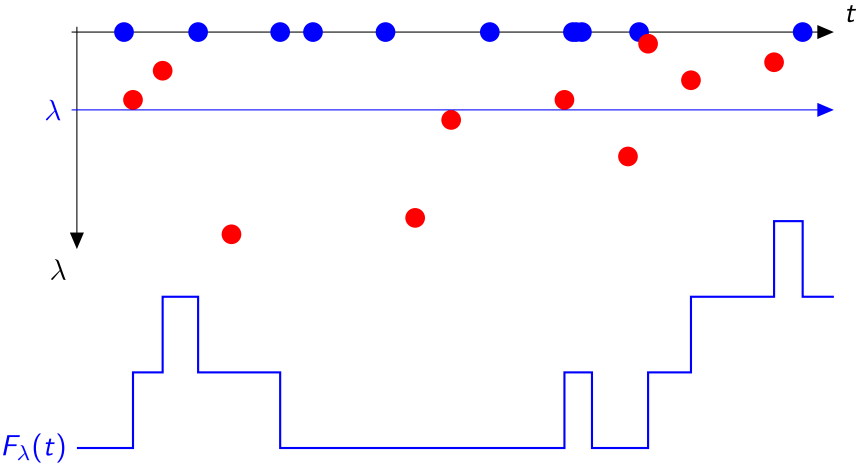

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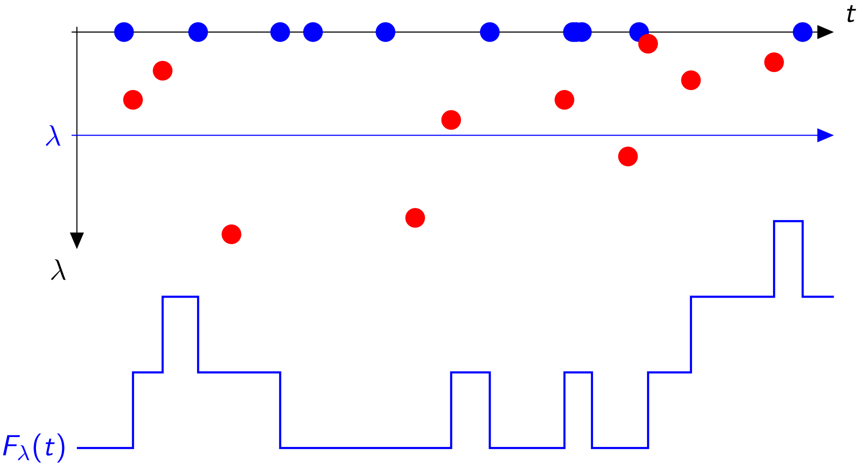
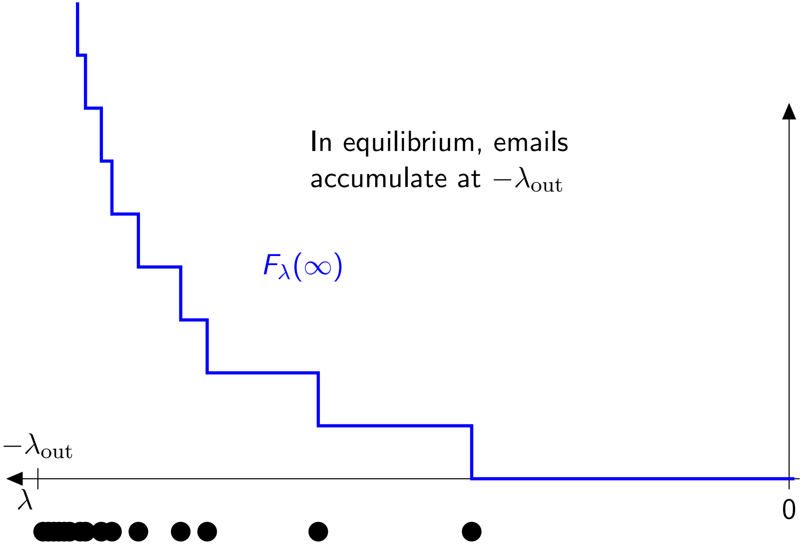


The equilibrium distribution

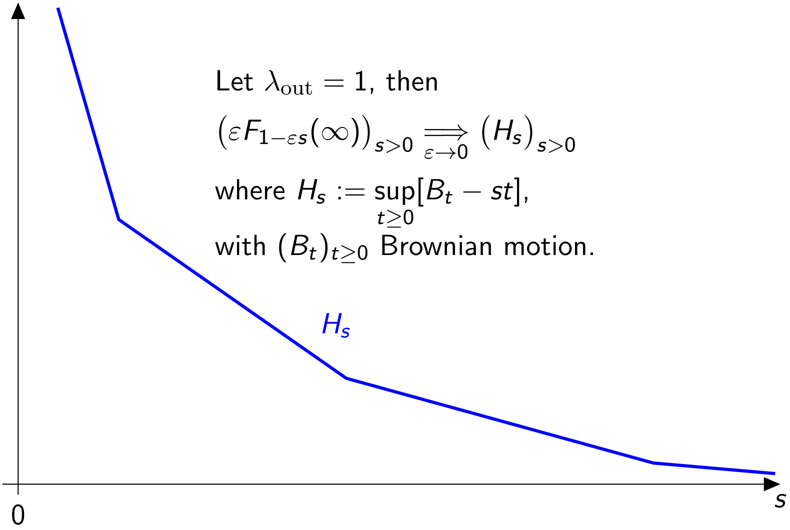


Let $\lambda_{\text{out}} = 1$, then

$$\big(\varepsilon F_{1-\varepsilon s}(\infty)\big)_{s>0} \underset{\varepsilon\to 0}{\Longrightarrow} \big(H_s\big)_{s>0}$$

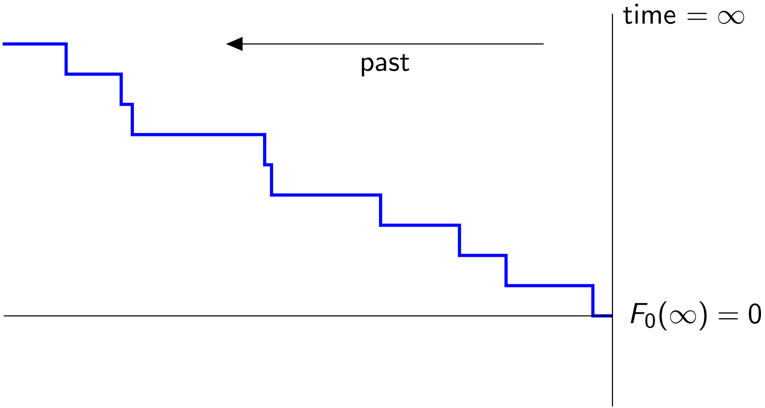
where $H_s := \sup_{t\geq 0}[B_t - st]$,

with $(B_t)_{t\geq 0}$ Brownian motion.



Groeneboom (1983): the concave majorant of Brownian motion.

Coupling from the past



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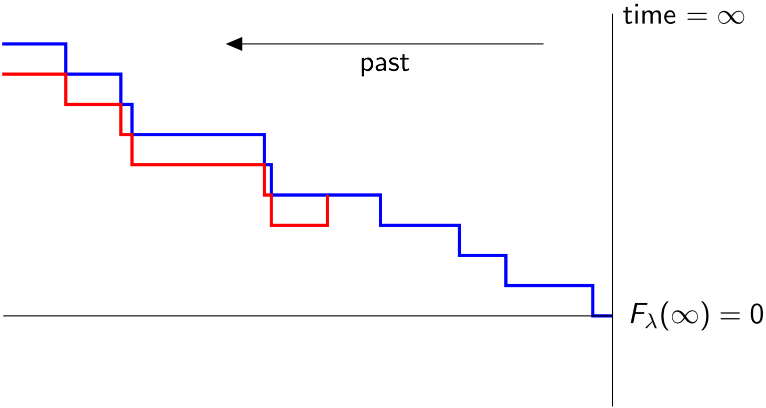

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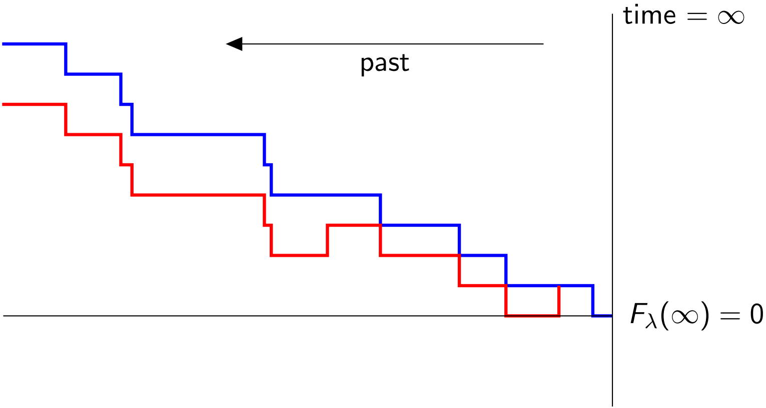

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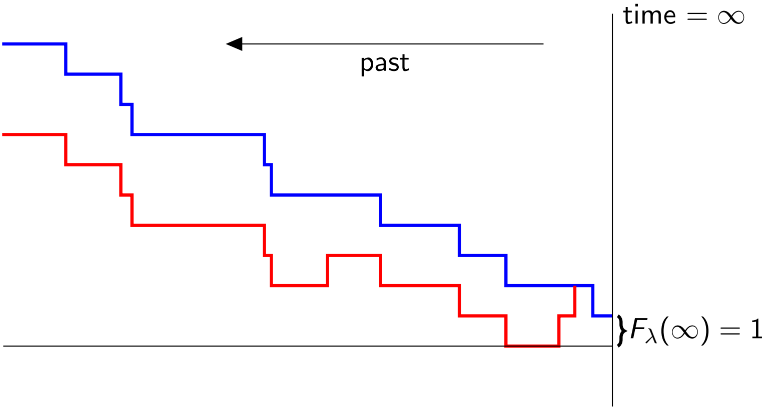

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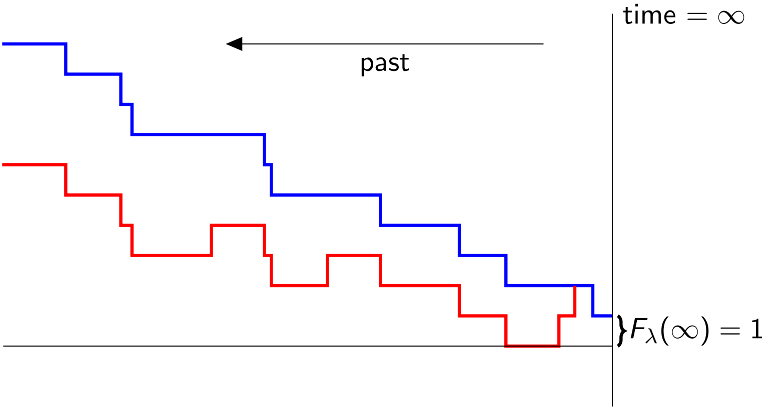

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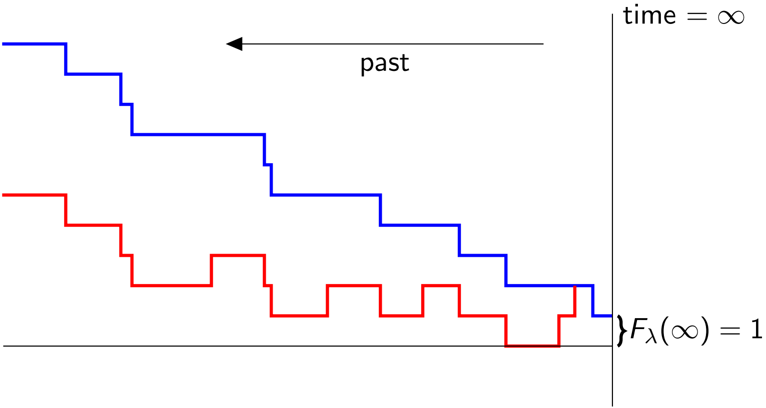

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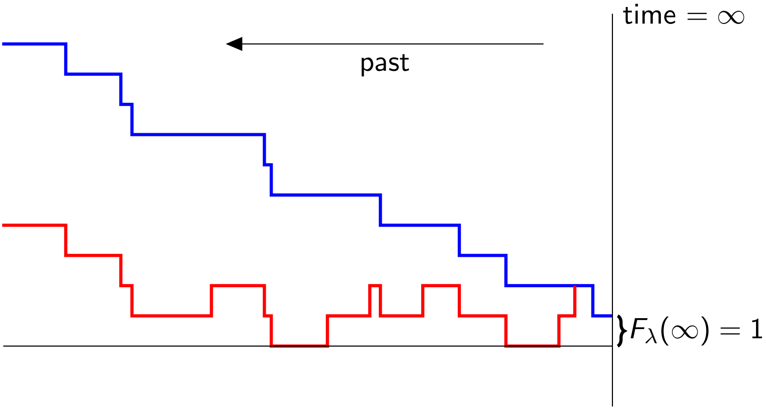

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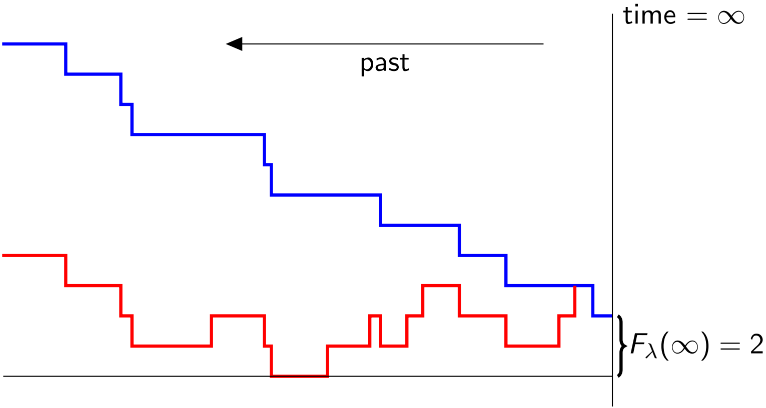

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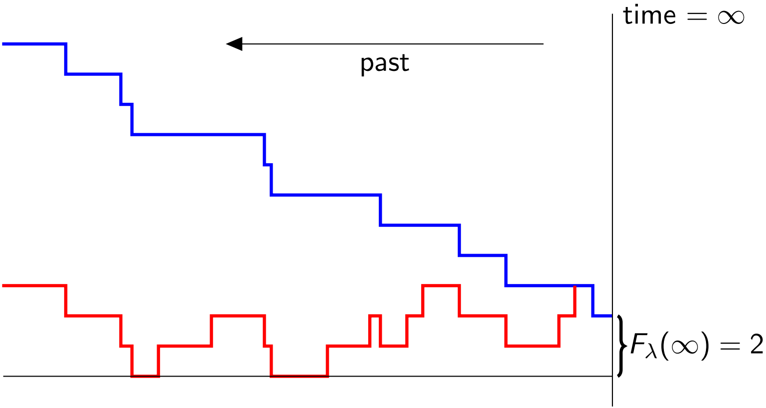

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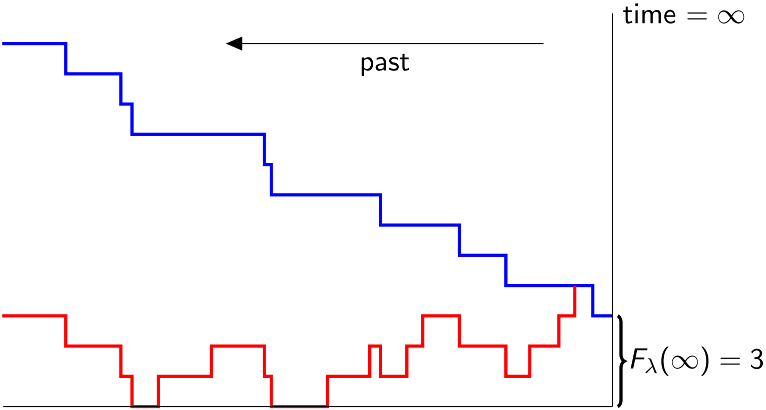
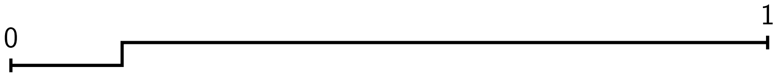


A one-sided canyon model

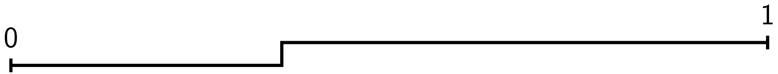
A river flows on the left.

A one-sided canyon model



The river either cuts deeped into the rock.

A one-sided canyon model



Or the shore is eroded down, starting from a random point.

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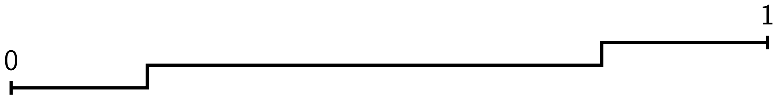
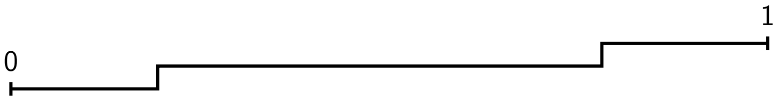


We either make the river deeper. . .

A one-sided canyon model

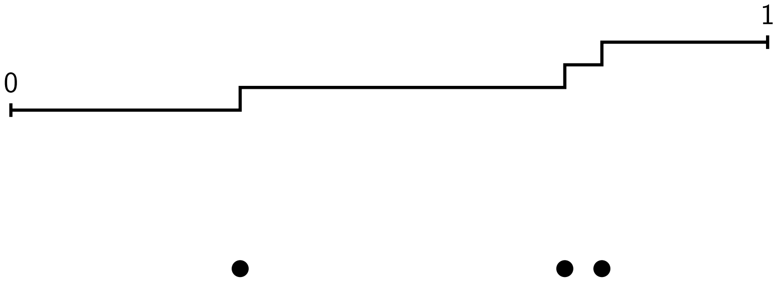


. . . or we erode the shore,

A one-sided canyon model

. . . depending on where the new point falls.

A one-sided canyon model



Points on the left of all others are simply added.

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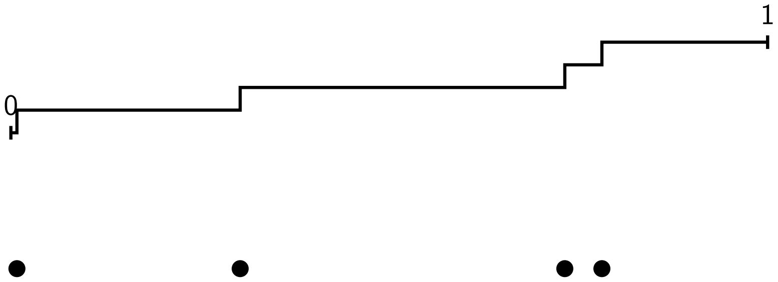

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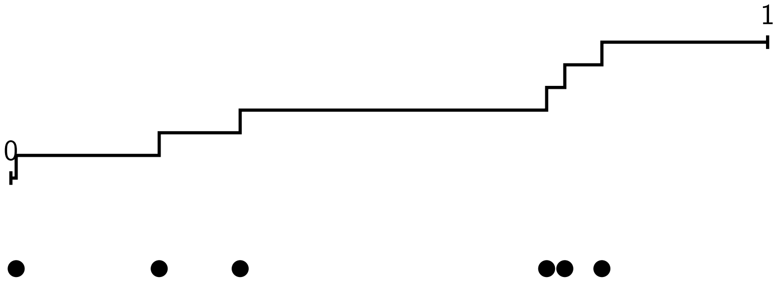
A one-sided canyon model

Otherwise, we remove the left-most point.

A one-sided canyon model

In other words, we always add the new point.

A one-sided canyon model



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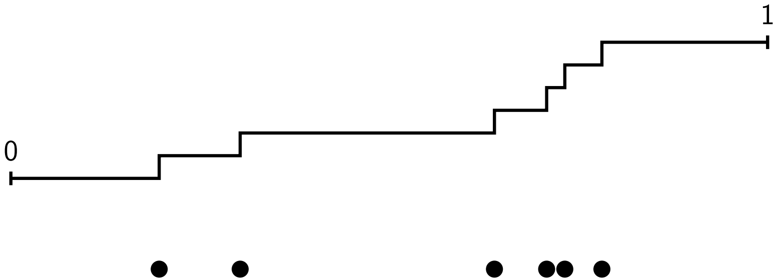
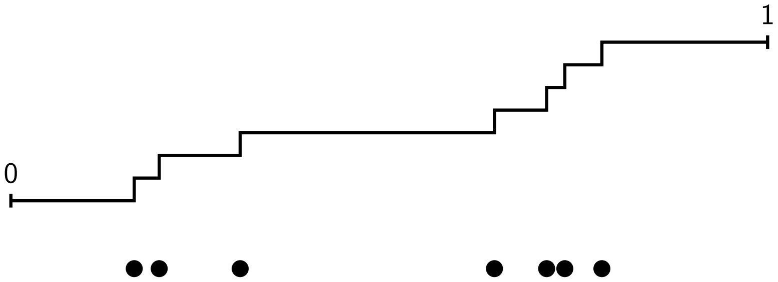


If the new point is not the left-most, then we remove the left-most.

A one-sided canyon model



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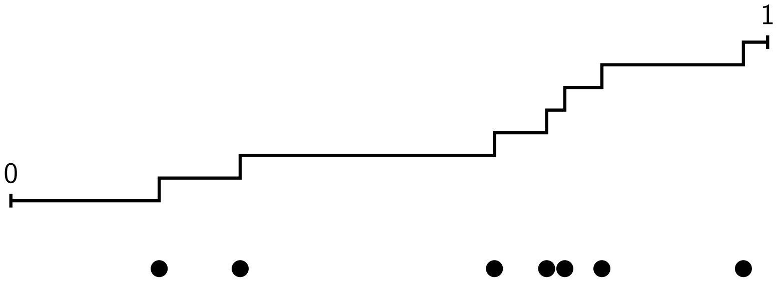

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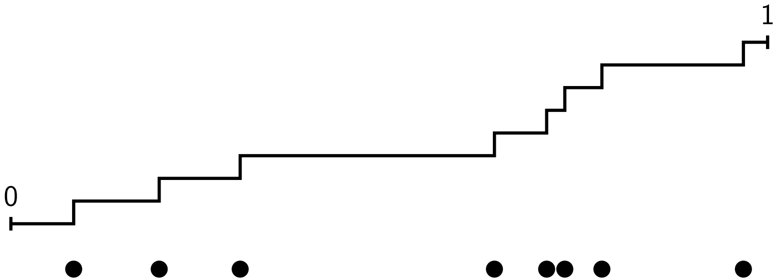

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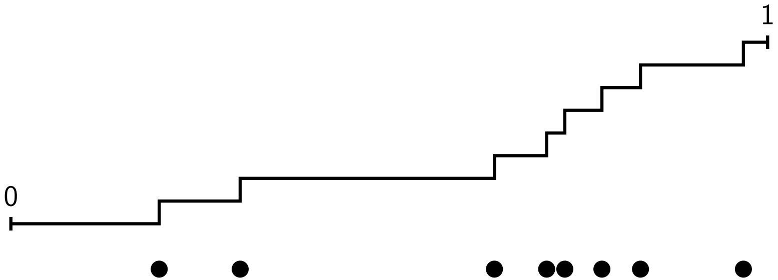

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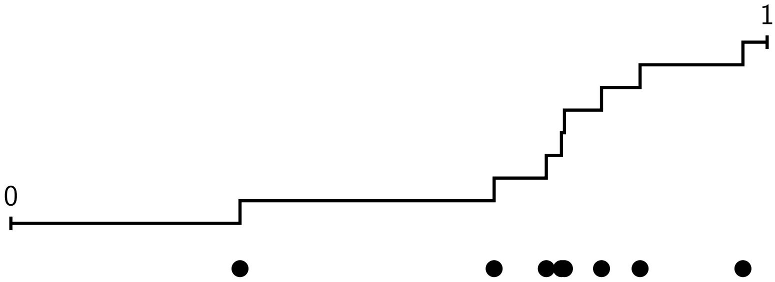

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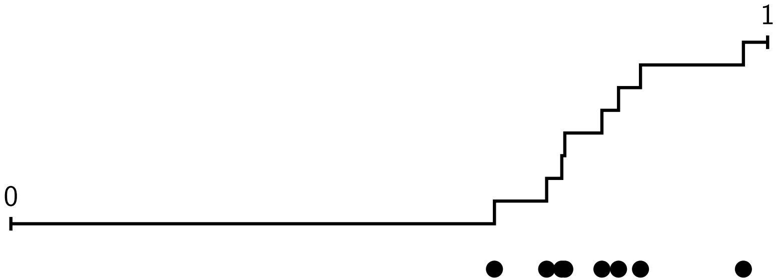

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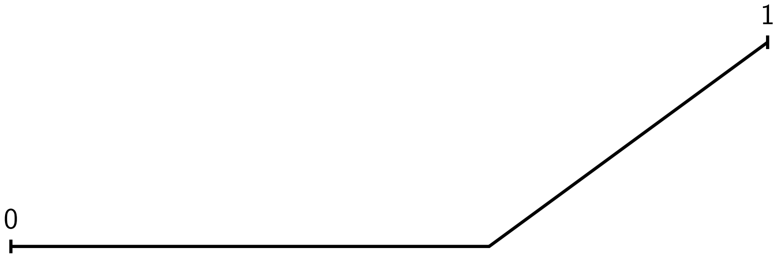


In this model, the critical point is $p_c = 1 - e^{-1} \approx 0.63212$.

The one-sided canyon model

The process just described defines a Markov chain $(X_k)_{k\geq 0}$ where $X_k \subset [0,1]$ is a finite set.

Consistency: For each $0 < q < 1$, we observe that the *restricted process*

$$\big(X_k \cap [0,q]\big)_{k\geq 0}$$

is a Markov chain.

Theorem The restricted process is positively recurrent for $q < 1 - e^{-1}$ and transient for $q > 1 - e^{-1}$.

Open problem Behavior at the critical point.

The critical point

Proof of the theorem Since only the relative order of the points matters, transforming space we may assume that the $(U_k)_{k\geq 1}$ are i.i.d. *exponentially* distributed with mean one and $X_k \subset [0,\infty]$.

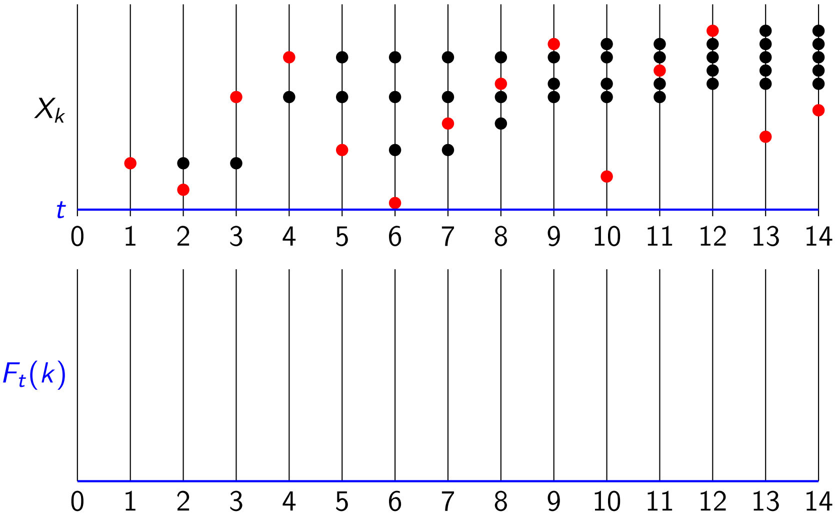
For the modified model, we must prove that $p_c = 1$.

Start with $X_0 = \emptyset$ and define

$$F_t(k) := \big|X_k \cap [0,t]\big| \qquad (k \geq 0,\ t \geq 0).$$

Claim $(F_t)_{t\geq 0}$ is a continuous-time Markov process taking values in the functions $F : \mathbb{N} \to \mathbb{N}$.

The point-counting function



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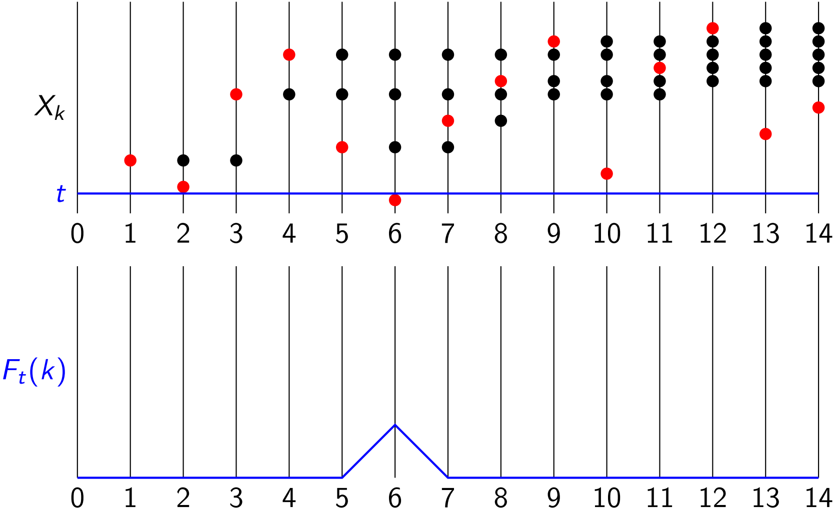

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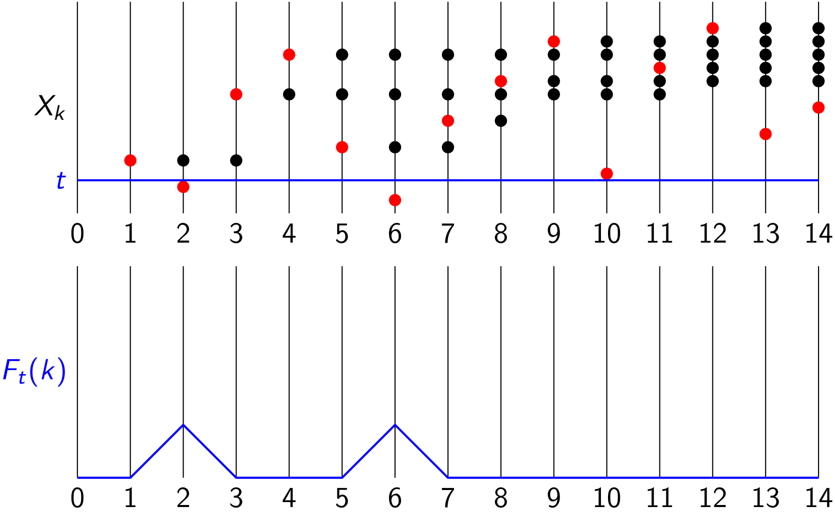

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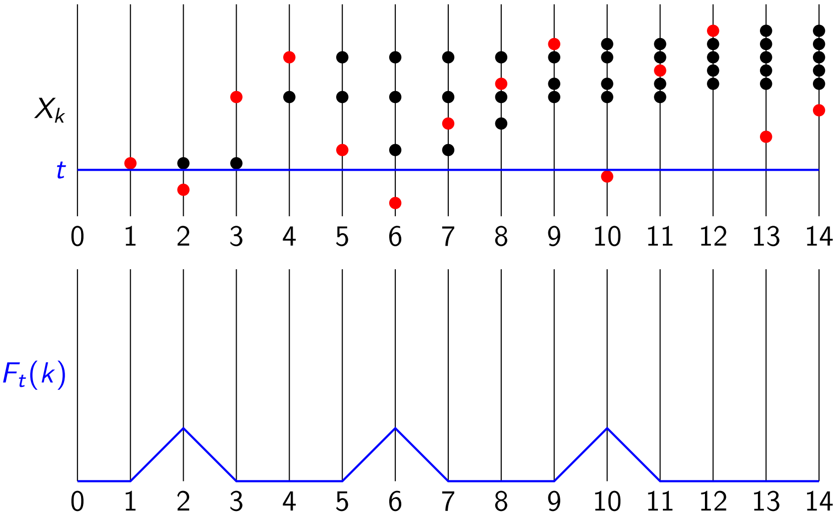

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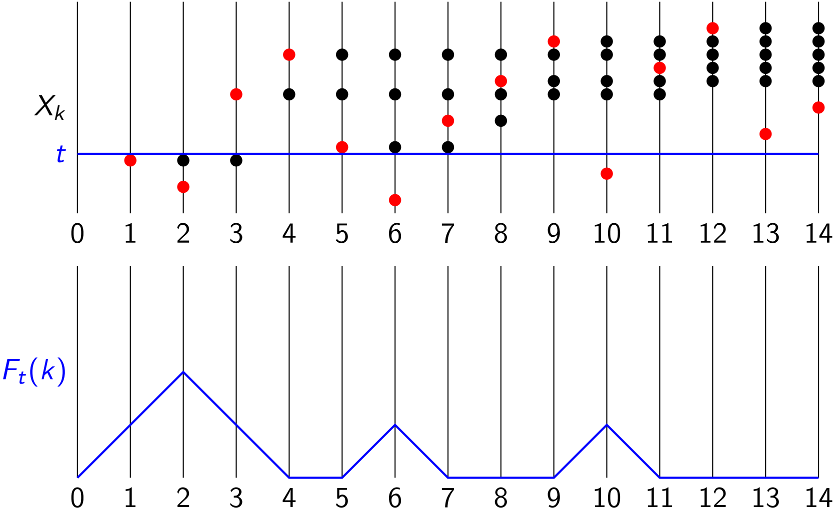

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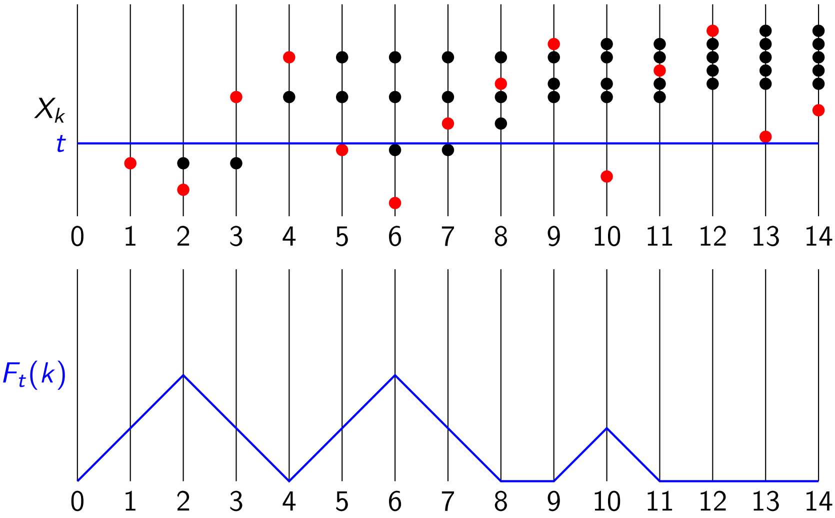

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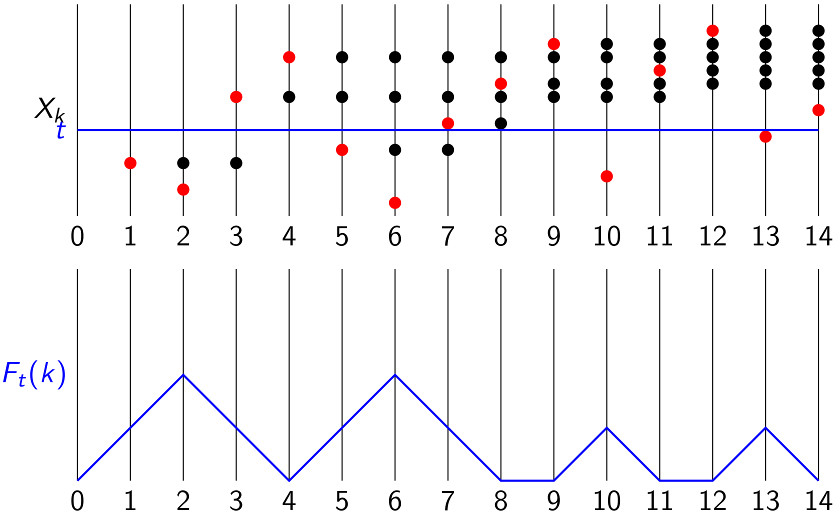

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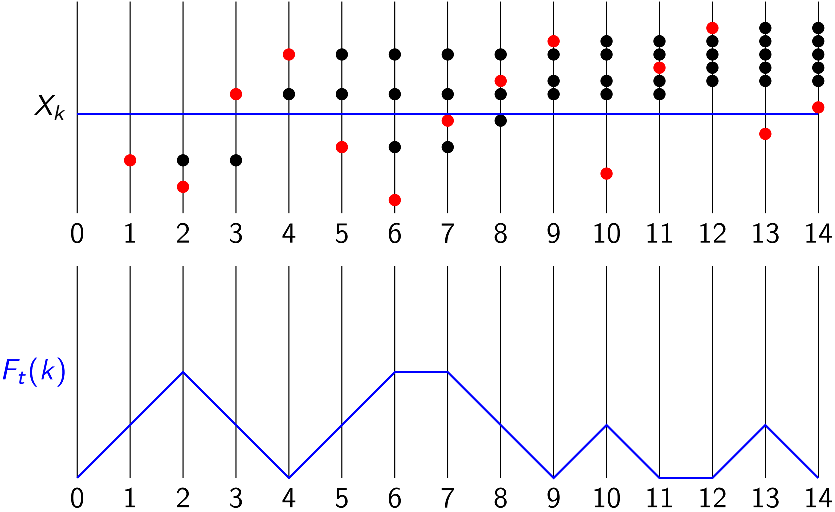

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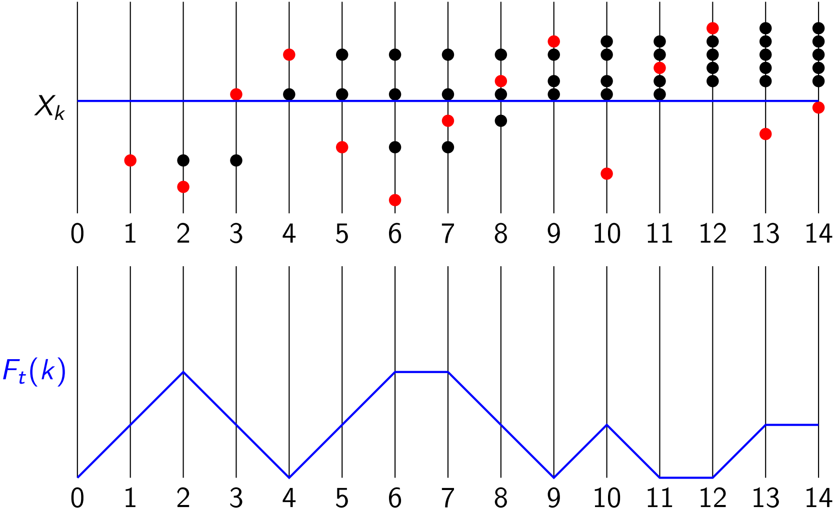

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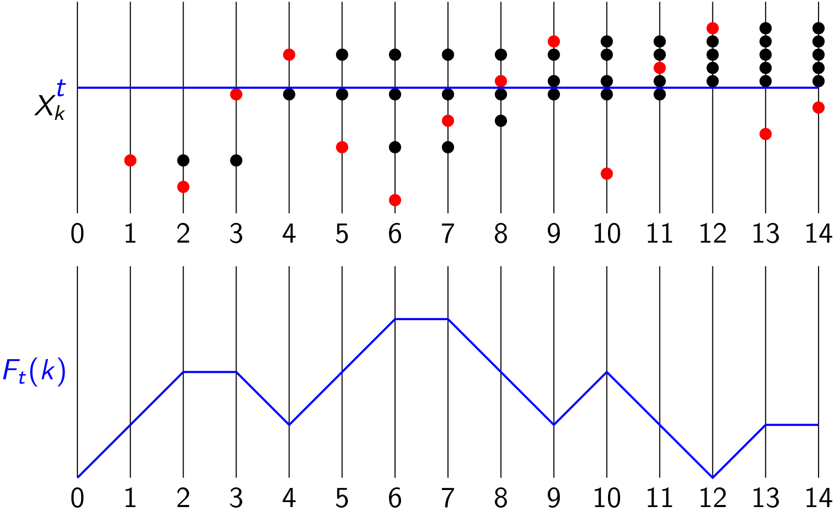

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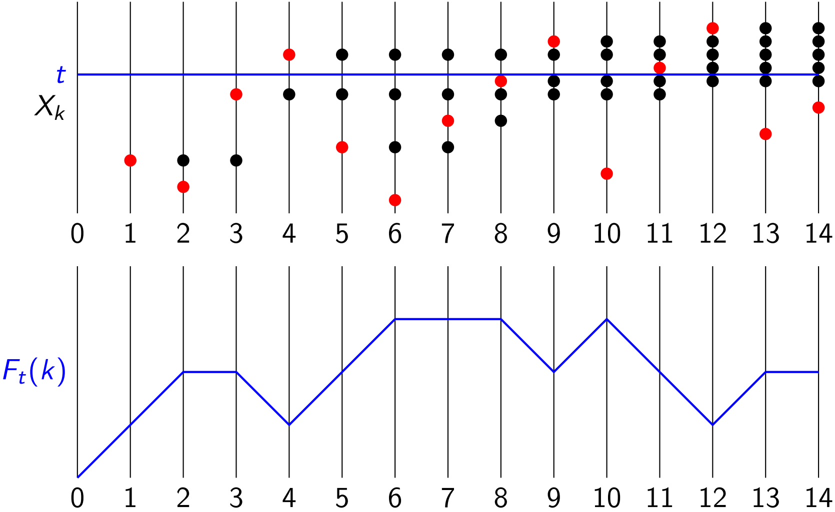

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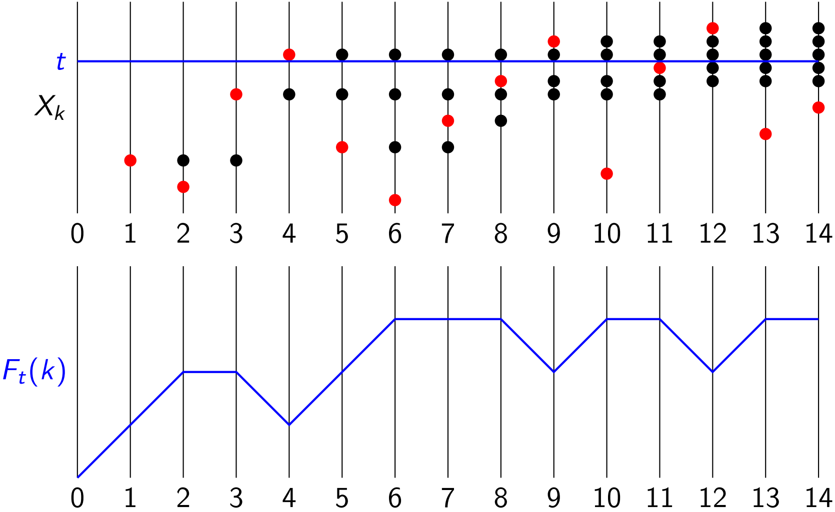

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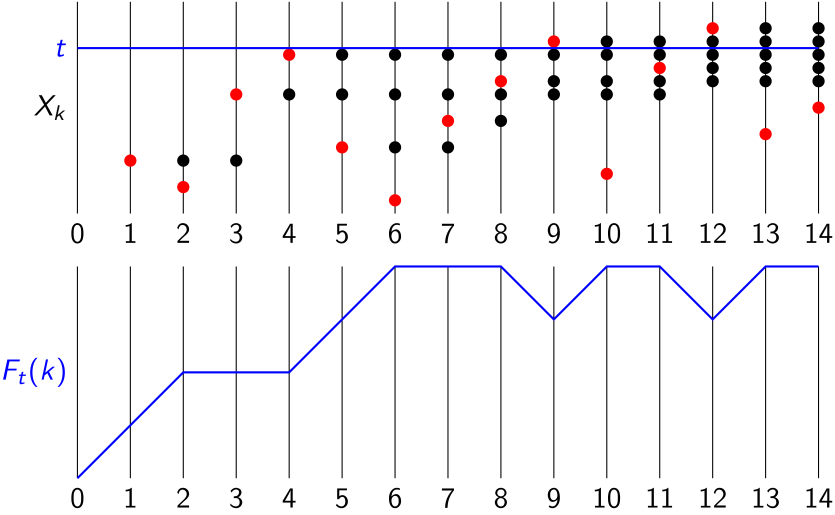

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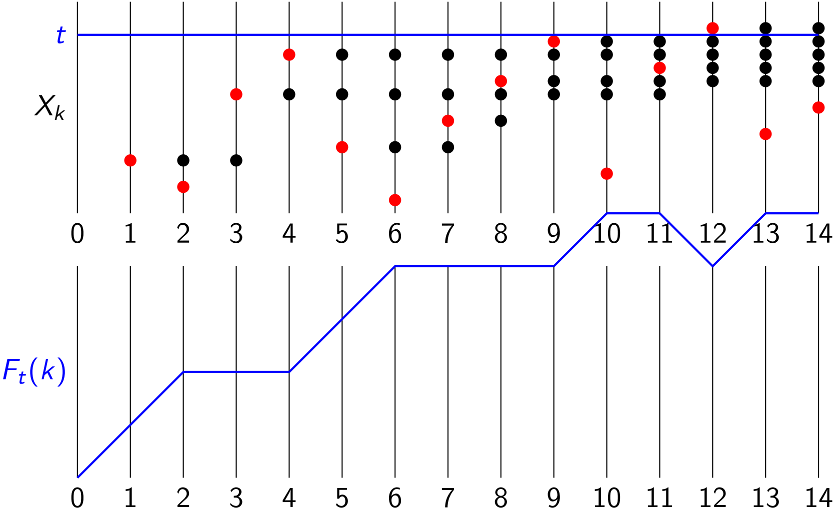

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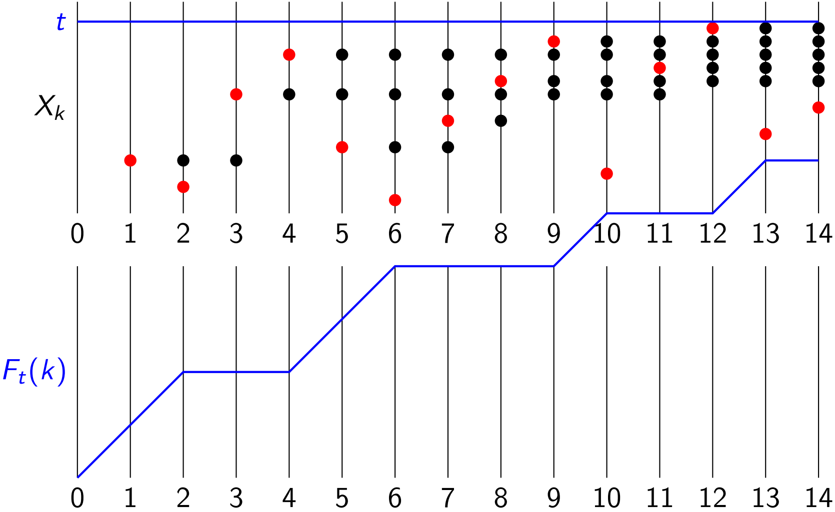


Increments

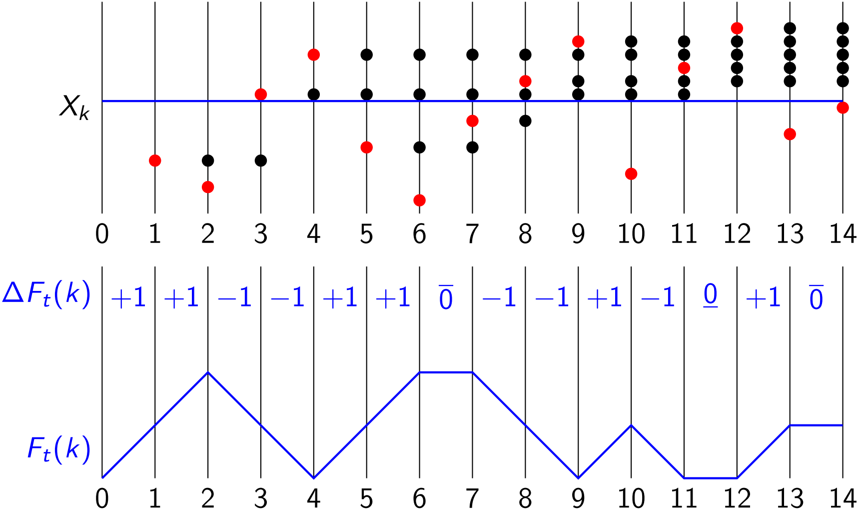
Define

$$\Delta F_t(k) := \begin{cases} \underline{0} & \text{if } F_t(k) = F_t(k-1) = 0, \\ \overline{0} & \text{if } F_t(k) = F_t(k-1) > 0, \\ -1 & \text{if } F_t(k) = F_t(k-1) - 1, \\ +1 & \text{if } F_t(k) = F_t(k-1) + 1. \end{cases}$$

At the exponentially distributed time $t = U_k$, the increment $\Delta F_t(k)$ changes from $\underline{0}$ to $+1$ or from -1 to $\overline{0}$.

At the same time, the next $\underline{0}$ to the right of k, if there is one, is changed into a -1.

The point-counting function



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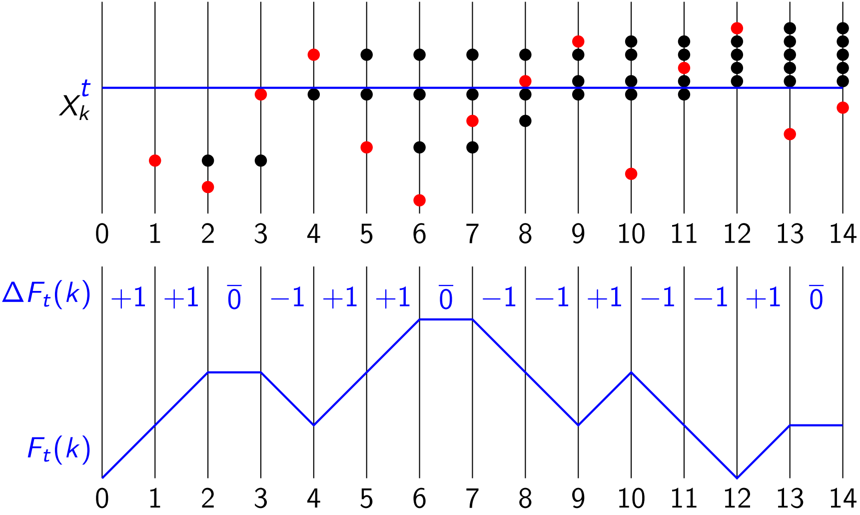


A stationary increment process

We can define the Markov process $(\Delta F_t)_{t\geq 0}$ also on $\mathbb{Z}$ instead of on $\mathbb{N}_+$.

As long as the density of $\underline{0}$'s is nonzero, the process started in $\Delta F_0(k) = \underline{0}$ $(k \in \mathbb{Z})$ satisfies

$$\begin{aligned}
\tfrac{\partial}{\partial t}\mathbb{P}[\Delta F_t(k) = \underline{0}] &= -2\mathbb{P}[\Delta F_t(k) = \underline{0}] - \mathbb{P}[\Delta F_t(k) = -1], \\
\tfrac{\partial}{\partial t}\mathbb{P}[\Delta F_t(k) = -1] &= \mathbb{P}[\Delta F_t(k) = \underline{0}],
\end{aligned}$$

from which we derive that the $\underline{0}$'s run out at $t_{\mathrm{c}} = 1$ and

$$\mathbb{P}[\Delta F_t(1) = \underline{0}] = (1-t)e^{-t} \quad \text{and} \quad \mathbb{P}[\Delta F_t(1) = -1] = te^{-t}$$

$(0 \leq t \leq 1)$.

The increment process

The process $(F_t)_{t\geq 0}$, both on $\mathbb{N}_+$ and $\mathbb{Z}$, makes i.i.d. excursions away from 0.

For the process started in $X_0 = \emptyset$, define the return time

$$\tau_t^{\emptyset} := \inf\big\{k \geq 1 : X_k \cap [0,t] = \emptyset\big\}.$$

From the density of $\underline{0}$'s for the process $(F_t)_{t\geq 0}$ on $\mathbb{Z}$ we deduce that

$$\mathbb{E}[\tau_t^{\emptyset}] = (1-t)^{-1} \qquad (0 \leq t < 1).$$

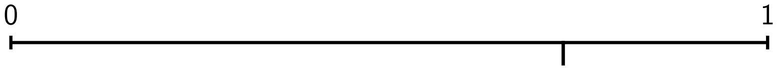
This proves positive recurrence $\Leftrightarrow$ $t < 1$.
It is not hard to derive from this that the restricted process $\big(X_k \cap [0,t]\big)_{k\geq 0}$ is transient for $t > 1$.

Null recurrence at $t = 1$ is so far an open problem.

A two-sided canyon model

We start with a flat rock profile.

A two-sided canyon model



The river cuts into the rock at a uniformly chosen point.

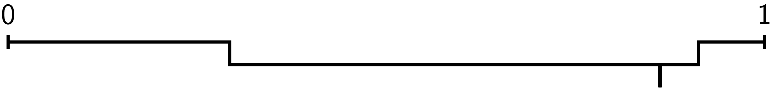
A two-sided canyon model

Rock between a next point and the river is eroded one step down.

A two-sided canyon model

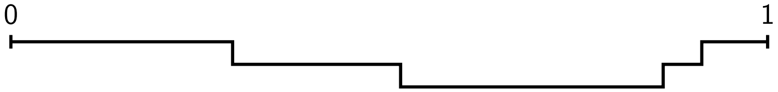
We continue in this way.

A two-sided canyon model



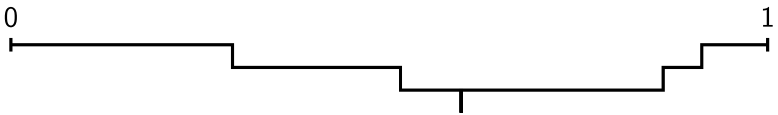
Either the river cuts deeper in the rock.

A two-sided canyon model



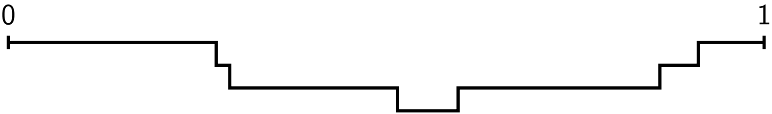
Or one side of the river is eroded down.

A two-sided canyon model



We are interested in the limit profile.

A two-sided canyon model



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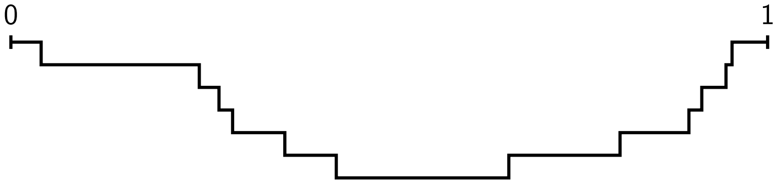

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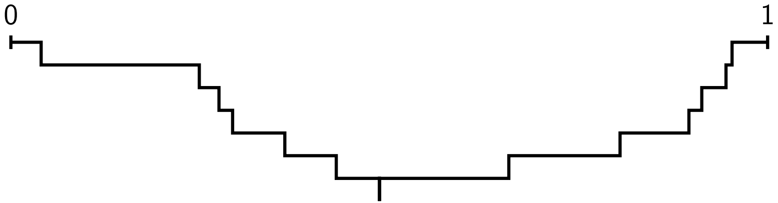

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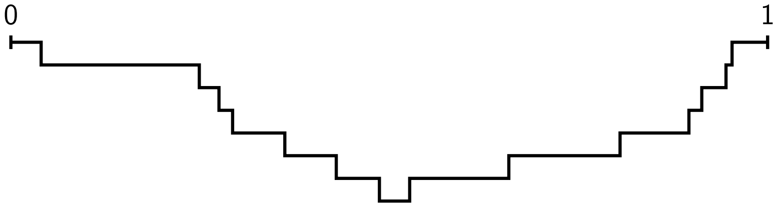

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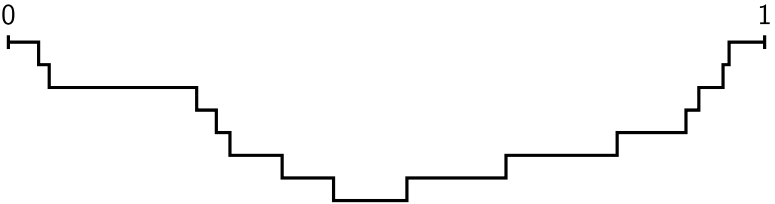

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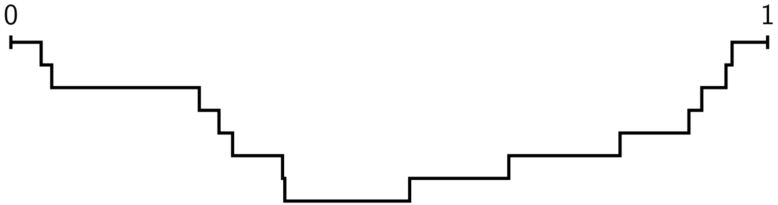

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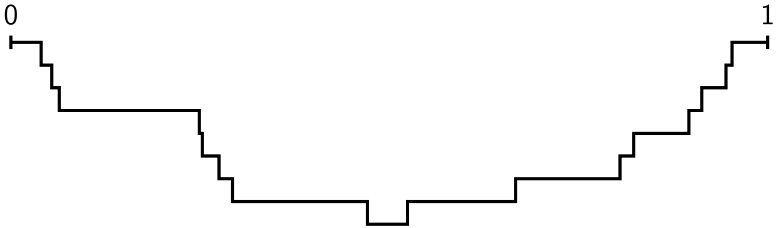

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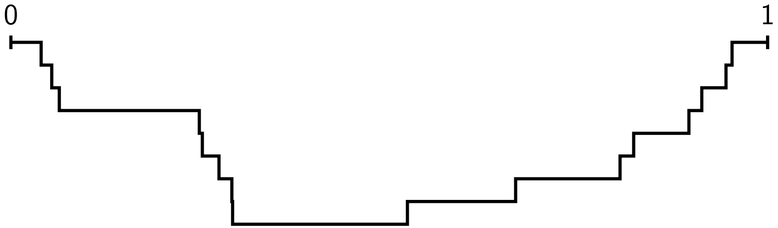

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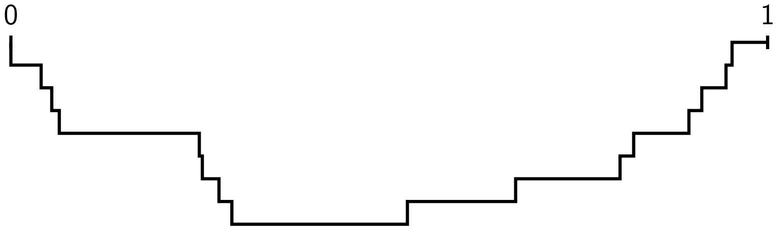

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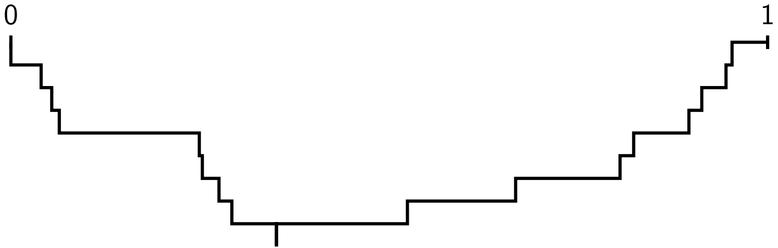

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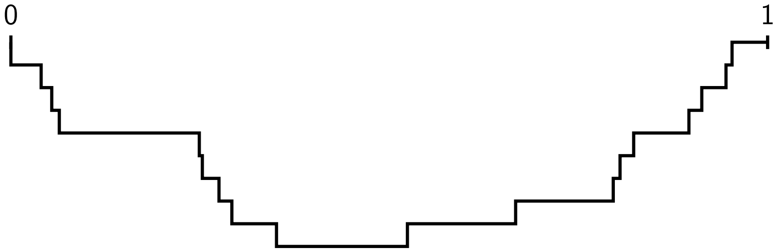

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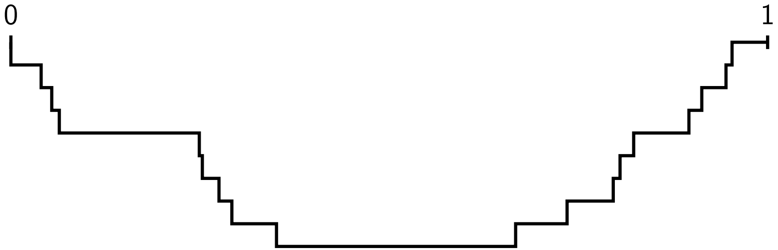

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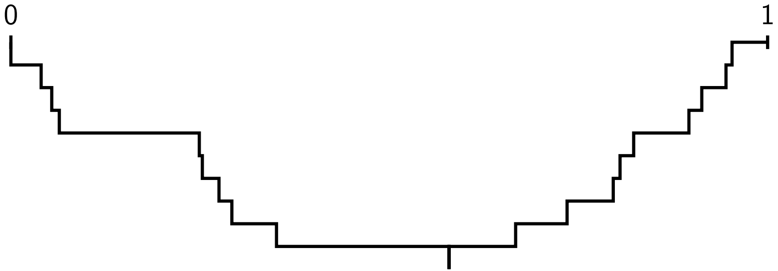

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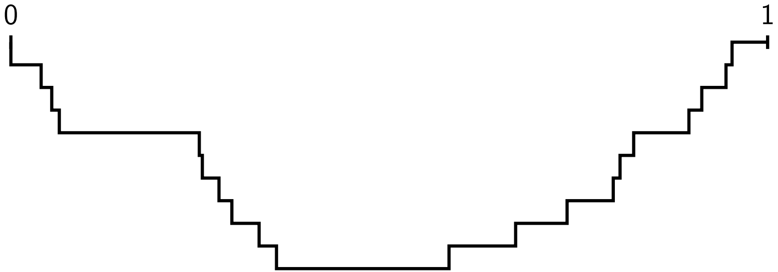

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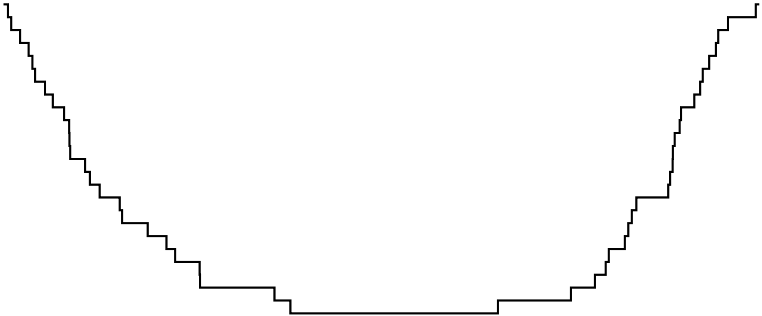
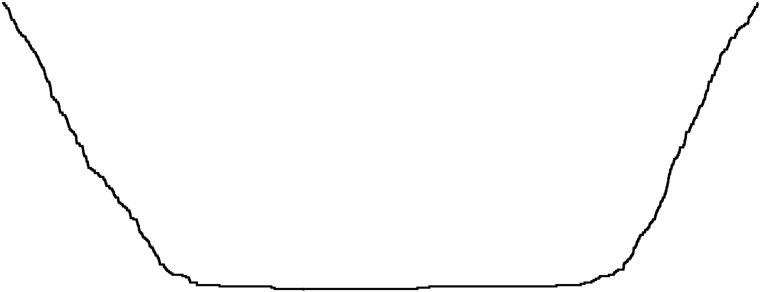


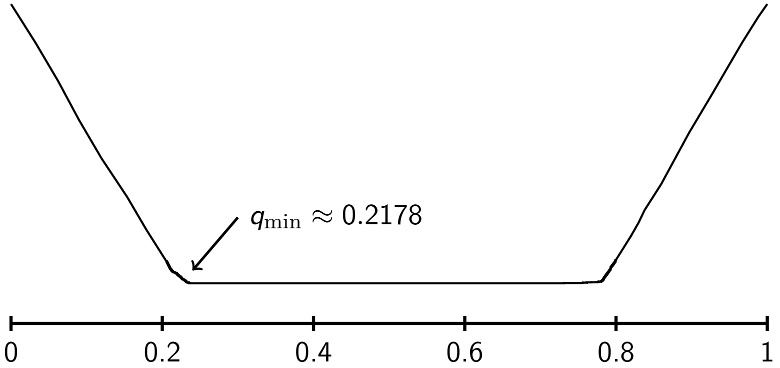
The profile after 100 steps.

A two-sided canyon model



The profile after 1000 steps.

A two-sided canyon model



The profile after 10,000 steps.

A two-sided canyon model

We find the same critical point $q_{\rm min}$ as for the Stigler-Luckock model.

In fact, the models are very similar:

- In the Stigler-Luckock model, interpret a buy limit order as an increment -1 and interpret a sell limit order as an increment $+1$.
- Assume that each trader places *both* a buy and sell limit order, at the (almost) same price, but with the sell order infinitesimally on the right of the buy order.

Then we obtain the canyon model.

The Bak Sneppen model

Introduced by Bak & Sneppen (1993).

Consider an ecosystem with N species. Each species has a fitness in $[0, 1]$.

In each step, the species $i \in \{1, \ldots, N\}$ with the lowest fitness dies out, together with its neighbors $i - 1$ and $i + 1$ (with periodic b.c.), and all three are replaced by species with new, i.i.d. uniformly distributed fitnesses.

There is a critical fitness $f_{\mathrm{c}} \approx 0.6672(2)$ such that when N is large, after sufficiently many steps, the fitnesses are approximately uniformly distributed on $(f_{\mathrm{c}}, 1]$ with only a few smaller fitnesses. Moreover, for each $\varepsilon > 0$, the lowest fitness spends a positive fraction of time above $f_{\mathrm{c}} - \varepsilon$, uniformly as $N \to \infty$.

The modified Bak Sneppen model

Introduced by Meester & Sarkar (2012).

Instead of the neighbors of the least fit species, choose one arbitrary other species from the population that dies together with the least fit species.

Critical point exactly $f_c = 1/2$.

Critical behavior at f_c: intervals between times when all individuals have a fitness $> f_c$ have a power-law distribution with $\mathbb{P}[\tau \geq k] \sim k^{-1/2}$.

Proof based on coupling to a branching process.

Self-organized criticality

All these models share some common features:

- Only the relative order of the limit orders, points of increase, priorities, or fitnesses matter. As a result, replacing the uniform distribution with any other atomless law basically yields the same model (up to a transformation of space).
- All models use some version of the rule "kill the minimal element".
- All models exhibit *self-organized criticality*.

Self-organized criticality

Physical systems with second order phase transitions exhibit *critical behavior* at the point of the phase transition, which is characterized by:

- Scale invariance.
- Power law decay of quantities.
- Critical exponents.

Usually, critical behavior is only observed when the parameter(s) of the system, such as the temperature, have just the right value so that we are at the point of the phase transition, also called (in this context) the *critical point*.

Self-organized criticality

Some physical systems show critical behavior even without the necessity to tune a parameter to exactly the right value.

In particular, this happens for systems whose dynamics find the critical point themselves. Such systems are said to exhibit *self-organized criticality*.

A classical example are sandpiles, which automatically find the maximal slope that is still stable. Adding a single grain to such a sandpile causes an avalanche whose size has a power-law distribution.

The Bak Sneppen model is another classical example of self-organized criticality and a cornerstone of Bak's (1996) book.

In the email model, the distribution of serving times (of answered emails) has a power-law tail. (As opposed to the more usual exponential tails in queueing theory.)