Necessary and sufficient conditions for strong R-positivity

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The Perron-Frobenius theorem

Let $A = (A(x,y))_{x,y \in S}$ be a nonnegative matrix indexed by a countable set S.
We say that A is *irreducible* if $\forall x, y \in S\ \exists n \geq 0$ s.t. $A^n(x,y) > 0$.
We say that A is *aperiodic* if the greatest common divisor of $\{n \geq 1 : A^n(x,x) > 0\}$ is one.

> **[Perron-Frobenius (1912)]** *Let A be irreducible and let S be finite. Then there exist a unique constant $c > 0$ and a function $h : S \to (0,\infty)$ that is unique up to scalar multiples, such that $Ah = ch$.*

How about infinite S?
Observation Let A be a nonnegative matrix, $c > 0$ a constant, and $h : S \to (0,\infty)$. Then the following conditions are equivalent.

1. $Ah = ch$.
2. $P(x,y) := c^{-1}h(x)^{-1}A(x,y)h(y)$ defines a probability kernel.

Gibbs measures

Let Ω^n denote the space of all functions $\omega : \{0, \dots, n\} \to S$. Let $\mu^{A,n}_{x,y}$ be the probability measure(!) on Ω^n defined by

$$\mu^{A,n}_{x,y}(\omega) := \frac{1_{\{\omega_0 = x,\ \omega_n = y\}}}{A^n(x,y)} \prod_{k=1}^{n} A(\omega_{k-1}, \omega_k).$$

We call this the *Gibbs measure* on Ω^n with *transfer matrix* A and *boundary conditions* x, y.

[Equivalence of transfer matrices] *Let A, B be irreducible with $A(x,y) > 0$ iff $B(x,y) > 0$. Then the following conditions are equivalent.*

1. $\mu^{A,n}_{x,y} = \mu^{B,n}_{x,y}$ *for all x, y, n such that $A^n(x,y) > 0$.*
2. *There exists a $c > 0$ and $h : S \to (0, \infty)$ such that $B(x,y) := c^{-1} h(x)^{-1} A(x,y) h(y)$.*

Moreover, in 2., the constant c is unique and h is unique up to a multiplicative constant.

The infinite-volume limit

Write $A \sim_c B$ or $A \sim B$ and call A, B *equivalent* if

$$B(x,y) := c^{-1} h(x)^{-1} A(x,y) h(y) \qquad (x, y \in S)$$

for some $c > 0$ and $h : S \to (0, \infty)$.

Perron-Frobenius A irreducible and S finite $\Rightarrow$ $\exists$ unique probability kernel P such that $A \sim P$.

> **[Infinite-volume limit]** *The Gibbs measures* $\mu^{A,n}_{x,y}$ *converge weakly as* $n \to \infty$ *to the law of the Markov chain with initial state* x *and transition kernel* P*.*

Note A and P determine c uniquely, and h uniquely up to a multiplicative constant.

R-recurrence

[David Vere-Jones 1962, 1967] Let A be a countable, irreducible, nonnegative matrix. Then there exists *at most one* recurrent probability kernel P such that $A \sim P$. Call such A *R-recurrent*.

Consequence If A is R-recurrent, then there exists a unique $c > 0$ and a $h : S \to (0, \infty)$ that is unique up to constant multiples, such that

$$P(x, y) := c^{-1} h(x)^{-1} A(x, y) h(y)$$

is a *recurrent* probability kernel.

Call A *R-transient* if A is not R-recurrent.
Call A *R-positive* if P is positive recurrent.
The result about the infinite-volume limit holds more generally for all R-positive A.

The spectral radius

[Kingman 1963] Let A be aperiodic and irreducible. Then the limit

$$\rho(A) := \lim_{n\to\infty} \big(A^n(x,y)\big)^{1/n}$$

exists and does not depend on $x, y \in S$. We call this the *spectral radius*.

- $A \sim_c B \quad \Rightarrow \quad \rho(A) = c\rho(B)$,
- P recurrent probability kernel $\quad \Rightarrow \rho(P) = 1$.

[David Vere-Jones 1962, 1967] A is R-recurrent if and only if

$$\sum_{n=1}^{\infty} \rho(A)^{-n} A^n(x,x) = \infty$$

for some, and hence for all $x \in S$.

The spectral radius

Proof of the necessity: If $A \sim_c P$ with P a recurrent probability kernel, then $c = \rho(A)$ and

$$\sum_{n=1}^{\infty} \rho(A)^{-n} A^n(x,x) = \sum_{n=1}^{\infty} P^n(x,x) = \infty.$$

Here we used $P^n(x,y) = c^{-1} h(x)^{-1} A^n(x,y) h(y)$ and in particular $P^n(x,x) = c^{-1} A^n(x,x)$. ∎

Similarly: A is R-positive if and only if

$$\lim_{n\to\infty} \rho(A)^{-n} A^n(x,x) > 0.$$

A generalization of Perron-Frobenius

[David Vere-Jones 1962, 1967] If A is irreducible and R-recurrent, then there exists a function $h : S \to (0, \infty)$, unique up to scalar multiples, such that $Ah = \rho(A)h$. If some function $f : S \to [0, \infty)$ satisfies $Af \leq \rho(A)f$, then f is a scalar multiple of h.

Consequence If $Af = cf$ for some $f : S \to (0, \infty)$, then $\rho(A) \leq c$. If $\rho(A) < c$, then $P(x, y) := c^{-1} f(x)^{-1} A(x, y) f(y)$ is a transient probability kernel.

Note 1 In the infinite dimensional case, $\rho(A)$ need not be the largest eigenvalue.

Note 2 The theory of R-recurrence is just one way to generalize the Perron-Frobenius theorem to infinite dimensions. A functional analytic generalization is the Krein-Rutman theorem (1948).

Geometric ergodicity

Let $(X_k)_{k\geq 0}$ be a Markov chain with irreducible transition kernel P. Let $\sigma_x := \inf\{k > 0 : X_k = x\}$ denote the first return time to x.

Def P is *strongly positive recurrent* if for some, and hence for all $x \in S$, there exists an $\varepsilon > 0$ s.t. $\mathbb{E}^x\big[e^{\varepsilon\sigma_x}\big] < \infty$.

[Kendall '59, Vere-Jones '62] Let P be irreducible and aperiodic with invariant law π. Then P is strongly positive recurrent if and only if it is *geometrically ergodic* in the sense that

$$\exists\varepsilon > 0,\ M_{x,y} < \infty \text{ s.t. } \big|P^n(x,y) - \pi(y)\big| \leq M_{x,y}e^{-\varepsilon n}.$$

$(x, y \in S,\ n \geq 0)$.

Def A is *strongly R-positive* if $A \sim P$ for some (necessarily unique) strongly positive recurrent P.

Conditions for R-recurrence

Problem: it is often not easy to check whether

$$\sum_{n=1}^{\infty} \rho(A)^{-n} A^n(x,x) = \infty.$$

In fact, not using the Perron-Frobenius theorem, it is not even clear how to prove this is always true for finite matrices.

Intuition: A should be R-recurrent if the Gibbs measures $\mu^{A,n}_{x,y}$ put, for large n, most of their mass on walks ω that stay close to x.

Intuition

Recall $\rho(A) := \lim_{n\to\infty} \big(A^n(x,y)\big)^{1/n}$.
The unnormalized measures

$$\nu^{A,n}_{x,y}(\omega) := 1_{\{\omega_0 = x,\ \omega_n = y\}} \prod_{k=1}^{n} A(\omega_{k-1}, \omega_k)$$

have total mass $A^n(x,y)$, which grows exponentially as

$$A^n(x,y) = e^{\,n \log \rho(A) + o(n)}.$$

We need to know if most of this mass lies on walks that stay close to x, or on paths that wander far away.

Local modifications

[Swart 2017] Let $A \leq B$ be irreducible with $\rho(B) < \infty$. Assume that $A(x, y) > 0$ whenever $B(x, y) > 0$ and that $\{(x, y) : A(x, y) < B(x, y)\}$ is finite and nonempty.

(a) B is strongly R-positive if and only if $\rho(A) < \rho(B)$.

(b) A is R-transient if and only if $\rho(A) = \rho\big(A + \varepsilon(B - A)\big)$ for some $\varepsilon > 0$.

Note: This implies in particular that finite matrices are strongly R-positive.

Pinning models

Let Q denote the transition kernel of nearest-neighbor random walk on $\mathbb{Z}^d$. Define

$$A_\beta(x,y) := \begin{cases} e^\beta Q(x,y) & \text{if } x = 0, \\ Q(x,y) & \text{otherwise.} \end{cases}$$

[Giacomin, Caravenna, Zambotti 2006] There exists a $-\infty < \beta_{\rm c} < \infty$ such that:

- A_β is R-transient for $\beta < \beta_{\rm c}$.
- A_β is R-null recurrent or weakly R-positive for $\beta = \beta_{\rm c}$.
- A_β is strongly R-positive for $\beta > \beta_{\rm c}$.

Moreover, $\beta \mapsto \rho(A_\beta)$ is constant on $(-\infty, \beta_{\rm c}]$ and stricty increasing on $[\beta_{\rm c}, \infty)$.
One has $\beta_{\rm c} = 0$ in dimensions $d = 1, 2$ and $\beta_{\rm c} > 0$ in dimensions $d \geq 3$. In fact, $e^{-\beta_{\rm c}}$ is the return probability of the random walk.

Bounds on the spectral radius

Sharp upper bounds on $\rho(A)$ can (in principle) be obtained from

$$\rho(A) = \inf\big\{K < \infty : \exists f : S \to (0,\infty) \text{ s.t. } Af \leq Kf\big\}.$$

Let (π, Q) be a pair such that 1. π is a probability measure on some finite $S' \subset S$, 2. Q is a transition kernel on S' with invariant law π. Define a large deviations *rate function*

$$I_A(\pi, Q) := \sum_{x,y} \pi(x) Q(x,y) \log\Big(\frac{Q(x,y)}{A(x,y)}\Big).$$

Sharp lower bounds on $\rho(A)$ can be obtained from

$$\rho(A) = \sup_{(\pi,Q)} e^{-I_A(\pi,Q)}.$$

Open problems

Define

$$\rho_\infty(A) := \inf\big\{\rho(B) : B \leq A \text{ a finite modification of } A\big\}.$$

Then A strongly R-positive $\Leftrightarrow \rho_\infty(A) < \rho(A)$.

Open problem Prove that $\rho_\infty(A^n) = \rho_\infty(A)^n$.

Open problem Develop the theory for semigroups $(A_t)_{t\geq 0}$ of nonnegative matrices.

Open problem Show that the contact process modulo translations is strongly R-positive for all $\lambda \neq \lambda_{\rm c}$.

[Sturm & Swart 2014] The contact process modulo translations is R-positive for all $\lambda < \lambda_{\rm c}$.

Proof of the main result

The most interesting part of the main theorem is:

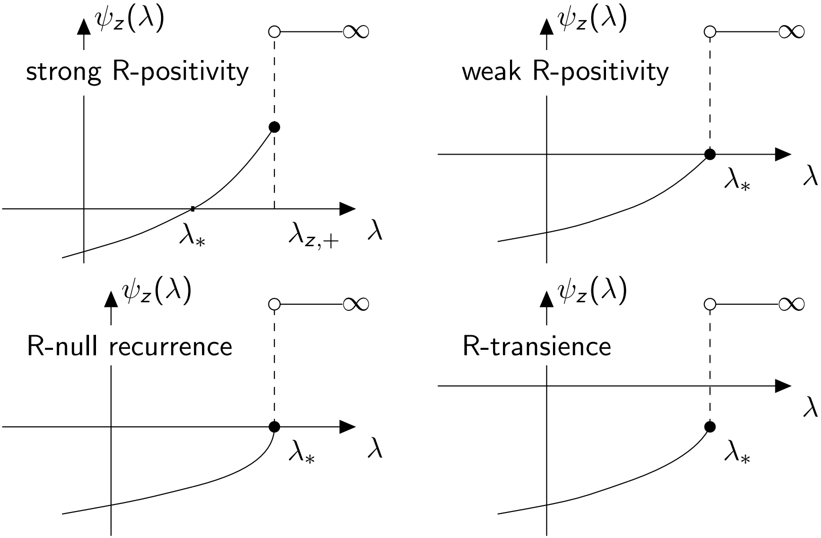
> *Let A be irreducible. If there exists a finite modification $B \leq A$ such that $\rho(B) < \rho(A)$, then $A \sim P$ for some strongly positive recurrent probability kernel P.*

Sketch of the proof Fix a reference point $z \in S$. Let $\widehat{\Omega}_z$ denote the space of all *excursions* away from z, i.e., functions $\omega : \{0, \ldots, n\} \to S$ with $\omega_0 = z = \omega_n$ and $\omega_k \neq z$ for all $0 < k < n$. Let $\ell_\omega := n$ denote the length of ω. Define

$$e^{\psi_z(\lambda)} := \sum_{\omega \in \widehat{\Omega}_z} \nu_\lambda(\omega) \quad \text{with} \quad \nu_\lambda(\omega) := e^{\lambda \ell_\omega} \prod_{k=1}^{\ell_\omega} A(\omega_{k-1}, \omega_k)$$

STEP I: If there exists some λ such that $\psi_z(\lambda) = 0$, then the process that makes i.i.d. excursions away from z with law ν_λ is a recurrent Markov chain with transition kernel $P \sim A$.

The logarithmic moment generating function



Sketch of the proof (continued)

Let G be the directed graph with vertex set S and edge set $\{(x,y) : A(x,y) > 0\}$. For any subgraph $F \subset G$ and vertices $x, y \in F$, define $\psi^F_{x,y}(\lambda) = \log \phi^F_{x,y}(\lambda)$ with

$$\phi^F_{x,y}(\lambda) := \sum_{\omega \in \widehat{\Omega}_{x,y}(F)} e^{\lambda \ell_\omega} \prod_{k=1}^{\ell_\omega} A(\omega_{k-1}, \omega_k),$$

where $\widehat{\Omega}_{x,y}(F)$ denotes the space of excursions away from F starting in x and ending in y.

Removal of an edge Let $F' = F \backslash \{e\}$ be obtained from F by the removal of an edge e. Then

$$\phi^{F'}_{x,y}(\lambda) = \begin{cases} \phi^F_{x,y}(\lambda) + e^\lambda A(x,y) & \text{if } e = (x,y), \\ \phi^F_{x,y}(\lambda) & \text{otherwise} \end{cases} \qquad (\lambda \in \mathbb{R}).$$

Sketch of the proof

Removal of an isolated vertex Let $F' = F \backslash \{z\}$ be obtained from F by the removal of an isolated vertex z. Then

$$\phi^{F'}_{x,y}(\lambda) = \phi^{F}_{x,y}(\lambda) + \sum_{k=0}^{\infty} \phi^{F}_{x,z}(\lambda)\phi^{F}_{z,z}(\lambda)^k \phi^{F}_{z,y}(\lambda).$$

Proof: Set $\mathcal{A}(\omega) := \prod_{k=1}^{\ell_\omega} A(\omega_{k-1}, \omega_k)$. Distinguishing excursions away from F' according to how often they visit the vertex z, we have

$$\begin{aligned}\phi^{F'}_{x,y}(\lambda) &= \sum_{\omega_{x,y}} e^{\lambda \ell_{\omega_{x,y}}} \mathcal{A}(\omega_{x,y}) \\ &+ \sum_{k=0}^{\infty} \sum_{\omega_{x,z}} \sum_{\omega_{z,y}} \sum_{\omega^1_{z,z}} \cdots \sum_{\omega^k_{z,z}} e^{\lambda(\ell_{\omega_{x,z}} + \ell_{\omega_{z,y}} + \ell_{\omega^1_{z,z}} + \cdots + \ell_{\omega^k_{z,z}})} \\ &\qquad \times \mathcal{A}(\omega_{x,z}) \mathcal{A}(\omega_{z,y}) \mathcal{A}(\omega^1_{z,z}) \cdots \mathcal{A}(\omega^k_{z,z}),\end{aligned}$$

where we sum over $\omega_{x,y} \in \widehat{\Omega}_{x,y}(F)$ etc.

Sketch of the proof

Rewriting gives

$$\phi^{F'}_{x,y}(\lambda) = \sum_{\omega_{x,y}} e^{\lambda \ell_{\omega_{x,y}}} \mathcal{A}(\omega_{x,y}) +$$

$$\Big(\sum_{\omega_{x,z}} e^{\lambda \ell_{\omega_{x,z}}} \mathcal{A}(\omega_{x,z})\Big)\Big(\sum_{\omega_{z,y}} e^{\lambda \ell_{\omega_{z,y}}} \mathcal{A}(\omega_{z,y})\Big) \sum_{k=0}^{\infty} \Big(\sum_{\omega_{z,z}} e^{\lambda \ell_{\omega_{z,z}}} \mathcal{A}(\omega_{z,z})\Big)^k.$$

Sketch of the proof

Lemma (Exponential moments of excursions) Let P be an irreducible subprobability kernel. Set

$$\lambda^F_{x,y} := \sup\{\lambda : \psi^F_{x,y}(\lambda) < \infty\}.$$

Then, if

$$\lambda^F_{x,y,+} > 0 \text{ for all } x, y \in F \cap S$$

holds for some finite nonempty subgraph F of G, it holds for all such subgraphs.

Proof: By induction, removing edges and isolated vertices.

Sketch of the proof

Lemma Set $\lambda_* := \sup\{\lambda : \psi_z(\lambda) = 0\}$. Then $\lambda_* = -\log \rho(A)$.

Proof of the theorem: Assume that A is not strongly positive recurrent. Let $A' \leq A$ be a finite modification. We must show that $\rho(A') = \rho(A)$. By a similarity transformation, we may assume w.l.o.g. that A is a subprobability kernel and $\lambda_* = 0$. We need to show $\lambda'_* = 0$. It suffices to show that for the subgraph $F = \{z\}$, we have $\lambda'_{z,+} = 0$. Since A is not strongly positive, we have $\lambda_{z,+} = 0$. Since B is a finite modification, we can choose a finite subgraph F such that $\lambda^F_{x,y,+}$ is the same for A and A'. Now

$$\lambda_{z,+} \leq 0 \quad \Leftrightarrow \quad \lambda^F_{x,y,+} \leq 0 \text{ for some } x, y \in F \quad \Leftrightarrow \quad \lambda'_{z,+} \leq 0.$$

$\blacksquare$