Necessary and sufficient conditions for strong R-positivity

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Let $A = (A(x, y))_{x, y \in S}$ be a nonnegative matrix indexed by a countable set $S$.
We say that $A$ is *irreducible* if $\forall x, y \in S \ \exists n \geq 0$ s.t. $A^n(x, y) > 0$.
We say that $A$ is *aperiodic* if the greatest common divisor of $\{n \geq 1 : A^n(x, x) > 0\}$ is one.

**[Perron-Frobenius (1912)]** Let $A$ be irreducible and let $S$ be finite. Then there exist a unique constant $c > 0$ and a function $h : S \rightarrow (0, \infty)$ that is unique up to scalar multiples, such that $Ah = ch$.

How about infinite $S$?

**Observation** Let $A$ be a nonnegative matrix, $c > 0$ a constant, and $h : S \rightarrow (0, \infty)$. Then the following conditions are equivalent.

1. $Ah = ch$.
2. $P(x, y) := c^{-1}h(x)^{-1}A(x, y)h(y)$ defines a probability kernel.
Let $\Omega^n$ denote the space of all functions $\omega : \{0, \ldots, n\} \rightarrow S$. Let $\mu_{x,y}^{A,n}$ be the probability measure(!) on $\Omega^n$ defined by

$$
\mu_{x,y}^{A,n}(\omega) := \frac{1_{\{\omega_0=x, \omega_n=y\}}}{A^n(x, y)} \prod_{k=1}^{n} A(\omega_{k-1}, \omega_{k}).
$$

We call this the **Gibbs measure** on $\Omega^n$ with **transfer matrix** $A$ and **boundary conditions** $x, y$.

**[Equivalence of transfer matrices]** Let $A, B$ be irreducible with $A(x, y) > 0$ iff $B(x, y) > 0$. Then the following conditions are equivalent.

1. $\mu_{x,y}^{A,n} = \mu_{x,y}^{B,n}$ for all $x, y, n$ such that $A^n(x, y) > 0$.
2. There exists a $c > 0$ and $h : S \rightarrow (0, \infty)$ such that $B(x, y) := c^{-1} h(x)^{-1} A(x, y) h(y)$.

Moreover, in 2., the constant $c$ is unique and $h$ is unique up to a multiplicative constant.
Write $A \sim_c B$ or $A \sim B$ and call $A, B$ equivalent if

$$B(x, y) := c^{-1}h(x)^{-1}A(x, y)h(y) \quad (x, y \in S)$$

for some $c > 0$ and $h : S \to (0, \infty)$.

**Perron-Frobenius** $A$ irreducible and $S$ finite $\Rightarrow \exists$ unique probability kernel $P$ such that $A \sim P$.

**[Infinite-volume limit]** The Gibbs measures $\mu_{x,y}^{A,n}$ converge weakly as $n \to \infty$ to the law of the Markov chain with initial state $x$ and transition kernel $P$.

**Note** $A$ and $P$ determine $c$ uniquely, and $h$ uniquely up to a multiplicative constant.
[David Vere-Jones 1962, 1967] Let $A$ be a countable, irreducible, nonnegative matrix. Then there exists at most one recurrent probability kernel $P$ such that $A \sim P$. Call such $A$ R-recurrent.

**Consequence** If $A$ is R-recurrent, then there exists a unique $c > 0$ and a $h : S \to (0, \infty)$ that is unique up to constant multiples, such that

$$P(x, y) := c^{-1} h(x)^{-1} A(x, y) h(y)$$

is a recurrent probability kernel.

Call $A$ R-transient if $A$ is not R-recurrent.

Call $A$ R-positive if $P$ is positive recurrent.

The result about the infinite-volume limit holds more generally for all R-positive $A$. 

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The spectral radius

[Kingman 1963] Let $A$ be aperiodic and irreducible. Then the limit

$$
\rho(A) := \lim_{{n \to \infty}} (A^n(x, y))^{1/n}
$$

exists and does not depend on $x, y \in S$. We call this the spectral radius.

- $A \sim_c B \Rightarrow \rho(A) = c\rho(B)$,
- $P$ recurrent probability kernel $\Rightarrow \rho(P) = 1$.

[David Vere-Jones 1962, 1967] $A$ is $R$-recurrent if and only if

$$
\sum_{{n=1}}^{\infty} \rho(A)^{-n}A^n(x, x) = \infty
$$

for some, and hence for all $x \in S$. 
**Proof of the necessity:** If $A \sim_c P$ with $P$ a recurrent probability kernel, then $c = \rho(A)$ and

$$\sum_{n=1}^{\infty} \rho(A)^{-n} A^n(x, x) = \sum_{n=1}^{\infty} P^n(x, x) = \infty.$$ 

Here we used $P^n(x, y) = c^{-1} h(x)^{-1} A^n(x, y) h(y)$ and in particular $P^n(x, x) = c^{-1} A^n(x, x).$

**Similarly:** $A$ is R-positive if and only if

$$\lim_{n \to \infty} \rho(A)^{-n} A^n(x, x) > 0.$$
[David Vere-Jones 1962, 1967] If $A$ is irreducible and R-recurrent, then there exists a function $h : S \rightarrow (0, \infty)$, unique up to scalar multiples, such that $Ah = \rho(A)h$. If some function $f : S \rightarrow [0, \infty)$ satisfies $Af \leq \rho(A)f$, then $f$ is a scalar multiple of $h$.

**Consequence** If $Af = cf$ for some $f : S \rightarrow (0, \infty)$, then $\rho(A) \leq c$. If $\rho(A) < c$, then $P(x, y) := c^{-1}f(x)^{-1}A(x, y)f(y)$ is a transient probability kernel.

*Note 1* In the infinite dimensional case, $\rho(A)$ need not be the largest eigenvalue.

*Note 2* The theory of R-recurrence is just one way to generalize the Perron-Frobenius theorem to infinite dimensions. A functional analytic generalization is the Krein-Rutman theorem (1948).
Geometric ergodicity

Let \((X_k)_{k \geq 0}\) be a Markov chain with irreducible transition kernel \(P\). Let \(\sigma_x := \inf\{ k > 0 : X_k = x \}\) denote the first return time to \(x\).

**Def** \(P\) is **strongly positive recurrent** if for some, and hence for all \(x \in S\), there exists an \(\varepsilon > 0\) s.t. \(\mathbb{E}^x[e^{\varepsilon\sigma_x}] < \infty\).

**[Kendall '59, Vere-Jones '62]** Let \(P\) be irreducible and aperiodic with invariant law \(\pi\). Then \(P\) is strongly positive recurrent if and only if it is **geometrically ergodic** in the sense that

\[
\exists \varepsilon > 0, \; M_{x,y} < \infty \text{ s.t. } |P^n(x,y) - \pi(y)| \leq M_{x,y} e^{-\varepsilon n}.
\]

\((x,y \in S, \; n \geq 0)\).

**Def** \(A\) is **strongly R-positive** if \(A \sim P\) for some (necessarily unique) strongly positive recurrent \(P\).
Problem: it is often not easy to check whether

\[ \sum_{n=1}^{\infty} \rho(A)^{-n} A^n(x, x) = \infty. \]

In fact, not using the Perron-Frobenius theorem, it is not even clear how to prove this is always true for finite matrices.

*Intuition:* A should be R-recurrent if the Gibbs measures \( \mu_{x,y}^{A,n} \) put, for large \( n \), most of their mass on walks \( \omega \) that stay close to \( x \).
Recall $\rho(A) := \lim_{n \to \infty} (A^n(x, y))^{1/n}$.

The unnormalized measures

$$\nu_{x,y}^{A,n}(\omega) := 1\{\omega_0=x, \omega_n=y\} \prod_{k=1}^{n} A(\omega_{k-1}, \omega_k)$$

have total mass $A^n(x, y)$, which grows exponentially as

$$A^n(x, y) = e^{n \log \rho(A) + o(n)}.$$

We need to know if most of this mass lies on walks that stay close to $x$, or on paths that wander far away.
[Swart 2017] Let $A \leq B$ be irreducible with $\rho(B) < \infty$. Assume that $A(x, y) > 0$ whenever $B(x, y) > 0$ and that
$\{(x, y) : A(x, y) < B(x, y)\}$ is finite and nonempty.

(a) $B$ is strongly R-positive if and only if $\rho(A) < \rho(B)$.

(b) $A$ is R-transient if and only if $\rho(A) = \rho(A + \varepsilon(B - A))$ for some $\varepsilon > 0$.

Note: This implies in particular that finite matrices are strongly R-positive.
Pinning models

Let $Q$ denote the transition kernel of nearest-neighbor random walk on $\mathbb{Z}^d$. Define

$$A_\beta(x, y) := \begin{cases} e^{\beta} Q(x, y) & \text{if } x = 0, \\ Q(x, y) & \text{otherwise.} \end{cases}$$

**[Giacomin, Caravenna, Zambotti 2006]** There exists a $-\infty < \beta_c < \infty$ such that:

- $A_\beta$ is R-transient for $\beta < \beta_c$.
- $A_\beta$ is R-null recurrent or weakly R-positive for $\beta = \beta_c$.
- $A_\beta$ is strongly R-positive for $\beta > \beta_c$.

Moreover, $\beta \mapsto \rho(A_\beta)$ is constant on $(-\infty, \beta_c]$ and strictly increasing on $[\beta_c, \infty)$.

One has $\beta_c = 0$ in dimensions $d = 1, 2$ and $\beta_c > 0$ in dimensions $d \geq 3$. In fact, $e^{-\beta_c}$ is the return probability of the random walk.
Bounds on the spectral radius

Sharp upper bounds on $\rho(A)$ can (in principle) be obtained from

$$\rho(A) = \inf \{ K < \infty : \exists f : S \to (0, \infty) \text{ s.t. } Af \leq Kf \}.$$

Let $(\pi, Q)$ be a pair such that 1. $\pi$ is a probability measure on some finite $S' \subset S$, 2. $Q$ is a transition kernel on $S'$ with invariant law $\pi$. Define a large deviations rate function

$$I_A(\pi, Q) := \sum_{x,y} \pi(x) Q(x, y) \log \left( \frac{Q(x, y)}{A(x, y)} \right).$$

Sharp lower bounds on $\rho(A)$ can be obtained from

$$\rho(A) = \sup_{(\pi, Q)} e^{-I_A(\pi, Q)}.$$
Open problems

Define

$$
\rho_\infty(A) := \inf \{ \rho(B) : B \leq A \text{ a finite modification of } A \}.
$$

Then $A$ strongly R-positive $\iff \rho_\infty(A) < \rho(A)$.

**Open problem** Prove that $\rho_\infty(A^n) = \rho_\infty(A)^n$.

**Open problem** Develop the theory for semigroups $(A_t)_{t \geq 0}$ of nonnegative matrices.

**Open problem** Show that the contact process modulo translations is strongly R-positive for all $\lambda \neq \lambda_c$.

**[Sturm & Swart 2014]** The contact process modulo translations is R-positive for all $\lambda < \lambda_c$. 

Jan M. Swart (ÚTIA AV ČR) Strong R-positivity
Proof of the main result

The most interesting part of the main theorem is:

Let $A$ be irreducible. If there exists a finite modification $B \leq A$ such that $\rho(B) < \rho(A)$, then $A \sim P$ for some strongly positive recurrent probability kernel $P$.

Sketch of the proof Fix a reference point $z \in S$. Let $\hat{\Omega}_z$ denote the space of all excursions away from $z$, i.e., functions $\omega : \{0, \ldots, n\} \to S$ with $\omega_0 = z = \omega_n$ and $\omega_k \neq z$ for all $0 < k < n$. Let $\ell_\omega := n$ denote the length of $\omega$. Define

$$e^{\psi_z(\lambda)} := \sum_{\omega \in \hat{\Omega}_z} \nu_\lambda(\omega) \quad \text{with} \quad \nu_\lambda(\omega) := e^{\lambda \ell_\omega} \prod_{k=1}^{\ell_\omega} A(\omega_{k-1}, \omega_k)$$

STEP I: If there exists some $\lambda$ such that $\psi_z(\lambda) = 0$, then the process that makes i.i.d. excursions away from $z$ with law $\nu_\lambda$ is a recurrent Markov chain with transition kernel $P \sim A$. 
The logarithmic moment generating function

\[ \psi_z(\lambda) \]

strong R-positivity

weak R-positivity

R-null recurrence

R-transience

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Let $G$ be the directed graph with vertex set $S$ and edge set
$\{(x, y) : A(x, y) > 0\}$. For any subgraph $F \subset G$ and vertices $x, y \in F$, define $\psi^F_{x,y}(\lambda) = \log \phi^F_{x,y}(\lambda)$ with

$$
\phi^F_{x,y}(\lambda) := \sum_{\omega \in \tilde{\Omega}_{x,y}(F)} e^{\lambda \ell_\omega} \prod_{k=1}^{\ell_\omega} A(\omega_{k-1}, \omega_k),
$$

where $\tilde{\Omega}_{x,y}(F)$ denotes the space of excursions away from $F$ starting in $x$ and ending in $y$.

**Removal of an edge** Let $F' = F \setminus \{e\}$ be obtained from $F$ by the removal of an edge $e$. Then

$$
\phi^{F'}_{x,y}(\lambda) = \begin{cases} 
\phi^F_{x,y}(\lambda) + e^{\lambda} A(x, y) & \text{if } e = (x, y), \\
\phi^F_{x,y}(\lambda) & \text{otherwise}
\end{cases} (\lambda \in \mathbb{R}).
$$
Sketch of the proof

**Removal of an isolated vertex** Let $F' = F \setminus \{z\}$ be obtained from $F$ by the removal of an isolated vertex $z$. Then

$$
\phi_{x,y}^{F'}(\lambda) = \phi_{x,y}^{F}(\lambda) + \sum_{k=0}^{\infty} \phi_{x,z}^{F}(\lambda) \phi_{z,z}^{F}(\lambda)^k \phi_{z,y}^{F}(\lambda).
$$

**Proof:** Set $A(\omega) := \prod_{k=1}^{\ell} A(\omega_{k-1}, \omega_k)$. Distinguishing excursions away from $F'$ according to how often they visit the vertex $z$, we have

$$
\phi_{x,y}^{F'}(\lambda) = \sum_{\omega_{x,y}} e^{\lambda \ell_{x,y}} A(\omega_{x,y})
$$

$$
+ \sum_{k=0}^{\infty} \sum_{\omega_{x,z}} \sum_{\omega_{z,y}} \sum_{\omega_{1,z,z}} \cdots \sum_{\omega_{k,z,z}} e^{\lambda (\ell_{x,z} + \ell_{z,y} + \ell_{1,z,z} + \cdots + \ell_{k,z,z})}
$$

$$
\times A(\omega_{x,z}) A(\omega_{z,y}) A(\omega_{1,z,z}) \cdots A(\omega_{k,z,z}),
$$

where we sum over $\omega_{x,y} \in \hat{\Omega}_{x,y}(F)$ etc.
Sketch of the proof

Rewriting gives

\[ \phi_{x,y}^F(\lambda) = \sum_{\omega_{x,y}} e^{\lambda \ell_{\omega_{x,y}}} A(\omega_{x,y}) + \]

\[ (\sum_{\omega_{x,z}} e^{\lambda \ell_{\omega_{x,z}}} A(\omega_{x,z})) (\sum_{\omega_{z,y}} e^{\lambda \ell_{\omega_{z,y}}} A(\omega_{z,y})) \sum_{k=0}^{\infty} (\sum_{\omega_{z,z}} e^{\lambda \ell_{\omega_{z,z}}} A(\omega_{z,z}))^k. \]
Lemma (Exponential moments of excursions) Let $P$ be an irreducible subprobability kernel. Set

$$\lambda_{x,y}^F := \sup\{\lambda : \psi_{x,y}^F(\lambda) < \infty\}.$$ 

Then, if

$$\lambda_{x,y,+}^F > 0 \text{ for all } x, y \in F \cap S$$

holds for some finite nonempty subgraph $F$ of $G$, it holds for all such subgraphs.

Proof: By induction, removing edges and isolated vertices.
**Lemma** Set $\lambda_* := \sup\{\lambda : \psi_z(\lambda) = 0\}$. Then $\lambda_* = - \log \rho(A)$.

**Proof of the theorem:** Assume that $A$ is not strongly positive recurrent. Let $A' \leq A$ be a finite modification. We must show that $\rho(A') = \rho(A)$. By a similarity transformation, we may assume w.l.o.g. that $A$ is a subprobability kernel and $\lambda_* = 0$. We need to show $\lambda'_* = 0$. It suffices to show that for the subgraph $F = \{z\}$, we have $\lambda'_{z,+} = 0$. Since $A$ is not strongly positive, we have $\lambda_{z,+} = 0$. Since $B$ is a finite modification, we can choose a finite subgraph $F$ such that $\lambda^F_{x,y,+}$ is the same for $A$ and $A'$. Now

$$\lambda_{z,+} \leq 0 \iff \lambda^F_{x,y,+} \leq 0 \text{ for some } x, y \in F \iff \lambda'_{z,+} \leq 0.$$