Recursive tree processes and the mean-field limit of stochastic flows

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Mean-field particle systems

Basic ingredients

(i) Polish space S *local state space.*

(ii) $(\Omega, \mathcal{B}, \mathbf{r})$ Polish space with Borel σ-field and finite measure: *source of external randomness.*

(iii) $\kappa : \Omega \to \mathbb{N}$ measurable function.

(iv) For each $\omega \in \Omega$, a measurable function $\gamma[\omega] : S^{\kappa(\omega)} \to S$.

Def $\mathcal{P}(S) :=$ the space of probability measures on S.
Def $\mathbf{T} : \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$\mathbf{T}(\mu) := \text{ the law of } \gamma[\boldsymbol{\omega}](X_1, \ldots, X_{\kappa(\boldsymbol{\omega})}),$$

where $\boldsymbol{\omega}$ is an Ω-valued random variable with law $|\mathbf{r}|^{-1}\mathbf{r}$ and $(X_i)_{i \geq 1}$ are i.i.d. with law μ. We are interested in *mean-field equations* of the form

$$\frac{\partial}{\partial t}\mu_t = |\mathbf{r}|\big\{\mathbf{T}(\mu_t) - \mu_t\big\} \qquad (t \geq 0). \tag{1}$$

Cooperative branching

Let $G = (V, E)$ be a graph.
Let $X = (X_t)_{t \geq 0}$ be a Markov process with state space $\{0, 1\}^V$ such that $X_t(i)$ jumps:

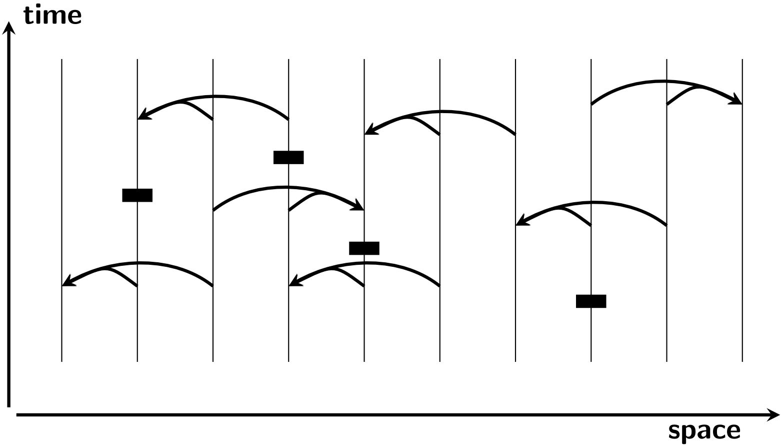
- (*cooperative branching*) $0 \mapsto 1$ with rate $\alpha \times$ the fraction of pairs j, k with $i \sim j \sim k$ such that $X_t(j) = X_t(k) = 1$.
- (*death*) $1 \mapsto 0$ with rate 1.

In this example, $S = \{0, 1\}$ and the dynamics of X can be described in terms of the maps:

$$\begin{aligned} \texttt{cob} : S^3 \to S \quad &\text{with} \quad \texttt{cob}(x_1, x_2, x_3) := x_1 \vee (x_2 \wedge x_3), \\ \texttt{dth} : S^0 \to S \quad &\text{with} \quad \texttt{dth}(\varnothing) := 0. \end{aligned}$$

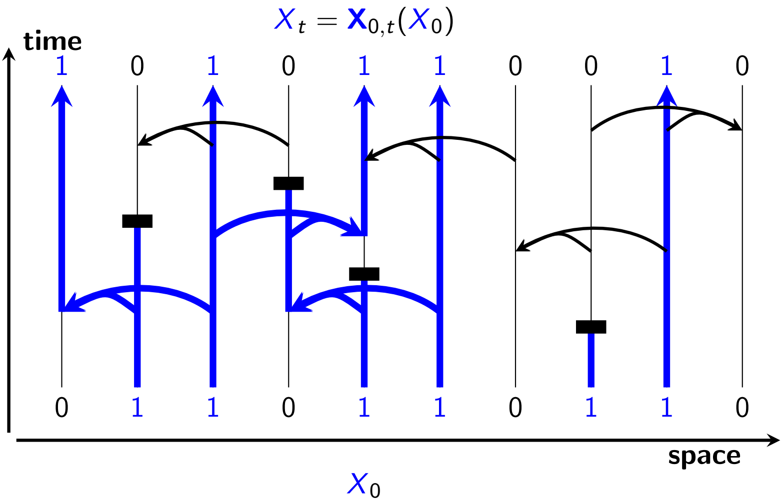
For each $i \sim j \sim k$, with Poisson rate α, we replace $X_t(i)$ by $\texttt{cob}\big(X_t(i), X_t(j), X_t(k)\big)$.
For each i, with Poisson rate 1, we replace $X_t(i)$ by 0.

A graphical representation



We denote `cob` and `dth` by suitable symbols.

A graphical representation



The Poisson events define a random map $x \mapsto \mathbf{X}_{0,t}(x)$.

A stochastic flow

The random maps $(\mathbf{X}_{s,t})_{s\leq t}$ form a *stochastic flow*

$$\mathbf{X}_{s,s} = 1 \quad \text{and} \quad \mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u}$$

with *independent increments*, in the sense that

$$\mathbf{X}_{t_0,t_1}, \ldots, \mathbf{X}_{t_{n-1},t_n}$$

are independent for each $t_0 < \cdots < t_n$.

If X_0 is independent of $(\mathbf{X}_{s,t})_{s\leq t}$, then setting

$$X_t := \mathbf{X}_{0,t}(X_0) \qquad (t \geq 0)$$

defines a Markov process $(X_t)_{t\geq 0}$ with the right jump rates.

Different stochastic flows can define the same Markov process, as there may be many different ways of expressing given jump rates in terms of random maps.

The mean-field limit

We are interested in the process on the *complete graph* with N vertices. Thus, for each site $i \in \{1, \ldots, N\}$,

- With rate α, we pick two other sites j, k at random and replace $X_t(i)$ by $\texttt{cob}\big(X_t(i), X_t(j), X_t(k)\big)$.
- With rate 1, we replace $X_t(i)$ by 0.

Let $\mu_t^N := \sum_{i=1}^N \delta_{X_t(i)}$ denote the empirical measure.
In the limit $N \to \infty$, the empirical measure solves

$$\frac{\partial}{\partial t}\mu_t = |\mathbf{r}|\big\{\mathbf{T}(\mu_t) - \mu_t\big\} \qquad (t \geq 0)$$

where the objects of our general setting are $S = \{0, 1\}$, $\Omega = \{1, 2\}$,

$$\begin{aligned} \gamma[1] &= \texttt{cob} : S^3 \to S, &\quad \kappa(1) &= 3, &\quad \mathbf{r}(\{1\}) &= \alpha, \\ \gamma[2] &= \texttt{dth} : S^0 \to S, &\quad \kappa(2) &= 0, &\quad \mathbf{r}(\{2\}) &= 1. \end{aligned}$$

The mean-field limit

For any deterministic map $g : S^k \to S$, let us write

$$\mathbf{T}_g(\mu) := \text{ the law of } g(X_1, \ldots, X_{\kappa(\omega)}),$$

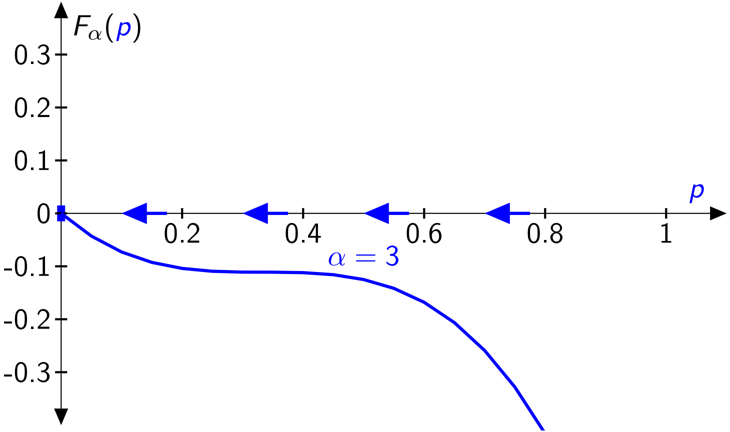
where $(X_i)_{i \geq 1}$ are i.i.d. with law μ. Then we can rewrite the mean-field equation as

$$\tfrac{\partial}{\partial t}\mu_t = \alpha\big\{\mathbf{T}_{\mathtt{cob}}(\mu_t) - \mu_t\big\} + \big\{\mathbf{T}_{\mathtt{dth}}(\mu_t) - \mu_t\big\}.$$

Rewriting this in terms of $p_t := \mu_t(\{1\})$ yields

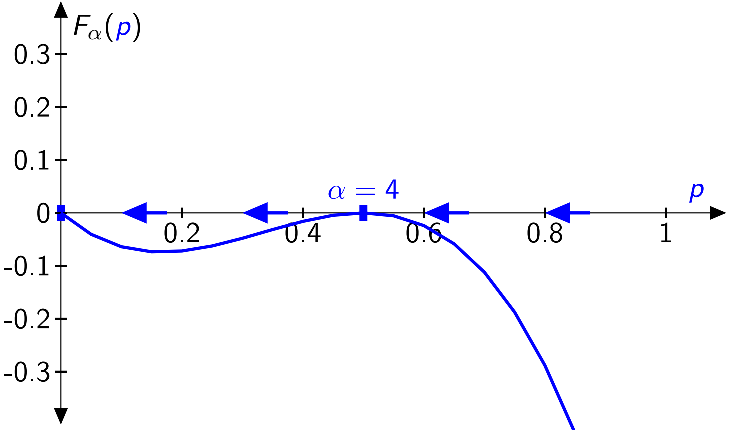
$$\tfrac{\partial}{\partial t}p_t = \alpha p_t^2(1 - p_t) - p_t =: F_\alpha(p_t) \qquad (t \geq 0).$$

Cooperative branching



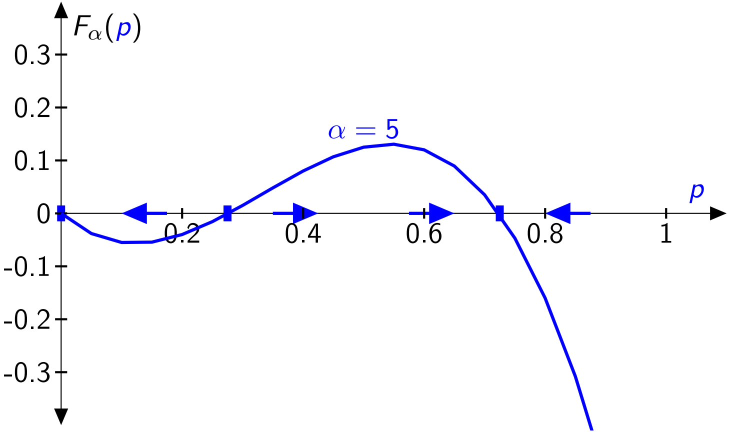
For $\alpha < 4$, the equation $\frac{\partial}{\partial t} p_t = F_\alpha(p_t)$ has a single, stable fixed point $p = 0$.

Cooperative branching



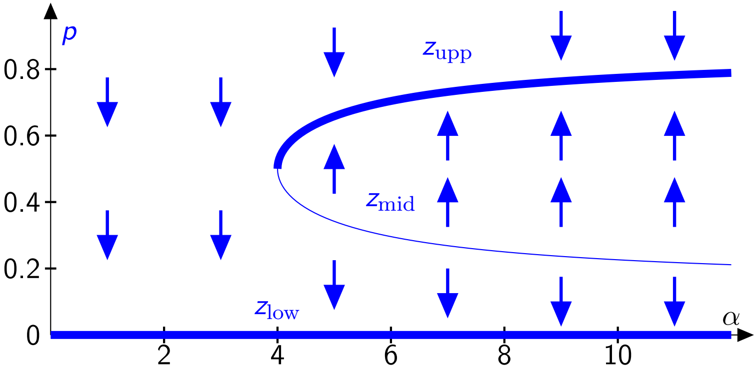
For $\alpha = 4$, a second fixed point appears at $p = 0.5$.

Cooperative branching



For $\alpha > 4$, there are two stable fixed points and one unstable fixed point, which separates the domains of attraction of the other two.

Cooperative branching



Fixed points of $\frac{\partial}{\partial t} p_t = F_\alpha(p_t)$ for different values of α.

Theorem [Mach, Sturm, S. '18] Assume that

$$\int_\Omega \mathbf{r}(\mathrm{d}\omega)\,\kappa(\omega) < \infty \tag{2}$$

Then for each initial state, the mean-field equation (1) has a unique solution.

Define a (nonlinear) semigroup $(\mathbf{T}_t)_{t\geq 0}$ of operators acting on probability measures by

$$\mathbf{T}_t(\mu) := \mu_t \quad \text{where } (\mu_t)_{t\geq 0} \text{ solves (1) with } \mu_0 = \mu.$$

Proposition [Mach, Sturm, S. '18] Assume that $\forall k,\ x \in S^k$

$$\mathbf{r}\big(\{\omega : \kappa(\omega) = k,\ \gamma[\omega] \text{ is discontinuous at x}\}\big) = 0. \tag{3}$$

Then the operators $\mathbf{T}_t$ are continuous w.r.t. weak convergence.

The mean-field equation

Let $\mu_t^N := \sum_{i=1}^N \delta_{X_t(i)}$ denote the empirical measure.
Let d be any metric that generates the topology of weak convergence and let $\|\cdot\|$ denote the total variation norm.

Theorem [Mach, Sturm, S. '18] Assume (2) and at least one of the following conditions:

(i) $\mathbb{P}\big[d(\mu_0^N, \mu_0) \geq \varepsilon\big] \underset{N\to\infty}{\longrightarrow} 0$ for all $\varepsilon > 0$, and (3) holds.

(ii) $\big\|\mathbb{E}[(\mu_0^N)^{\otimes n}] - \mu_0^{\otimes n}\big\| \underset{N\to\infty}{\longrightarrow} 0$ for all $n \geq 1$.

Then

$$\mathbb{P}\Big[\sup_{0\leq t\leq T} d\big(\mu_t^N, \mathbf{T}_t(\mu_0)\big) \geq \varepsilon\Big] \underset{N\to\infty}{\longrightarrow} 0 \qquad (\varepsilon > 0,\ T < \infty).$$

A recursive tree representation

Note that $\gamma[\omega] : S^{\kappa(\omega)} \to S$ is a random map. We call

$$\mathbf{T}(\mu) := \text{ the law of } \gamma[\omega](X_1, \ldots, X_{\kappa(\omega)}),$$

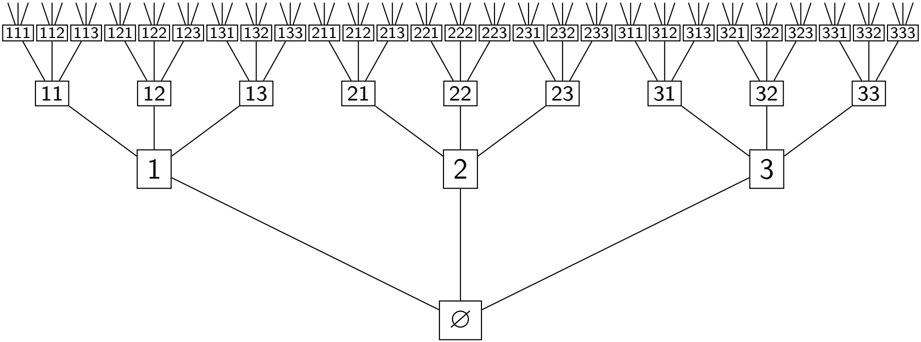
a *random mapping representation* of the operator $\mathbf{T}$.

Our aim is to find a similar random mapping representation for the operators $(\mathbf{T}_t)_{t \geq 0}$.

In a discrete-time setting, something similar has been done by Aldous and Bandyopadyay (2005) for iterates $\mathbf{T}^n$ of the map $\mathbf{T}$.

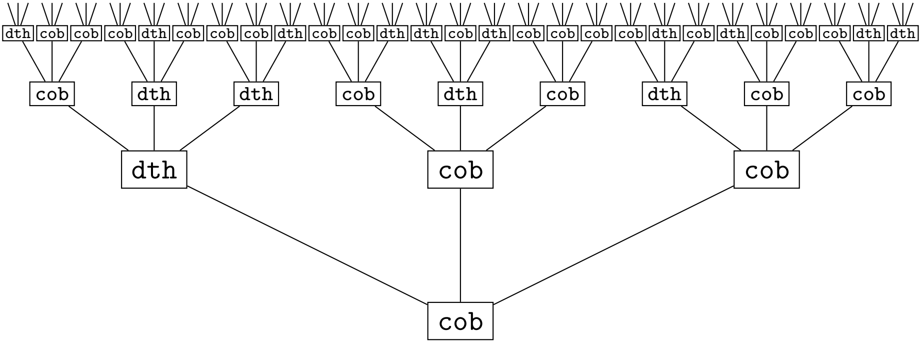
In what follows, we fix $d \in \mathbb{N}_+ \cup \{\infty\}$ such that $\kappa(\omega) \leq d$ for all $\omega \in \Omega$. We let $\mathbb{T}^d$ denote the space of all words $\mathbf{i} = i_1 \cdots i_n$ made from the alphabet $\{1, \ldots, d\}$ (if $d < \infty$) resp. $\mathbb{N}_+$ (if $d = \infty$).

A recursive tree representation



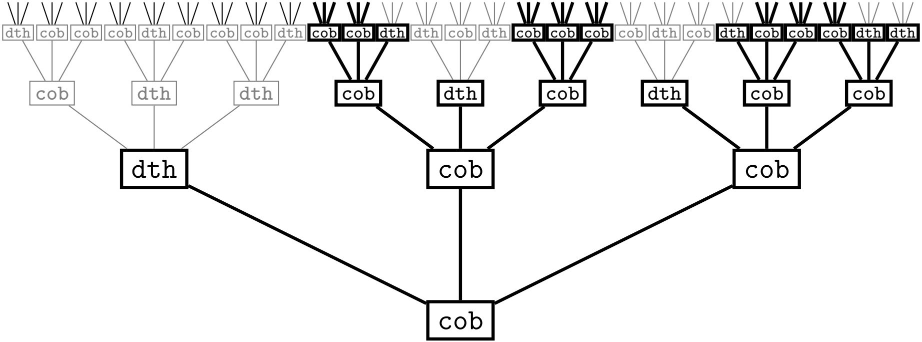
We view $\mathbb{T}^d$ as a tree with root $\varnothing$, the word of length zero.

A recursive tree representation



We attach i.i.d. $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ with law $|\mathbf{r}|^{-1}\mathbf{r}$ to each node, which translate into maps $(\gamma[\omega_{\mathbf{i}}])_{\mathbf{i}\in\mathbb{T}}$.

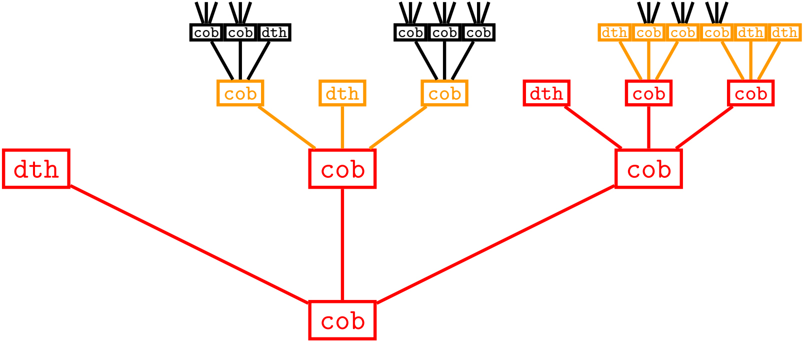
A recursive tree representation



Let $\mathbb{S}$ be the random subtree of $\mathbb{T}$ defined as

$$\mathbb{S} := \big\{ i_1 \cdots i_n \in \mathbb{T} : i_m \leq \kappa(\omega_{i_1 \cdots i_{m-1}}) \ \forall 1 \leq m \leq n \big\}.$$

A recursive tree representation

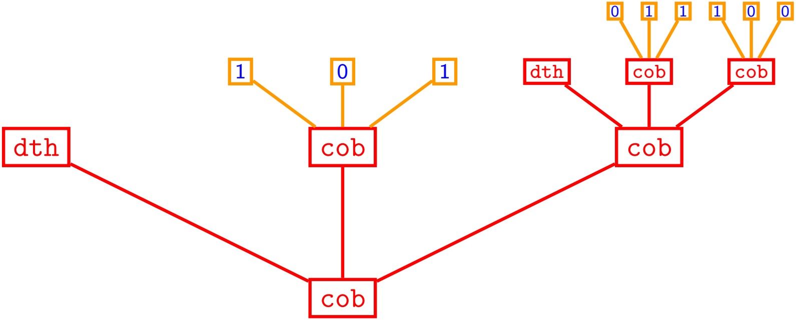


For any rooted subtree $\mathbb{U} \subset \mathbb{S}$, let

$$\nabla\mathbb{U} := \{ i_1 \cdots i_n \in \mathbb{S} : i_1 \cdots i_{n-1} \in \mathbb{U},\ i_1 \cdots i_n \notin \mathbb{U} \}$$

denote the boundary of $\mathbb{U}$ relative to $\mathbb{S}$.

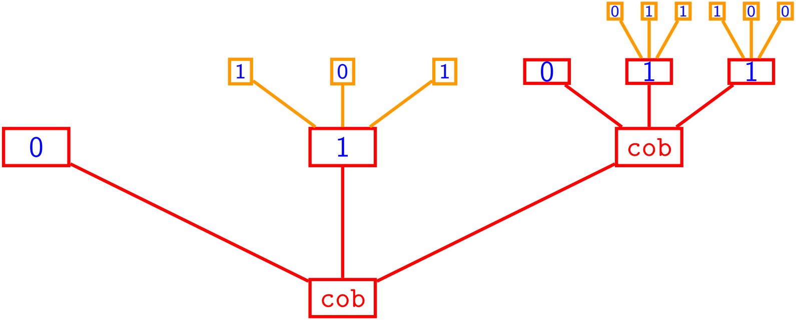
A recursive tree representation



Given $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}$, we inductively define $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$ by

$$X_{\mathbf{i}} = \gamma[\omega_{\mathbf{i}}]\big(X_{\mathbf{i}1},\ldots,X_{\mathbf{i}\kappa(\omega)}\big) \qquad (\mathbf{i}\in\mathbb{U}).$$

A recursive tree representation



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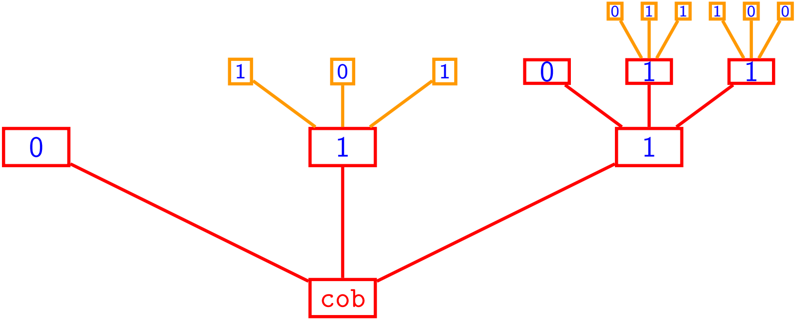

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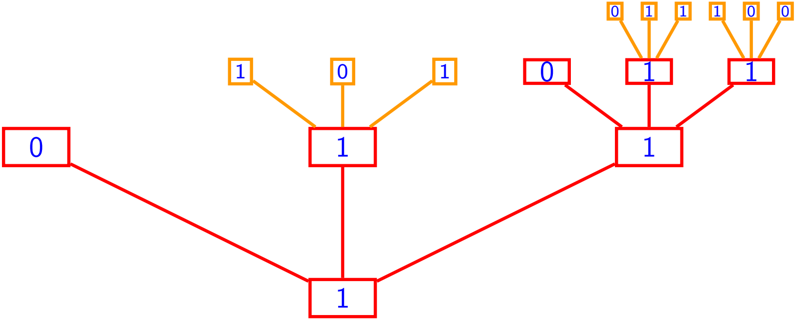

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$$X_{\mathbf{i}} = \gamma[\omega_{\mathbf{i}}]\big(X_{\mathbf{i}1},\ldots,X_{\mathbf{i}\kappa(\omega)}\big) \qquad (\mathbf{i}\in\mathbb{U}).$$

A recursive tree representation

Setting

$$G_{\mathbb{U}}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}\big) := X_\varnothing$$

defines a random map

$$G_{\mathbb{U}} : \mathbb{S}^{\nabla\mathbb{U}} \to \mathbb{S}$$

that is the concatenation of the maps $\big(\gamma[\omega_{\mathbf{i}}]\big)_{\mathbf{i}\in\mathbb{U}}$ according to the tree structure of $\mathbb{U}$.

Let $|i_1\cdots i_n| := n$ denote the length of a word $\mathbf{i}$ and set

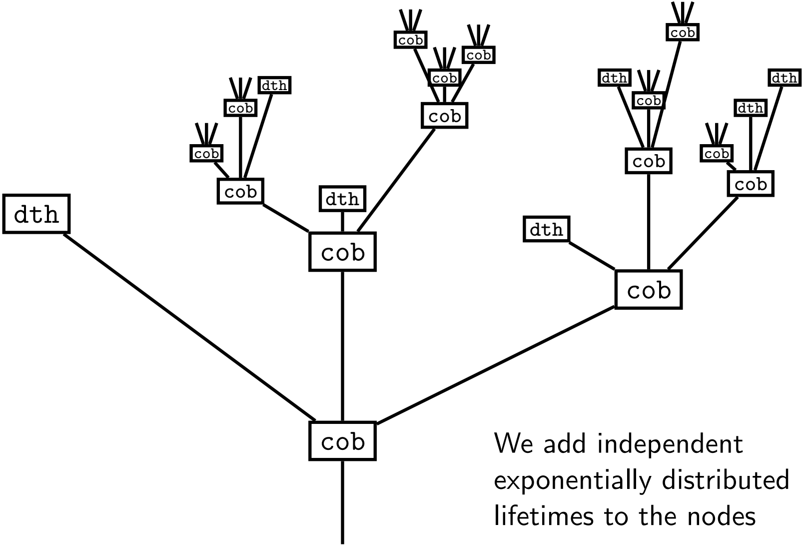
$$\mathbb{S}_{(n)} := \{\mathbf{i}\in\mathbb{S} : |\mathbf{i}| < n\} \quad \text{and} \quad \nabla\mathbb{S}_{(n)} = \{\mathbf{i}\in\mathbb{S} : |\mathbf{i}| = n\}.$$

Aldous and Bandyopadyay (2005) observed that

$$\mathbf{T}^n(\mu) := \text{ the law of } G_{\mathbb{S}_{(n)}}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_{(n)}}\big),$$

where $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_{(n)}}$ are i.i.d. with law μ and independent of $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}_{(n)}}$.

A recursive tree representation



We add independent exponentially distributed lifetimes to the nodes

A recursive tree representation

Let $(\sigma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be i.i.d. exponentially distributed with mean $|\mathbf{r}|^{-1}$, independent of $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, and set

$$\tau^*_{\mathbf{i}} := \sum_{m=1}^{n-1} \sigma_{i_1\cdots i_m} \quad \text{and} \quad \tau^\dagger_{\mathbf{i}} := \tau^*_{\mathbf{i}} + \sigma_{\mathbf{i}} \qquad (\mathbf{i} = i_1\cdots i_n),$$
$$\mathbb{S}_t := \{\mathbf{i}\in\mathbb{S} : \tau^\dagger_{\mathbf{i}} \leq t\} \quad \text{and} \quad \nabla\mathbb{S}_t = \{\mathbf{i}\in\mathbb{S} : \tau^*_{\mathbf{i}} \leq t < \tau^\dagger_{\mathbf{i}}\}.$$

Let $\mathcal{F}_t$ be the filtration

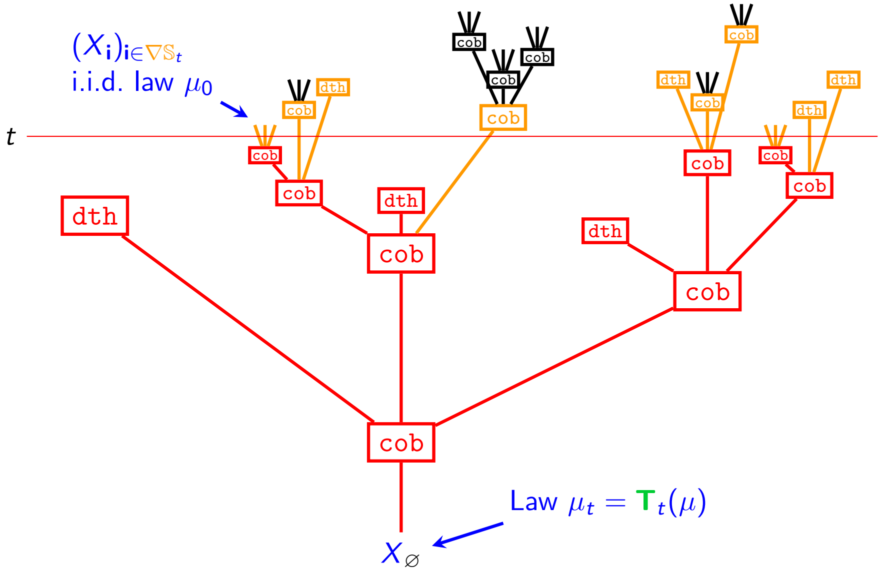
$$\mathcal{F}_t := \sigma\big(\nabla\mathbb{S}_t, (\omega_{\mathbf{i}}, \sigma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}_t}\big) \qquad (t \geq 0).$$

Theorem [Mach, Sturm, S. '18]

$$\mathbf{T}_t(\mu) := \text{ the law of } G_{\mathbb{S}_t}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_t}\big),$$

where $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_t}$ are i.i.d. with law μ and independent of $\mathcal{F}_t$.

A recursive tree representation



A *Recursive Distributional Equation* is an equation of the form

$$X \stackrel{\mathrm{d}}{=} \gamma[\omega](X_1, \ldots, X_{\kappa(\omega)}) \qquad \text{(RDE)},$$

where $X_1, X_2, \ldots$ are i.i.d. copies of X, independent of ω.

A law ν solves (RDE) iff

$$\text{(i)} \quad \mathbf{T}_t(\nu) = \nu \quad (t \geq 0) \qquad \text{or} \qquad \text{(ii)} \quad \mathbf{T}(\nu) = \nu.$$

We can view ν as the "invariant law" of a "Markov chain" where time has a tree-like structure.

In our example, solutions to the RDE are the Bernoulli distributions $\nu_{\rm low}$, $\nu_{\rm mid}$, $\nu_{\rm upp}$ with density $z_{\rm low}$, $z_{\rm mid}$, $z_{\rm upp}$.

Recursive Tree Processes

For any rooted subtree $\mathbb{U} \subset \mathbb{T}$, let

$$\partial\mathbb{U} := \big\{ i_1 \cdots i_n \in \mathbb{T} : i_1 \cdots i_{n-1} \in \mathbb{U},\ i_1 \cdots i_n \notin \mathbb{U} \big\}$$

denote the boundary of $\mathbb{U}$ relative to $\mathbb{T}$.

For each solution ν of (RDE), there exists a *Recursive Tree Process (RTP)* $(\boldsymbol{\omega}_\mathbf{i}, \mathbf{X}_\mathbf{i})_{\mathbf{i}\in\mathbb{T}}$, unique in law, such that:

(i) $(\boldsymbol{\omega}_\mathbf{i})_{\mathbf{i}\in\mathbb{T}}$ are i.i.d. with law $|\mathbf{r}|^{-1}\mathbf{r}$.

(ii) For finite $\mathbb{U} \subset \mathbb{T}$, the r.v.'s $(\mathbf{X}_\mathbf{i})_{\mathbf{i}\in\partial\mathbb{U}}$ are i.i.d. with ν and independent of $(\boldsymbol{\omega}_\mathbf{i})_{\mathbf{i}\in\mathbb{U}}$.

(iii) $\mathbf{X}_\mathbf{i} = \gamma[\boldsymbol{\omega}_\mathbf{i}]\big(\mathbf{X}_{\mathbf{i}1}, \ldots, \mathbf{X}_{\mathbf{i}\kappa(\boldsymbol{\omega}_\mathbf{i})}\big) \qquad (\mathbf{i} \in \mathbb{T}).$

If we add independent exponentially distributed lifetimes, then:

- Conditional on $\mathcal{F}_t$, the r.v.'s $(\mathbf{X}_\mathbf{i})_{\mathbf{i}\in\nabla\mathbb{S}_t}$ are i.i.d. with law ν.

n-Variate processes

For each $n \geq 1$, a measurable map $g : S^k \to S$ gives rise to *n-variate map* $g^{(n)} : (S^n)^k \to S^n$ defined as

$$g^{(n)}(x_1, \ldots, x_k) = g^{(n)}(x^1, \ldots, x^n) := \big(g(x^1), \ldots, g(x^n)\big),$$

with $x = (x_i^m)_{i=1,\ldots,k}^{m=1,\ldots,n}$, $x_i = (x_i^1, \ldots, x_i^n)$, $x^m = (x_1^m, \ldots, x_k^m)$.

We define an *n-variate map*

$$\mathbf{T}^{(n)}(\mu^{(n)}) := \text{ the law of } \gamma^{(n)}[\boldsymbol{\omega}](X_1, \ldots, X_{\kappa(\boldsymbol{\omega})}),$$

which acts on probability measures $\mu^{(n)}$ on S^n.
The *n-variate mean-field equation*

$$\tfrac{\partial}{\partial t}\mu_t^{(n)} = |\mathbf{r}|\big\{\mathbf{T}^{(n)}(\mu_t^{(n)}) - \mu_t^{(n)}\big\} \qquad (t \geq 0).$$

describes the mean-field limit of n coupled processes that are constructed using the same stochastic flow $(\mathbf{X}_{s,u})_{s \leq u}$.

n-Variate processes

$\mathcal{P}(S)$	space of probability measures on S.
$\mathcal{P}_{\rm sym}(S^n)$	space of probability measures on S^n that are symmetric under a permutation of the coordinates.
$S^n_{\rm diag}$	$\{x \in S^n : x_1 = \cdots = x_n\}$
$\mathcal{P}(S^n)_\mu$	space of probability measures on S^n whose one-dimensional marginals are all equal to μ.

- If $(\mu^{(n)}_t)_{t\geq 0}$ solves the n-variate equation, then its m-dimensional marginals solve the m-variate equation.
- $\mu^{(n)}_0 \in \mathcal{P}_{\rm sym}(S^n)$ implies $\mu^{(n)}_t \in \mathcal{P}_{\rm sym}(S^n)$ $(t \geq 0)$.
- $\mu^{(n)}_0 \in \mathcal{P}(S^n_{\rm diag})$ implies $\mu^{(n)}_t \in \mathcal{P}(S^n_{\rm diag})$ $(t \geq 0)$.
- If $\mathbf{T}(\nu) = \nu$, then $\mu^{(n)}_0 \in \mathcal{P}(S^n)_\nu$ implies $\mu^{(n)}_t \in \mathcal{P}(S^n)_\nu$.

n-Variate processes

If $\nu = \mathbb{P}[X \in \cdot\,]$ solves the RDE $\mathbf{T}(\nu) = \nu$, then

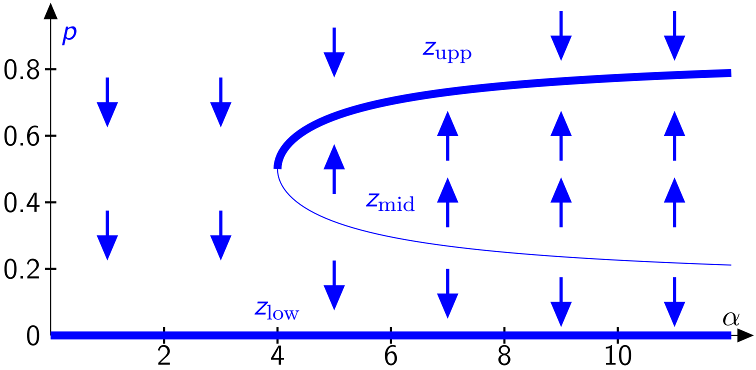
$$\overline{\nu}^{(n)} := \mathbb{P}\big[(\underbrace{X, \ldots, X}_{n \text{ times}}) \in \cdot\,\big]$$

solves the *n-variate RDE* $\mathbf{T}^{(n)}(\nu^{(n)}) = \nu^{(n)}$.

Questions:

- Is $\overline{\nu}^{(n)}$ a stable fixed point of the *n*-variate equation?
- Is $\overline{\nu}^{(n)}$ the only solution in $\mathcal{P}_{\rm sym}(S^n)_\nu$ of the *n*-variate RDE?

n-Variate processes



Fixed points of $\frac{\partial}{\partial t} p_t = F_\alpha(p_t)$ for different values of α.

Cooperative branching with branching rate $\alpha > 4$

The RDE $\mathbf{T}(\nu) = \nu$ has three solutions $\nu_{\mathrm{low}}, \nu_{\mathrm{mid}}$, and ν_{upp}, where $\nu_{\dots}$ is the probability measure on $\{0,1\}$ with mean $\nu_{\dots}(\{1\}) = z_{\dots}$ $(\dots = \mathrm{low}, \mathrm{mid}, \mathrm{upp})$, which

give rise to solutions $\overline{\nu}^{(2)}_{\mathrm{low}}, \overline{\nu}^{(2)}_{\mathrm{mid}}$, and $\overline{\nu}^{(2)}_{\mathrm{upp}}$ of the *bivariate RDE.*

Proposition [Mach, Sturm, S. '18] Apart from $\overline{\nu}^{(2)}_{\mathrm{low}}, \overline{\nu}^{(2)}_{\mathrm{mid}}, \overline{\nu}^{(2)}_{\mathrm{upp}}$, the *bivariate RDE* has one more solution $\underline{\nu}^{(2)}_{\mathrm{mid}}$ in $\mathcal{P}_{\mathrm{sym}}(S^2)$. The domains of attraction are:

$$
\begin{array}{ll}
\overline{\nu}^{(2)}_{\mathrm{low}}: & \big\{\mu^{(2)}_0 : \mu^{(1)}_0(\{1\}) < z_{\mathrm{mid}}\big\},\\
\underline{\nu}^{(2)}_{\mathrm{mid}}: & \big\{\mu^{(2)}_0 : \mu^{(1)}_0(\{1\}) = z_{\mathrm{mid}},\ \mu^{(2)}_0 \neq \overline{\nu}^{(2)}_{\mathrm{mid}}\big\},\\
\overline{\nu}^{(2)}_{\mathrm{mid}}: & \big\{\overline{\nu}^{(2)}_{\mathrm{mid}}\big\},\\
\overline{\nu}^{(2)}_{\mathrm{upp}}: & \big\{\mu^{(2)}_0 : \mu^{(1)}_0(\{1\}) > z_{\mathrm{mid}}\big\}.
\end{array}
$$

n-Variate processes

Let $\big(\mathbf{X}(t)\big)_{t\geq 0}$ be a process in S^N.
Initial law: $\big(\mathbf{X}_i(0)\big)_{1\leq i\leq N}$ i.i.d. with mean $z_{\rm mid}$.
Let $\big(\mathbf{X}'(t)\big)_{t\geq 0}$ be a process with modified initial state:
$\mathbf{X}'_i(0) = \mathbf{X}_i(0)$ except for an ε-fraction of sites i, which are redrawn using independent randomness.

In the mean-field limit when N is large, if ε is small, then the fraction of sites where $\mathbf{X}'_i(t) \neq \mathbf{X}_i(t)$ initially increases and then tends to a limit.
The joint empirical law of $\mathbf{X}(t), \mathbf{X}'(t)$ converges as first $N \to \infty$ and then $t \to \infty$ to $\underline{\nu}^{(2)}_{\rm mid}$.

Let $(\omega_{\mathbf{i}}, \mathbf{X}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to a solution ν of the RDE.

Aldous and Bandyopadyay say that an RTP is *endogenous* if

$\mathbf{X}_\varnothing$ is measurable w.r.t. the σ-field generated by $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$.

Theorem [AB '05 & MSS '18] The following statements are equivalent:

(i) The RTP corresponding to ν is endogenous.

(ii) $\mathbf{T}_t^{(n)}(\mu) \underset{t\to\infty}{\Longrightarrow} \overline{\nu}^{(n)}$ for all $\mu \in \mathcal{P}(S^n)_\nu$ and $n \geq 1$.

(iii) $\overline{\nu}^{(2)}$ is the only solution in $\mathcal{P}_{\rm sym}(S^2)_\nu$ of the bivariate RDE.

In our example, the RTPs for $\nu_{\rm low}, \nu_{\rm upp}$ are endogenous, but the RTP corresponding to $\nu_{\rm mid}$ is not.

The higher-level equation

The n-variate map $\mathbf{T}^{(n)}$ is defined even for $n = \infty$, and $\mathbf{T}^{(\infty)}$ maps $\mathcal{P}_{\rm sym}(S^{\mathbb{N}_+})$ into itself.

By De Finetti's theorem, $(X_i)_{i\in\mathbb{N}_+}$ have a law in $\mathcal{P}_{\rm sym}(S^{\mathbb{N}_+})$ if and only if there exists a random probability measure ξ on S such that conditional on ξ, the $(X_i)_{i\in\mathbb{N}_+}$ are i.i.d. with law ξ.

Let $\rho := \mathbb{P}[\xi \in \cdot\,]$ the law of ξ. Then $\rho \in \mathcal{P}(\mathcal{P}(S))$.
In view of this, $\mathcal{P}_{\rm sym}(S^{\mathbb{N}_+}) \cong \mathcal{P}(\mathcal{P}(S))$.

The map $\mathbf{T}^{(\infty)} : \mathcal{P}_{\rm sym}(S^{\mathbb{N}_+}) \to \mathcal{P}_{\rm sym}(S^{\mathbb{N}_+})$
corresponds to a *higher-level map* $\check{\mathbf{T}} : \mathcal{P}(\mathcal{P}(S)) \to \mathcal{P}(\mathcal{P}(S))$.

The higher-level equation

For any measurable map $g : S^k \to S$, define $\check{g} : \mathcal{P}(S)^k \to \mathcal{P}(S)$ by

$$\check{g} := \text{ the law of } g(X_1, \ldots, X_k), \\ \text{where } (X_1, \ldots, X_k) \text{ are independent with laws } \mu_1, \ldots, \mu_k.$$

Then

$$\check{\mathbf{T}}(\rho) := \text{ the law of } \check{\gamma}[\boldsymbol{\omega}](\xi_1, \ldots, \xi_{\kappa(\boldsymbol{\omega})}),$$

with $\boldsymbol{\omega}$ as before and $\xi_1, \xi_2, \ldots$ i.i.d. with law ρ.

Define *n-th moment measures*

$$\rho^{(n)} := \mathbb{E}\big[\underbrace{\xi \otimes \cdots \otimes \xi}_{n \text{ times}}\big] \quad \text{where } \xi \text{ has law } \rho.$$

Proposition [MSS '18] If $(\rho_t)_{t \geq 0}$ solves the *higher-level mean-field equation*, then its n-th moment measures $(\rho_t^{(n)})_{t \geq 0}$ solve the n-variate equation.

The higher-level equation

Equip $\mathcal{P}(\mathcal{P}(S))_\nu = \{\rho : \rho^{(1)} = \nu\}$ with the *convex order*

$$\rho_1 \leq_{\rm cv} \rho_2 \quad \text{iff} \quad \int \phi\,\mathrm{d}\rho_1 \leq \int \phi\,\mathrm{d}\rho_2 \quad \forall \text{ convex } \phi.$$

[Strassen '65] $\rho_1 \leq_{\rm cv} \rho_2$ iff there exist a r.v. X with law ν and σ-fields $\mathcal{H}_1 \subset \mathcal{H}_2$ s.t. $\rho_i = \mathbb{P}\big[\mathbb{P}[X \in \cdot\,|\mathcal{H}_i] \in \cdot\,\big]$ $(i = 1, 2)$.

Maximal and minimal elements: $\mathcal{H}_1 = \{\Omega, \emptyset\} \;\Rightarrow\; \rho_1 = \delta_\nu$.
$\mathcal{H}_2 = \sigma(X) \;\Rightarrow\; \rho_2 = \overline{\nu} := \mathbb{P}[\delta_X \in \cdot\,]$ with $\mathbb{P}[X \in \cdot\,] = \nu$.

$$\delta_\nu \leq_{\rm cv} \rho \leq_{\rm cv} \overline{\nu} \qquad \forall \rho \in \mathcal{P}(\mathcal{P}(S))_\nu.$$

Proposition [MSS '18] $\check{\mathsf{T}}$ is monotone w.r.t. the convex order. There exists a solution $\underline{\nu}$ to the higher-level RDE s.t.

$$\check{\mathsf{T}}^n(\delta_\nu) \underset{n\to\infty}{\Longrightarrow} \underline{\nu} \quad \text{and} \quad \check{\mathsf{T}}_t(\delta_\nu) \underset{t\to\infty}{\Longrightarrow} \underline{\nu}$$

and any solution $\rho \in \mathcal{P}(\mathcal{P}(S))_\nu$ to the higher-level RDE satisfies

$$\underline{\nu} \leq_{\rm cv} \rho \leq_{\rm cv} \overline{\nu} \qquad \forall \rho \in \mathcal{P}(\mathcal{P}(S))_\nu.$$

The higher-level equation

Proposition [MSS '18]
Let $(\omega_{\mathbf{i}}, \mathbf{X}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to γ and ν. Set

$$\xi_{\mathbf{i}} := \mathbb{P}\big[X_{\mathbf{i}} \in \cdot \,|\, (\omega_{\mathbf{ij}})_{\mathbf{j}\in\mathbb{T}}\big].$$

Then $(\omega_{\mathbf{i}}, \xi_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is an RTP corresponding to $\check{\gamma}$ and $\underline{\nu}$.
Also, $(\omega_{\mathbf{i}}, \delta_{X_{\mathbf{i}}})_{\mathbf{i}\in\mathbb{T}}$ is an RTP corresponding to $\check{\gamma}$ and $\overline{\nu}$.

Corollary The RTP is endogenous iff $\underline{\nu} = \overline{\nu}$.

The higher-level equation

Theorem [Mach, Sturm, S. '18] One has

$$\underline{\nu}_{\rm low} = \overline{\nu}_{\rm low}, \quad \underline{\nu}_{\rm upp} = \overline{\nu}_{\rm upp}, \quad \text{but} \quad \underline{\nu}_{\rm mid} \neq \overline{\nu}_{\rm mid}.$$

These are all solutions to the higher-level RDE.
Any solution $(\rho_t)_{t\geq 0}$ to the higher-level mean-field equation converges to one of these fixed points.
The domains of attraction are:

$$\begin{array}{ll}
\overline{\nu}_{\rm low}: & \big\{\rho_0 : \rho_0^{(1)}(\{1\}) < z_{\rm mid}\big\},\\
\underline{\nu}_{\rm mid}: & \big\{\rho_0 : \rho_0^{(1)}(\{1\}) = z_{\rm mid},\ \rho_0 \neq \overline{\nu}_{\rm mid}\big\},\\
\overline{\nu}_{\rm mid}: & \big\{\overline{\nu}_{\rm mid}\big\},\\
\overline{\nu}_{\rm upp}: & \big\{\rho_0 : \rho_0^{(1)}(\{1\}) > z_{\rm mid}\big\}.
\end{array}$$

The higher-level equation

The map $\mu \mapsto \mu(\{1\})$ defines a bijection $\mathcal{P}(\{0,1\}) \cong [0,1]$, and hence $\mathcal{P}(\mathcal{P}(\{0,1\})) \cong \mathcal{P}[0,1]$.

Then the higher-level RDE takes the form

$$\eta \stackrel{\mathrm{d}}{=} \chi \cdot \big(\eta_1 + (1-\eta_1)\eta_2\eta_3\big),$$

where η takes values in $[0,1]$, η_1, η_2, η_3 are independent copies of η and χ is an independent Bernoulli r.v. with $\mathbb{P}[\chi = 1] = \alpha/(\alpha+1)$.

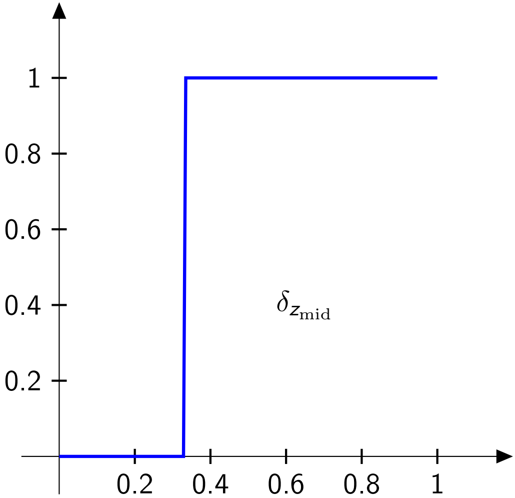
This RDE has three "trivial" solutions

$$\overline{\nu}_{\ldots} = (1 - z_{\ldots})\delta_0 + z_{\ldots}\delta_1 \qquad (\ldots = \mathrm{low}, \mathrm{mid}, \mathrm{upp}),$$

and a nontrivial solution

$$\underline{\nu}_{\mathrm{mid}} = \lim_{n\to\infty} \check{\mathsf{T}}^n(\delta_{z_{\mathrm{mid}}}).$$

Numerical results



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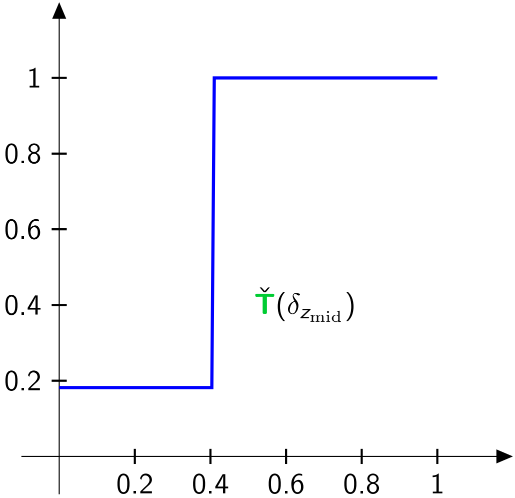

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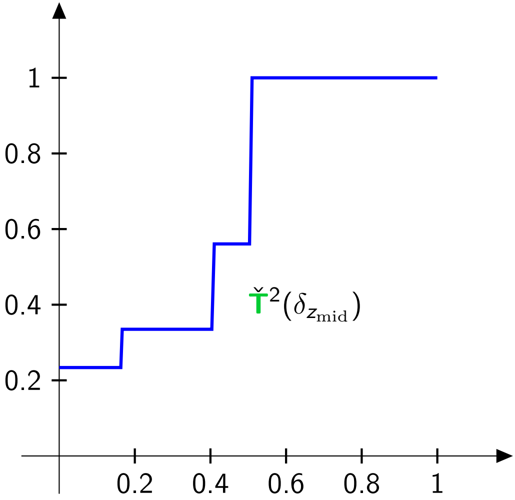

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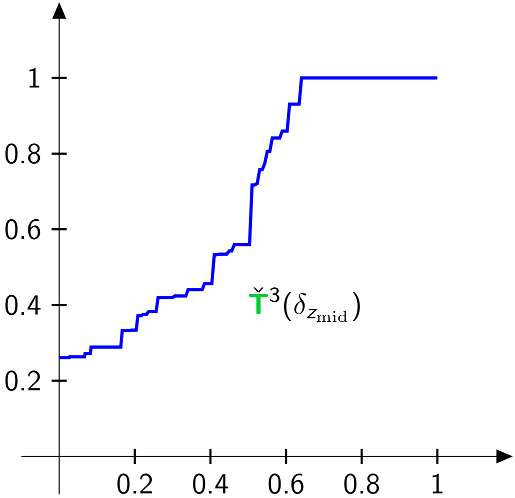

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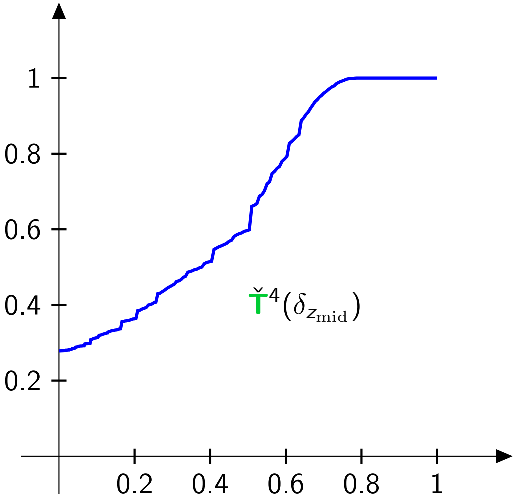

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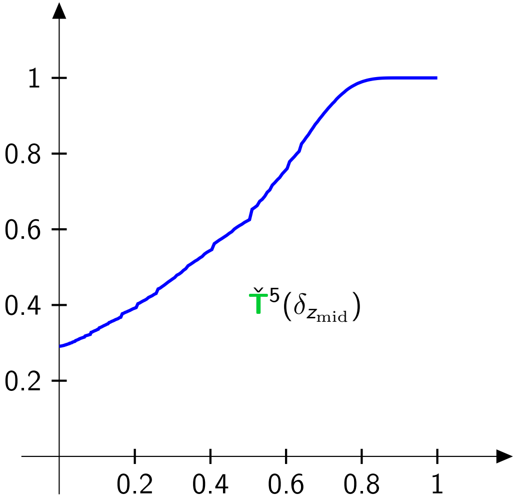

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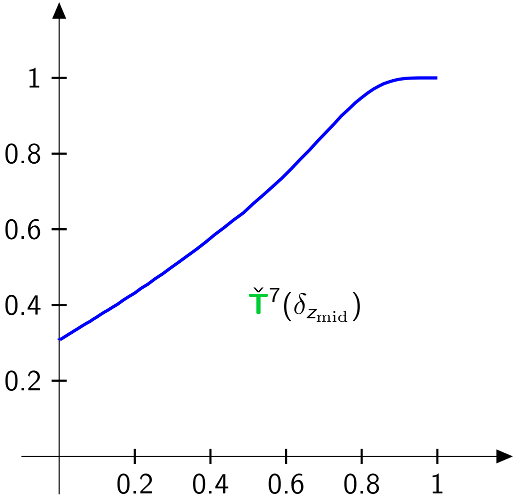

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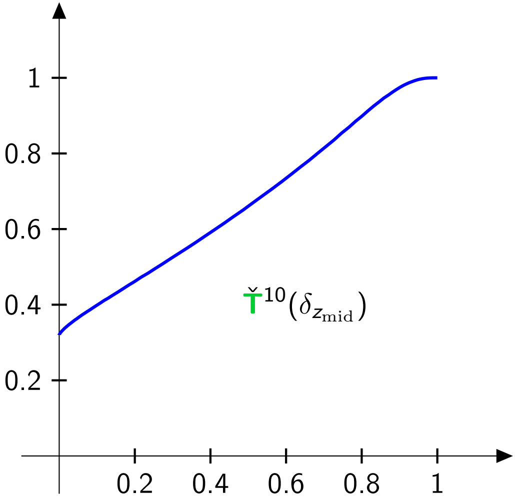

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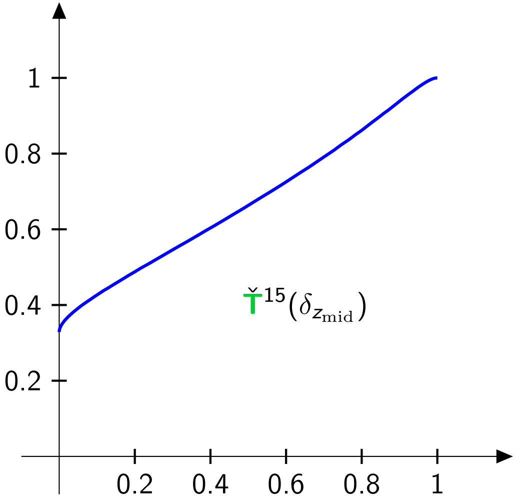

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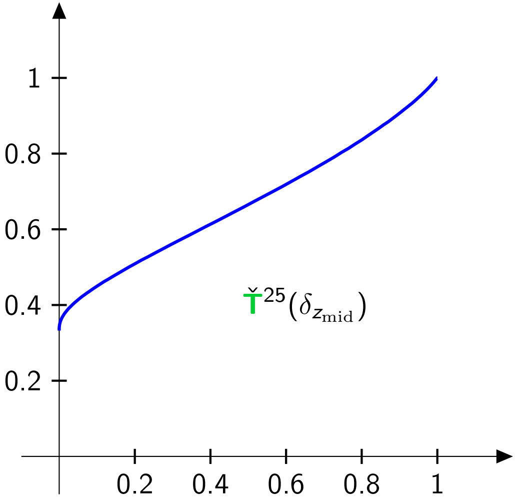

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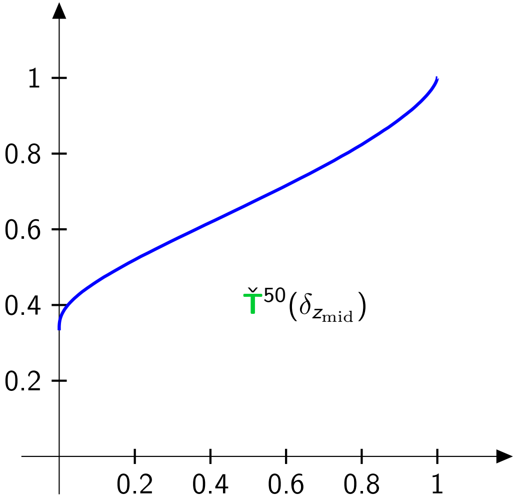

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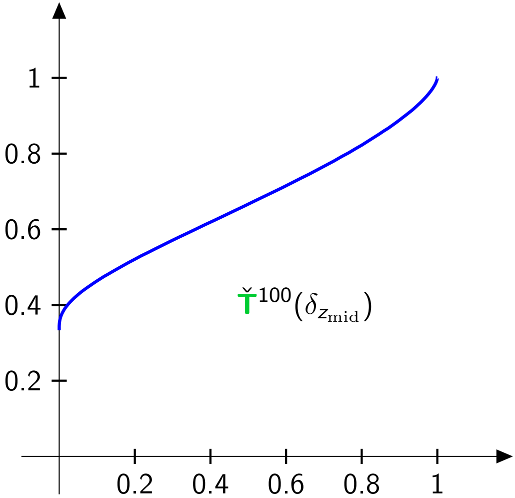

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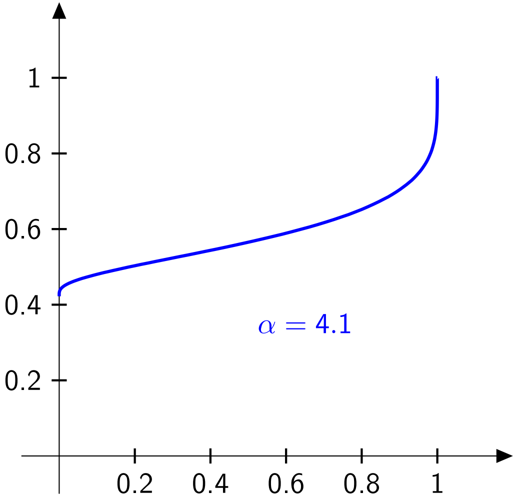

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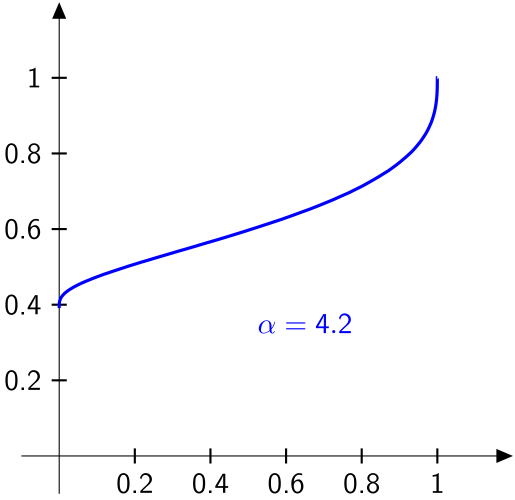

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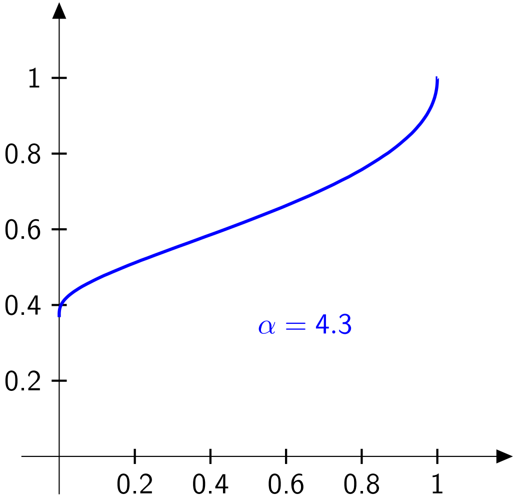

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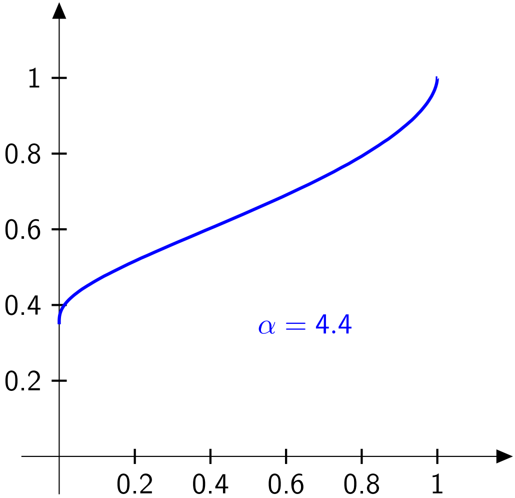

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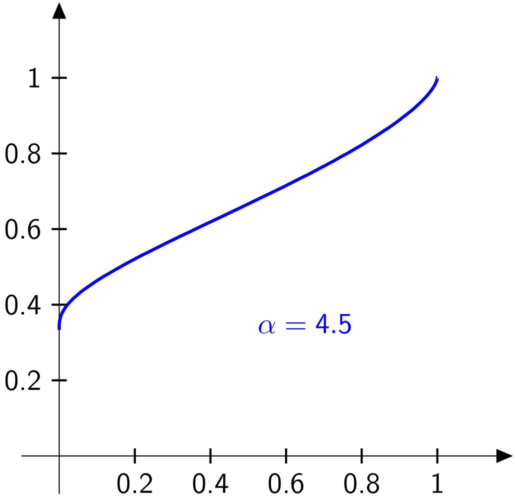

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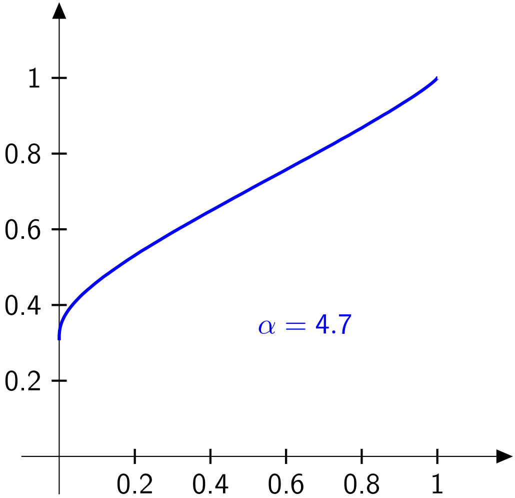

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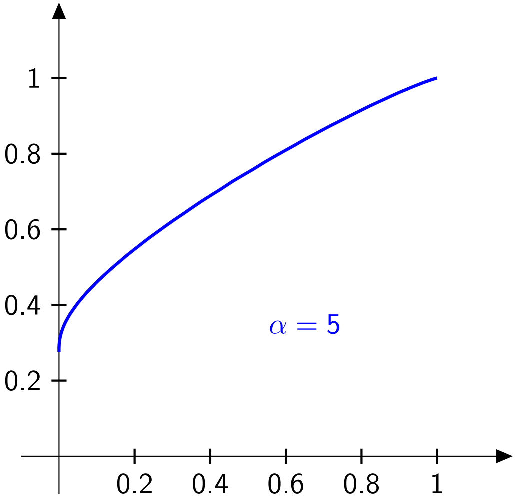

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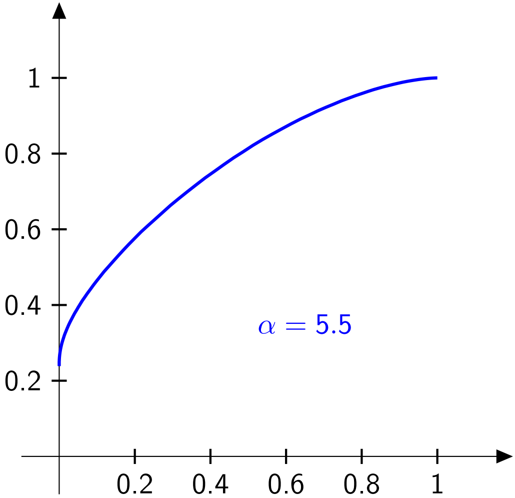

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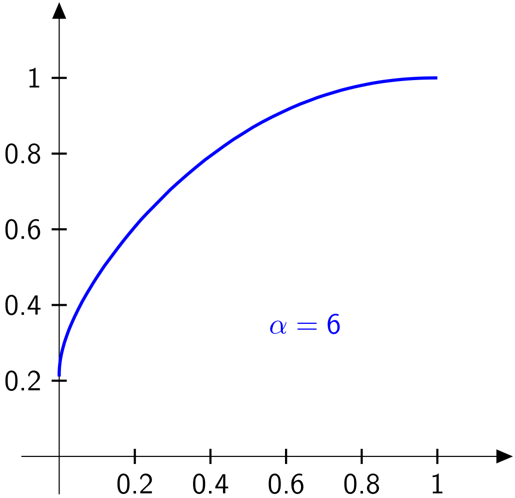

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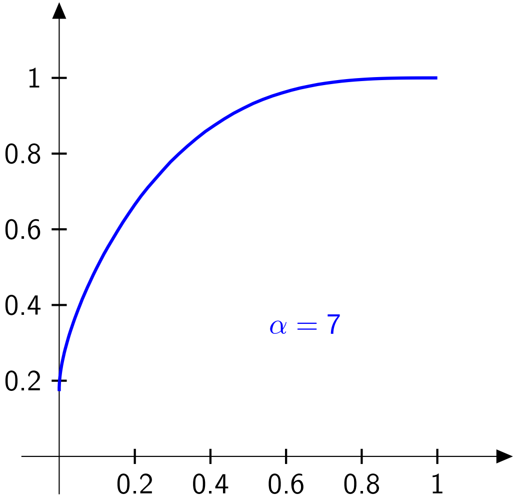

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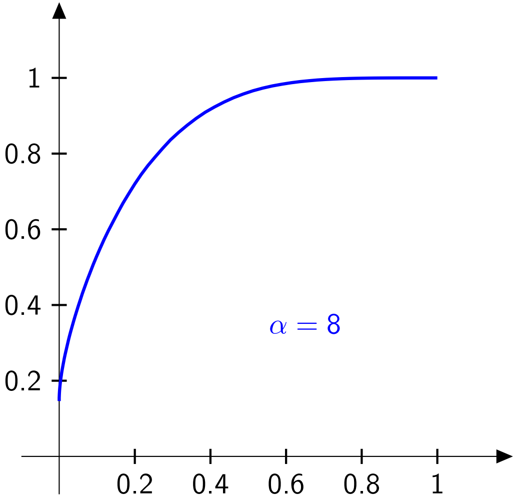

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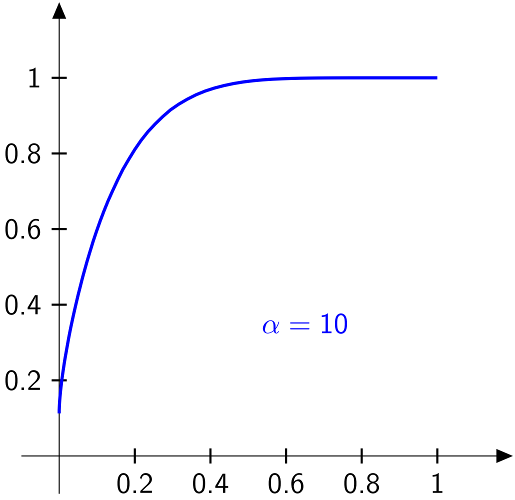

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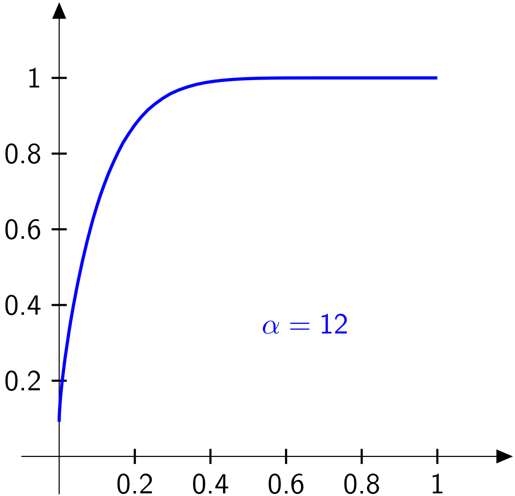

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