Weaves, webs and flows I

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joint with Nic Freeman

Stochastic flows

Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{X}$ be a probability space and a measurable space. By definition, a *stochastic flow* on $\mathcal{X}$ is a collection $(\mathbb{X}_{s,t})_{s\leq t}$ of random maps $\mathbb{X}_{s,t} : \mathcal{X} \to \mathcal{X}$, such that:

(i) $(s,t,\omega,x) \mapsto \mathbb{X}_{s,t}[\omega](x)$ is jointly measurable as a function on $\{(s,t) \in \mathbb{R}^2 : s \leq t\} \times \Omega \times \mathcal{X}$.

(ii) $\mathbb{X}_{s,s} = \mathrm{Id}$ and $\mathbb{X}_{t,u} \circ \mathbb{X}_{s,t} = \mathbb{X}_{s,u}$ $(s \leq t \leq u)$.

Sometimes (ii) is required only for deterministic $s \leq t \leq u$, i.e.,

(ii)' $\mathbb{X}_{s,s} = \mathrm{Id}$ and $\mathbb{X}_{t,u} \circ \mathbb{X}_{s,t} = \mathbb{X}_{s,u}$ a.s. $(s \leq t \leq u)$.

A stochastic flow $(\mathbb{X}_{s,t})_{s\leq t}$ is *stationary* if:

(iii) $(\mathbb{X}_{s,t})_{s\leq t}$ is equal in law to $(\mathbb{X}_{s+r,t+r})_{s\leq t}$ for all $r \in \mathbb{R}$,

and we say that $(\mathbb{X}_{s,t})_{s\leq t}$ has *independent increments* if:

(iv) $\mathbb{X}_{t_0,t_1}, \ldots, \mathbb{X}_{t_{n-1},t_n}$ are independent for all $t_0 \leq \cdots \leq t_n$.

Stochastic flows

If $(\mathbb{X}_{s,t})_{s\leq t}$ is a stochastic flow with independent increments, $s\in\mathbb{R}$, and X_0 is an independent $\mathcal{X}$-valued random variable, then setting

$$X_t := \mathbb{X}_{s,s+t}(X_0) \qquad (t\geq 0)$$

defines a Markov process $(X_t)_{t\geq 0}$. If $(\mathbb{X}_{s,t})_{s\leq t}$ is stationary, then $(X_t)_{t\geq 0}$ is time-homogeneous.

Many Markov processes can be constructed from stochastic flows. Examples:

- Markov processes constructed from Poisson point processes
- Solutions to stochastic differential equations, with the driving Brownian motion playing the role of white noise.

Stochastic flows

More generally, if $\vec{X}_0 = (X_0^1, \ldots, X_0^n)$ is an independent $\mathcal{X}^n$-valued random variable, then setting

$$X_t^i := \mathbb{X}_{s,s+t}(X_0^i) \qquad (t \geq 0,\ 1 \leq i \leq n)$$

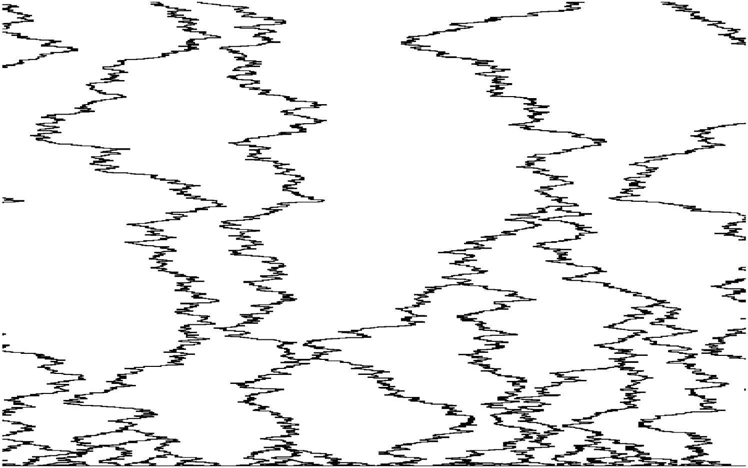
defines a stochastic process $(\vec{X}_t)_{t\geq 0}$. These *n-point motions* satisfy a natural consistency property.

Le Jan and Raimond (AoP 2004) have shown that each consistent family of Feller processes gives rise to a stationary stochastic flow (in the weak sense of (ii)') with independent increments.

We will be interested in stochastic flows on $\mathbb{R}$ with *non-crossing* n-point motions.

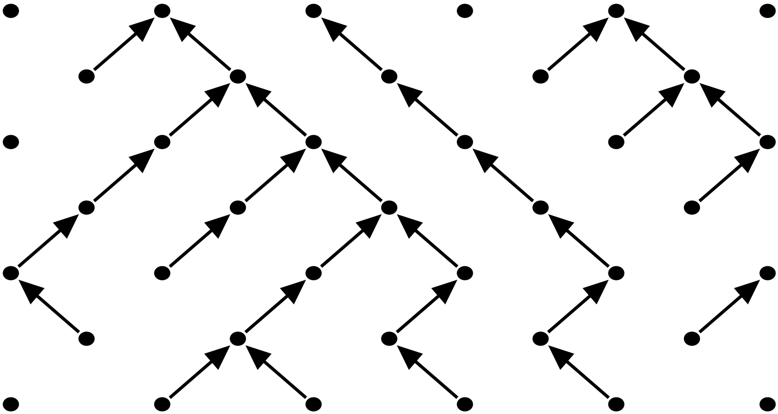
Alternatively, this says that the maps $\mathbb{X}_{s,t} : \mathbb{R} \to \mathbb{R}$ are *monotone* in the sense that $x \leq y$ implies $\mathbb{X}_{s,t}(x) \leq \mathbb{X}_{s,t}(y)$.

The Arratia flow



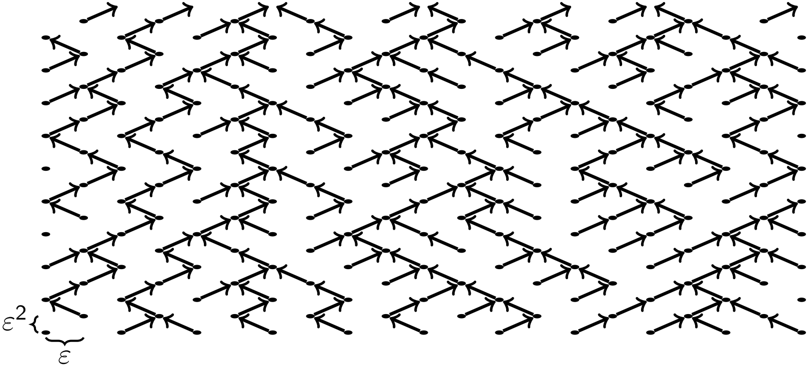
The *n*-point motions of the *Arratia flow* are *coalescing Brownian motions*.

Discrete approximation



We can define random maps $X_{k,k+1} : (\mathbb{Z} + k) \to (\mathbb{Z} + k + 1)$ such that $X_{k,k+1}(i) = i - 1$ or $i + 1$ with equal probabilies, independently for all $(i, k) \in \mathbb{Z}^2_{\rm even} := \{(x, t) \in \mathbb{Z}^2 : x + t \text{ is even}\}$.

Discrete approximation



If we rescale space by ε and time by ε^2 and let $\varepsilon \to 0$, then these discrete maps converge to the Arratia flow.

Harris flows

By definition, a *correlation function* is a continuous function $\rho : \mathbb{R} \to [-1, 1]$ such that:

(i) ρ is continuous with $\rho(0) = 1$,

(ii) for each $x_1, \ldots, x_n \in \mathbb{R}$, setting $M_{ij} := \rho(x_i - x_j)$ defines a positive semidefinite matrix.

By Bochner's theorem, if μ is a symmetric probability measure on $\mathbb{R}$, then

$$\rho(x) := \int_{-\infty}^{\infty} \mu(\mathrm{d}y) e^{-2\pi i x y} \qquad (x \in \mathbb{R})$$

defines a correlation function, and each correlation function is of this form.

Harris flows

Harris (SPA 1984) proved that if ρ is a correlation function, then for each initial state in $\mathbb{R}^n$, there exists a unique solution to the martingale problem for the operator

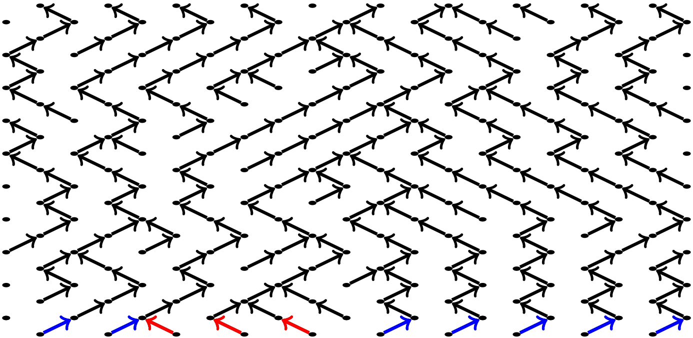
$$Gf(x_1,\ldots,x_n) := \tfrac{1}{2}\sum_{i,j=1}^{n} \rho(x_i - x_j)\tfrac{\partial^2}{\partial x_i \partial x_j} f(x),$$

with the additional condition that paths coalesce once they meet.[1] The solutions to this martingale problem are correlated Brownian motions that form the n-point motions for a *Harris flow* $(\mathbb{X}_{s,t})_{s\leq t}$.

$$\lim_{t\to 0} t^{-1}\mathbb{E}\big[(\mathbb{X}_{0,t}(x)-x)(\mathbb{X}_{0,t}(y)-y)\big] = \rho(x-y).$$

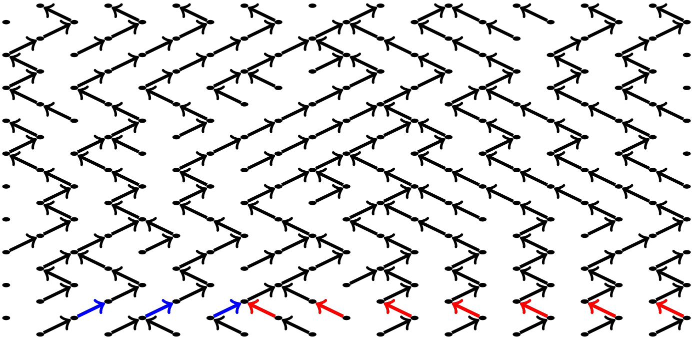
[1] Paths meet a.s. iff $\displaystyle\int_{0+} \frac{x}{1-\rho(x)}\mathrm{d}x < \infty$.

Wright-Fisher correlations



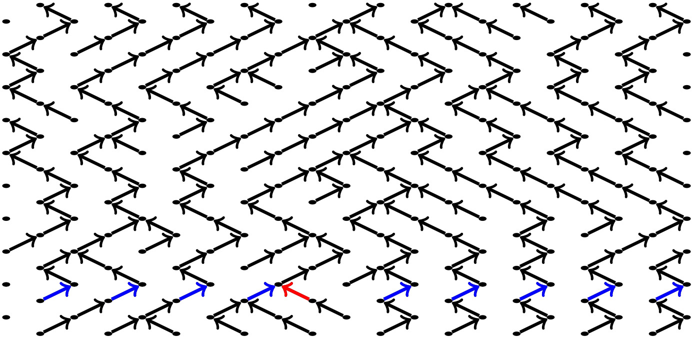
At even times we divide $0, 2, \ldots, 2n-2$ into two intervals.

Wright-Fisher correlations



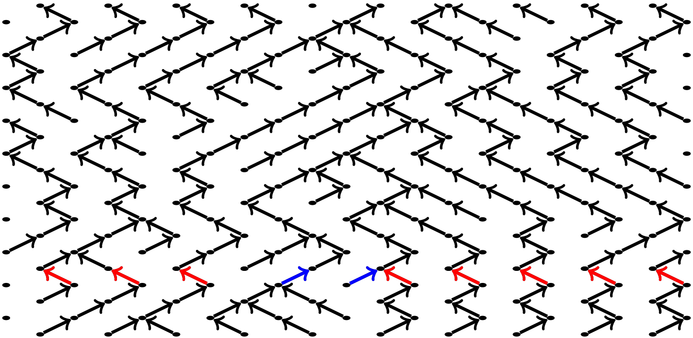
At odd times we divide $1, 3, \ldots, 2n-1$ randomly into two intervals.

Wright-Fisher correlations



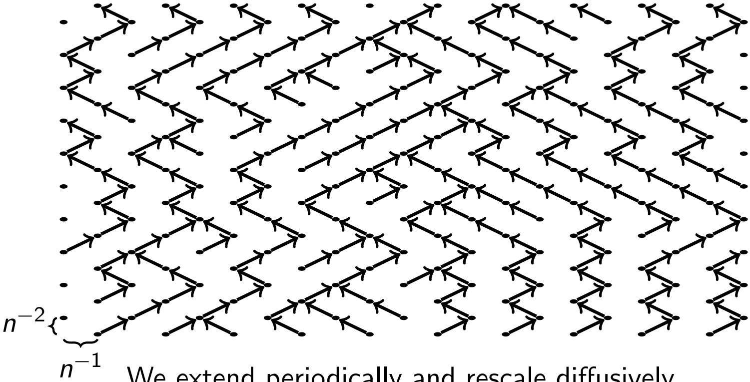
In one interval (with periodic b.c.) we draw arrows to the left.

Wright-Fisher correlations



And in the other interval we draw arrows to the right.

Wright-Fisher correlations



We extend periodically and rescale diffusively.

Wright-Fisher correlations

The result is a Harris flow with correlation function

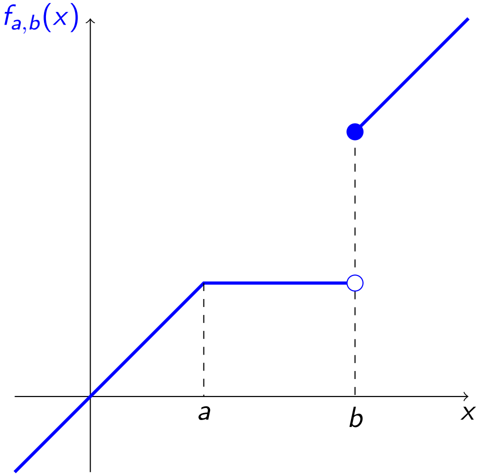
$$\rho(x) = 1 - 4\dot{x}(1 - \dot{x}) \qquad (x \in \mathbb{R}),$$

where $\dot{x} := x - \lfloor x \rfloor$.

Note $\rho(x) := 1 - c\dot{x}(1 - \dot{x})$ is positive definite iff $c \in [0, 6]$.

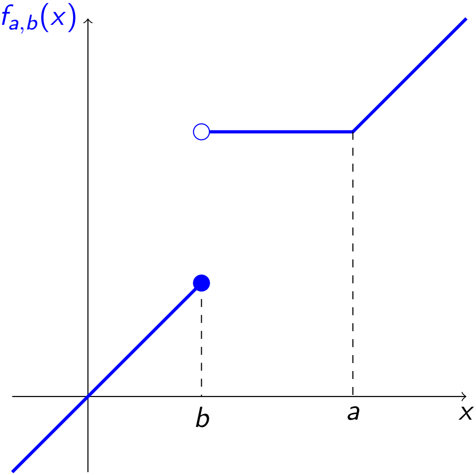
Note Bertoin and Le Gall (AIHP 05) have shown that this Harris flow is closely related to the Kingman coalescent.

Jump-type n-point motions



$$f_{a,b}(x) := \begin{cases} a & \text{if } a < x < b \text{ or } b < x < a \\ x & \text{otherwise.} \end{cases}$$

Jump-type n-point motions



$$f_{a,b}(x) := \begin{cases} a & \text{if } a < x < b \text{ or } b < x < a \\ x & \text{otherwise.} \end{cases}$$

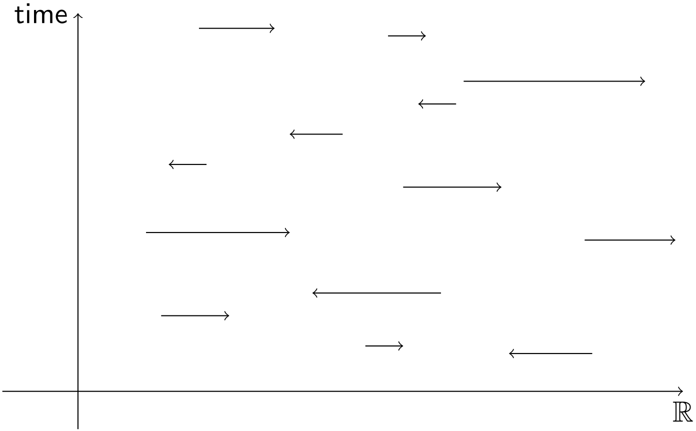
Jump-type n-point motions

Let ν be a locally finite measure on $\mathcal{A} := \mathbb{R}^2 \backslash \{(a,a) : a \in \mathbb{R}\}$ and let ℓ denote the Lebesgue measure on $\mathbb{R}$.
Let $\omega \subset \mathcal{A} \times \mathbb{R}$ be a Poisson point set with intensity $\nu \otimes \ell$.

We would like to define a stochastic flow with the following informal description:

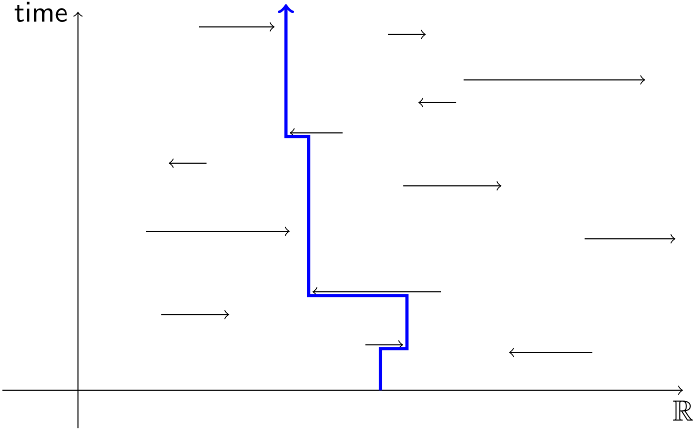
For each $(a, b, t) \in \omega$, we apply the map $f_{a,b}$ at time t.

Jump-type n-point motions



The n-point motions would be coalescing Lévy processes.

Jump-type n-point motions



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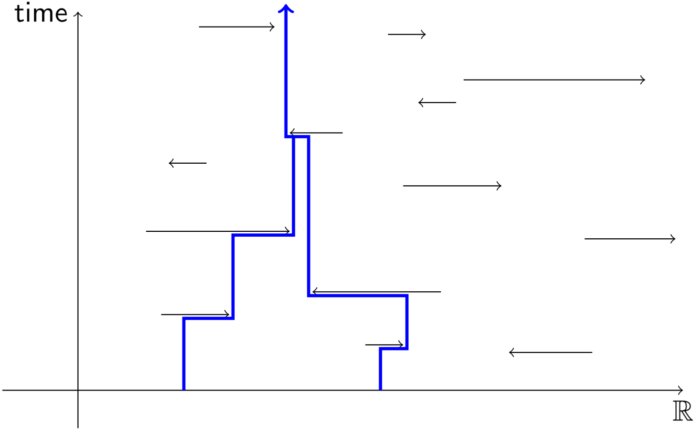
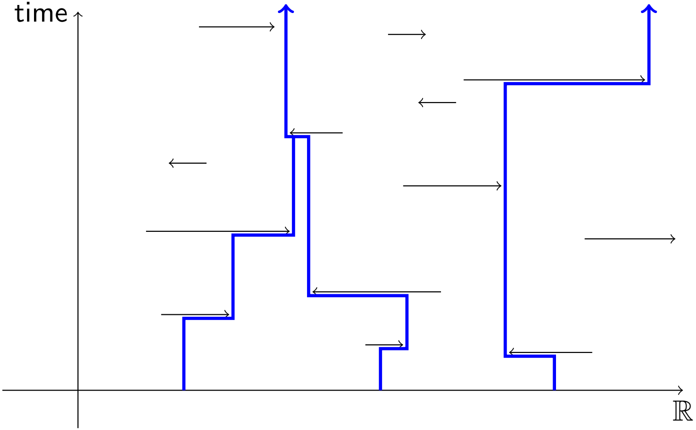


The n-point motions would be coalescing Lévy processes.

Jump-type n-point motions



The n-point motions would be coalescing Lévy processes.

The Brownian web

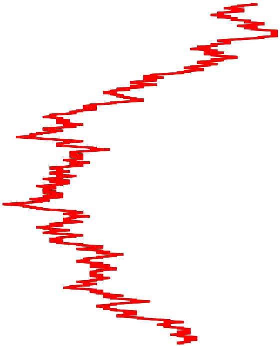
Let $(z_i)_{i\geq 1} = (x_i, t_i)_{i\geq 1}$ be points in $\mathbb{R}^2$.
Let $(B_i)_{i\geq 1}$ with $B_i = \big(B_i(t)\big)_{t\geq t_i}$ be independent Brownian motions started from $B_i(t_i) = x_i$.

Define inductively $\tau_i := \inf\{t \geq t_i : \big(B_i(t), t\big) \in \bigcup_{k=1}^{i-1} A_k\big\}$ with $A_i := \big\{\big(B_i(t), t\big) : t_i \leq t < \tau_i\big\}$.
For $i \geq 2$ define $\kappa(i) < i$ by $\big(B_i(\tau_i), \tau_i\big) \in A_{\kappa(i)}$.

Then we can inductively define
coalescing Brownian motions $(P_i)_{i\geq 1}$ *started from* $(z_i)_{i\geq 1}$ by:

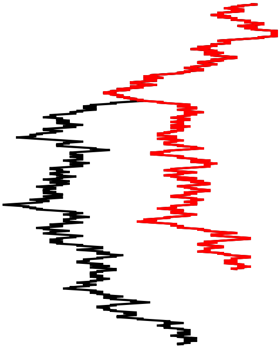
$P_i(t) := B_i(t)\ (t_i \leq t < \tau_i)$
$P_i(t) := P_{\kappa(i)}(t)\ (\tau_i \leq t < \infty)$

The Brownian web



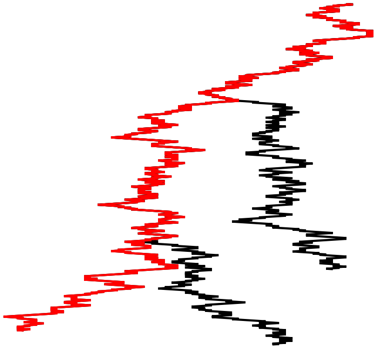
Coalescing Brownian motions.

The Brownian web



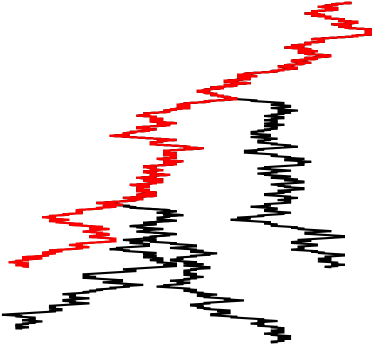
Coalescing Brownian motions.

The Brownian web



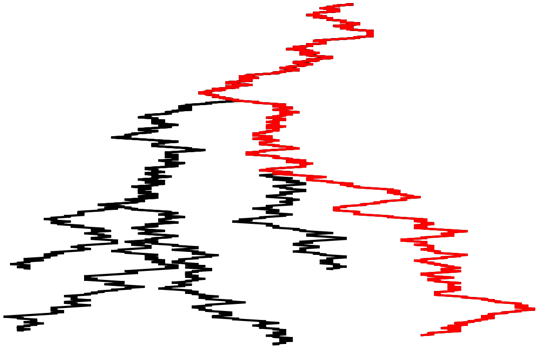
Coalescing Brownian motions.

The Brownian web



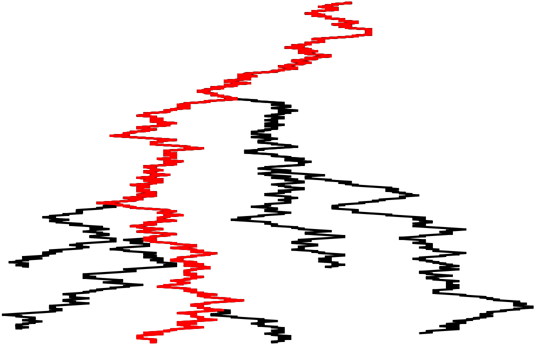
Coalescing Brownian motions.

The Brownian web



Coalescing Brownian motions.

The Brownian web



Coalescing Brownian motions.

The Brownian web

Let $\Pi_c^{\uparrow}$ be the space of continuous, upward paths.
We equip $\Pi_c^{\uparrow}$ with a suitable topology (convergence of starting times and locally uniform convergence of paths -details later.)

Let $(P_i)_{i\geq 1}$ be coalescing Brownian motions started from $(z_i)_{i\geq 1}$.
Assume that $\{z_i : i \in \mathbb{N}_+\}$ is dense in $\mathbb{R}^2$.

[Fontes, Isopi, Newman & Ravishankar (AoP 2004)]
The set $\{P_i : i \in \mathbb{N}_+\}$ is precompact and the law of

$$\mathcal{W} := \overline{\{P_i : i \in \mathbb{N}_+\}}$$

does not depend on $\{z_i : i \in \mathbb{N}_+\}$.

The compact set $\mathcal{W}$ is called the *Brownian web*.

Discrete approximation

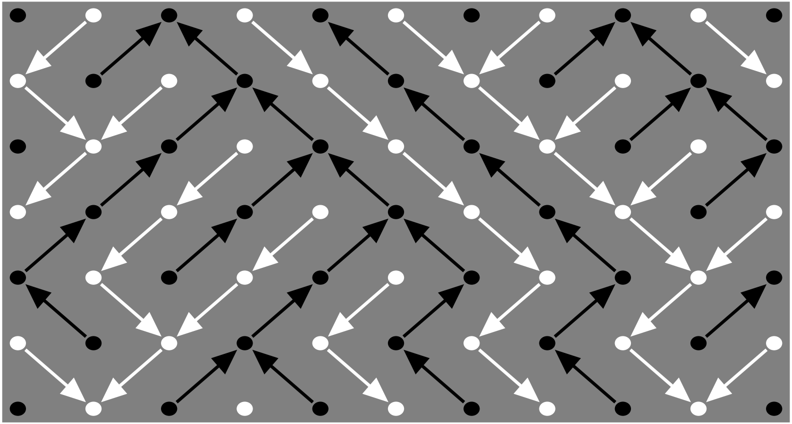
Let $\mathcal{K}(\Pi_{\rm c}^{\uparrow})$ denote the space of all compact sets of paths, equipped with the *Hausdorff metric* (details later).
Let $\mathcal{W}^{\varepsilon}$ be the collection of all paths in an arrow configuration, diffusively rescaled by ε.

Fontes, Isopi, Newman & Ravishankar (AoP 2004) have shown that

$$\mathbb{P}\big[\mathcal{W}^{\varepsilon}\in\,\cdot\,\big]\underset{\varepsilon\to 0}{\Longrightarrow}\mathbb{P}\big[\mathcal{W}\in\,\cdot\,\big]$$

with respect to the topology on $\mathcal{K}(\Pi_{\rm c}^{\uparrow})$.

Dual Brownian web



Associated to each arrow configuration is a dual arrow configuration.

Dual Brownian web

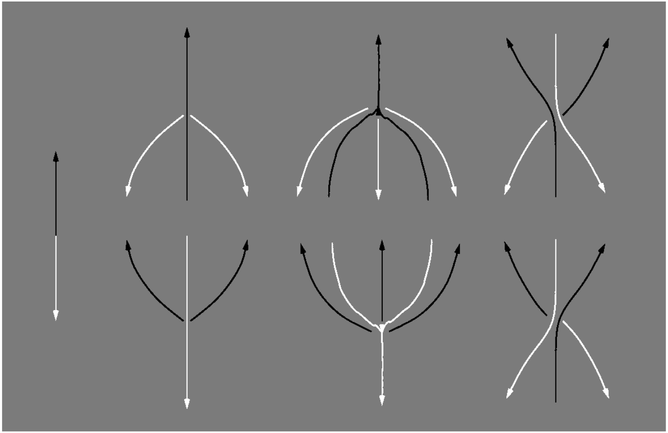
Let $P_1, \ldots, P_n$ be coalescing Brownian motions started from $z_1, \ldots, z_n$,
Let $\hat{P}_1, \ldots, \hat{P}_n$ be downward coalescing Brownian motions started from $z_1, \ldots, z_n$, with Shorohod reflection off the paths $P_1, \ldots, P_n$.

This is consistent in the sense of Kolmogorov!

Setting $\mathcal{W} := \overline{\{P_i : i \in \mathbb{N}_+\}}$ and $\hat{\mathcal{W}} := \overline{\{\hat{P}_i : i \in \mathbb{N}_+\}}$ yields a Brownian web $\mathcal{W}$ together with its *dual Brownian web* $\hat{\mathcal{W}}$.

If we concatenate the forward and dual paths coming out of $\{z_i : i \in \mathbb{N}_+\}$ and then take the closure, we obtain an object known as the *full Brownian web*, which is a random compact subset of the space $\Pi_c^{\updownarrow}$ of bi-infinite paths.

Dual Brownian web



Special points of the Brownian web are distinguished according to the numbers $(m_{\rm in}, m_{\rm out})$ of incoming and outgoing paths.

$$\hat{m}_{\rm out} = m_{\rm in} + 1 \quad \text{and} \quad m_{\rm out} = \hat{m}_{\rm in} + 1.$$

General webs

Aim We want to extend the theory of the Brownian web to more general stochastic flows with non-crossing *n*-point motions, including such that make jumps.

Weaves, webs and flows II

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Cadlag functions

The *split real line* is the set $\mathbb{R}_{\mathfrak{s}}$ consisting of all pairs $t\pm$ consisting of a real number $t \in \mathbb{R}$ and a sign $\pm \in \{-, +\}$.

For an element $\tau = t\pm$ of $\mathbb{R}_{\mathfrak{s}}$ we let $\underline{\tau} := t$ denote its real part and $\mathfrak{s}(\tau) := \pm$ its sign.

We equip $\mathbb{R}_{\mathfrak{s}}$ with the lexographic order, in which $\sigma \leq \tau$ if and only if $\underline{\sigma} < \underline{\tau}$ or $\underline{\sigma} = \underline{\tau}$ and $\mathfrak{s}(\sigma) \leq \mathfrak{s}(\tau)$.

We write $\sigma < \tau$ iff $\sigma \leq \tau$ and $\sigma \neq \tau$ and define intervals

$$
\begin{aligned}
(\sigma, \rho) &:= \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma < \tau < \rho\}, & [\sigma, \rho) &:= \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma \leq \tau < \rho\}, \\
(\sigma, \rho] &:= \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma < \tau \leq \rho\}, & [\sigma, \rho] &:= \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma \leq \tau \leq \rho\}.
\end{aligned}
$$

There is some redundency, e.g., $(s-, r+] = [s+, r+]$. We also write

$$
(\sigma, \infty) := \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma < \tau\}, \quad [\sigma, \infty) := \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma \leq \tau\}, \text{ etc.}
$$

Cadlag functions

We equip the split real line $\mathbb{R}_\mathfrak{s}$ with the *order topology*.
A basis for the topology is formed by all open intervals (σ, ρ) with $\sigma, \rho \in \mathbb{R}_\mathfrak{s}$, $\sigma < \rho$.

(i) $\tau_n \to t+$ iff $\underline{\tau}_n \to t$ and $\tau \geq t+$ for n sufficiently large.

(ii) $\tau_n \to t-$ iff $\underline{\tau}_n \to t$ and $\tau \leq t-$ for n sufficiently large.

Lemma $\mathbb{R}_\mathfrak{s}$ is first countable, Hausdorff and separable, but not second countable and not metrisable.

Lemma For $C \subset \mathbb{R}_\mathfrak{s}^d$, the following are equivalent:
(i) C is compact, (ii), C is sequentially compact,
(iii) C is closed and bounded.

Cadlag functions

Lemma Let $\mathcal{I} \subset \mathbb{R}_\mathfrak{s}$ be an interval and let $\mathcal{X}$ be a Hausdorff topological space. Then a function $f : \mathcal{I} \to \mathcal{X}$ is continuous iff:

(i) $f(\tau_n) \to f(t+)$ for all $\tau_n \in \mathcal{I}$ such that $\tau_n \to t$ and $\tau_n > t$ for all n.

(ii) $f(\tau_n) \to f(t-)$ for all $\tau_n \in \mathcal{I}$ such that $\tau_n \to t$ and $\tau_n < t$ for all n.

Let $\mathcal{I}^\pm := \{t \in \mathbb{R} : t\pm \in \mathcal{I}\}$ and define $f^\pm : \mathcal{I}^\pm \to \mathcal{X}$ by $f^\pm(t) := f(t\pm)$ $(t \in \mathcal{I}^\pm)$. Then f^+ is *cadlag* (right continuous with left limits) and f^- is *caglad* (left continuous with right limits).

Corollary A function $f : [0+, \infty) \to \mathcal{X}$ is continuous iff $f^+ : [0, \infty) \to \mathcal{X}$ is cadlag and $f^- : (0, \infty) \to \mathcal{X}$ is its caglad modification.

Remark Continuous functions $f : [0-, \infty) \to \mathcal{X}$ are similar, except that they can also jump at time zero.

Let $(\mathcal{X}, d)$ be a metric space and let

$$\mathcal{R}(\mathcal{X}) := (\mathcal{X} \times \mathbb{R}) \cup \big\{(*, -\infty), (*, \infty)\big\}.$$

Let $\overline{\mathbb{R}} := [-\infty, \infty]$, let $d_{\overline{\mathbb{R}}}$ generate the topology on $\overline{\mathbb{R}}$.
Let $\varphi : \overline{\mathbb{R}} \to [0, \infty)$ be continuous with $\varphi(t) > 0$ iff $t \in \mathbb{R}$.

Lemma

$$\begin{aligned} d_{\rm sqz}\big((x, s), (y, t)\big) := {} & \big(\varphi(s) \wedge \varphi(t)\big)\big(d(x, y) \wedge 1\big) \\ & + \big|\varphi(s) - \varphi(t)\big| + d_{\overline{\mathbb{R}}}(s, t) \end{aligned}$$

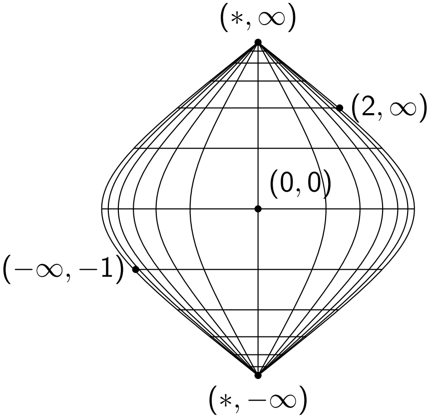
is a metric on $\mathcal{R}(\mathcal{X})$ such that $d_{\rm sqz}\big((x_n, t_n), (x, t)\big) \underset{n\to\infty}{\longrightarrow} 0$ iff

(i) $t_n \to t$,

(ii) if $t \in \mathbb{R}$, then $x_n \to x$.

The topology on $\mathcal{R}(\mathcal{X})$ does not depend on the choice of the metric on $\mathcal{X}$.

Squeezed space



A picture of $\mathcal{R}(\overline{\mathbb{R}})$.

Squeezed space

Lemma If $(\mathcal{X}, d)$ is separable, then so is $(\mathcal{R}(\mathcal{X}), d_{\rm sqz})$.
If $(\mathcal{X}, d)$ is complete, then so is $(\mathcal{R}(\mathcal{X}), d_{\rm sqz})$.

Lemma $A \subset \mathcal{R}(\mathcal{X})$ is compact iff $\forall T < \infty\ \exists K \in \mathcal{K}(\mathcal{X})$ s.t.

$$A \cap \big(\mathcal{X} \times [-T, T]\big) \subset K \times [-T, T].$$

Path space

Def A *path* π in a metrisable space $\mathcal{X}$ with *starting time* σ_π and *final time* τ_π is a continuous function $\pi : I_{\mathfrak{s}}(\pi) \to \mathcal{X}$, where
$\bar{I}(\pi) := [\sigma_\pi, \tau_\pi]$, $I(\pi) := \bar{I}(\pi) \cap \mathbb{R}$,
$I_{\mathfrak{s}}(\pi) := \big\{ t\pm : t \in I(\pi),\ \pm \in \{-,+\} \big\}$.

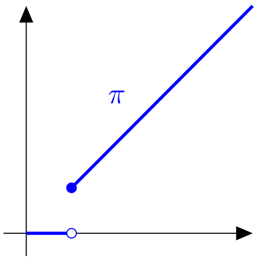
$\Pi = \Pi(\mathcal{X}) :=$ the set of all paths in $\mathcal{X}$.
$\Pi_{\rm c} := \big\{ \pi \in \Pi : \pi(t-) = \pi(t+)\ \forall t \in I(\pi) \big\}$,
$\Pi^{\uparrow} := \big\{ \pi \in \Pi : \tau_\pi = \infty \big\}$, $\Pi^{\downarrow} := \big\{ \pi \in \Pi : \sigma_\pi = -\infty \big\}$.

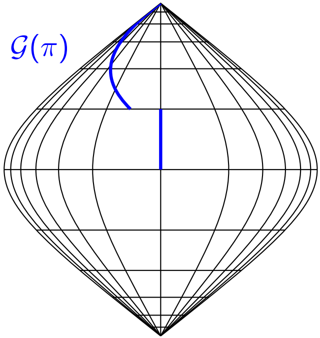
Def The *closed graph* of π is the set

$$\mathcal{G}(\pi) := \big\{ \big(\pi(t\pm), t\big) : t \in I(\pi) \big\} \cup \big\{ (*, \pm\infty) : \pm\infty \in \bar{I}(\pi) \backslash I(\pi) \big\}.$$

Lemma $\mathcal{G}(\pi)$ is a compact subset of $\mathcal{R}(\mathcal{X})$.

Path space





A path and its graph.

Path space

For $a, b \in \overline{\mathbb{R}}$, let

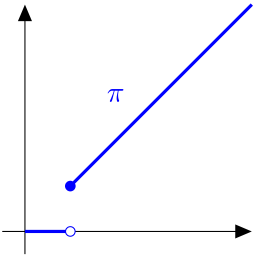
$$\langle a, b \rangle := \begin{cases} [a, b] & \text{if } a \le b, \\ [b, a] & \text{if } b \le a. \end{cases}$$

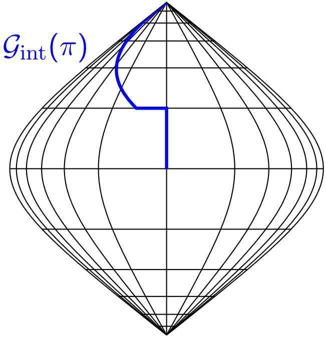
Def The *interpolated graph* of $\pi \in \Pi(\overline{\mathbb{R}})$ is the set

$$\begin{aligned} \mathcal{G}_{\rm int}(\pi) := \big\{(x, t) : t \in I(\pi),\ x \in \langle \pi(t-), \pi(t+) \rangle\big\} \\ \cup \big\{(*, \pm\infty) : \pm\infty \in \overline{I}(\pi) \backslash I(\pi)\big\}. \end{aligned}$$

Lemma $\mathcal{G}_{\rm int}(\pi)$ is a compact subset of $\mathcal{R}(\overline{\mathbb{R}})$.

Path space





A path and its interpolated graph.

The Hausdorff metric

Let $(\mathcal{X}, d)$ be a metric space. Let $d(x, A) := \inf\{d(x, y) : y \in A\}$.

Let $\mathcal{K}(\mathcal{X})$ be the set of all compact subsets of $\mathcal{X}$ and let $\mathcal{K}_+(\mathcal{X}) := \{K \in \mathcal{K}(\mathcal{X}) : K \neq \emptyset\}$.

The *Hausdorff metric* on $\mathcal{K}_+(\mathcal{X})$ is defined as

$$d_{\mathrm{H}}(K_1, K_2) := \sup_{x_1 \in K_1} d(x_1, K_2) \vee \sup_{x_2 \in K_2} d(x_2, K_1).$$

A *correspondence* between A_1, A_2 is a set $R \subset A_1 \times A_2$ such that:

$$\forall (i, j) \in \{(1, 2), (2, 1)\},\ x_i \in A_i\ \exists x_j \in A_j \text{ s.t. } (x_i, x_j) \in R.$$

Let $\mathrm{Cor}(A_1, A_2)$ denote the set of all correspondences between A_1, A_2. Then

$$d_{\mathrm{H}}(K_1, K_2) = \inf_{R \in \mathrm{Cor}(K_1, K_2)} \sup_{(x_1, x_2) \in R} d(x_1, x_2).$$

The Hausdorff metric

Lemma If $(\mathcal{X}, d)$ is separable, then so is $(\mathcal{K}_+(\mathcal{X}), d_{\rm H})$.
If $(\mathcal{X}, d)$ is complete, then so is $(\mathcal{K}_+(\mathcal{X}), d_{\rm H})$.

Lemma A set $\mathcal{A} \subset \mathcal{K}_+(\mathcal{X})$ is compact iff there exists a compact $C \subset \mathcal{X}$ such that $K \subset C$ for all $K \in \mathcal{A}$.

Lemma Let $K_n \in \mathcal{K}_+(\mathcal{X})$ and let

$$\begin{aligned}
\mathrm{Lim}\big((K_n)\big) &:= \big\{x \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } x_n \underset{n\to\infty}{\longrightarrow} x\big\},\\
\mathrm{Clus}\big((K_n)\big) &:= \big\{x \in \mathcal{X} : \exists\ n(k)\to\infty,\ x_{n(k)} \in K_{n(k)} \text{ s.t. } x_{n(k)} \underset{k\to\infty}{\longrightarrow} x\big\}.
\end{aligned}$$

Then $d_{\rm H}(K_n, K) \underset{n\to\infty}{\longrightarrow} 0$ iff

(i) $\exists C \subset \mathcal{K}_+(\mathcal{X})$ s.t. $K_n \subset C$ $\forall n$,

(ii) $\mathrm{Lim}(K) = K = \mathrm{Clus}(K)$.

The topology on $\mathcal{K}_+(\mathcal{X})$ does not depend on the choice of the metric on $\mathcal{X}$.

Topologies on path space

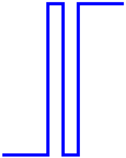
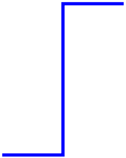
Ideas (not so good) Skorohod's J2 topology is generated by the metric on $\Pi(\overline{\mathbb{R}})$ defined as:

$$d_{\mathrm{J2}}(\pi_1,\pi_2) := d_{\mathrm{H}}\big(\mathcal{G}(\pi_1),\mathcal{G}(\pi_2)\big).$$

Skorohod's M2 topology is generated by:

$$d_{\mathrm{M2}}(\pi_1,\pi_2) := d_{\mathrm{H}}\big(\mathcal{G}_{\mathrm{int}}(\pi_1),\mathcal{G}_{\mathrm{int}}(\pi_2)\big).$$

Problem of these topologies:

We do not want  to be close to 

Topologies on path space

Idea Both $\mathcal{G}(\pi)$ and $\mathcal{G}_{\mathrm{int}}(\pi)$ are naturally equipped with a total order:

$$(x_1,t_1) \leq (x_2,t_2) \quad \Leftrightarrow$$

$$t_1 < t_2 \text{ or } t_1 = t_2 =: t \text{ and } x_2 \text{ lies closer to } \pi(t+) \text{ than } x_1.$$

Let R be a correspondence between totally ordered sets A_1, A_2.

Def R is *monotone* if there do not exist $(x_1,x_2) \in R$ and $(y_1,y_2) \in R$ such that $x_1 < y_1$ but $y_2 < x_2$.

Let $\mathrm{Cor}_+(A_1,A_2)$ denote the set of monotone correspondences and set

$$d_{\mathrm{H}+}(A_1,A_2) := \inf_{R \in \mathrm{Cor}_+(A_1,A_2)} \sup_{(x_1,x_2)\in R} d(x_1,x_2).$$

Topologies on path space

Alternative idea For a totally ordered set A, define $A^{\leq} \subset \mathcal{X}^2$ by

$$A^{\leq} := \big\{(x, y) \in A^2 : x \leq y\big\},$$

equip $\mathcal{X}^2$ with a metric that generates the product topology and set

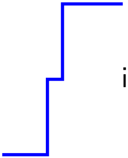
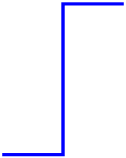
$$d_{\mathrm{H}\leq}(A_1, A_2) := d_{\mathrm{H}}(A_1^{\leq}, A_2^{\leq}).$$

It seems both approaches yield the same topology. The metrics

$$d_{\mathrm{J1}}(\pi_1, \pi_2) := d_{\mathrm{H+}}\big(\mathcal{G}(\pi_1), \mathcal{G}(\pi_2)\big),$$
$$d_{\mathrm{M1}}(\pi_1, \pi_2) := d_{\mathrm{H+}}\big(\mathcal{G}_{\mathrm{int}}(\pi_1), \mathcal{G}_{\mathrm{int}}(\pi_2)\big),$$

generate topologies on Π that correspond to Skorohod's J1 and M1 topologies.

Note

 is d_{M1}-close but not d_{J1}-close to 

Topologies on path space

Recall that a Polish space is a separable topological space such that there exists a complete metric generating the topology.

Theorem If $\mathcal{X}$ is a Polish space, then $\Pi(\mathcal{X})$, equipped with the J1-topology, is also a Polish space. Moreover, $\Pi(\overline{\mathbb{R}})$ and $\Pi(\mathbb{R})$, equipped with the M1-topogy, are Polish spaces.

Lemma The subset $\Pi_{\rm c}(\overline{\mathbb{R}})$ of $\Pi(\overline{\mathbb{R}})$ is closed w.r.t. the J1 topology, but not w.r.t. the M1 topology. The J1 and M1 topologies induce the same topology on $\Pi_{\rm c}(\overline{\mathbb{R}})$, which corresponds to locally uniform convergence.

Topologies on path space

The *modulus of continuity* is defined as

$$m_{T,\delta}(\pi) := \sup\big\{d\big(\pi(\sigma),\pi(\tau)\big) : -T < \underline{\sigma} < \underline{\tau} < T,\ \underline{\tau} - \underline{\sigma} \leq \delta\big\}.$$

The *J1-modulus of continuity* is defined as

$$\begin{aligned} m^{\mathrm{J1}}_{T,\delta}(\pi) := \sup\big\{\, & d\big(\pi(\sigma),\pi(\tau)\big) \wedge d\big(\pi(\tau),\pi(\rho)\big) : \\ & -T < \underline{\sigma} < \underline{\tau} < \underline{\rho} < T,\ \underline{\rho} - \underline{\sigma} \leq \delta\big\}. \end{aligned}$$

The *M1-modulus of continuity* is defined as

$$\begin{aligned} m^{\mathrm{M1}}_{T,\delta}(\pi) := \sup\big\{\, & d\big(\pi(\tau),\langle\pi(\sigma),\pi(\rho)\rangle\big) : \\ & -T < \underline{\sigma} < \underline{\tau} < \underline{\rho} < T,\ \underline{\rho} - \underline{\sigma} \leq \delta\big\}. \end{aligned}$$

where as before

$$\langle a, b\rangle := \begin{cases} [a,b] & \text{if } a \leq b, \\ [b,a] & \text{if } b \leq a. \end{cases}$$

Topologies on path space

Def A set $\mathcal{A} \subset \Pi(\mathcal{X})$ is *compactly contained* if

$$\forall T < \infty \; \exists C \in \mathcal{K}(\mathcal{X}) \text{ s.t. } \pi(t\pm) \in C \; \forall t \in [-T, T] \cap I(\pi).$$

A set $\mathcal{A} \subset \Pi_{\rm c}(\mathcal{X})$ is *equicontinuous* if

$$\lim_{\delta \to 0} \sup_{\pi \in \mathcal{A}} m_{T,\delta}(\pi) = 0 \qquad (T < \infty).$$

We define *J1-equicontinuity* and *M1-equicontinuity* similarly.

Arzela-Ascoli A set $\mathcal{A} \subset \Pi_{\rm c}(\mathcal{X})$ is precompact iff it is compactly contained and equicontinuous.

Theorem A set $\mathcal{A} \subset \Pi_{\rm c}(\mathcal{X})$ is precompact w.r.t. the J1-topology iff it is compactly contained and J1-equicontinuous.

Theorem A set $\mathcal{A} \subset \Pi_{\rm c}(\mathbb{R})$ is precompact w.r.t. the M1-topology iff it is compactly contained and M1-equicontinuous.