Pathwise duality for monotone interacting particle systems.

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Transitive graphs

Let $G = (V, E)$ be a graph with vertex set V and edge set E.

An *automorphism* of G is a bijection $\psi : V \to V$ such that $\{\psi(v), \psi(w)\} \in E \;\Leftrightarrow\; \{v, w\} \in E$.

We say that G is *vertex transitive* if for each $v, w \in V$ there exists an automorphism ψ such that $\psi(v) = w$.

If G is vertex transitive then each vertex v has the same degree

$$\Delta := \#\big\{w \in V : \{v, w\} \in E\big\}.$$

We call Δ the degree of G.

A cooperative contact process

Let $G = (V, E)$ be a vertex transitive graph of degree $\Delta \geq 2$.
Let $S := \{0,1\}^V$ be the space of all functions $x : V \to \{0,1\}$.
Vertices with $x(i) = 0$ and $= 1$ are called *vacant* and *occupied*.
Let $\alpha \in [0,1]$ and $\delta \geq 0$ be constants.

Consider the Markov process $(X_t)_{t\geq 0}$ with state space S that evolves as follows:

- With rate $1 - \alpha$, each vacant vertex selects one neighbouring vertex uniformly at random and becomes occupied if this neighbour is occupied.
- With rate α, each vacant vertex selects two neighbouring vertices uniformly at random from all ways to do so, and becomes occupied if both these neighbours are occupied.
- With rate δ, each occupied vertex becomes vacant.

A cooperative contact process

Let $S_{\rm fin} := \{x \in S : |x| < \infty\}$ with $|x| := \sum_{i \in S} x(i)$.

Let $\underline{0}, \underline{1} \in S$ denote the all zero and all one configurations.
We say that the process with parameters α and δ *survives* if

$$\mathbb{P}^x\big[X_t \neq \underline{0}\ \forall t \geq 0\big] > 0 \quad \text{for some } x \in S_{\rm fin}.$$

Standard theory for monotone systems implies that

$$\mathbb{P}^{\underline{1}}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu},$$

where $\overline{\nu}$ is the *upper invariant law*. For all values of α and δ,

- either $\overline{\nu}$ is *trivial* in the sense that $\overline{\nu} = \delta_{\underline{0}}$,
- or $\overline{\nu}$ is *nontrivial* in the sense that $\overline{\nu}(\{\underline{0}\}) = 0$.

If $\overline{\nu}$ is nontrivial, then we say the process is *stable*.

The noncooperative case

For $\alpha = 0$ the process is a *contact process* with infection rate 1 and death rate δ.

Self-duality of the contact process implies:

$$\text{stability} \Leftrightarrow \text{survival}.$$

In fact

$$\mathbb{P}^x\big[X_t \neq \underline{0}\ \forall t \geq 0\big] = \int \overline{\nu}(\mathrm{d}y) 1_{\{x \wedge y \neq \underline{0}\}}.$$

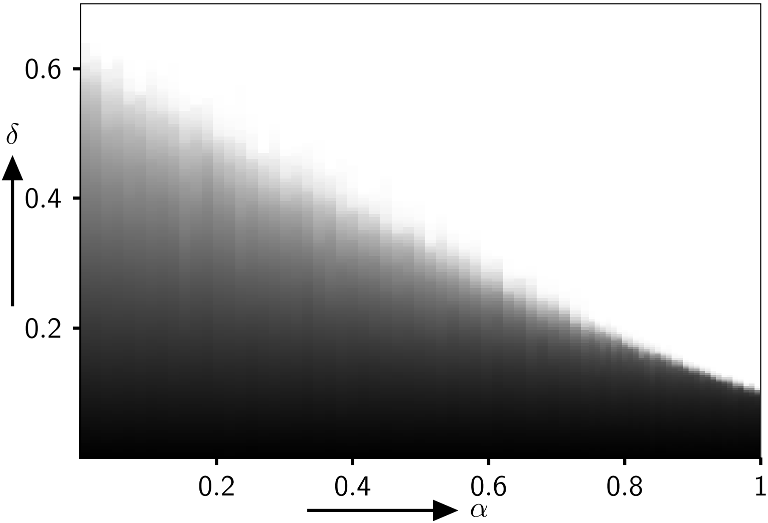
The cooperative case

For $\alpha = 1$ the process has only *cooperative branching*.

On $G = \mathbb{Z}^d$ with nearest-neighbour edges, the process cannot escape (hyper-) rectangles and hence does not survive for any $\delta > 0$.

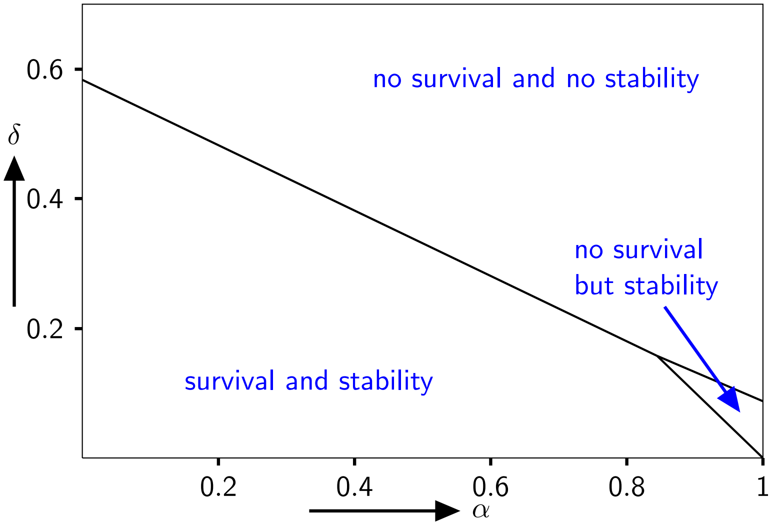
On the other hand, a result of Lawrence Gray (1999) shows that in dimensions $d \geq 2$ the upper invariant law is nontrivial for δ small enough.

Numerical data



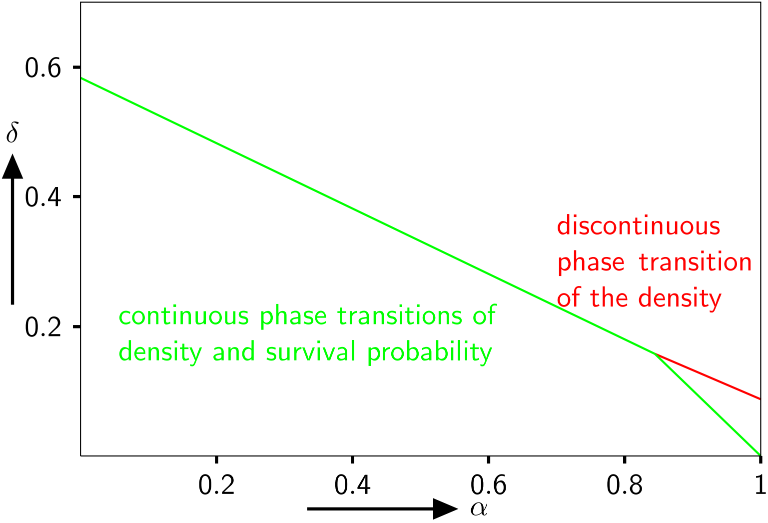
Density of the upper invariant law for the process on $\mathbb{Z}^2$.

Conjectured phase diagram



Conjectured phase diagram for the process on $\mathbb{Z}^2$.

Conjectured phase diagram



Conjectured phase diagram for the process on $\mathbb{Z}^2$.

Pathwise construction

Define *branching maps*, *cooperative branching maps*, and *death maps* by

$$\mathtt{bra}_{ij}(x)(k) := \begin{cases} x(i) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

$$\mathtt{cob}_{ii'j}(x)(k) := \begin{cases} \big(x(i) \wedge x(i')\big) \vee x(j) & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

$$\mathtt{dth}_{j}(x)(k) := \begin{cases} 0 & \text{if } k = j, \\ x(k) & \text{otherwise.} \end{cases}$$

Pathwise construction

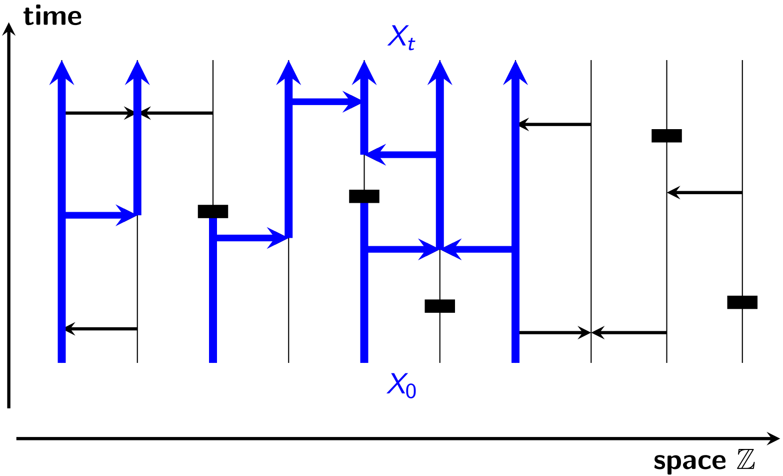
Let $\mathcal{N}_j := \{i \in V : \{i, j\} \in E\}$
and $\mathcal{N}_j^2 := \{(i, i') \in V : \{i, j\} \in E,\ \{i, j\} \in E,\ i \neq i'\}$.

Then the generator of the process can be written as

$$\begin{aligned}Lf(x) := (1-\alpha)\sum_{j\in V}\frac{1}{|\mathcal{N}_j|}\sum_{i\in\mathcal{N}_j}\big\{f\big(\mathtt{bra}_{ij}(x)\big) - f\big(x\big)\big\}\\ +\alpha\sum_{j\in V}\frac{1}{|\mathcal{N}_j^2|}\sum_{(i,i')\in\mathcal{N}_j^2}\big\{f\big(\mathtt{cob}_{ii'j}(x)\big) - f\big(x\big)\big\}\\ +\delta\sum_{j\in V}\big\{f\big(\mathtt{dth}_j(x)\big) - f\big(x\big)\big\}.\end{aligned}$$

We wish to construct the process from a *graphical representation* by applying these maps at Poisson times.

Pathwise construction



Pathwise construction

We equip $S = \{0,1\}^V$ with the product topology.
Let T be a finite set and $f : S \to T$.
We define the set of *relevant* vertices for $f : S \to T$ as

$$\mathcal{R}(f) := \big\{ i \in V : \exists x \in S \text{ s.t. } f(x^{i,0}) \neq f(x^{i,1}) \big\}.$$

with

$$x^{i,\sigma}(k) := \begin{cases} \sigma & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Lemma $f : S \to T$ is continuous if and only if

(i) $\mathcal{R}(f)$ is finite,

(ii) $x = y$ on $\mathcal{R}(f)$ implies $f(x) = f(y)$.

Pathwise construction

For a map $m : S \to S$ and $i \in V$, we define $m[i] : S \to \{0, 1\}$ by

$$m[i](x) := m(x)(i) \qquad (x \in S,\ i \in V).$$

We denote the set of vertices that can be affected by m by

$$\mathcal{D}(m) := \big\{i \in V : \exists x \in S \text{ s.t. } m(x)(i) \neq x(i)\big\}.$$

Def A map $m : S \to S$ is *local* if

(i) m is continuous,

(ii) $\mathcal{D}(m)$ is finite.

Based on information from finitely many vertices, finitely many vertices change their value.

Pathwise construction

Consider a generator of the form

$$Lf(x) := \sum_{m\in\mathcal{G}} r_m\big\{f\big(m(x)\big) - f\big(x\big)\big\},$$

where $\mathcal{G}$ is a countable collection of local maps
and $(r_m)_{m\in\mathcal{G}}$ are nonnegative rates.

Let μ be the measure on $\mathcal{G}\times\mathbb{R}$ defined by

$$\mu\big(\{m\}\times[s,t]\big) := r_m(t-s) \qquad (m\in\mathcal{G},\ s\leq t).$$

Let ω be a Poisson point set on $\mathcal{G}\times\mathbb{R}$ with intensity μ.

Elements of ω are pairs (m,t) that have the interpretation that at time t, the local map m should be applied.

Pathwise construction

Let $\omega_{s,u} := \{(m,t) \in \omega : s < t \leq u\} \qquad (s \leq u)$.
If μ is a finite measure, then we can define random maps $(\mathbb{X}_{s,u})_{s \leq u}$ by

$$\mathbb{X}_{s,u} := m_n \circ \cdots \circ m_1$$
$$\text{with} \quad \omega_{s,u} := \{(m_1,t_1),\ldots,(m_n,t_n)\}$$
$$\text{and} \quad t_1 < \cdots < t_n.$$

The random maps $(\mathbb{X}_{s,u})_{s \leq u}$ form a *stochastic flow:*

$$\mathbb{X}_{s,s} = 1 \quad \text{and} \quad \mathbb{X}_{t,u} \circ \mathbb{X}_{s,t} = \mathbb{X}_{s,u} \qquad (s \leq t \leq u).$$

Let X_0 be independent of ω, and let $s \in \mathbb{R}$. Then

$$X_t := \mathbb{X}_{s,s+t}(X_0) \qquad (t \geq 0)$$

defines a Markov process $(X_t)_{t \geq 0}$ with generator L.

Pathwise construction

More generally, assume that

$$\sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G}\\ \mathcal{D}(m)\ni i}} r_m\big(1+|\mathcal{R}(m([i]))|\big) < \infty.$$

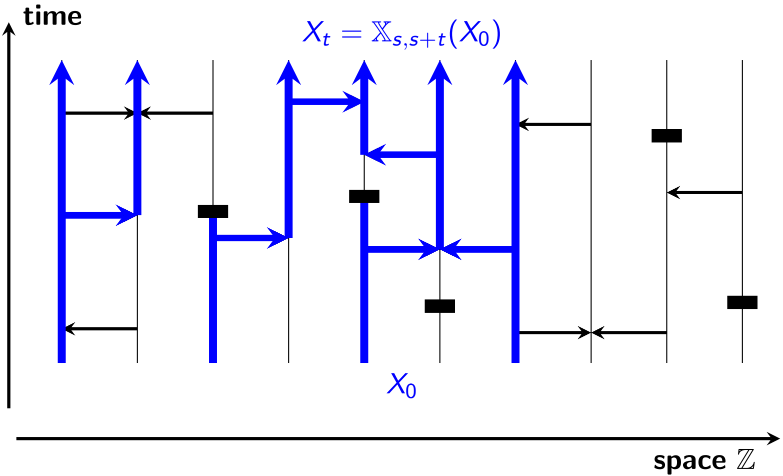
Then almost surely, for each $i \in V$ and $u \in \mathbb{R}$, there exists a piecewise constant, left-continuous process

$$\big(\mathbb{X}_{u-t,u}[i]\big)_{t\geq 0}$$

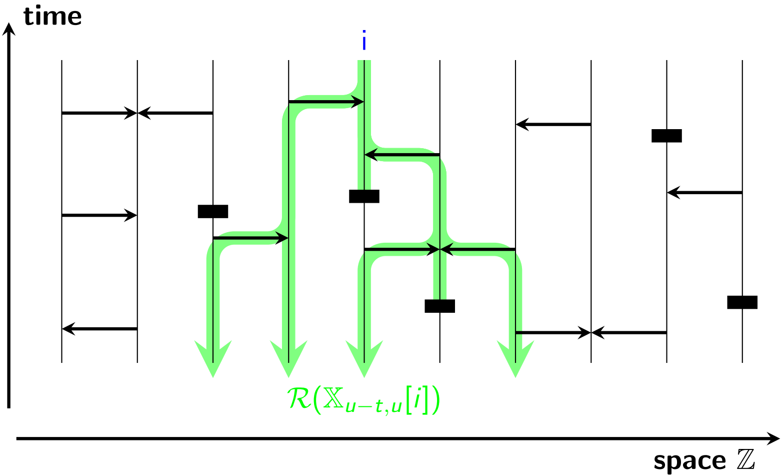
taking values in the (countable!) space of continuous functions $\mathcal{C}\big(S,\{0,1\}\big)$, such that for all $\varepsilon > 0$ small enough

$$\mathbb{X}_{u-t-\varepsilon,u}[i] = \begin{cases} \mathbb{X}_{u-t,u}[i]\circ m & \text{if } \exists (m,t)\in\omega \text{ s.t.} \\ & \mathcal{D}(m)\cap\mathcal{R}(\mathbb{X}_{u-t,u}[i]) \neq \emptyset, \\ \mathbb{X}_{u-t,u}[i] & \text{otherwise.} \end{cases}$$

Pathwise construction



Pathwise construction



The dual process

Recall that $S := \{0,1\}^V$. We set $T := \{0,1\}$ and let $\mathcal{L}_+(S,T)$ denote the set of functions $f : S \to T$ that are lower semi-continuous, monotone, and satisfy $f(\underline{0}) = \underline{0}$.

Let $S^+_{\rm fin} := \big\{x \in S : 0 < |x| < \infty\big\}$. A *minimal element* of a set $Y \subset S^+_{\rm fin}$ is an $y \in Y$ such that $\nexists y' \neq y,\ y' \leq y,\ y' \in Y$. We set

$$Y^\circ := \big\{y \in Y : y \text{ is minimal}\big\},$$

$$\mathcal{H} := \big\{Y \subset S^+_{\rm fin} : Y^\circ = Y\big\}.$$

Proposition Each $Y \in \mathcal{H}$ defines a function $f_Y \in \mathcal{L}_+(S,T)$ via

$$f_Y(x) := 1_{\{\exists y \in Y \text{ s.t. } x \geq y\}} \qquad (x \in S),$$

and $\mathcal{H} \ni Y \mapsto f_Y \in \mathcal{L}_+(S,T)$ is a bijection.

The dual process

For any $f \in \mathcal{L}_+(S,T)$, let

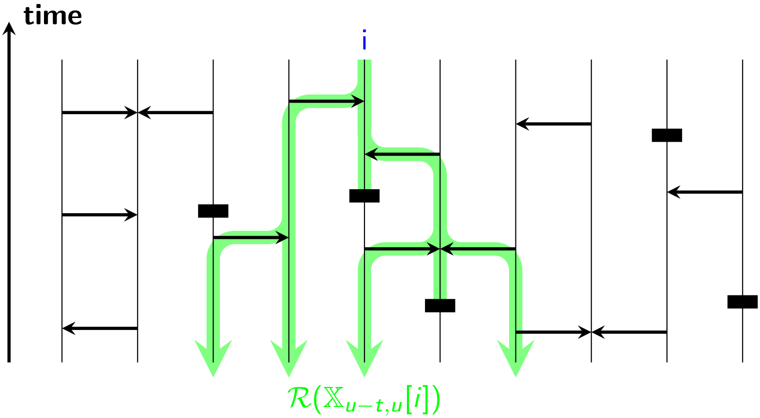
$$Y(f) := \big\{x \in S^+_{\rm fin} : f(x) = 1\big\}^\circ$$

denote the set of minimal configurations y such that $f(y) = 1$. Then $f \mapsto Y(f)$ is the inverse of $Y \mapsto f_Y$.

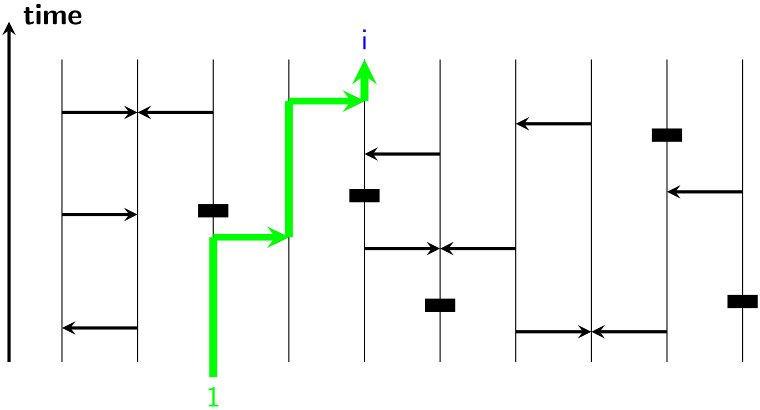
Set $\mathcal{H}_{\rm fin} := \big\{Y \in \mathcal{H} : |Y| < \infty\big\}$
and $\mathcal{C}_+(S,T) := \big\{f \in \mathcal{L}_+(S,T) : f \text{ is continuous}\big\}$.

Then $f \in \mathcal{C}_+(S,T) \;\Leftrightarrow\; Y(f) \in \mathcal{H}_{\rm fin}$.

Percolation picture



Percolation picture



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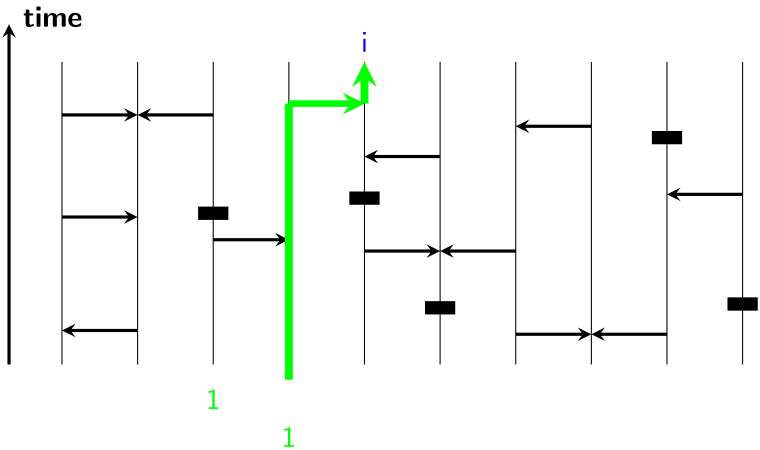
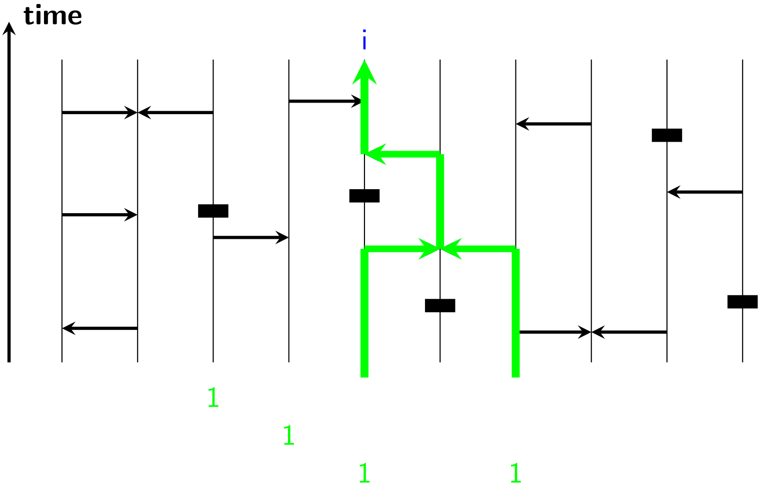


Percolation picture



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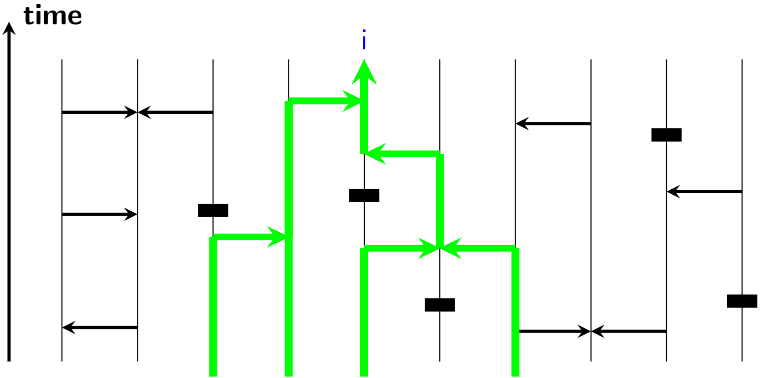


y_1	0	0	1	0	0	0	0	0	0	0
y_2	0	0	0	1	0	0	0	0	0	0
y_3	0	0	0	0	1	0	1	0	0	0

$$Y(\mathbb{X}_{u-t,u}[i]) = \{y_1, y_2, y_3\}$$

Duality

Define $\psi : S \times \mathcal{H} \to T$ by

$$\psi(x, Y) := 1_{\{\exists y \in Y \text{ s.t. } x \geq y\}} \qquad (x \in S,\ \ Y \in \mathcal{H}).$$

There exists random maps $\mathbb{Y}_{u,s} : \mathcal{H} \to \mathcal{H}$ $(u \geq s)$ such that

$$\psi\big(\mathbb{X}_{s,u}(x), Y\big) = \psi\big(x, \mathbb{Y}_{u,s}(Y)\big) \qquad (x \in S,\ \ Y \in \mathcal{H},\ \ s \leq u).$$

These form a *dual stochastic flow*

$$\mathbb{Y}_{s,s} = 1 \quad \text{and} \quad \mathbb{Y}_{t,s} \circ \mathbb{Y}_{u,t} = \mathbb{Y}_{u,s} \qquad (u \geq t \geq s).$$

If Y_0 is independent of $(\mathbb{Y}_{u,s})_{u \geq s}$ and $u \in \mathbb{R}$, then

$$Y_t := \mathbb{Y}_{u-t,u}(Y_0) \qquad (t \geq 0)$$

defines a Markov process $(Y_t)_{t \geq 0}$ with state space $\mathcal{H}$.

Duality

There exists a metrisable topology on $\mathcal{H}$ such that

$$Y_n \to Y \quad \Leftrightarrow \quad \psi(x, Y_n) \to \psi(x, Y) \quad \forall x \in S_{\rm fin}.$$

The space $\mathcal{H}$ is compact under this topology.

We define a partial order on $\mathcal{H}$ by

$$Y_1 \leq Y_2 \quad \Leftrightarrow \quad \psi(x, Y_1) \leq \psi(x, Y_2).$$

The least element of $\mathcal{H}$ in this order is $\emptyset$ and the greatest element is

$$\top := \big\{ 1_{\{i\}} : i \in V \big\}.$$

One has

$$\mathbb{P}^\top \big[Y_t \in \cdot \, \big] \underset{t \to \infty}{\Longrightarrow} \overline{\mu},$$

where $\overline{\mu}$ is the *upper invariant law* of the dual process.

Survival and stability

- Either $\overline{\mu}$ is *trivial* in the sense that $\overline{\mu} = \delta_\emptyset$,
- or $\overline{\mu}$ is *nontrivial* in the sense that $\overline{\mu}(\{\emptyset\}) = 0$.

If $\overline{\mu}$ is nontrivial, then we say the dual process $(Y_t)_{t\geq 0}$ is *stable*.

We say that the dual process *survives* if

$$\mathbb{P}^Y\big[Y_t \neq \emptyset \ \forall t \geq 0\big] > 0 \quad \text{for some } Y \in \mathcal{H}_{\rm fin}.$$

Theorem [Gray '86, Latz & S. '23] One has

$$\begin{aligned} X \text{ is stable} \quad &\Leftrightarrow \quad Y \text{ survives},\\ X \text{ survives} \quad &\Leftrightarrow \quad Y \text{ is stable} \end{aligned}$$

The main novelty of our work is the construction of $(Y_t)_{t\geq 0}$ for infinite initial states.

Additive duality

Let $\mathcal{H}_1 := \{Y \in \mathcal{H} : |y| = 1 \ \forall y \in Y\}$.
We can naturally identify $Y \in \mathcal{H}_1$ with $x \in S$ defined as

$$x(i) = 1 \quad \Leftrightarrow \quad 1_{\{i\}} \in Y.$$

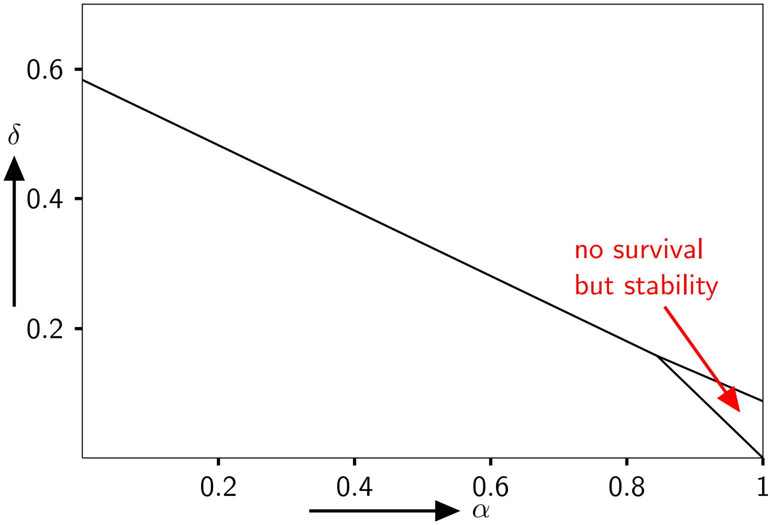
If all maps $m \in \mathcal{G}$ are *additive* in the sense that

$$m(x \vee y) = m(x) \vee m(y) \qquad (x, y \in S),$$

then the dual stochastic flow $(\mathbb{Y}_{u,s})_{u \geq s}$ maps $\mathcal{H}_1$ into itself and the corresponding Markov process on $\mathcal{H}_1 \cong S$ is itself an additive particle system.

In particular, the contact process is additive and its dual is also a contact process.

Conjectured phase diagram



We conjectured a regime where X is stable but does not survive.

Stability without survival

If X is stable but does not survive, then Y is unstable but survives. We believe that for each $i \in \mathbb{Z}^2$,

$$\mathbb{P}^{\{1_{\{i\}}\}}\big[Y_t \neq \emptyset \ \forall t \geq 0\big] > 0,$$
$$\mathbb{P}^{\{1_{\{i\}}\}}\big[Y_t \underset{t\to\infty}{\longrightarrow} \emptyset\big] = 1.$$

On the event $\{Y_t \neq \emptyset \ \forall t \geq 0\}$, we conjecture that

$$\inf\big\{|y| : y \in Y_t\big\} \underset{t\to\infty}{\longrightarrow} \infty \qquad \text{a.s.}$$

This means that as $t \to \infty$, the maps $\mathbb{X}_{u-t,t}[i] \in \mathcal{C}_+(S, T)$ have the property that $\mathbb{X}_{u-t,t}[i](x) = 1$ requires $|x|$ to be ever larger.

The mean-field limit

Let $G = (V, E)$ be the *complete graph* $G = K_N$ with N vertices.

In the *mean-field limit* $N \to \infty$, the frequency of ones

$$P_t := \frac{1}{N} \sum_{i \in V} X_t(i)$$

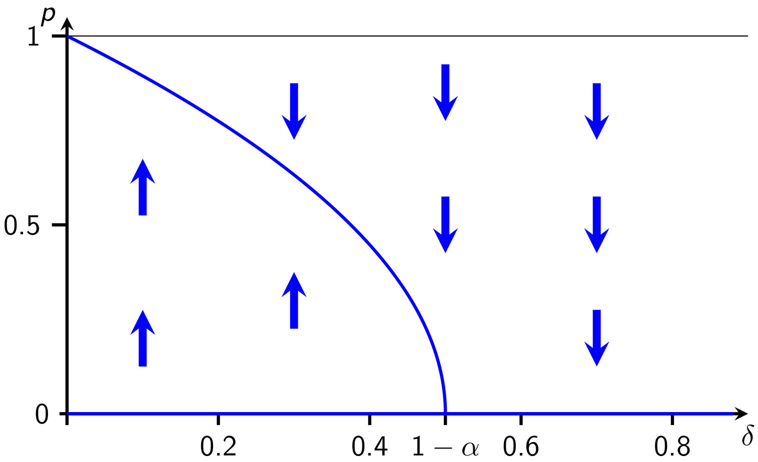
solves the *mean-field equation*

$$\frac{\partial}{\partial t} p_t = \alpha p_t^2 (1 - p_t) + (1 - \alpha) p_t (1 - p_t) - \delta p_t. \qquad (1)$$

Define a (nonlinear) semigroup $(\mathbf{T}_t)_{t \geq 0}$ of operators acting on $[0, 1]$ by

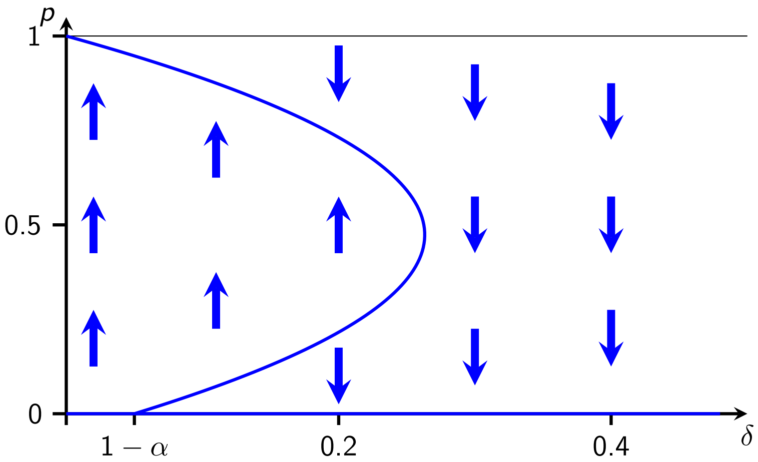
$$\mathbf{T}_t(p) := p_t \quad \text{where } (p_t)_{t \geq 0} \text{ solves (1) with } p_0 = p.$$

The mean-field limit



Fixed points of the mean-field equation for $\alpha = 0.5$ and their domains of attraction, as a function of δ.

The mean-field limit



Fixed points of the mean-field equation for $\alpha = 0.95$ and their domains of attraction, as a function of δ.

A mean-field dual

Recall $\mathbf{T}_t(p) := p_t$ where $(p_t)_{t\geq 0}$ solves (1) with $p_0 = p$.

Our aim is to show that there exist random maps $F_t : \{0,1\}^{\nabla\mathbb{S}_t} \to \{0,1\}$ such that

$$\mathbf{T}_t(p) := \mathbb{P}\big[F_t\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_t}\big) = 1\big],$$

where $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_t}$ are i.i.d. with intensity p.

We can think of F_t as the mean-field limit of $\mathbb{X}_{u-t,t}[i]$.

A mean-field dual

For (notational) simplicity we concentrate on the case $\alpha = 1$ (only cooperative branching).

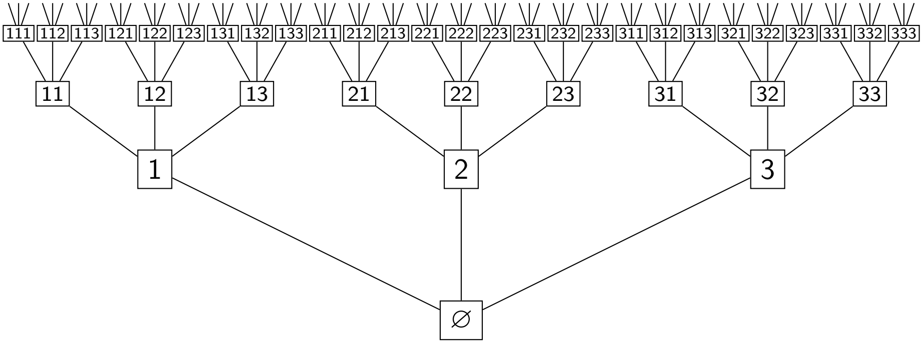
Define *cob* : $\{0,1\}^3 \to \{0,1\}$, and *dth* : $\{0,1\}^0 \to \{0,1\}$ by

$$cob(x_1, x_2, x_3) := x_1 \vee (x_2 \wedge x_3),$$
$$dth(\varnothing) := 0.$$

We set $\kappa(cob) := 3$ and $\kappa(dth) := 0$.

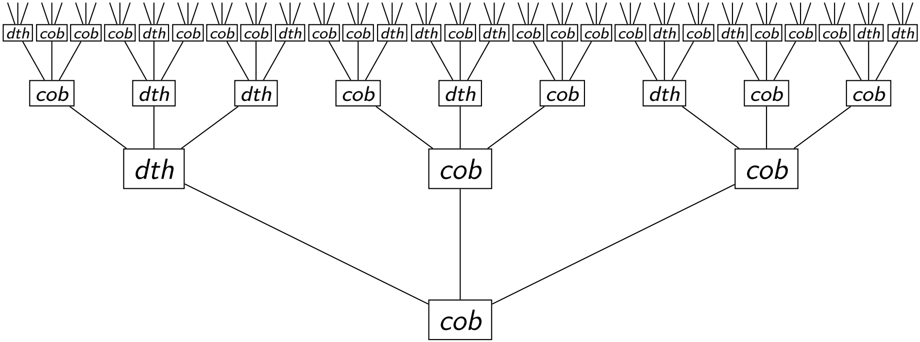
Let $\mathbb{T} = \mathbb{T}^3$ denote the space of all words $\mathbf{i} = i_1 \cdots i_n$ made from the alphabet $\{1, 2, 3\}$.

A recursive tree representation



We view $\mathbb{T}^3$ as a tree with root $\varnothing$, the word of length zero.

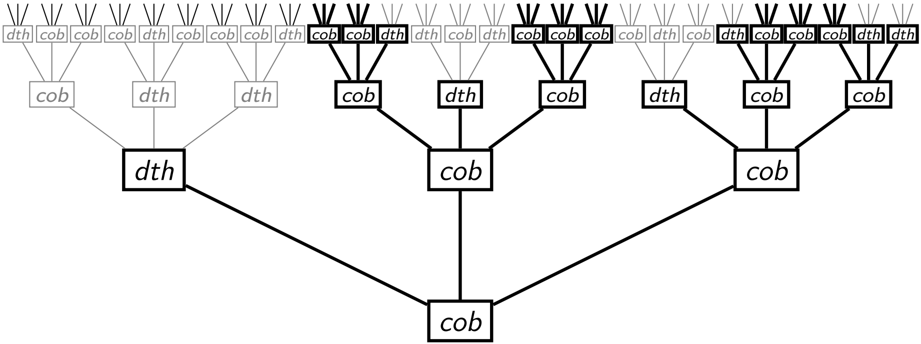
A recursive tree representation



We attach i.i.d. maps $(\gamma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ to the nodes, with

$$\mathbb{P}\big[\gamma_{\mathbf{i}} = cob\big] = \frac{1}{1+\delta} \quad \text{and} \quad \mathbb{P}\big[\gamma_{\mathbf{i}} = dth\big] = \frac{\delta}{1+\delta}.$$

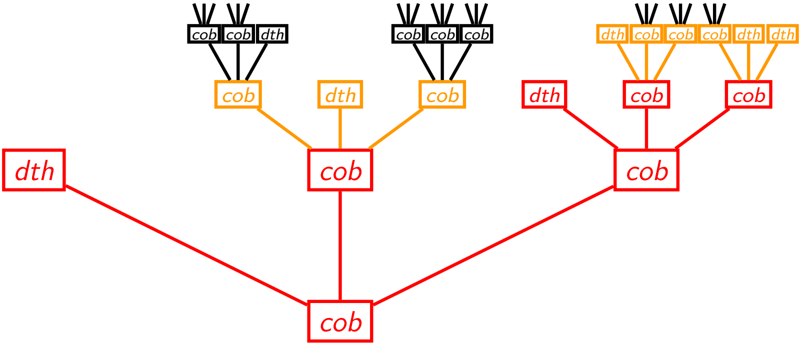
A recursive tree representation



Let $\mathbb{S}$ be the random subtree of $\mathbb{T}$ defined as

$$\mathbb{S} := \big\{ i_1 \cdots i_n \in \mathbb{T} : i_m \leq \kappa(\gamma_{i_1 \cdots i_{m-1}}) \; \forall 1 \leq m \leq n \big\}.$$

A recursive tree representation

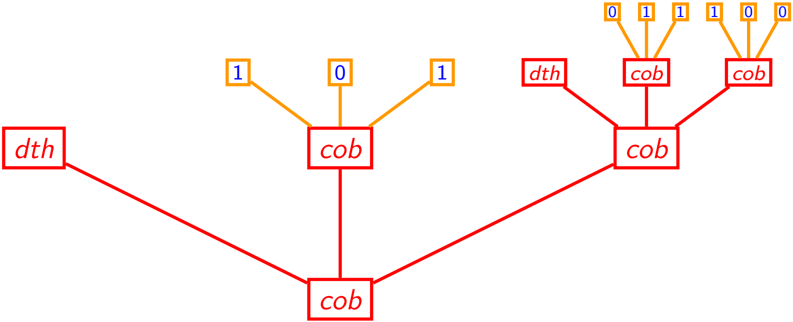


For any rooted subtree $\mathbb{U} \subset \mathbb{S}$, let

$$\nabla\mathbb{U} := \{i_1 \cdots i_n \in \mathbb{S} : i_1 \cdots i_{n-1} \in \mathbb{U},\ i_1 \cdots i_n \notin \mathbb{U}\}$$

denote the boundary of $\mathbb{U}$ relative to $\mathbb{S}$.

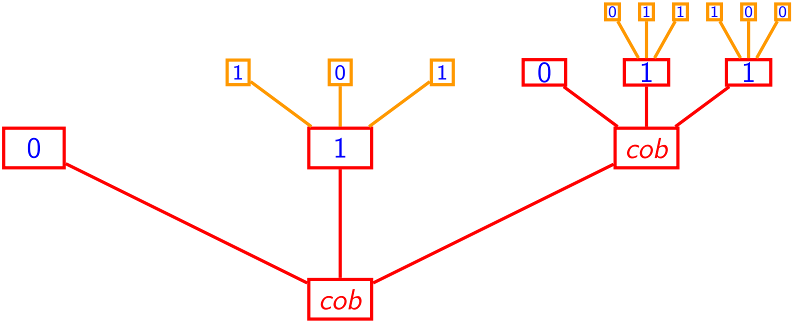
A recursive tree representation



Given $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}$, we inductively define $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$ by

$$X_{\mathbf{i}} = \gamma_{\mathbf{i}}\big(X_{\mathbf{i}1},\ldots,X_{\mathbf{i}\kappa(\gamma_{\mathbf{i}})}\big) \qquad (\mathbf{i}\in\mathbb{U}),$$

and define $F_{\mathbb{U}} : \{0,1\}^{\nabla\mathbb{U}} \to \{0,1\}$ by $F_{\mathbb{U}}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}\big) := X_{\varnothing}$.

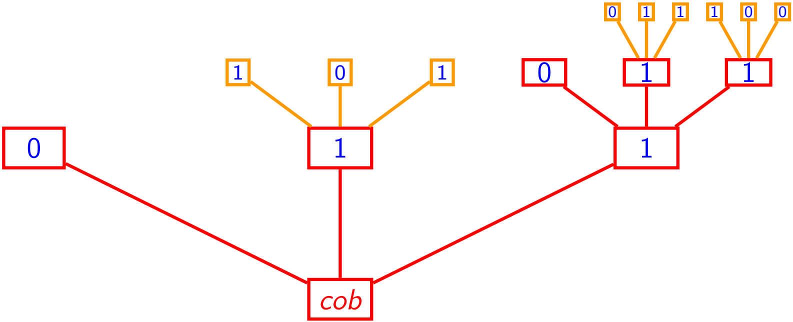


Given $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}$, we inductively define $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$ by

$$X_{\mathbf{i}} = \gamma_{\mathbf{i}}\big(X_{\mathbf{i}1},\ldots,X_{\mathbf{i}\kappa(\gamma_{\mathbf{i}})}\big) \qquad (\mathbf{i}\in\mathbb{U}),$$

and define $F_{\mathbb{U}}:\{0,1\}^{\nabla\mathbb{U}}\to\{0,1\}$ by $F_{\mathbb{U}}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}\big) := X_{\varnothing}$.

A recursive tree representation

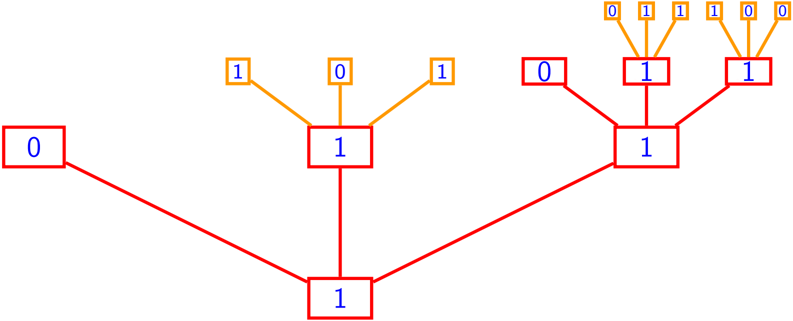


Given $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}$, we inductively define $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$ by

$$X_{\mathbf{i}} = \gamma_{\mathbf{i}}\big(X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}\kappa(\gamma_{\mathbf{i}})}\big) \qquad (\mathbf{i} \in \mathbb{U}),$$

and define $F_{\mathbb{U}} : \{0,1\}^{\nabla\mathbb{U}} \to \{0,1\}$ by $F_{\mathbb{U}}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}\big) := X_{\varnothing}$.

A recursive tree representation



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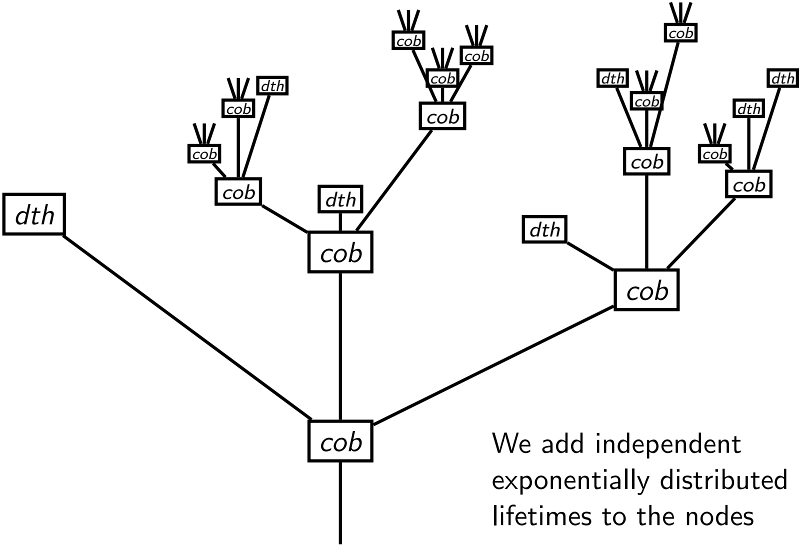
A recursive tree representation



We add independent exponentially distributed lifetimes to the nodes

A recursive tree representation

Let $(\sigma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be i.i.d. exponentially distributed with mean $(1+\delta)^{-1}$, independent of $(\gamma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, and set

$$\tau^*_{\mathbf{i}} := \sum_{m=1}^{n-1} \sigma_{i_1\cdots i_m} \quad \text{and} \quad \tau^\dagger_{\mathbf{i}} := \tau^*_{\mathbf{i}} + \sigma_{\mathbf{i}} \qquad (\mathbf{i} = i_1 \cdots i_n),$$
$$\mathbb{S}_t := \big\{\mathbf{i} \in \mathbb{S} : \tau^\dagger_{\mathbf{i}} \leq t\big\} \quad \text{and} \quad \nabla\mathbb{S}_t = \big\{\mathbf{i} \in \mathbb{S} : \tau^*_{\mathbf{i}} \leq t < \tau^\dagger_{\mathbf{i}}\big\}.$$

Let $\mathcal{F}_t$ be the filtration

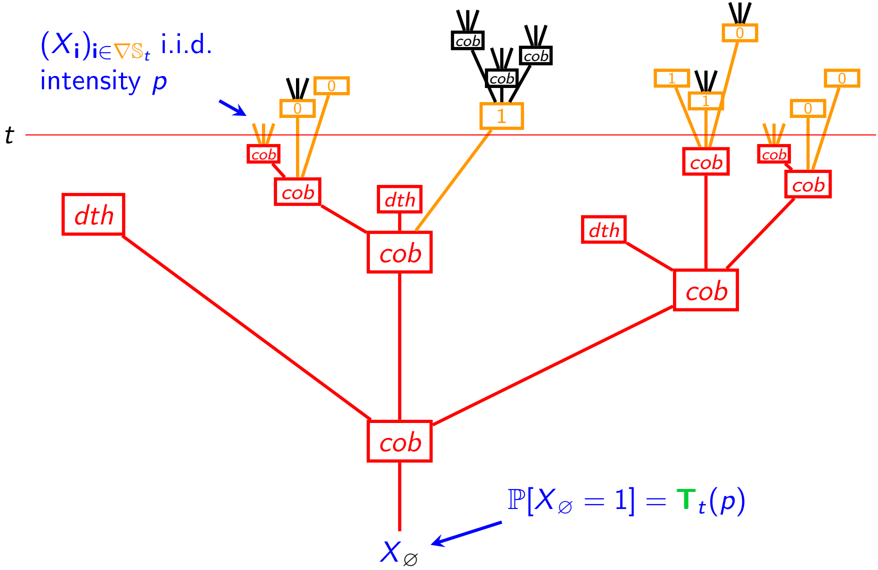
$$\mathcal{F}_t := \sigma\big(\nabla\mathbb{S}_t, (\gamma_{\mathbf{i}}, \sigma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}_t}\big) \qquad (t \geq 0).$$

Theorem [Mach, Sturm, S. '20]

$$\mathbf{T}_t(p) := \mathbb{P}\big[F_{\mathbb{S}_t}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_t}\big) = 1\big]$$

where $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_t}$ are i.i.d. with intensity p and independent of $\mathcal{F}_t$.

A recursive tree representation



For any rooted subtree $\mathbb{U} \subset \mathbb{T}$, let

$$\partial\mathbb{U} := \big\{ i_1 \cdots i_n \in \mathbb{T} : i_1 \cdots i_{n-1} \in \mathbb{U},\ \ i_1 \cdots i_n \notin \mathbb{U} \big\}$$

denote the boundary of $\mathbb{U}$ relative to $\mathbb{T}$.

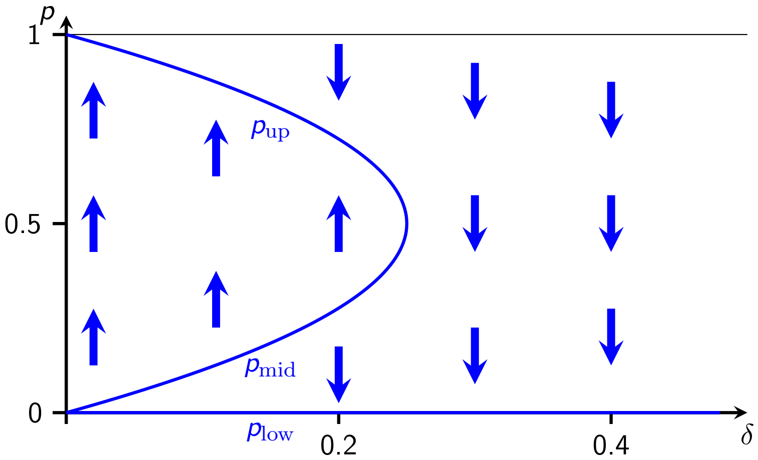
For each fixed point p of the mean-field equation, there exists a *Recursive Tree Process (RTP)* $(\gamma_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, unique in law, such that:

(i) $(\gamma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ are i.i.d. with law as before.

(ii) For finite $\mathbb{U} \subset \mathbb{T}$, the r.v.'s $(X_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{U}}$ are i.i.d. with intensity p and independent of $(\gamma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$.

(iii) $X_{\mathbf{i}} = \gamma_{\mathbf{i}}\big(X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}\kappa(\gamma_{\mathbf{i}})}\big) \qquad (\mathbf{i} \in \mathbb{T}).$

Fixed points



Fixed points of the mean-field equation for $\alpha = 1$ and their domains of attraction, as a function of δ.

Endogeny

Aldous and Bandyopadyay (2005) say that an RTP is *endogenous* if

$$X_\varnothing \text{ is measurable w.r.t. the } \sigma\text{-field generated by } (\gamma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}.$$

In our case, it has been shown that the RTP's corresponding to the stable fixed points $p_{\rm low}$ and $p_{\rm up}$ are endogenous but the RTP corresponding to the unstable fixed point $p_{\rm mid}$ is not.