Intertwining of Markov processes

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- First passage times of birth and death processes

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- The contact process on the hierarchical group

A change point problem

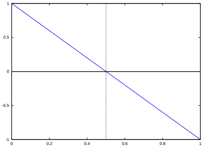
Let τ be exponentially distributed with mean one and let $Y_t := 1_{\{t \geq \tau\}}$. Let X be a diffusion in $[0,1]$ such that while $Y_t = y$, X_t evolves according to the generator

$$G_y f(x) := \tfrac{1}{2} x(1-x) \frac{\partial^2}{\partial x^2} f(x) + b_y(x) \frac{\partial}{\partial x} f(x) \qquad (y = 0, 1),$$

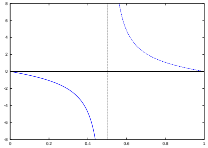
where

$$b_0(x) = 2(\tfrac{1}{2} - x), \quad b_1(x) = \frac{8x(1-x)(x-\frac{1}{2})}{1 - 4x(1-x)}.$$

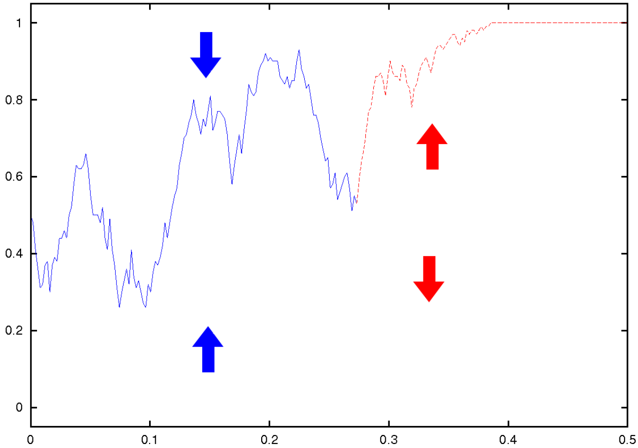
b_0



b_1

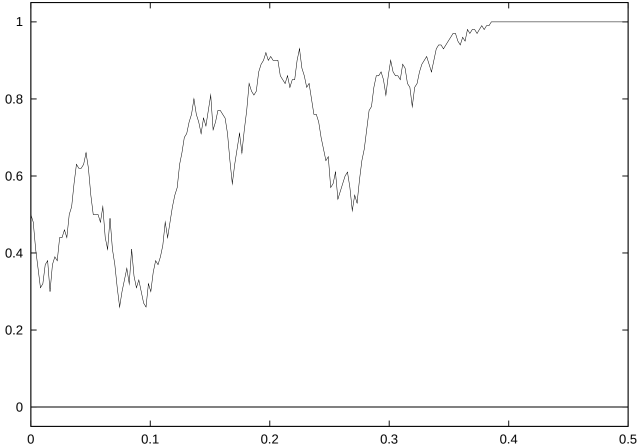


Wright-Fisher diffusion with drift



As long as $Y_t = 0$, X cannot reach the boundary. When $Y_t = 1$, X cannot cross the middle.

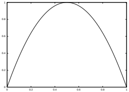
Wright-Fisher diffusion



If we forget about the colors, then X is just a Wright-Fished diffusion without drift!

Explanation

The process (X, Y) is Markov, but X is not autonomous, i.e., its dynamics depend on the state of Y. So how is it possible that X, on its own, is Markov?



In fact, one has

$$\mathbb{P}[Y_t = 0 \,|\, (X_s)_{0\le s\le t}] = 4X_t(1 - X_t) \quad \text{a.s.}$$

In particular, this probability depends only on the endpoint of the path $(X_s)_{0\le s\le t}$, and

$$\mathbb{E}[b_{Y_t}(X_t) \,|\, X_t = x] = \\ 4x(1-x)b_0(x) + \big(1 - 4x(1-x)\big)\, b_1(x) = 0.$$

General principle

[Rogers & Pitman '81] Let (X, Y) be a Markov process with state space $S \times T$ and generator $\hat{G}$, and let K be a probability kernel from S to T. Define $K : \mathbb{R}^{S \times T} \to \mathbb{R}^S$ by

$$Kf(x) := \sum_{y \in S} K(x, y) f(x, y).$$

Let G be the generator of a Markov process in S and assume that

$$GK = K\hat{G}.$$

Then

$$\mathbb{P}[Y_0 = y \,|\, X_0] = K(X_0, y) \quad \text{a.s.}$$

implies

$$\mathbb{P}[Y_t = y \,|\, (X_s)_{0 \le s \le t}] = K(X_t, y) \quad \text{a.s.} \qquad (t \ge 0).$$

and X, on its own, is a Markov process with generator G.

Intertwining of semigroups

[Fill '92] Let X and Y be Markov processes with state spaces S and T, semigroups $(P_t)_{t\geq 0}$ and $(P'_t)_{t\geq 0}$, and generators G and G'. Let K be a probability kernel from S to T and define

$$Kf(x) := \sum_y K(x,y)f(y).$$

Then $GK = KG'$ implies the *intertwining relation*

$$P_t K = K P'_t \quad (t \geq 0)$$

and the processes X and Y can be coupled such that

$$\mathbb{P}[Y_t = y \,|\, (X_s)_{0\leq s\leq t}] = K(X_t, y) \quad \text{a.s.} \qquad (t \geq 0).$$

We call Y an *averaged Markov process on* X.

Example: Wright-Fisher diffusion

Generalization Let X be a Wright-Fisher diffusion without drift. Let Y be a process with state space $\{0,1,\ldots,\infty\}$ that jumps $k \mapsto k+1$ with rate $(k+1)(2k+1)$ and gets absorbed in ∞ in finite time. Define a kernel $K : [0,1] \to \{0,1,\ldots,\infty\}$ by

$$K(x,y) = (1-x^2)x^{2y}.$$

Then the processes started in $X_0 = \frac{1}{2}$ and $Y_0 = 0$ can be coupled such that

$$\mathbb{P}[Y_t = y \,|\, (X_s)_{0\leq s\leq t}] = K(X_t, y) \quad \text{a.s.} \quad (t \geq 0)$$

and

$$\inf\big\{t \geq 0 : X_t \in \{0,1\}\big\} = \inf\{t \geq 0 : Y_t = \infty\}.$$

First passage times of birth and death processes

[Karlin & McGregor '59] Let Z be a Markov process with state space $\{0,1,2,\ldots\}$, started in $Z_0 = 0$, that jumps $k-1 \mapsto k$ with rate $b_k > 0$ and $k \mapsto k-1$ with rate $d_k > 0$ $(k \geq 1)$. Then

$$\tau_N := \inf\{t \geq 0 : Z_t = N\}$$

is distributed as a sum of independent exponentially distributed random variables whose parameters $\lambda_1 < \cdots < \lambda_N$ are the negatives of the eigenvalues of the generator of the process stopped in N.

Coupling of birth and death processes

[Diaconis & Miclos '09] Let $X_t := Z_{t \wedge \tau_N}$ be the stopped process and let $0 > -\lambda_1 > \cdots > -\lambda_N$ be its eigenvalues. Let X^+ be a pure birth process with birth rates $b_1, \ldots, b_N$ given by $\lambda_N, \ldots, \lambda_1$. Then it is possible to couple the processes X and X^+, both started in zero, in such a way that $X_t \leq X_t^+$ for all $t \geq 0$ and both processes arrive in N at the same time.

A probabilistic proof

Let G, G^+ be the generators of X, X^+. We claim that there exists a kernel K^+ such that

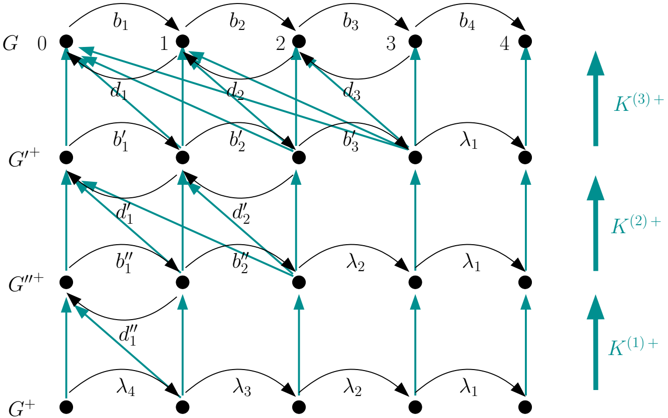
$$\begin{aligned} &K^+(x, \{0, \ldots, x\}) = 1 \qquad (0 \le x \le N), \\ &K^+(N, N) = 1, \end{aligned}$$

and moreover

$$K^+ G = G^+ K^+.$$

This can be proved by induction, using the Perron-Frobenius theorem in each step.

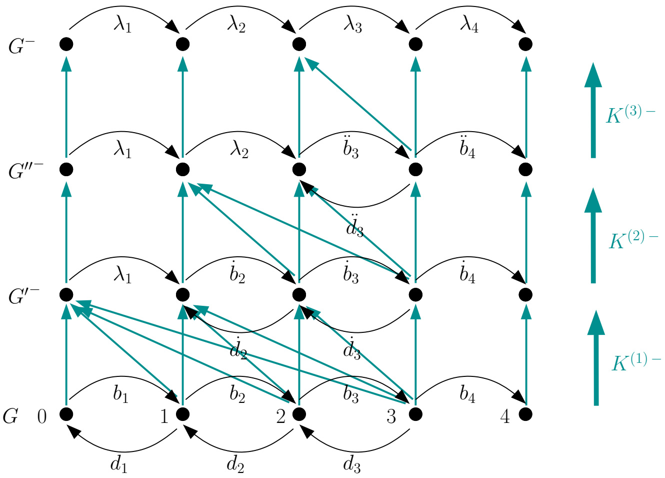
A probabilistic proof



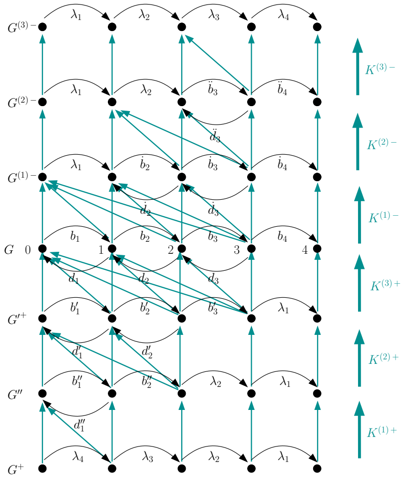
Coupling of birth and death processes

[S. '10] Let X_t and $\lambda_1, \ldots, \lambda_N$ be as before. Let X^- be a pure birth process with birth rates $b_1, \ldots, b_N$ given by $\lambda_1, \ldots, \lambda_N$. Then it is possible to couple the processes X and X^-, both started in zero, in such a way that $X^-_t \leq X_t$ for all $t \geq 0$ and both processes arrive in N at the same time.

A probabilistic proof



The complete figure

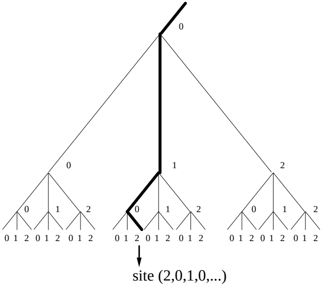


The hierarchical group

By definition, the *hierarchical group with freedom N* is the set

$$\Omega_N := \Big\{ i = (i_0, i_1, \ldots) : \begin{array}{l} i_k \in \{0, \ldots, N-1\}, \\ i_k \neq 0 \text{ for finitely many } k \Big\}, \end{array}$$

equipped with componentwise addition modulo N. Think of sites $i \in \Omega_N$ as the leaves of an infinite tree. Then $i_0, i_1, i_2, \ldots$ are the labels of the branches on the unique path from i to the root of the tree.

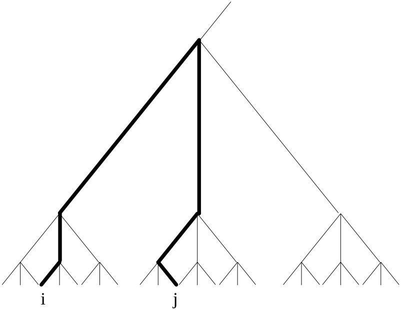


The hierarchical distance

Set

$$|i| := \inf\{k \geq 0 : i_m = 0 \ \forall m \geq k\} \qquad (i \in \Omega_N).$$

Then $|i - j|$ is the *hierarchical distance* between two elements $i, j \in \Omega_N$. In the tree picture, $|i - j|$ measures how high we must go up the tree to find the last common ancestor of i and j.



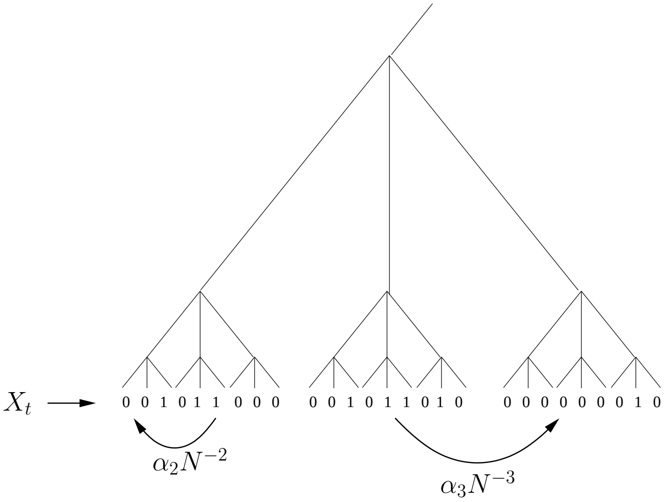
two sites at hierarchical distance 3

Hierarchical contact processes

Fix a *recovery rate* $\delta \geq 0$ and *infection rates* $\alpha_k \geq 0$ such that $\sum_{k=1}^{\infty} \alpha_k < \infty$. The *contact process* on Ω_N with these rates is the $\{0, 1\}^{\Omega_N}$-valued Markov process $(X_t)_{t \geq 0}$ with the following description:

If $X_t(i) = 0$ (resp. $X_t(i) = 1$), then we say that the site $i \in \Omega_N$ is *healthy* (resp. *infected*) at time $t \geq 0$. An infected site i infects a healthy site j at hierarchical distance $k := |i - j|$ with rate $\alpha_k N^{-k}$, and infected sites become healthy with rate $\delta \geq 0$.

Hierarchical contact processes



Infection rates on the hierarchical group.

The critical recovery rate

We say that a contact process $(X_t)_{t\geq 0}$ on Ω_N with given recovery and infection rates *survives* if there is a positive probability that the process started with only one infected site never recovers completely, i.e., there are infected sites at any $t \geq 0$. For given infection rates, we let

$$\delta_{\rm c} := \sup\Big\{\delta \geq 0 : \text{the contact process with infection rates } (\alpha_k)_{k\geq 1} \text{ and recovery rate } \delta \text{ survives}\Big\}$$

denote the *critical recovery rate*. A simple monotone coupling argument shows that X survives for $\delta < \delta_{\rm c}$ and dies out for $\delta > \delta_{\rm c}$. It is not hard to show that $\delta_{\rm c} < \infty$. The question whether $\delta_{\rm c} > 0$ is more subtle.

(Non)triviality of the critical recovery rate

[Athreya & S. '10] Assume that $\alpha_k = e^{-\theta^k}$ $(k \geq 1)$. Then:

(a) If $N < \theta$, then $\delta_{\rm c} = 0$.

(b) If $1 < \theta < N$, then $\delta_{\rm c} > 0$.

More generally, we show that $\delta_{\rm c} = 0$ if

$$\liminf_{k\to\infty} N^{-k} \log(\beta_k) = -\infty, \quad \text{where} \quad \beta_k := \sum_{n=k}^{\infty} \alpha_n \quad (k \geq 1),$$

while $\delta_{\rm c} > 0$ if

$$\sum_{k=m}^{\infty} (N')^{-k} \log(\alpha_k) > -\infty,$$

for some $m \geq 1$ and $N' < N$.

Proof of survival

We use added-on Markov processes to inductively derive bounds on the finite-time survival probability of finite systems. Let

$$\Omega_2^n := \big\{ i = (i_0, \ldots, i_{n-1}) : i_k \in \{0,1\} \big\}$$

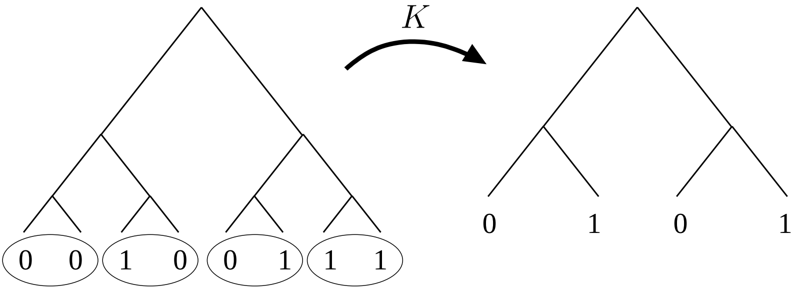
and let $S_n := \{0,1\}^{\Omega_2^n}$. We define a kernel from S_n to S_{n-1} by independently replacing blocks consisting of two spins by a single spin according to the stochastic rules:

$$00 \longrightarrow 0, \qquad 11 \longrightarrow 1,$$

$$\text{and} \quad 01 \text{ or } 10 \longrightarrow \begin{cases} 0 & \text{with probability } \xi, \\ 1 & \text{with probability } 1-\xi, \end{cases}$$

where $\xi \in (0, \frac{1}{2}]$ is a constant, to be determined later.

Renormalization kernel



The probability of this transition is $1 \cdot (1-\xi) \cdot \xi \cdot 1$.

An added-on process

Let X be a contact process on Ω_2^n with infection rates $\alpha_1, \ldots, \alpha_n$ and recover rate δ. Then X can be coupled to a process Y such that

$$\mathbb{P}[Y_t = y \,|\, (X_s)_{0 \leq s \leq t}] = K(X_t, y) \quad \text{a.s.} \qquad (t \geq 0),$$

where K is the kernel defined before, and

$$\xi := \gamma - \sqrt{\gamma^2 - \tfrac{1}{2}} \quad \text{with} \quad \gamma := \frac{1}{4}\Big(3 + \frac{\alpha_1}{2\delta}\Big).$$

Moreover, the process Y can be coupled to a finite contact process Y' on Ω_2^{n-1} with recovery rate $\delta' := 2\xi\delta$ and infection rates $\alpha'_1, \ldots, \alpha'_{n-1}$ given by $\alpha'_k := \frac{1}{2}\alpha_{k+1}$, in such a way that $Y'_t \leq Y_t$ for all $t \geq 0$.

Renormalization

We may view the map $(\delta, \alpha_1, \ldots, \alpha_n) \mapsto (\delta', \alpha'_1, \ldots, \alpha'_{n-1})$ as an (approximate) renormalization transformation. By iterating this map n times, we get a sequence of recovery rates $\delta, \delta', \delta'', \ldots$, the last of which gives a upper bound on the spectral gap of the finite contact process X on Ω_2^n. Under suitable assumptions on the α_k's, we can show that this spectral gap tends to zero as $n \to \infty$, and in fact, we can derive explicit lower bounds on the probability that finite systems survive till some fixed time t.

A question

Question Can we find an *exact* renormalization map $(\delta, \alpha_1, \ldots, \alpha_n) \mapsto (\delta', \alpha'_1, \ldots, \alpha'_{n-1})$ for hierarchical contact processes, i.e., for each hierarchical contact process X on Ω_2^n, can we find an *averaged Markov process* Y that is itself a hierarchical contact process on Ω_2^{n-1}?