Systems of branching, annihilating, and coalescing particles

Jan M. Swart (ÚTIA)

January 3rd, 2013
Underlying random walk

**Lattice** $\Lambda$ is a countable set.

**Random walk kernel** $q : \Lambda \times \Lambda \to \mathbb{R}$ jump rates satisfying

1. \textit{nonnegative} $q(i, j) \geq 0$
2. \textit{summable} $\sup_i \sum_j q(i, j) < \infty$
3. \textit{weakly irreducible} $\forall \Delta \subset \Lambda, \Delta \neq \emptyset, \Lambda$ there exists $i \in \Delta, j \in \Lambda \setminus \Delta$ such that either $q(i, j) > 0$ or $q(j, i) > 0$ (or both).
4. \textit{influx equals outflux} $|q| := \sum_j q(i, j) = \sum_j q(j, i)$.

We say that $\xi = (\xi_t)_{t \geq 0}$ is a random walk with kernel $q$ if $\xi$ is a Markov process in $\Lambda$ that stays in a state $i$ for an exponential time with mean $|q|^{-1}$ and then jumps to the state $j$ with probability $|q|^{-1} q(i, j)$. Condition 4 says that counting measure is an invariant measure for this process. The process is reversible if and only if $q^\dagger = q$ where $q^\dagger(i, j) := q(j, i)$ are the time-reversed jump rates.
Translation invariance

Let $(\Lambda, q)$ be a lattice equipped with a random walk kernel. An automorphism of $(\Lambda, q)$ is a bijection $g : \Lambda \to \Lambda$ such that $q(gi, gj) = q(i, j)$ for all $i, j \in \Lambda$. Let $\text{Aut}(\Lambda, q)$ be the group of all automorphisms of $(\Lambda, q)$. We say that a subgroup $G \subset \text{Aut}(\Lambda, q)$ is transitive if for each $i, j \in \Lambda$ there exists a $g \in G$ such that $gi = j$.

**Example** $\Lambda = \mathbb{Z}^d$, $q(i, j) = 1$ if $|i - j| = 1$ and 0 otherwise. $\text{Aut}(\Lambda, q)$ contains, e.g., translations, rotations by 90° along an axis, mirroring in a point, axis, or plane spanned by two axes, etc. The translations form a transitive subgroup.

**Other examples** Nearest neighbor random walks on regular trees or Cayley graphs. Random walks on groups.
A particle system

Fix \((\Lambda, q)\) and rates \(a, b, c, d \geq 0\). Consider a system of particles such that

- Each particle jumps, independently of the others, from site \(i\) to site \(j\) with rate \(q(i, j)\).
- Each pair of particles, present on the same site, annihilates with rate \(2a\), resulting in the disappearance of both particles.
- Each particle branches with rate \(b\) into two new particles, created on the position of the old one.
- Each pair of particles, present on the same site, coalesces with rate \(2c\), resulting in the creation of one new particle on the position of the two old ones.
- Each particle dies (disappears) with rate \(d\).
Fix $(\Lambda, q)$ and rates $a, b, c, d \geq 0$. Consider a system of particles such that

- Each particle jumps, independently of the others, from site $i$ to site $j$ with rate $q(i, j)$.
- Each pair of particles, present on the same site, annihilates with rate $2a$, resulting in the disappearance of both particles.
- Each particle branches with rate $b$ into two new particles, created on the position of the old one.
- Each pair of particles, present on the same site, coalesces with rate $2c$, resulting in the creation of one new particle on the position of the two old ones.
- Each particle dies (disappears) with rate $d$. 
A particle system

Fix $(\Lambda, q)$ and rates $a, b, c, d \geq 0$. Consider a system of particles such that

- Each particle jumps, independently of the others, from site $i$ to site $j$ with rate $q(i, j)$.
- Each pair of particles, present on the same site, *annihilates* with rate $2a$, resulting in the disappearance of both particles.
- Each particle *branches* with rate $b$ into two new particles, created on the position of the old one.
- Each pair of particles, present on the same site, *coalesces* with rate $2c$, resulting in the creation of one new particle on the position of the two old ones.
- Each particle *dies* (disappears) with rate $d$. 

Jan M. Swart (ÚTIA)
Fix \((\Lambda, q)\) and rates \(a, b, c, d \geq 0\). Consider a system of particles such that

- Each particle jumps, independently of the others, from site \(i\) to site \(j\) with rate \(q(i, j)\).
- Each pair of particles, present on the same site, *annihilates* with rate \(2a\), resulting in the disappearance of both particles.
- Each particle *branches* with rate \(b\) into two new particles, created on the position of the old one.
- Each pair of particles, present on the same site, *coalesces* with rate \(2c\), resulting in the creation of one new particle on the position of the two old ones.
- Each particle dies (disappears) with rate \(d\).
A particle system

Fix $(\Lambda, q)$ and rates $a, b, c, d \geq 0$. Consider a system of particles such that

- Each particle jumps, independently of the others, from site $i$ to site $j$ with rate $q(i, j)$.
- Each pair of particles, present on the same site, annihilates with rate $2a$, resulting in the disappearance of both particles.
- Each particle branches with rate $b$ into two new particles, created on the position of the old one.
- Each pair of particles, present on the same site, coalesces with rate $2c$, resulting in the creation of one new particle on the position of the two old ones.
- Each particle dies (disappears) with rate $d$. 
Let $X_t(i)$ be the number of particles at site $i \in \Lambda$ at time $t \geq 0$. Then $X = (X_t)_{t \geq 0}$ with $X_t = (X_t(i))_{i \in \Lambda}$ is a Markov process in $\mathbb{N}^\Lambda$ with generator

$$Gf(x) := \sum_{ij} q(i,j)x(i)\{f(x + \delta_j - \delta_i) - f(x)\}$$

$$+ a \sum_i x(i)(x(i) - 1)\{f(x - 2\delta_i) - f(x)\}$$

$$+ b \sum_i x(i)\{f(x + \delta_i) - f(x)\}$$

$$+ c \sum_i x(i)(x(i) - 1)\{f(x - \delta_i) - f(x)\}$$

$$+ d \sum_i x(i)\{f(x - \delta_i) - f(x)\},$$

where $\delta_i(j) := 1$ if $i = j$ and 0 otherwise. We call $X$ the $(q, a, b, c, d)$-branco-process. $X$ is well-defined for initial states with finitely many particles and also for some infinite initial states.
Assume that $\text{Aut}(\Lambda, q)$ is transitive.

We define shift operators $T_g : \mathbb{N}^\Lambda \to \mathbb{N}^\Lambda$ by

$$T_g x(i) := x(g^{-1}i) \quad (i \in \Lambda, \ x \in \mathbb{N}^\Lambda, \ g \in \text{Aut}(\Lambda, q)).$$

If $G$ is a subgroup of $\text{Aut}(\Lambda, q)$, then we say that a probability measure $\nu$ on $\mathbb{N}^\Lambda$ is $G$-homogeneous if $\nu \circ T_g^{-1} = \nu$ for all $g \in G$.

We say that $\nu$ is nontrivial if $\nu(0) = 0$, where $0$ denotes the configuration with no particles.

We say that the $(q, a, b, c, d)$-branco-process survives if

$$\mathbb{P}^{\delta_i} [X_t \neq 0 \ \forall \ t \geq 0] > 0 \quad (i \in \Lambda).$$

The process survives locally if

$$\liminf_{t \to \infty} \mathbb{P}^{\delta_i} [X_t(i) > 0] > 0.$$
Let $\Lambda = \mathbb{Z}^d$, let $G$ be the group of translations, and assume $G \subset \text{Aut}(\Lambda, q)$. Assume $a = 0$ and let $c, d > 0$ be fixed. Thm 1.3 in Shiga & Uchiyama (1986) and Thm 1 in Athreya & S. (2005) imply:

**Theorem** There exists a $0 < b_c < \infty$ such that:

- For $b > b_c$, the process survives and has a unique nontrivial, $G$-homogeneous invariant law $\nu$.  
- For $b < b_c$, the process dies out and the delta measure on 0 is the only invariant law.
Let $\Lambda = \mathbb{Z}^d$, let $G$ be the group of translations, and assume $G \subset \text{Aut}(\Lambda, q)$.
Assume $a = 0$ and let $c, d > 0$ be fixed.
Thm 1.3 in Shiga & Uchiyama (1986) and Thm 1 in Athreya & S. (2005) imply:

**Theorem** There exists a $0 < b_c < \infty$ such that:

- For $b > b_c$, the process survives and has a unique nontrivial, $G$-homogeneous invariant law $\overline{\nu}$.
- For $b < b_c$, the process dies out and the delta measure on $0$ is the only invariant law.
Cayley graphs

Let $\Lambda$ be a group with symmetric finite generating set $\Delta$. Draw the associated (left) Cayley graph which has a vertex between $i, j$ if and only if $j = ki$ for some $k \in \Delta$. Let $d$ denote the graph distance and let $0$ denote the unit element (origin).

The Cayley graph has subexponential growth if

$$\lim_{n \to \infty} \frac{1}{n} \log |\{i : d(i, 0) \leq n\}| = 0.$$ 

The Cayley graph is amenable if

$$\forall \varepsilon > 0 \exists \text{ finite nonempty } A \text{ s.t. } \frac{|\partial A|}{|A|} \leq \varepsilon,$$

where $\partial A := \{i : d(i, A) = 1\}$. Subexponential growth implies amenability but not vice versa.
Assume $a = d = 0$ and let $c > 0$ be fixed. Then survival is trivial. Frank Schirmeier (Erlangen) has shown:

**Theorem**

- If the Cayley graph has subexponential growth, then one has local survival for all $b > 0$.

- If the Cayley graph is nonamenable, then one has local extinction for all $b$ sufficiently small.

Open problems:

- What if the Cayley graph is amenable but has exponential growth?
- For positive death rate, on subexponential graphs, does survival imply local survival?
Assume $a = d = 0$ and let $c > 0$ be fixed. Then survival is trivial. Frank Schirmeier (Erlangen) has shown:

**Theorem**
- If the Cayley graph has subexponential growth, then one has local survival for all $b > 0$.
- If the Cayley graph is nonamenable, then one has local extinction for all $b$ sufficiently small.

**Open problems:**
- What if the Cayley graph is amenable but has exponential growth?
- For positive death rate, on subexponential graphs, does survival imply local survival?
Let $X^{(n)}$ be $(q, a, b, c, d)$-branco-processes started in initial states $x^{(n)}$ such that

$$x^{(n)}(i) \uparrow \infty \quad (i \in \Lambda).$$

Assume $a + c > 0$. Then Athreya & S. (2012) have shown (Thm. 1.4) that $X^{(n)}$ converges in law to a process $(X_t^{(\infty)})_{t>0}$ with

$$
\mathbb{E}[X_t^{(\infty)}(i)] \leq \begin{cases} 
\frac{\gamma}{(2a + c)(1 - e^{-\gamma t})} & \text{if } \gamma \neq 0, \\
\frac{1}{(2a + c)t} & \text{if } \gamma = 0
\end{cases} \quad (i \in \Lambda),
$$

where $\gamma := a + b + c - d$. Moreover,

$$
\mathbb{P}[X_t^{(\infty)} \in \cdot] \xrightarrow{t \to \infty} \nu,
$$

where $\nu$ is an invariant law.

If $a = 0$, then it is known that $\nu$ is the maximal invariant law w.r.t. the stochastic order (Thm 2 in Athreya & S. (2005)).
Starting from a single site

Consider the case \( a = b = d = 0 \) and \( c > 0 \) (pure coalescent). Let \( q \) be a nearest-neighbor kernel on an infinite graph of bounded degree and fix some element \( 0 \in \Lambda \). Let \( X^{(n)} \) be the process started with \( n \) particles at 0.

**Theorem** (Angel, Berestycki & Limic 2010): For fixed \( t > 0 \),

\[
|X_t^{(n)}| \approx \left| \{ i : d(0, i) \leq \log^*(n) \} \right| \quad \text{as} \; n \to \infty,
\]

where

\[
\log^*(n) := \inf \{ m \geq 0 : \underbrace{\exp \circ \cdots \circ \exp(1)}_{m \text{ times}} \geq n \}.
\]
Assume that \( \text{Aut}(\Lambda, q) \) has a transitive subgroup \( G \).
Assume that \( \mathbb{P}[X_0 \in \cdot] \) is nontrivial and \( G \)-homogeneous.

**Theorem** (Athreya & S. ’05, ’12) Assume \( a + c > 0 \). Then

\[
\mathbb{P}[X_t \in \cdot] \xrightarrow{t \to \infty} \nu.
\]
An infinite system of diffusions

Let \((\Lambda, q)\) be as before and let \(r, s, m \geq 0\). Let \(\mathcal{X} = (\mathcal{X}_t)_{t \geq 0}\) be the \([0, 1]^{\Lambda}\)-valued diffusion given by the infinite-dimensional SDE

\[
\begin{align*}
\mathrm{d}\mathcal{X}_t(i) &= \sum_j q(j, i)(\mathcal{X}_t(j) - \mathcal{X}_t(i)) \, \mathrm{d}t + s\mathcal{X}_t(i)(1 - \mathcal{X}_t(i)) \, \mathrm{d}t \\
&\quad - m\mathcal{X}_t(i) \, \mathrm{d}t + \sqrt{2r\mathcal{X}_t(i)(1 - \mathcal{X}_t(i))} \, \mathrm{d}B_t(i),
\end{align*}
\]

where \((B(i))_{i \in \Lambda}\) is a collection of independent Brownian motions.

\(\mathcal{X}\) models local gene frequencies in the presence of resampling (rate \(r\)), positive selection (rate \(s\)), and negative mutation (rate \(m\)). We call \(\mathcal{X}\) the \((q, r, s, m)\)-resem-process.
Moment duality

For $\phi \in [0, 1]^\Lambda$ and $x \in \mathbb{N}^\Lambda$, define

$$\phi^x := \prod_i \phi(i)^{x(i)}.$$ 

**Proposition** (Athreya & S. 2012) Assume $a + c > 0$. Let

$$\alpha = a/(a + c), \quad r = a + c, \quad s = (1 + \alpha)b, \quad \text{and} \quad m = \alpha b + d.$$ 

Let $X$ and $X^\dagger$ be a $(q, a, b, c, d)$-branco-process and $(q^\dagger, r, s, m)$-resem-process, independent of each other. Then

$$\mathbb{E}[(1 - (1 + \alpha)X^\dagger_0)X_t] = \mathbb{E}[(1 - (1 + \alpha)X_t^\dagger)X_0] \quad (t \geq 0),$$

provided one or more of the following conditions are satisfied:

(i) $\alpha < 1$,  
(ii) $|X_0| < \infty$ a.s.,  
(iii) $|X^\dagger_0| < \infty$ a.s.
The case without annihilation

If $a = 0$ and hence $\alpha = 0$ the duality reads

$$\mathbb{E}[(1 - \chi_0^*)X_t] = \mathbb{E}[(1 - \chi_t^*)X_0] \quad (t \geq 0).$$

For $\phi \in [0, 1]^\Lambda$ and $x \in \mathbb{N}^\Lambda$, let $\text{Thin}_\phi(x)$ be a random particle configuration obtained from $x$ by keeping a particle at $i$ with probability $\phi(i)$, independently for each particle.

If $\phi$ and / or $x$ are random, we construct $\text{Thin}_\phi(x)$ so that its conditional law given $\phi$ and $x$ is as described.

Then the duality can be written as

$$\mathbb{P}[\text{Thin}_0(x^*_t) = 0] = \mathbb{P}[\text{Thin}_t(x^*_0) = 0] \quad (t \geq 0).$$

Interpretation: $X_t$ are the potential ancestors of $X_0$ (Krone & Neuhauser '97).
The case without coalescence

If $c = 0$ and hence $\alpha = 1$ the duality reads

$$
\mathbb{E}[(1 - 2\chi_0^\dagger)X_t] = \mathbb{E}[(1 - 2\chi_t^\dagger)X_0] \quad (t \geq 0).
$$

Since

$$
\mathbb{E}[-1^{\text{Thin}_\phi(x)}] = \mathbb{E}[(1 - 2\phi)^X],
$$

the duality can be rewritten as

$$
\mathbb{P}[|\text{Thin}_\chi_0^\dagger(X_t)| \text{ is odd}] = \mathbb{P}[|\text{Thin}_\chi_t^\dagger(X_0)| \text{ is odd}] \quad (t \geq 0).
$$

General case: Lloyd & Sudbury ’97 and local mean field limit S. ’06.
Assume $r > 0$. It has been proved in Athreya & S. (2005) Thm 1 that the $(q, r, s, m)$-resem-process $\mathcal{X}$ and the $(q^\dagger, r, s, m)$-resem-process $\mathcal{X}^\dagger$ are dual in the sense that

$$
\mathbb{E}\left[e^{-s\mathcal{X}_0^\dagger}\mathcal{X}_t^\dagger\right] = \mathbb{E}\left[e^{-s\mathcal{X}_t\mathcal{X}_0^\dagger}\right] \quad (t \geq 0).
$$

This can be rewritten as

$$
P\left[\text{Pois}\left(\frac{s}{r}|\mathcal{X}_0^\dagger\mathcal{X}_t^\dagger|\right) = 0\right] = P\left[\text{Pois}\left(\frac{s}{r}|\mathcal{X}_t\mathcal{X}_0^\dagger|\right) = 0\right] \quad (t \geq 0),
$$

where $\text{Pois}(\phi)$ denotes a configuration which has at site $i$ a Poisson number of particles with mean $\phi(i)$, independently for each site (conditional given $\phi$).

Application: critical points for survival and nontriviality of $\nu$ are the same.
Fix $s, m \geq 0, r > 0$ and $0 \leq \alpha \leq 1$.
Let $\mathcal{X}$ be the $(q, r, s, m)$-resem-process.
Let $X$ be the $(q, \alpha r, \frac{1}{1+\alpha} s, (1 - \alpha)r, m - \frac{\alpha}{1+\alpha} s)$-branco-process.
Then
\[
\mathbb{P}[X_0 \in \cdot] = \mathbb{P}[\text{Pois}(\frac{s}{(1+\alpha)r} X_0) \in \cdot]
\]
implies
\[
\mathbb{P}[X_t \in \cdot] = \mathbb{P}[\text{Pois}(\frac{s}{(1+\alpha)r} X_t) \in \cdot]
\]

The invariant law $\bar{\nu}$ is a Poissonization of the upper invariant law of the $(q, r, s, m)$-resem-process.
Fix $s, m \geq 0, r > 0$ and $0 \leq \beta \leq \alpha \leq 1$.

Let $\overline{X}$ be the $(q, \alpha r, \frac{1}{1+\alpha} s, (1 - \alpha) r, m - \frac{\alpha}{1+\alpha} s)$-branco-process.

Let $X$ be the $(q, \beta r, \frac{1}{1+\beta} s, (1 - \beta) r, m - \frac{\beta}{1+\beta} s)$-branco-process.

Then

$$\mathbb{P}[X_0 \in \cdot] = \mathbb{P}[\text{Thin}_{\frac{1+\beta}{1+\alpha}}(\overline{X}_0) \in \cdot]$$

implies

$$\mathbb{P}[X_t \in \cdot] = \mathbb{P}[\text{Thin}_{\frac{1+\beta}{1+\alpha}}(\overline{X}_t) \in \cdot]$$

Every process with $a > 0$ can be obtained a the thinning of some process with $a = 0$.

The process $X$ started at infinity is a $(1 + \beta)/(1 + \alpha)$-thinning of $\overline{X}$ started at infinity.

Proof: Lloyd & Sudbury have shown that when two particle processes have the same dual, one is a thinning of the other.
Assume that $\mathbb{P}[X_0 \in \cdot]$ is nontrivial and $G$-homogeneous. To prove that

$$\mathbb{P}[X_t \in \cdot] \xrightarrow{t \to \infty} \bar{\nu},$$

by duality, it suffices to show that

$$\mathbb{E}[(1 - (1 + \alpha)X_0^\dagger)X_t] = \mathbb{E}[(1 - (1 + \alpha)X_t^\dagger)X_0]$$

$$\xrightarrow{t \to \infty} \mathbb{P}[|X_t^\dagger| = 0 \text{ for some } t \geq 0],$$

where $|X_t^\dagger| := \sum_i X_t^\dagger(i)$.

It has been shown in Athreya & S. (’05) that with probability one either $|X_t^\dagger| = 0$ at some $t \geq 0$ (and hence thereafter), or $|X_t^\dagger| \to \infty$ as $t \to \infty$.

This and a sufficient amount of ‘local randomness’ does the job.