# Systems of branching, annihilating, and coalescing particles

Jan M. Swart (ÚTIA)

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# Underlying random walk

**Lattice** $\Lambda$ is a countable set.

**Random walk kernel** $q : \Lambda \times \Lambda \to \mathbb{R}$ jump rates satisfying

1. *nonnegative* $q(i, j) \geq 0$
2. *summable* $\sup_i \sum_j q(i, j) < \infty$
3. *weakly irreducible* $\forall \Delta \subset \Lambda,\ \Delta \neq \emptyset, \Lambda$ there exists $i \in \Delta$, $j \in \Lambda \backslash \Delta$ such that either $q(i, j) > 0$ or $q(j, i) > 0$ (or both).
4. *influx equals outflux* $|q| := \sum_j q(i, j) = \sum_j q(j, i)$.

We say that $\xi = (\xi_t)_{t \geq 0}$ is a random walk with kernel $q$ if $\xi$ is a Markov process in $\Lambda$ that stays in a state $i$ for an exponential time with mean $|q|^{-1}$ and then jumps to the state $j$ with probability $|q|^{-1} q(i, j)$. Condition 4 says that counting measure is an invariant measure for this process. The process is reversible if and only if $q^\dagger = q$ where $q^\dagger(i, j) := q(j, i)$ are the time-reversed jump rates.

# Translation invariance

Let $(\Lambda, q)$ be a lattice equipped with a random walk kernel.
An *automorphism* of $(\Lambda, q)$ is a bijection $g : \Lambda \to \Lambda$ such that $q(gi, gj) = q(i, j)$ for all $i, j \in \Lambda$.
Let $\mathrm{Aut}(\Lambda, q)$ be the group of all automorphisms of $(\Lambda, q)$.
We say that a subgroup $G \subset \mathrm{Aut}(\Lambda, q)$ is *transitive* if for each $i, j \in \Lambda$ there exists a $g \in G$ such that $gi = j$.

**Example** $\Lambda = \mathbb{Z}^d$, $q(i, j) = 1$ if $|i - j| = 1$ and 0 otherwise.
$\mathrm{Aut}(\Lambda, q)$ contains, e.g., translations, rotations by $90^\circ$ along an axis, mirroring in a point, axis, or plane spanned by two axes, etc.
The translations form a transitive subgroup.
**Other examples** Nearest neighbor random walks on regular trees or Cayley graphs. Random walks on groups.

# A particle system

Fix $(\Lambda, q)$ and rates $a, b, c, d \geq 0$. Consider a system of particles such that

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- Each pair of particles, present on the same site, *coalesces* with rate $2c$, resulting in the creation of one new particle on the position of the two old ones.

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- Each particle *dies* (disappears) with rate $d$.

# Generator description

Let $X_t(i)$ be the number of particles at site $i \in \Lambda$ at time $t \geq 0$. Then $X = (X_t)_{t\geq 0}$ with $X_t = (X_t(i))_{i\in\Lambda}$ is a Markov process in $\mathbb{N}^\Lambda$ with generator

$$
\begin{aligned}
Gf(x) := \sum_{ij} & q(i,j)x(i)\big\{f(x+\delta_j-\delta_i) - f(x)\big\} \\
& + a\sum_i x(i)(x(i)-1)\big\{f(x-2\delta_i) - f(x)\big\} \\
& + b\sum_i x(i)\big\{f(x+\delta_i) - f(x)\big\} \\
& + c\sum_i x(i)(x(i)-1)\big\{f(x-\delta_i) - f(x)\big\} \\
& + d\sum_i x(i)\big\{f(x-\delta_i) - f(x)\big\},
\end{aligned}
$$

where $\delta_i(j) := 1$ if $i = j$ and 0 otherwise. We call $X$ the $(q, a, b, c, d)$-*branco-process*. $X$ is well-defined for initial states with finitely many particles and also for some infinite initial states.

# Survival

Assume that $\mathrm{Aut}(\Lambda, q)$ is transitive.

We define shift operators $T_g : \mathbb{N}^\Lambda \to \mathbb{N}^\Lambda$ by

$$T_g x(i) := x(g^{-1} i) \qquad \big(i \in \Lambda,\ x \in \mathbb{N}^\Lambda,\ g \in \mathrm{Aut}(\Lambda, q)\big).$$

If $G$ is a subgroup of $\mathrm{Aut}(\Lambda, q)$, then we say that a probability measure $\nu$ on $\mathbb{N}^\Lambda$ is *$G$-homogeneous* if $\nu \circ T_g^{-1} = \nu$ for all $g \in G$. We say that $\nu$ is *nontrivial* if $\nu(\underline{0}) = 0$, where $\underline{0}$ denotes the configuration with no particles.

We say that the $(q, a, b, c, d)$-branco-process *survives* if

$$\mathbb{P}^{\delta_i}\big[X_t \neq \underline{0}\ \forall t \geq 0\big] > 0 \quad (i \in \Lambda).$$

The process *survives locally* if

$$\liminf_{t \to \infty} \mathbb{P}^{\delta_i}\big[X_t(i) > 0\big] > 0.$$

# Processes on $\mathbb{Z}^d$

Let $\Lambda = \mathbb{Z}^d$, let $G$ be the group of translations, and assume $G \subset \mathrm{Aut}(\Lambda, q)$.
Assume $a = 0$ and let $c, d > 0$ be fixed.
Thm 1.3 in Shiga & Uchiyama (1986) and Thm 1 in Athreya & S. (2005) imply:

**Theorem** There exists a $0 < b_{\mathrm{c}} < \infty$ such that:

- For $b > b_{\mathrm{c}}$, the process survives and has a unique nontrivial, $G$-homogeneous invariant law $\overline{\nu}$.

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- For $b > b_{\rm c}$, the process survives and has a unique nontrivial, $G$-homogeneous invariant law $\overline{\nu}$.
- For $b < b_{\rm c}$, the process dies out and the delta measure on $\underline{0}$ is the only invariant law.

# Cayley graphs

Let $\Lambda$ be a group with symmetric finite generating set $\Delta$. Draw the associated (left) Cayley graph which has a vertex between $i, j$ if and only if $j = ki$ for some $k \in \Delta$. Let $d$ denote the graph distance and let 0 denote the unit element (origin).

The Cayley graph has *subexponential growth* if

$$\lim_{n\to\infty} \frac{1}{n} \log \big|\{i : d(i,0) \leq n\}\big| = 0.$$

The Cayley graph is *amenable* if

$$\forall \varepsilon > 0 \ \exists \text{ finite nonempty } A \text{ s.t. } \frac{|\partial A|}{|A|} \leq \varepsilon,$$

where $\partial A := \{i : d(i, A) = 1\}$. Subexponential growth implies amenability but not vice versa.

# Zero death rate and local survival

Assume $a = d = 0$ and let $c > 0$ be fixed. Then survival is trivial. Frank Schirmeier (Erlangen) has shown:

**Theorem**

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**Theorem**

- If the Cayley graph has subexponential growth, then one has local survival for all $b > 0$.
- If the Cayley graph is nonamenable, then one has local extinction for all $b$ sufficiently small.

**Open problems:**

- What if the Cayley graph is amenable but has exponential growth?
- For positive death rate, on subexponential graphs, does survival imply local survival?

# Implosion

Let $X^{(n)}$ be $(q, a, b, c, d)$-branco-processes started in initial states $x^{(n)}$ such that

$$x^{(n)}(i) \uparrow \infty \qquad (i \in \Lambda).$$

Assume $a + c > 0$. Then Athreya & S. (2012) have shown (Thm. 1.4) that $X^{(n)}$ converges in law to a process $(X_t^{(\infty)})_{t>0}$ with

$$\mathbb{E}[X_t^{(\infty)}(i)] \leq \begin{cases} \dfrac{\gamma}{(2a+c)(1-e^{-\gamma t})} & \text{if } \gamma \neq 0, \\ \dfrac{1}{(2a+c)t} & \text{if } \gamma = 0 \end{cases} \qquad (i \in \Lambda),$$

where $\gamma := a + b + c - d$. Moreover,

$$\mathbb{P}\big[X_t^{(\infty)} \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu},$$

where $\overline{\nu}$ is an invariant law.

If $a = 0$, then it is known that $\overline{\nu}$ is the maximal invariant law w.r.t. the stochastic order (Thm 2 in Athreya & S. (2005)).

# Starting from a single site

Consider the case $a = b = d = 0$ and $c > 0$ (pure coalescent). Let $q$ be a nearest-neighbor kernel on an infinite graph of bounded degree and fix some element $0 \in \Lambda$.
Let $X^{(n)}$ be the process started with $n$ particles at 0.

**Theorem** (Angel, Berestycki & Limic 2010): For fixed $t > 0$,

$$\big|X_t^{(n)}\big| \approx \big|\{i : d(0,i) \leq \log^*(n)\}\big| \quad \text{as } n \to \infty,$$

where

$$\log^*(n) := \inf\{m \geq 0 : \underbrace{\exp \circ \cdots \circ \exp}_{m \text{ times}}(1) \geq n\}.$$

# Ergodicity

Assume that $\mathrm{Aut}(\Lambda, q)$ has a transitive subgroup $G$.
Assume that $\mathbb{P}[X_0 \in \cdot\,]$ is nontrivial and $G$-homogeneous.

**Theorem** (Athreya & S. '05, '12) Assume $a + c > 0$. Then

$$\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}.$$

# An infinite system of diffusions

Let $(\Lambda, q)$ be as before and let $r, s, m \geq 0$.
Let $\mathcal{X} = (\mathcal{X}_t)_{t\geq 0}$ be the $[0,1]^\Lambda$-valued diffusion given by the infinite-dimensional SDE

$$\begin{aligned}
\mathrm{d}\mathcal{X}_t(i) = & \sum_j q(j,i)(\mathcal{X}_t(j) - \mathcal{X}_t(i))\,\mathrm{d}t + s\mathcal{X}_t(i)(1-\mathcal{X}_t(i))\,\mathrm{d}t \\
& - m\mathcal{X}_t(i)\,\mathrm{d}t + \sqrt{2r\mathcal{X}_t(i)(1-\mathcal{X}_t(i))}\,\mathrm{d}B_t(i),
\end{aligned}$$

where $(B(i))_{i\in\Lambda}$ is a collection of independent Brownian motions.

$\mathcal{X}$ models local gene frequencies in the presence of *resampling* (rate $r$), positive *selection* (rate $s$), and negative *mutation* (rate $m$).
We call $\mathcal{X}$ the $(q, r, s, m)$-*resem-process*.

# Moment duality

For $\phi \in [0,1]^\Lambda$ and $x \in \mathbb{N}^\Lambda$, define

$$\phi^x := \prod_i \phi(i)^{x(i)}.$$

**Proposition** (Athreya & S. 2012) Assume $a + c > 0$. Let

$$\alpha = a/(a+c), \quad r = a + c, \quad s = (1+\alpha)b, \quad \text{and} \quad m = \alpha b + d.$$

Let $X$ and $\mathcal{X}^\dagger$ be a $(q, a, b, c, d)$-branco-process and $(q^\dagger, r, s, m)$-resem-process, independent of each other. Then

$$\mathbb{E}\big[(1 - (1+\alpha)\mathcal{X}_0^\dagger)^{X_t}\big] = \mathbb{E}\big[(1 - (1+\alpha)\mathcal{X}_t^\dagger)^{X_0}\big] \qquad (t \geq 0),$$

provided one or more of the following conditions are satisfied:

$$\text{(i) } \alpha < 1, \quad \text{(ii) } |X_0| < \infty \text{ a.s.}, \quad \text{(iii) } |\mathcal{X}_0^\dagger| < \infty \text{ a.s.}$$

# The case without annihilation

If $a = 0$ and hence $\alpha = 0$ the duality reads

$$\mathbb{E}\big[(1 - \mathcal{X}_0^\dagger)^{X_t}\big] = \mathbb{E}\big[(1 - \mathcal{X}_t^\dagger)^{X_0}\big] \qquad (t \geq 0).$$

For $\phi \in [0,1]^\Lambda$ and $x \in \mathbb{N}^\Lambda$, let $\mathrm{Thin}_\phi(x)$ be a random particle configuration obtained from $x$ by keeping a particle at $i$ with probability $\phi(i)$, independently for each particle.

If $\phi$ and / or $x$ are random, we construct $\mathrm{Thin}_\phi(x)$ so that its conditional law given $\phi$ and $x$ is as described.

Then the duality can be written as

$$\mathbb{P}\big[\mathrm{Thin}_{\mathcal{X}_0^\dagger}(X_t) = \underline{0}\big] = \mathbb{P}\big[\mathrm{Thin}_{\mathcal{X}_t^\dagger}(X_0) = \underline{0}\big] \qquad (t \geq 0).$$

Interpretation: $X_t$ are the potential ancestors of $X_0$ (Krone & Neuhauser '97).

# The case without coalescence

If $c = 0$ and hence $\alpha = 1$ the duality reads

$$\mathbb{E}\big[(1 - 2\mathcal{X}_0^\dagger)^{X_t}\big] = \mathbb{E}\big[(1 - 2\mathcal{X}_t^\dagger)^{X_0}\big] \qquad (t \geq 0).$$

Since

$$\mathbb{E}\big[(-1)^{\mathrm{Thin}_\phi(x)}\big] = \mathbb{E}\big[(1 - 2\phi)^x\big],$$

the duality can be rewritten as

$$\mathbb{P}\big[|\mathrm{Thin}_{\mathcal{X}_0^\dagger}(X_t)| \text{ is odd}\big] = \mathbb{P}\big[|\mathrm{Thin}_{\mathcal{X}_t^\dagger}(X_0)| \text{ is odd}\big] \qquad (t \geq 0).$$

General case: Lloyd & Sudbury '97 and local mean field limit S. '06.

# Self-duality

Assume $r > 0$. It has been proved in Athreya & S. (2005) Thm 1 that the $(q, r, s, m)$-resem-process $\mathcal{X}$ and the $(q^\dagger, r, s, m)$-resem-process $\mathcal{X}^\dagger$ are dual in the sense that

$$\mathbb{E}\big[e^{-\frac{s}{r}|\mathcal{X}_0\mathcal{X}^\dagger_t|}\big] = \mathbb{E}\big[e^{-\frac{s}{r}|\mathcal{X}_t\mathcal{X}^\dagger_0|}\big] \qquad (t \geq 0).$$

This can be rewritten as

$$\mathbb{P}\big[\mathrm{Pois}(\tfrac{s}{r}|\mathcal{X}_0\mathcal{X}^\dagger_t|) = \underline{0}\big] = \mathbb{P}\big[\mathrm{Pois}(\tfrac{s}{r}|\mathcal{X}_t\mathcal{X}^\dagger_0|) = \underline{0}\big] \qquad (t \geq 0),$$

where $\mathrm{Pois}(\phi)$ denotes a configuration which has at site $i$ a Poisson number of particles with mean $\phi(i)$, independently for each site (conditional given $\phi$).

Application: critical points for survival and nontriviality of $\overline{\nu}$ are the same.

# Poissonization

Fix $s, m \geq 0$, $r > 0$ and $0 \leq \alpha \leq 1$.
Let $\mathcal{X}$ be the $(q, r, s, m)$-resem-process.
Let $X$ be the $(q, \alpha r, \frac{1}{1+\alpha}s, (1-\alpha)r, m - \frac{\alpha}{1+\alpha}s)$-branco-process.
Then

$$\begin{aligned} & \mathbb{P}\big[X_0 \in \cdot\,\big] = \mathbb{P}\big[\mathrm{Pois}\big(\tfrac{s}{(1+\alpha)r}\mathcal{X}_0\big) \in \cdot\,\big] \\ \text{implies} \quad & \mathbb{P}\big[X_t \in \cdot\,\big] = \mathbb{P}\big[\mathrm{Pois}\big(\tfrac{s}{(1+\alpha)r}\mathcal{X}_t\big) \in \cdot\,\big] \end{aligned}$$

The invariant law $\overline{\nu}$ is a Poissonization of the upper invariant law of the $(q, r, s, m)$-resem-process.

# Thinning

Fix $s, m \geq 0$, $r > 0$ and $0 \leq \beta \leq \alpha \leq 1$.
Let $\overline{X}$ be the $(q, \alpha r, \frac{1}{1+\alpha}s, (1-\alpha)r, m - \frac{\alpha}{1+\alpha}s)$-branco-process.
Let $X$ be the $(q, \beta r, \frac{1}{1+\beta}s, (1-\beta)r, m - \frac{\beta}{1+\beta}s)$-branco-process.
Then

$$\begin{aligned} &\mathbb{P}\big[X_0 \in \cdot\,\big] = \mathbb{P}\big[\mathrm{Thin}_{\frac{1+\beta}{1+\alpha}}(\overline{X}_0) \in \cdot\,\big] \\ \text{implies} \quad &\mathbb{P}\big[X_t \in \cdot\,\big] = \mathbb{P}\big[\mathrm{Thin}_{\frac{1+\beta}{1+\alpha}}(\overline{X}_t) \in \cdot\,\big] \end{aligned}$$

Every process with $a > 0$ can be obtained a the thinning of some process with $a = 0$.

The process $X$ started at infinity is a $(1+\beta)/(1+\alpha)$-thinning of $\overline{X}$ started at infinity.

Proof: Lloyd & Sudbury have shown that when two particle processes have the same dual, one is a thinning of the other.

# Proof of ergodicity

Assume that $\mathbb{P}[X_0 \in \cdot\,]$ is nontrivial and $G$-homogeneous. To prove that

$$\mathbb{P}\big[X_t \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu},$$

by duality, it suffices to show that

$$\begin{aligned}\mathbb{E}\big[(1-(1+\alpha)\mathcal{X}_0^\dagger)^{X_t}\big] &= \mathbb{E}\big[(1-(1+\alpha)\mathcal{X}_t^\dagger)^{X_0}\big] \\ &\underset{t\to\infty}{\longrightarrow} \mathbb{P}\big[|\mathcal{X}_t^\dagger| = 0 \text{ for some } t \geq 0],\end{aligned}$$

where $|\mathcal{X}_t^\dagger| := \sum_i \mathcal{X}_t^\dagger(i)$.
It has been shown in Athreya & S. ('05) that with probability one either $|\mathcal{X}_t^\dagger| = 0$ at some $t \geq 0$ (and hence thereafter), or $|\mathcal{X}_t^\dagger| \to \infty$ as $t \to \infty$.
This and a sufficient amount of 'local randomness' does the job.