Recursive tree processes and the mean-field limit of stochastic flows

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Basic ingredients

(i) Polish space S *local state space.*

(ii) $(\Omega, \mathcal{B}, \mathbf{q})$ Polish space with Borel σ-field and finite measure: *source of external randomness.*

(iii) $\lambda : \Omega \to \mathbb{N}_+$ measurable function.

(iv) For each $\omega \in \Omega$, a function $\vec{\gamma}[\omega] : S^{\lambda(\omega)} \to S^{\lambda(\omega)}$.

We let $\Omega_l := \{\omega \in \Omega : \lambda(\omega) = l\}$ and assume

$$\Omega_l \times S^l \ni (\omega, x) \mapsto \vec{\gamma}[\omega](x) \in S^l$$

is measurable for each $l \geq 1$.

Mean-field particle systems

For $N \in \mathbb{N}_+$ let $[N] := \{1, \ldots, N\}$.
Let $[N]^{\langle l \rangle}$:= the set of all sequences $\mathbf{i} = (i_1, \ldots, i_l)$ for which $i_1, \ldots, i_l \in [N]$ are all different.
Let $X = (X(t))_{t \geq 0}$ be a Markov process with values in S^N that evolves as:

(i) With Poisson intensity $|\mathbf{q}|$, choose an element $\omega \in \Omega$ with law $|\mathbf{q}|^{-1}\mathbf{q}$.

(ii) If $\lambda(\omega) > N$, do nothing.

(iii) Otherwise, choose $\mathbf{i} \in [N]^{\langle \lambda(\omega) \rangle}$ uniformly and replace $\big(X_{i_1}(t-), \ldots, X_{i_{\lambda(\omega)}}(t-)\big)$ by $\big(X_{i_1}(t), \ldots, X_{i_{\lambda(\omega)}}(t)\big) := \vec{\gamma}[\omega]\big(X_{i_1}(t-), \ldots, X_{i_{\lambda(\omega)}}(t-)\big)$.

Mean-field particle systems

Write $\vec{\gamma}[\omega](x) = \big(\gamma_1[\omega](x), \ldots, \gamma_{\lambda(\omega)}[\omega](x)\big)$.
For $\omega \in \Omega$ with $\lambda(\omega) \leq N$ and $\mathbf{i} \in [N]^{\langle\lambda(\omega)\rangle}$, define
$m_{\omega,\mathbf{i}} : S^N \to S^N$ by

$$m_{\omega,\mathbf{i}}(x)_j := \begin{cases} \gamma_j[\omega](x_{i_1}, \ldots, x_{i_{\lambda(\omega)}}) & \text{if } j \in \{i_1, \ldots, i_{\lambda(\omega)}\}, \\ x_j & \text{otherwise,} \end{cases}$$

Interpretation: apply map $\vec{\gamma}[\omega]$ to coordinates in $\mathbf{i}$.
Let Π be a Poisson point set on

$$\big\{(\omega, \mathbf{i}, t) : \omega \in \Omega,\ \mathbf{i} \in [N]^{\langle\lambda(\omega)\rangle},\ t \in \mathbb{R}\big\}$$

with intensity

$$\mathbf{q}(\mathrm{d}\omega)\, \frac{1_{\{\lambda(\omega) \leq N\}}}{\big|[N]^{\langle\lambda(\omega)\rangle}\big|}\, \mathrm{d}t.$$

Interpretation: for each $(\omega, \mathbf{i}, t) \in \Pi$, at time t, apply $\vec{\gamma}[\omega]$ to coordinates in $\mathbf{i}$.

Mean-field particle systems

Order the elements of

$$\Pi_{s,u} = \big\{(\omega_1, \mathbf{i}_1, t_1), \ldots, (\omega_n, \mathbf{i}_n, t_n)\big\} \quad \text{with} \quad t_1 < \cdots < t_n$$

according to their times and define

$$\mathbf{X}_{s,u} := m_{\omega_n,\mathbf{i}_n} \circ \cdots \circ m_{\omega_1,\mathbf{i}_1}$$

The random maps $(\mathbf{X}_{s,u})_{s \leq u}$ form a *stochastic flow:*

$$\mathbf{X}_{s,s} = 1 \quad \text{and} \quad \mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u} \qquad (s \leq t \leq u)$$

with *independent increments*, i.e.,

$$\mathbf{X}_{t_1,t_2}, \ldots, \mathbf{X}_{t_{k-1},t_k} \quad \text{are independent} \quad (t_1 < \cdots < t_k).$$

If $X(0)$ is independent of Π, then

$$X(t) := \mathbf{X}_{0,t}\big(X(0)\big) \qquad (t \geq 0)$$

defines a Markov process $X = (X(t))_{t \geq 0}$ in S^N.

Mean-field particle systems

Consider Markov process $X^N = (X^N(t))_{t \geq 0}$ in S^N with $N \geq 1$. Let

$$\mu^N_t := \frac{1}{N} \sum_{i \in [N]} \delta_{X^N_i(t)} \qquad (t \geq 0)$$

denote the *empirical measure* of X^N_t.

Theorem [Mach, Sturm, S. '18] Under *certain technical assumptions*

$$\mathbb{P}\Big[\sup_{0 \leq t \leq T} d(\mu^N_{Nt}, \mu_t) \geq \varepsilon \Big] \underset{N \to \infty}{\longrightarrow} 0 \qquad (\varepsilon > 0,\ T < \infty),$$

where $(\mu_t)_{t \geq 0}$ is the unique solution to a *certain mean-field equation*.

Mean-field particle systems

For any measurable map $g : S^k \to S$, let

$$\mathbf{T}_g(\mu) := \text{ the law of } g(X_1, \ldots, X_k),$$

where $(X_i)_{i=1,\ldots,k}$ are i.i.d. with law μ. Then the *mean-field equation* reads:

$$\frac{\partial}{\partial t}\mu_t = \int_\Omega \mathbf{q}(\mathrm{d}\omega) \sum_{i=1}^{\lambda(\omega)} \big\{ \mathbf{T}_{\gamma_i[\omega]}(\mu_t) - \mu_t \big\}. \tag{1}$$

Let $\langle \mu, \phi \rangle := \int \phi \, \mathrm{d}\mu$. Then $(\mu_t)_{t \geq 0}$ solves (1) iff $t \mapsto \langle \mu_t, \phi \rangle$ is continuously differentiable for each bounded measurable function $\phi : S \to \mathbb{R}$, and

$$\frac{\partial}{\partial t}\langle \mu_t, \phi \rangle = \int_\Omega \mathbf{q}(\mathrm{d}\omega) \sum_{i=1}^{\lambda(\omega)} \big\{ \langle \mathbf{T}_{\gamma_i[\omega]}(\mu_t), \phi \rangle - \langle \mu_t, \phi \rangle \big\}.$$

Assume that for all $\omega \in \Omega$ and $1 \le i \le \lambda(\omega)$, there exists a finite set $K_i(\omega) \subset \{1, \ldots, \lambda(\omega)\}$ with cardinality $\kappa_i(\omega) := |K_i(\omega)|$, such that $\gamma_i[\omega](x_1, \ldots, x_{\lambda(\omega)}) = \gamma_i[\omega]\big((x_i)_{i \in K_i(\omega)}\big)$ depends only on the coordinates in $K_i(\omega)$.

Theorem [Mach, Sturm, S. '18] Assume that

$$\text{(i)} \int_\Omega \mathbf{q}(\mathrm{d}\omega)\, \lambda(\omega) < \infty \quad \text{and} \quad \text{(ii)} \int_\Omega \mathbf{q}(\mathrm{d}\omega) \sum_{i=1}^{\lambda(\omega)} \kappa_i(\omega) < \infty. \quad (2)$$

Then for each initial state, the mean-field equation (1) has a unique solution.

Mean-field particle systems

Define a (nonlinear) semigroup $(\mathbf{T}_t)_{t\geq 0}$ of operators acting on probability measures by

$$\mathbf{T}_t(\mu) := \mu_t \quad \text{where } (\mu_t)_{t\geq 0} \text{ solves (1) with } \mu_0 = \mu.$$

Proposition [Mach, Sturm, S. '18] Assume that

$$\mathbf{q}\big(\{\omega : \lambda(\omega) = l,\ \gamma_i[\omega] \text{ is discontinuous at x}\}\big) = 0 \qquad (3)$$

$(1 \leq i \leq l,\ x \in S^l)$.

Then the operators $\mathbf{T}_t$ are continuous w.r.t. weak convergence.

Mean-field particle systems

Let d be any metric that generates the topology of weak convergence.

Let $\|\cdot\|$ denote the total variation norm.

Theorem [Mach, Sturm, S. '18] (revisited) Assume (2) and at least one of the following conditions:

(i) $\mathbb{P}\big[d(\mu_0^N, \mu_0) \geq \varepsilon\big] \underset{N\to\infty}{\longrightarrow} 0$ for all $\varepsilon > 0$, and (3) holds.

(ii) $\big\|\mathbb{E}[(\mu_0^N)^{\otimes n}] - \mu_0^{\otimes n}\big\| \underset{N\to\infty}{\longrightarrow} 0$ for all $n \geq 1$.

Then

$$\mathbb{P}\Big[\sup_{0\leq t\leq T} d\big(\mu_{Nt}^N, \mathsf{T}_t(\mu_0)\big) \geq \varepsilon\Big] \underset{N\to\infty}{\longrightarrow} 0 \qquad (\varepsilon > 0,\ T < \infty).$$

Cooperative branching

Consider the case where $S = \{0,1\}$, $\Omega = \{1,2\}$,

$$\begin{aligned}
&\lambda(1) = 3 && \lambda(2) = 1, \\
&\mathbf{q}(\{1\}) = \alpha \geq 0 && \mathbf{q}(\{2\}) = 1, \\
&\vec{\gamma}[1](x_1, x_2, x_3) := \big(x_1 \vee (x_2 \wedge x_3), x_2, x_3\big) && \vec{\gamma}[2](x_1) := 0.
\end{aligned}$$

Then

$$\gamma_1[1] = \mathtt{cob}, \quad \gamma_2[1] = \mathrm{Id}, \quad \gamma_3[1] = \mathrm{Id}, \quad \text{and} \quad \gamma_1[2] = \mathtt{dth},$$

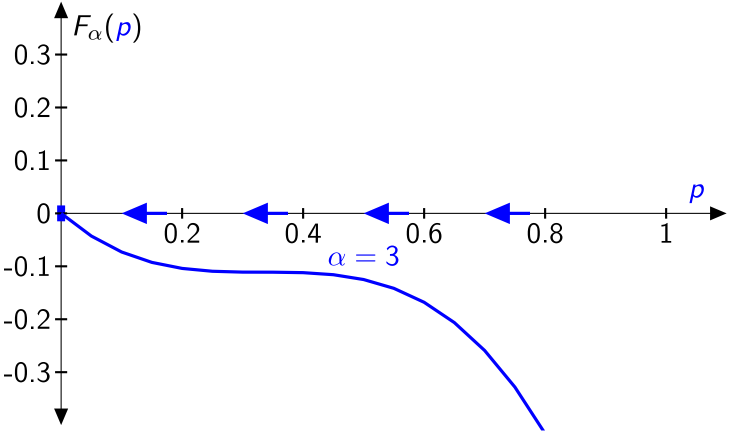
and the mean-field equation simplifies to

$$\tfrac{\partial}{\partial t}\mu_t = \alpha\big\{\mathbf{T}_{\mathtt{cob}}(\mu_t) - \mu_t\big\} + \big\{\mathbf{T}_{\mathtt{dth}}(\mu_t) - \mu_t\big\}.$$

Rewriting this in terms of $p_t := \mu_t(\{1\})$ yields

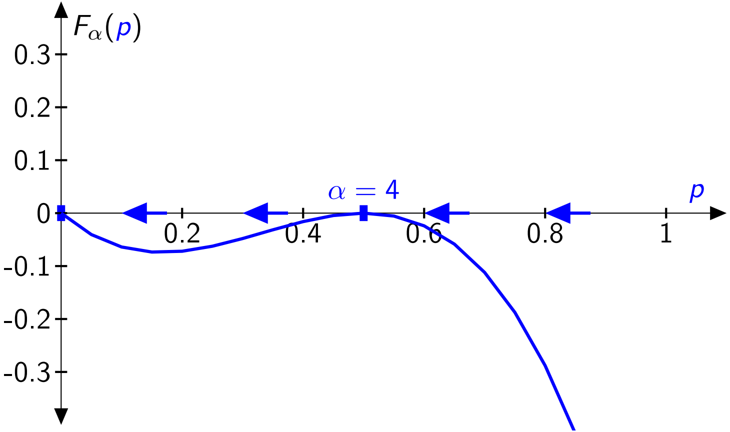
$$\tfrac{\partial}{\partial t}p_t = \alpha p_t^2(1 - p_t) - p_t =: F_\alpha(p_t) \qquad (t \geq 0).$$

Cooperative branching



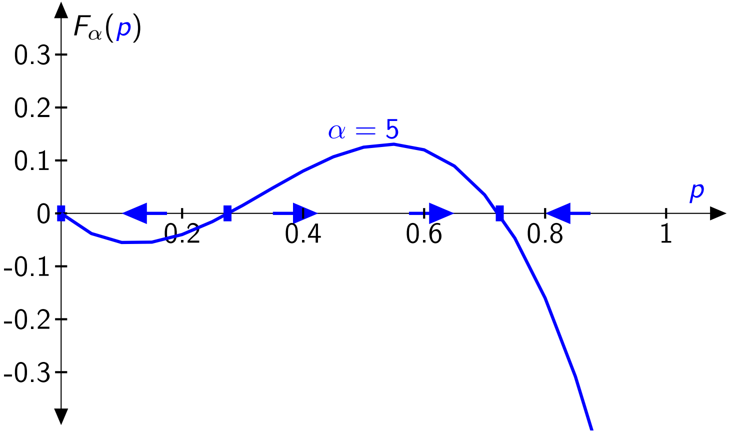
For $\alpha < 4$, the equation $\frac{\partial}{\partial t} p_t = F_\alpha(p_t)$ has a single, stable fixed point $\overline{x} = 0$.

Cooperative branching



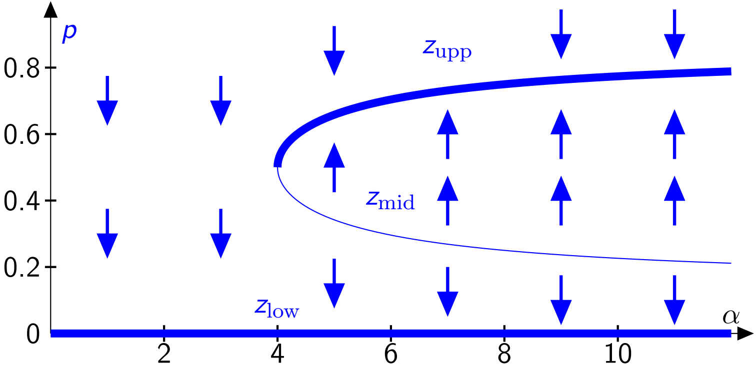
For $\alpha = 4$, a second fixed point appears at $\overline{x} = 0.5$.

Cooperative branching



For $\alpha > 4$, there are two stable fixed points and one unstable fixed point, which separates the domains of attraction of the other two.

Cooperative branching



Fixed points of $\frac{\partial}{\partial t} p_t = F_\alpha(p_t)$ for different values of α.

A recursive tree representation

Simplified ingredients

(i) Polish space S *local state space.*

(ii) $(\Omega, \mathcal{B}, \mathbf{r})$ Polish space with Borel σ-field and finite measure: *source of external randomness.*

(iii) $\kappa : \Omega \to \mathbb{N}$ measurable function.

(iv) For each $\omega \in \Omega$, a function $\gamma[\omega] : S^{\kappa(\omega)} \to S$.

Let

$$\mathbf{T}(\mu) := \text{ the law of } \gamma[\boldsymbol{\omega}](X_1, \ldots, X_{\kappa(\boldsymbol{\omega})}),$$

where $\boldsymbol{\omega}$ is an Ω-valued random variable with law $|\mathbf{r}|^{-1}\mathbf{r}$ and $(X_i)_{i\geq 1}$ are i.i.d. with law μ. Then each mean-field equation (1) can be rewritten in the simpler form

$$\frac{\partial}{\partial t}\mu_t = |\mathbf{r}|\big\{\mathbf{T}(\mu_t) - \mu_t\big\} \qquad (t \geq 0). \tag{4}$$

A recursive tree representation

Note that $\gamma[\omega] : S^{\kappa(\omega)} \to S$ is a random map. We call

$$\mathbf{T}(\mu) := \text{ the law of } \gamma[\omega](X_1, \ldots, X_{\kappa(\omega)}),$$

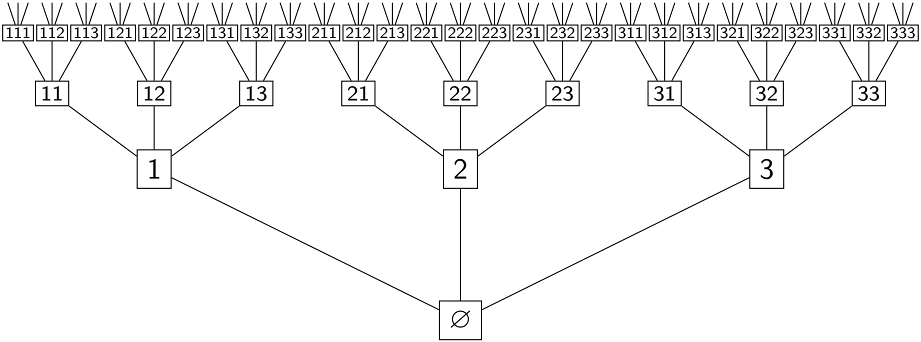
a *random mapping representation* of the operator $\mathbf{T}$.

Our aim is to find a similar random mapping representation for the operators $(\mathbf{T}_t)_{t \geq 0}$.

In a discrete-time setting, something similar has been done by Aldous and Bandyopadyay (2005) for iterates $\mathbf{T}^n$ of the map $\mathbf{T}$.

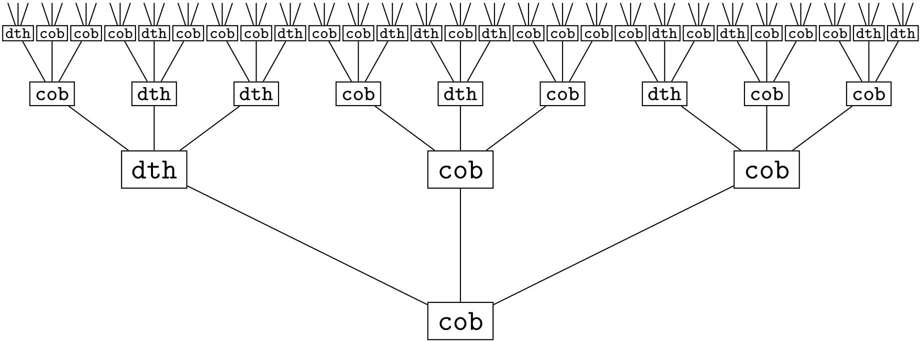
In what follows, we fix $d \in \mathbb{N}_+ \cup \{\infty\}$ such that $\kappa(\omega) \leq d$ for all $\omega \in \Omega$. We let $\mathbb{T}^d$ denote the space of all words $\mathbf{i} = i_1 \cdots i_n$ made from the alphabet $[d]$ (if $d < \infty$) resp. $\mathbb{N}_+$ (if $d = \infty$).

A recursive tree representation



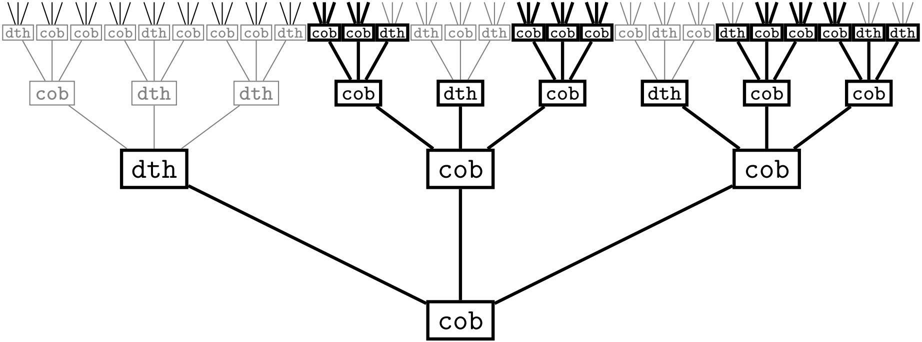
We view $\mathbb{T}^d$ as a tree with root $\varnothing$, the word of length zero.

A recursive tree representation



We attach i.i.d. $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ with law $|\mathbf{r}|^{-1}\mathbf{r}$ to each node, which translate into maps $(\gamma[\omega_{\mathbf{i}}])_{\mathbf{i}\in\mathbb{T}}$.

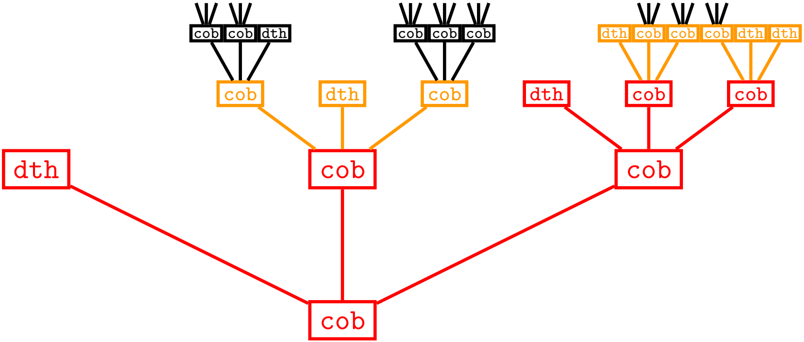
A recursive tree representation



Let $\mathbb{S}$ be the random subtree of $\mathbb{T}$ defined as

$$\mathbb{S} := \big\{ i_1 \cdots i_n \in \mathbb{T} : i_m \leq \kappa(\omega_{i_1 \cdots i_{m-1}}) \ \forall 1 \leq m \leq n \big\}.$$

A recursive tree representation

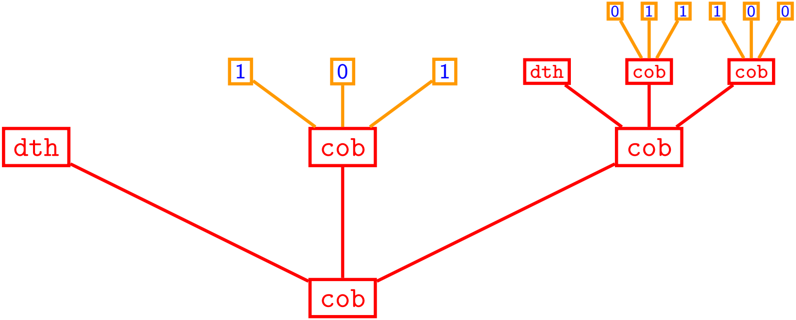


For any rooted subtree $\mathbb{U} \subset \mathbb{S}$, let

$$\nabla\mathbb{U} := \big\{ i_1 \cdots i_n \in \mathbb{S} : i_1 \cdots i_{n-1} \in \mathbb{U},\ \ i_1 \cdots i_n \notin \mathbb{U} \big\}$$

denote the boundary of $\mathbb{U}$ relative to $\mathbb{S}$.

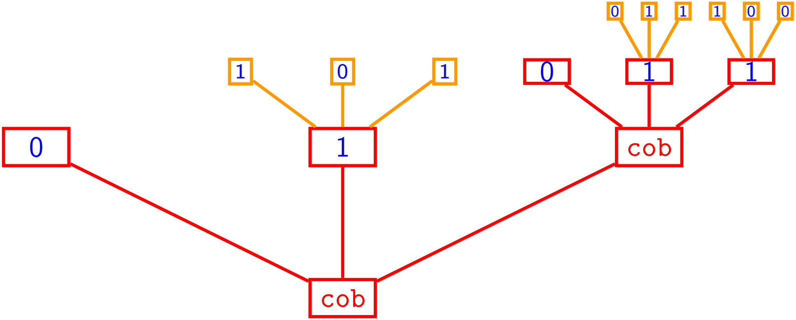
A recursive tree representation



Given $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}$, we inductively define $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$ by

$$X_{\mathbf{i}} = \gamma[\omega_{\mathbf{i}}]\big(X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}\kappa(\omega)}\big) \qquad (\mathbf{i}\in\mathbb{U}).$$

A recursive tree representation



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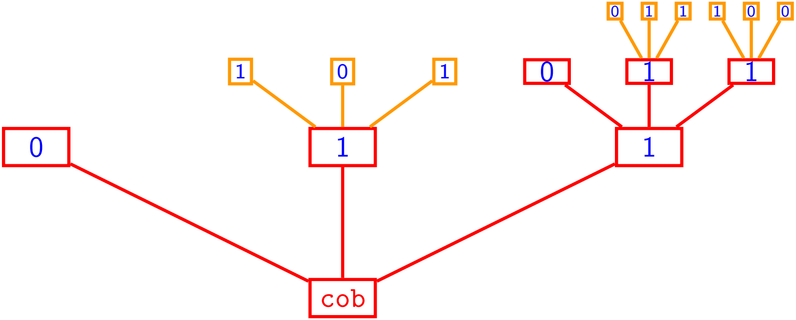
A recursive tree representation



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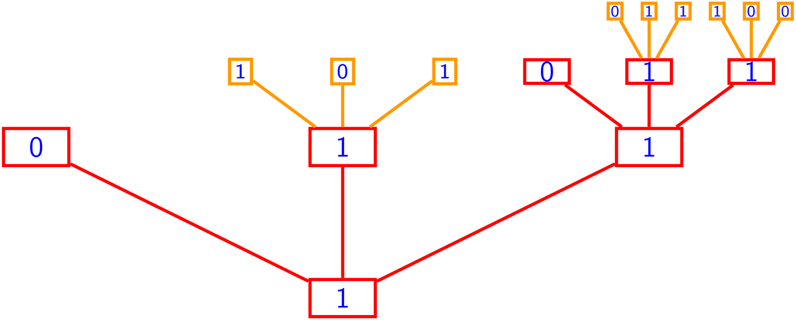


Given $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}$, we inductively define $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$ by

$$X_{\mathbf{i}} = \gamma[\omega_{\mathbf{i}}]\big(X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}\kappa(\omega)}\big) \qquad (\mathbf{i} \in \mathbb{U}).$$

A recursive tree representation

Setting

$$G_{\mathbb{U}}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{U}}\big) := X_{\varnothing}$$

defines a random map

$$G_{\mathbb{U}} : \mathbb{S}^{\nabla\mathbb{U}} \to \mathbb{S}$$

that is the concatenation of the maps $\big(\gamma[\omega_{\mathbf{i}}]\big)_{\mathbf{i}\in\mathbb{U}}$ according to the tree structure of $\mathbb{U}$.

Let $|i_1 \cdots i_n| := n$ denote the length of a word $\mathbf{i}$ and set

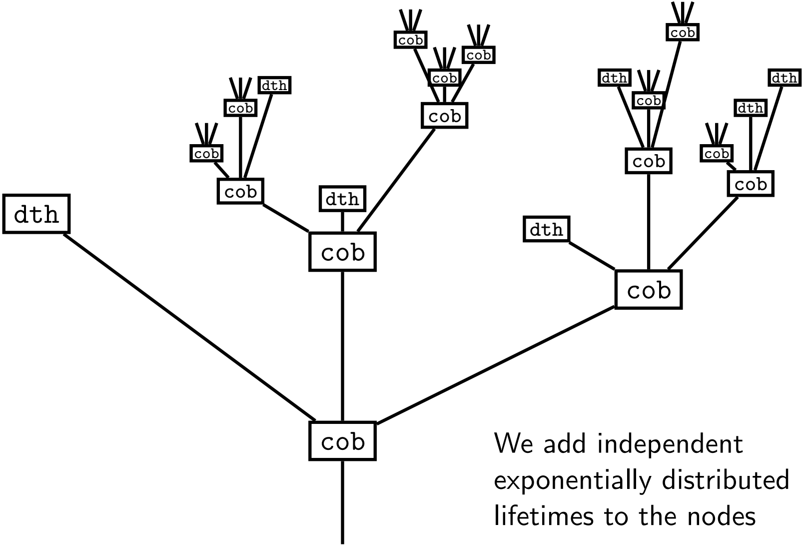
$$\mathbb{S}_{(n)} := \{\mathbf{i} \in \mathbb{S} : |\mathbf{i}| < n\} \quad \text{and} \quad \nabla\mathbb{S}_{(n)} := \{\mathbf{i} \in \mathbb{S} : |\mathbf{i}| = n\}.$$

Aldous and Bandyopadyay (2005) observed that

$$\mathbf{T}^n(\mu) := \text{ the law of } G_{\mathbb{S}_{(n)}}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_{(n)}}\big),$$

where $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_{(n)}}$ are i.i.d. with law μ and independent of $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}_{(n)}}$.

A recursive tree representation



We add independent exponentially distributed lifetimes to the nodes

A recursive tree representation

Let $(\sigma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be i.i.d. exponentially distributed with mean $|\mathbf{r}|^{-1}$, independent of $(\boldsymbol{\omega}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, and set

$$\tau^*_{\mathbf{i}} := \sum_{m=1}^{n-1} \sigma_{i_1\cdots i_m} \quad \text{and} \quad \tau^\dagger_{\mathbf{i}} := \tau^*_{\mathbf{i}} + \sigma_{\mathbf{i}} \qquad (\mathbf{i} = i_1\cdots i_n),$$
$$\mathbb{S}_t := \big\{\mathbf{i}\in\mathbb{S} : \tau^\dagger_{\mathbf{i}} \leq t\big\} \quad \text{and} \quad \nabla\mathbb{S}_t = \big\{\mathbf{i}\in\mathbb{S} : \tau^*_{\mathbf{i}} \leq t < \tau^\dagger_{\mathbf{i}}\big\}.$$

Let $\mathcal{F}_t$ be the filtration

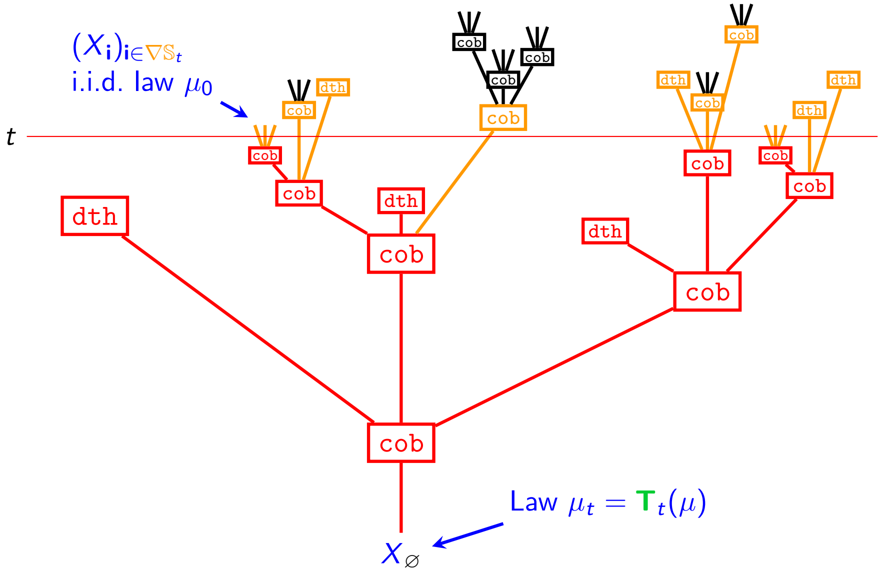
$$\mathcal{F}_t := \sigma\big(\nabla\mathbb{S}_t, (\boldsymbol{\omega}_{\mathbf{i}}, \sigma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}_t}\big) \qquad (t \geq 0).$$

Theorem [Mach, Sturm, S. '18]

$$\mathbf{T}_t(\mu) := \text{ the law of } G_{\mathbb{S}_t}\big((X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_t}\big),$$

where $(X_{\mathbf{i}})_{\mathbf{i}\in\nabla\mathbb{S}_t}$ are i.i.d. with law μ and independent of $\mathcal{F}_t$.

A recursive tree representation



Recursive Tree Processes

A *Recursive Distributional Equation* is an equation of the form

$$X \stackrel{\mathrm{d}}{=} \gamma[\omega](X_1,\ldots,X_{\kappa(\omega)}) \qquad \text{(RDE)},$$

where $X_1, X_2, \ldots$ are i.i.d. copies of X, independent of ω.

A law ν solves (RDE) iff

$$\text{(i)} \quad \mathbf{T}_t(\nu) = \nu \quad (t \geq 0) \qquad \text{or} \qquad \text{(ii)} \quad \mathbf{T}(\nu) = \nu.$$

We can view ν as the "invariant law" of a "Markov chain" where time has a tree-like structure.

In our example, solutions to the RDE are the Bernoulli distributions $\nu_{\rm low}$, $\nu_{\rm mid}$, $\nu_{\rm upp}$ with density $z_{\rm low}$, $z_{\rm mid}$, $z_{\rm upp}$.

Recursive Tree Processes

For any rooted subtree $\mathbb{U} \subset \mathbb{S}$, let

$$\partial\mathbb{U} := \{ i_1 \cdots i_n \in \mathbb{T} : i_1 \cdots i_{n-1} \in \mathbb{U},\ i_1 \cdots i_n \notin \mathbb{U} \}$$

denote the boundary of $\mathbb{U}$ relative to $\mathbb{T}$.

For each solution ν of (RDE), there exists a *Recursive Tree Process (RTP)* $(\boldsymbol{\omega}_{\mathbf{i}}, \mathbf{X}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$, unique in law, such that:

(i) $(\boldsymbol{\omega}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}}$ are i.i.d. with law $|\mathbf{r}|^{-1}\mathbf{r}$.

(ii) For finite $\mathbb{U} \subset \mathbb{T}$, the r.v.'s $(\mathbf{X}_{\mathbf{i}})_{\mathbf{i} \in \partial\mathbb{U}}$ are i.i.d. with ν and independent of $(\boldsymbol{\omega}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{U}}$.

(iii) $\mathbf{X}_{\mathbf{i}} = \gamma[\boldsymbol{\omega}_{\mathbf{i}}]\big(\mathbf{X}_{\mathbf{i}1}, \ldots, \mathbf{X}_{\mathbf{i}\kappa(\boldsymbol{\omega}_{\mathbf{i}})}\big) \qquad (\mathbf{i} \in \mathbb{T}).$

If we add independent exponentially distributed lifetimes, then:

- Conditional on $\mathcal{F}_t$, the r.v.'s $(\mathbf{X}_{\mathbf{i}})_{\mathbf{i} \in \nabla\mathbb{S}_t}$ are i.i.d. with law ν.

n-Variate processes

For each $n \geq 1$, a measurable map $g : S^k \to S$ gives rise to *n-variate map* $g^{(n)} : (S^n)^k \to S^n$ defined as

$$g^{(n)}(x_1, \ldots, x_k) = g^{(n)}(x^1, \ldots, x^n) := \big(g(x^1), \ldots, g(x^n)\big),$$

with $x = (x_i^m)_{i=1,\ldots,k}^{m=1,\ldots,n}$, $x_i = (x_i^1, \ldots, x_i^n)$, $x^m = (x_1^m, \ldots, x_k^m)$.

We define an *n-variate map*

$$\mathbf{T}^{(n)}(\mu^{(n)}) := \text{ the law of } \gamma^{(n)}[\boldsymbol{\omega}](X_1, \ldots, X_{\kappa(\boldsymbol{\omega})}),$$

which acts on probability measures $\mu^{(n)}$ on S^n.
The *n-variate mean-field equation*

$$\tfrac{\partial}{\partial t}\mu_t^{(n)} = |\mathbf{r}|\big\{\mathbf{T}^{(n)}(\mu_t^{(n)}) - \mu_t^{(n)}\big\} \qquad (t \geq 0).$$

describes the mean-field limit of n coupled processes that are constructed using the same stochastic flow $(\mathbf{X}_{s,u})_{s \leq u}$.

n-Variate processes

$\mathcal{P}(S)$	space of probability measures on S.
$\mathcal{P}_{\rm sym}(S^n)$	space of probability measures on S^n that are symmetric under a permutation of the coordinates.
$S^n_{\rm diag}$	$\{x \in S^n : x_1 = \cdots = x_n\}$
$\mathcal{P}(S^n)_\mu$	space of probability measures on S^n whose one-dimensional marginals are all equal to μ.

- If $(\mu^{(n)}_t)_{t\geq 0}$ solves the n-variate equation, then its m-dimensional marginals solve the m-variate equation.
- $\mu^{(n)}_0 \in \mathcal{P}_{\rm sym}(S^n)$ implies $\mu^{(n)}_t \in \mathcal{P}_{\rm sym}(S^n)$ $(t \geq 0)$.
- $\mu^{(n)}_0 \in \mathcal{P}(S^n_{\rm diag})$ implies $\mu^{(n)}_t \in \mathcal{P}(S^n_{\rm diag})$ $(t \geq 0)$.
- If $\mathbf{T}(\nu) = \nu$, then $\mu^{(n)}_0 \in \mathcal{P}(S^n)_\nu$ implies $\mu^{(n)}_t \in \mathcal{P}(S^n)_\nu$.

n-Variate processes

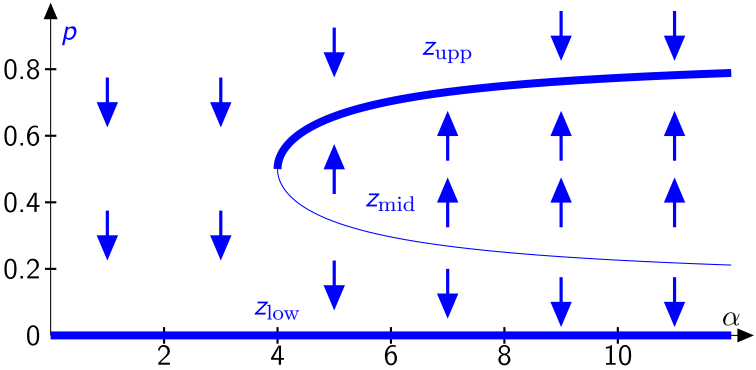
If $\nu = \mathbb{P}[X \in \cdot\,]$ solves the RDE $\mathbf{T}(\nu) = \nu$, then

$$\overline{\nu}^{(n)} := \mathbb{P}\big[(\underbrace{X, \ldots, X}_{n \text{ times}}) \in \cdot\,\big]$$

solves the *n-variate RDE* $\mathbf{T}^{(n)}(\nu^{(n)}) = \nu^{(n)}$.

Questions:

- Is $\overline{\nu}^{(n)}$ a stable fixed point of the *n*-variate equation?
- Is $\overline{\nu}^{(n)}$ the only solution in $\mathcal{P}_{\rm sym}(S^n)_\nu$ of the *n*-variate RDE?



Fixed points of $\frac{\partial}{\partial t} p_t = F_\alpha(p_t)$ for different values of α.

n-Variate processes

Cooperative branching with branching rate $\alpha > 4$

The RDE $\mathbf{T}(\nu) = \nu$ has three solutions $\nu_{\text{low}}, \nu_{\text{mid}}$, and ν_{upp}, where $\nu_{\dots}$ is the probability measure on $\{0,1\}$ with mean $\nu_{\dots}(\{1\}) = z_{\dots}$ $(\dots = \text{low}, \text{mid}, \text{upp})$, which

give rise to solutions $\overline{\nu}^{(2)}_{\text{low}}, \overline{\nu}^{(2)}_{\text{mid}}$, and $\overline{\nu}^{(2)}_{\text{upp}}$ of the *bivariate RDE.*

Proposition [Mach, Sturm, S. '18] Apart from $\overline{\nu}^{(2)}_{\text{low}}, \overline{\nu}^{(2)}_{\text{mid}}, \overline{\nu}^{(2)}_{\text{upp}}$, the *bivariate RDE* has one more solution $\underline{\nu}^{(2)}_{\text{mid}}$ in $\mathcal{P}_{\text{sym}}(S^2)$. The domains of attraction are:

$$
\begin{array}{ll}
\overline{\nu}^{(2)}_{\text{low}}: & \big\{\mu^{(2)}_0 : \mu^{(1)}_0(\{1\}) < z_{\text{mid}}\big\}, \\
\underline{\nu}^{(2)}_{\text{mid}}: & \big\{\mu^{(2)}_0 : \mu^{(1)}_0(\{1\}) = z_{\text{mid}},\ \mu^{(2)}_0 \neq \overline{\nu}^{(2)}_{\text{mid}}\big\}, \\
\overline{\nu}^{(2)}_{\text{mid}}: & \big\{\overline{\nu}^{(2)}_{\text{mid}}\big\}, \\
\overline{\nu}^{(2)}_{\text{upp}}: & \big\{\mu^{(2)}_0 : \mu^{(1)}_0(\{1\}) > z_{\text{mid}}\big\}.
\end{array}
$$

Let $\big(\mathbf{X}(t)\big)_{t\geq 0}$ be a process in S^N.
Initial law: $\big(\mathbf{X}_i(0)\big)_{i\in[N]}$ i.i.d. with mean $z_{\rm mid}$.
Let $\big(\mathbf{X}'(t)\big)_{t\geq 0}$ be a process with modified initial state:
$\mathbf{X}'_i(0) = \mathbf{X}_i(0)$ except for an ε-fraction of sites i, which are redrawn using independent randomness.

Then (in the mean-field limit N large), the fraction of sites where $\mathbf{X}'_i(t) \neq \mathbf{X}_i(t)$ increases as $t \to \infty$,
and the joint empirical law of $\mathbf{X}(t), \mathbf{X}'(t)$ converges to $\nu^{(2)}_{\rm mid}$.

Let $(\omega_{\mathbf{i}}, \mathbf{X}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to a solution ν of the RDE.

Aldous and Bandyopadyay (RDE) say that an RTP is *endogenous* if

$$\mathbf{X}_\varnothing \text{ is measurable w.r.t. the } \sigma\text{-field generated by } (\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}.$$

Theorem [AB '05 & MSS '18] The following statements are equivalent:

(i) The RTP corresponding to ν is endogenous.

(ii) $\mathbf{T}_t^{(n)}(\mu) \underset{t\to\infty}{\Longrightarrow} \overline{\nu}^{(n)}$ for all $\mu \in \mathcal{P}(S^n)_\nu$ and $n \geq 1$.

(iii) $\overline{\nu}^{(2)}$ is the only solution in $\mathcal{P}_{\rm sym}(S^2)_\nu$ of the bivariate RDE.

In our example, the RTPs for $\nu_{\rm low}, \nu_{\rm upp}$ are endogenous, but the RTP corresponding to $\nu_{\rm mid}$ is not.

The higher-level equation

The n-variate map $\mathbf{T}^{(n)}$ is defined even for $n = \infty$, and $\mathbf{T}^{(\infty)}$ maps $\mathcal{P}_{\rm sym}(S^{\mathbb{N}_+})$ into itself.

By De Finetti's theorem, $(X_i)_{i\in\mathbb{N}_+}$ have a law in $\mathcal{P}_{\rm sym}(S^{\mathbb{N}_+})$ if and only if there exists a random probability measure ξ on S such that conditional on ξ, the $(X_i)_{i\in\mathbb{N}_+}$ are i.i.d. with law ξ.

Let $\rho := \mathbb{P}[\xi \in \cdot\,]$ the law of ξ. Then $\rho \in \mathcal{P}(\mathcal{P}(S))$.
In view of this, $\mathcal{P}_{\rm sym}(S^{\mathbb{N}_+}) \cong \mathcal{P}(\mathcal{P}(S))$.

The map $\mathbf{T}^{(\infty)} : \mathcal{P}_{\rm sym}(S^{\mathbb{N}_+}) \to \mathcal{P}_{\rm sym}(S^{\mathbb{N}_+})$
corresponds to a *higher-level map* $\check{\mathbf{T}} : \mathcal{P}(\mathcal{P}(S)) \to \mathcal{P}(\mathcal{P}(S))$.

The higher-level equation

For any measurable map $g : S^k \to S$, define $\check{g} : \mathcal{P}(S)^k \to \mathcal{P}(S)$ by

$$\begin{aligned}\check{g} := \ &\text{the law of } g(X_1, \ldots, X_k),\\ &\quad \text{where } (X_1, \ldots, X_k) \text{ are independent with laws } \mu_1, \ldots, \mu_k.\end{aligned}$$

Then

$$\check{\mathbf{T}}(\rho) := \text{ the law of } \check{\gamma}[\boldsymbol{\omega}](\xi_1, \ldots, \xi_{\kappa(\boldsymbol{\omega})}),$$

with $\boldsymbol{\omega}$ as before and $\xi_1, \xi_2, \ldots$ i.i.d. with law ρ.

Define *n-th moment measures*

$$\rho^{(n)} := \mathbb{E}\big[\underbrace{\xi \otimes \cdots \otimes \xi}_{n \text{ times}}\big] \quad \text{where } \xi \text{ has law } \rho.$$

Proposition [MSS '18] If $(\rho_t)_{t \geq 0}$ solves the *higher-level mean-field equation*, then its n-th moment measures $(\rho_t^{(n)})_{t \geq 0}$ solve the n-variate equation.

The higher-level equation

Equip $\mathcal{P}(\mathcal{P}(S))_\nu = \{\rho : \rho^{(1)} = \nu\}$ with the *convex order*

$$\rho_1 \leq_{\mathrm{cv}} \rho_2 \quad \text{iff} \quad \int \phi \, \mathrm{d}\rho_1 \leq \int \phi \, \mathrm{d}\rho_2 \quad \forall \text{ convex } \phi.$$

[Strassen '65] $\rho_1 \leq_{\mathrm{cv}} \rho_2$ iff there exist a r.v. X and σ-fields $\mathcal{H}_1 \subset \mathcal{H}_2$ s.t. $\rho_i = \mathbb{P}\big[\mathbb{P}[X \in \cdot\,|\mathcal{H}_i] \in \cdot\,\big]$ $(i = 1, 2)$.

Define $\overline{\nu} := \mathbb{P}[\delta_X \in \cdot\,]$ with $\mathbb{P}[X \in \cdot\,] = \nu$. Maximal and minimal elements:

$$\delta_\nu \leq_{\mathrm{cv}} \rho \leq_{\mathrm{cv}} \overline{\nu} \qquad \forall \rho \in \mathcal{P}(\mathcal{P}(S))_\nu.$$

Proposition [MSS '18] $\check{\mathbf{T}}$ is monotone w.r.t. the convex order. There exists a solution $\underline{\nu}$ to the higher-level RDE s.t.

$$\check{\mathbf{T}}^n(\delta_\nu) \underset{n\to\infty}{\Longrightarrow} \underline{\nu} \quad \text{and} \quad \check{\mathbf{T}}_t(\delta_\nu) \underset{t\to\infty}{\Longrightarrow} \underline{\nu}$$

and any solution $\rho \in \mathcal{P}(\mathcal{P}(S))_\nu$ to the higher-level RDE satisfies

$$\underline{\nu} \leq_{\mathrm{cv}} \rho \leq_{\mathrm{cv}} \overline{\nu} \qquad \forall \rho \in \mathcal{P}(\mathcal{P}(S))_\nu.$$

The higher-level equation

Proposition [MSS '18]
Let $(\omega_{\mathbf{i}}, \mathbf{X}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to γ and ν. Set

$$\xi_{\mathbf{i}} := \mathbb{P}\big[X_{\mathbf{i}} \in \cdot \,|\, (\omega_{\mathbf{ij}})_{\mathbf{j}\in\mathbb{T}}\big].$$

Then $(\omega_{\mathbf{i}}, \xi_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is an RTP corresponding to $\check{\gamma}$ and $\underline{\nu}$.
Also, $(\omega_{\mathbf{i}}, \delta_{X_{\mathbf{i}}})_{\mathbf{i}\in\mathbb{T}}$ is an RTP corresponding to $\check{\gamma}$ and $\overline{\nu}$.

Corollary The RTP is endogenous iff $\underline{\nu} = \overline{\nu}$.

The higher-level equation

Theorem [Mach, Sturm, S. '18] One has

$$\underline{\nu}_{\rm low} = \overline{\nu}_{\rm low}, \quad \underline{\nu}_{\rm upp} = \overline{\nu}_{\rm upp}, \quad \text{but} \quad \underline{\nu}_{\rm mid} \neq \overline{\nu}_{\rm mid}.$$

These are all solutions to the higher-level RDE.
Any solution $(\rho_t)_{t\geq 0}$ to the higher-level mean-field equation converges to one of these fixed points.
The domains of attraction are:

$$\begin{array}{ll} \overline{\nu}_{\rm low}: & \big\{\rho_0 : \rho_0^{(1)}(\{1\}) < z_{\rm mid}\big\}, \\ \underline{\nu}_{\rm mid}: & \big\{\rho_0 : \rho_0^{(1)}(\{1\}) = z_{\rm mid},\ \rho_0 \neq \overline{\nu}_{\rm mid}\big\}, \\ \overline{\nu}_{\rm mid}: & \big\{\overline{\nu}_{\rm mid}\big\}, \\ \overline{\nu}_{\rm upp}: & \big\{\rho_0 : \rho_0^{(1)}(\{1\}) > z_{\rm mid}\big\}. \end{array}$$

The higher-level equation

The map $\mu \mapsto \mu(\{1\})$ defines a bijection $\mathcal{P}(\{0,1\}) \cong [0,1]$, and hence $\mathcal{P}(\mathcal{P}(\{0,1\})) \cong \mathcal{P}[0,1]$.

Then the higher-level RDE takes the form

$$\eta \stackrel{\mathrm{d}}{=} \chi \cdot \big(\eta_1 + (1-\eta_1)\eta_2\eta_3\big),$$

where η takes values in $[0,1]$, η_1, η_2, η_3 are independent copies of η and χ is an independent Bernoulli r.v. with $\mathbb{P}[\chi = 1] = \alpha/(\alpha+1)$.

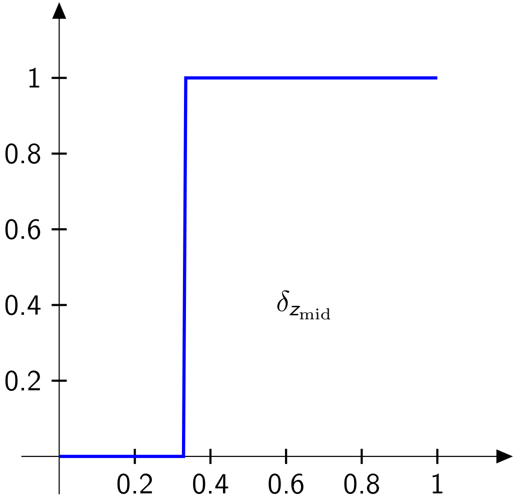
This RDE has three "trivial" solutions

$$\overline{\nu}_{\dots} = (1-z_{\dots})\delta_0 + z_{\dots}\delta_1 \qquad (\dots = \mathrm{low}, \mathrm{mid}, \mathrm{upp}),$$

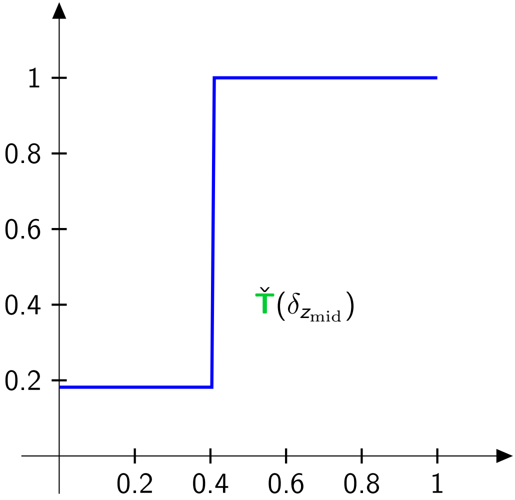
and a nontrivial solution

$$\underline{\nu}_{\mathrm{mid}} = \lim_{n\to\infty} \check{\mathsf{T}}^n(\delta_{z_{\mathrm{mid}}}).$$

Numerical results



Numerical results



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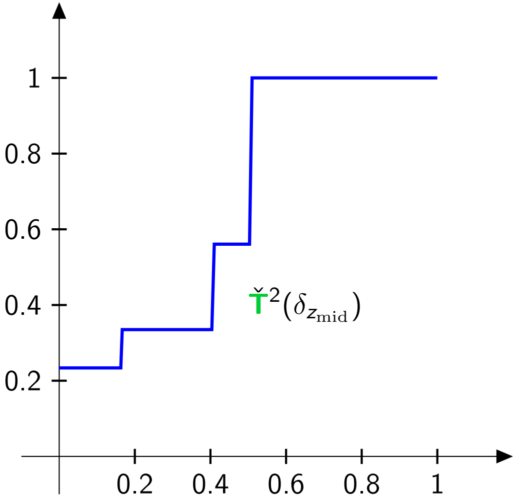

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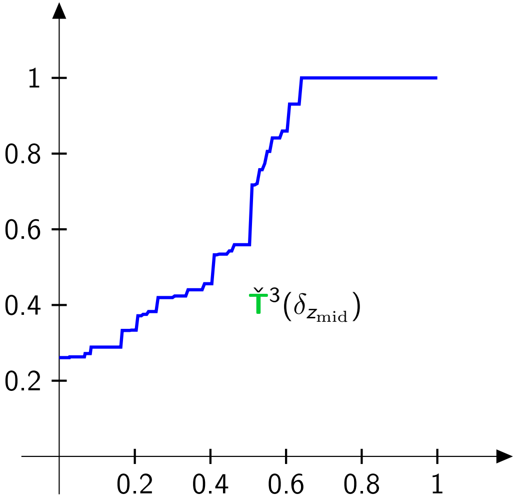

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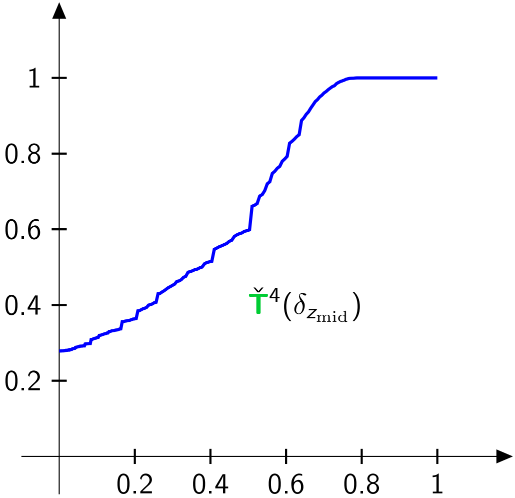

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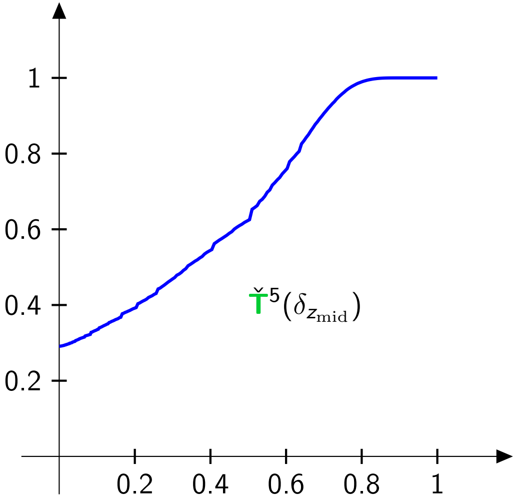

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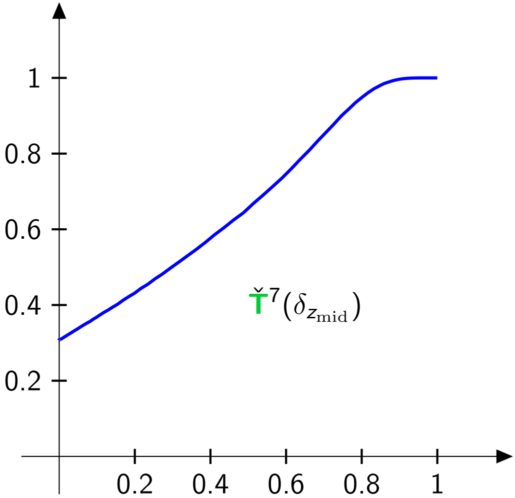

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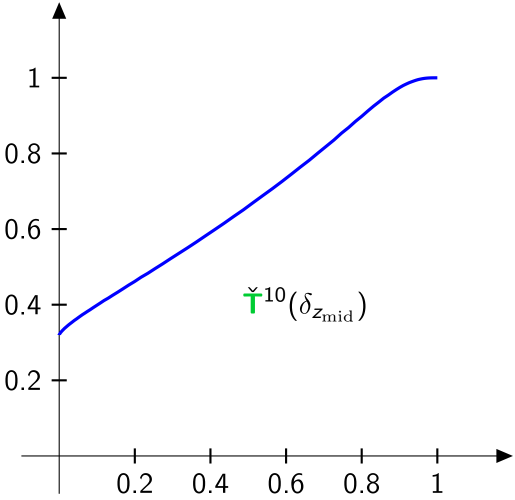

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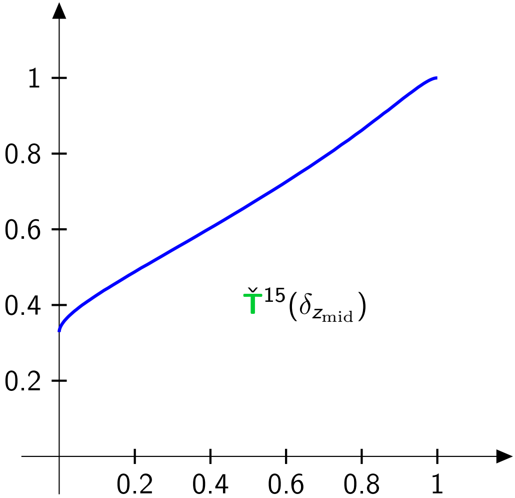

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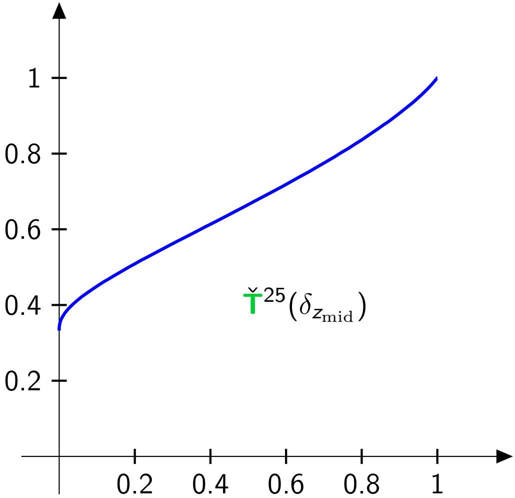

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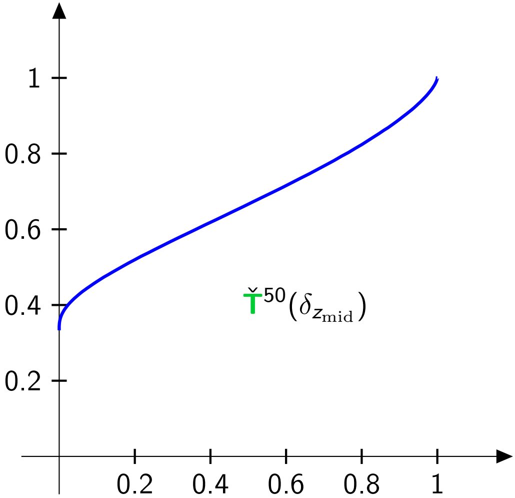

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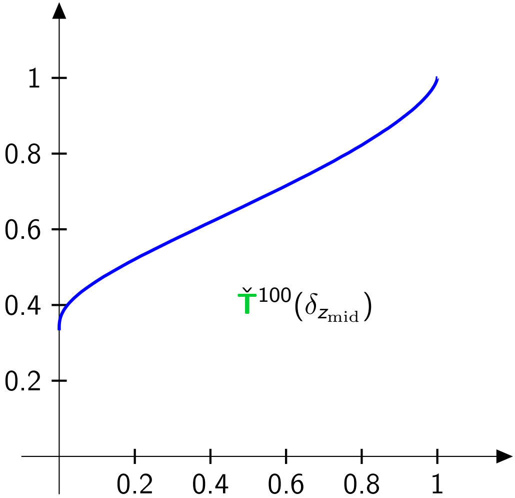

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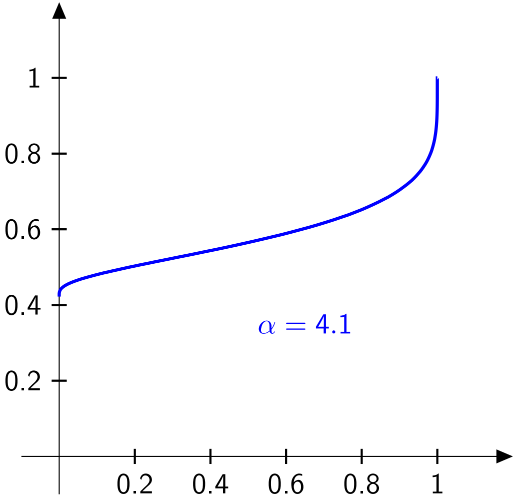

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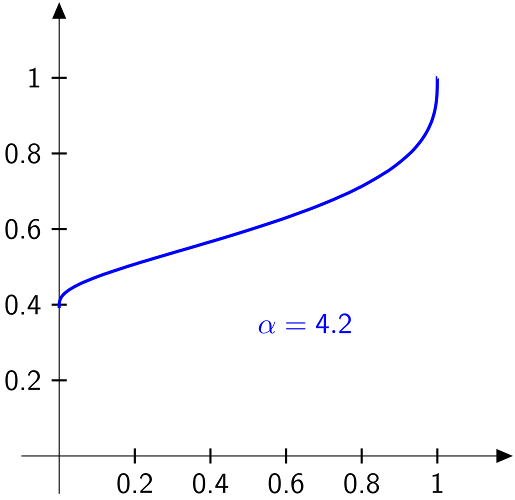

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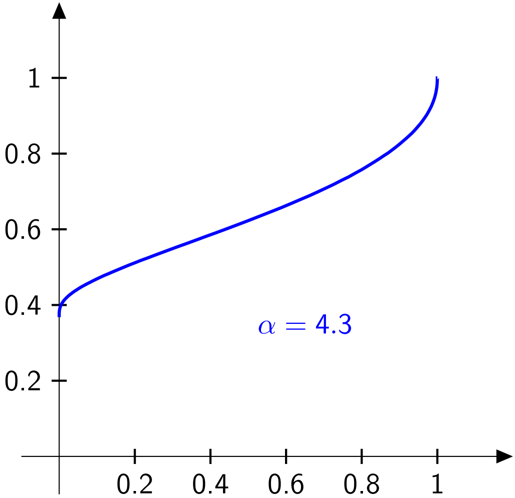

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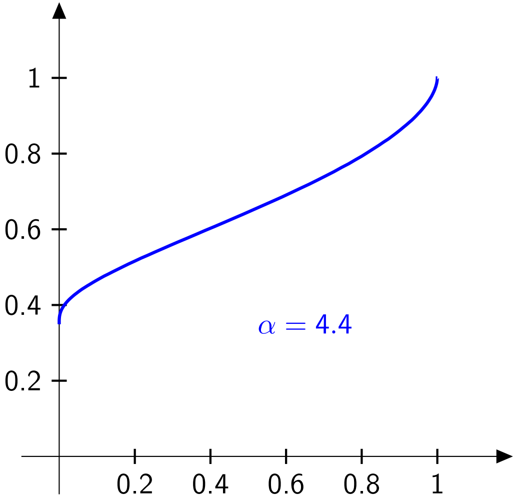

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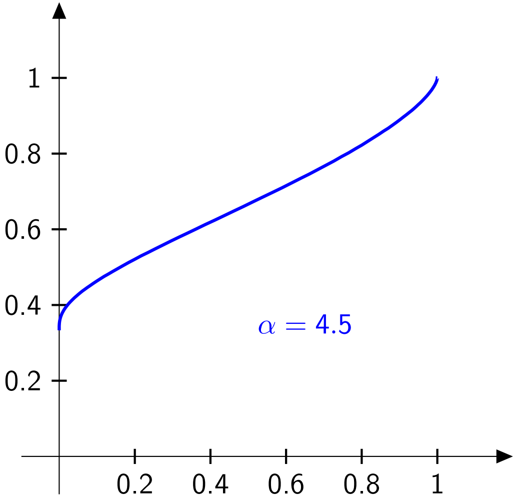

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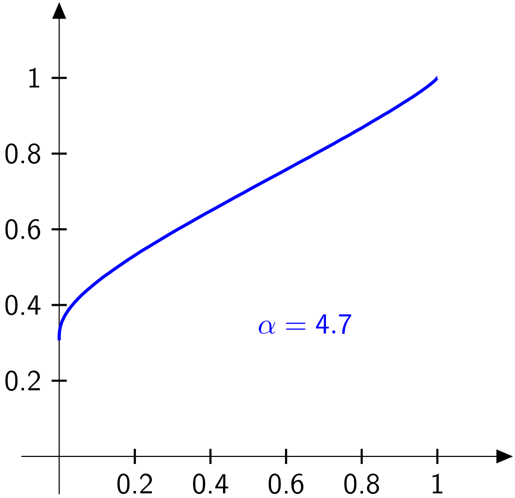

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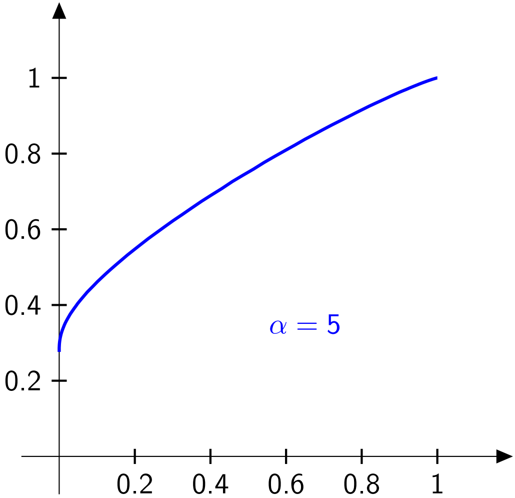

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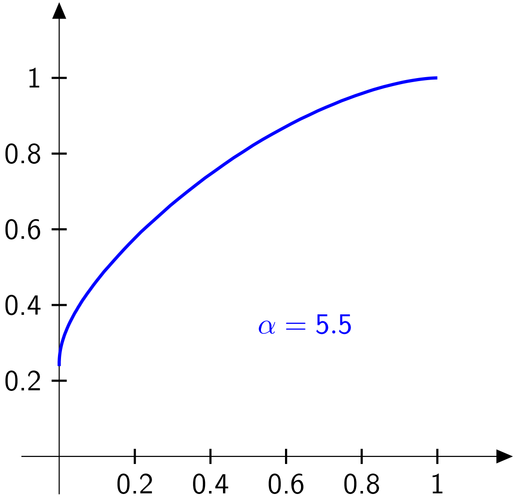

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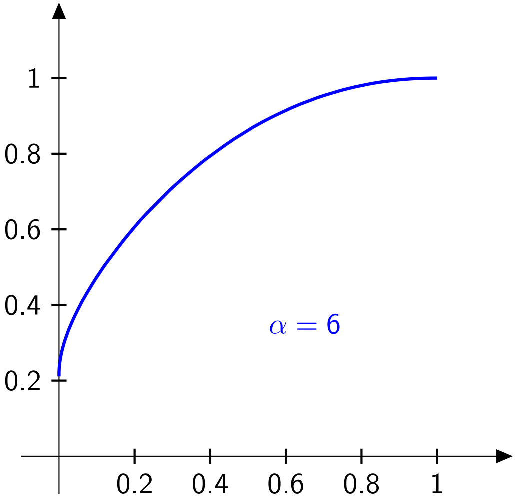

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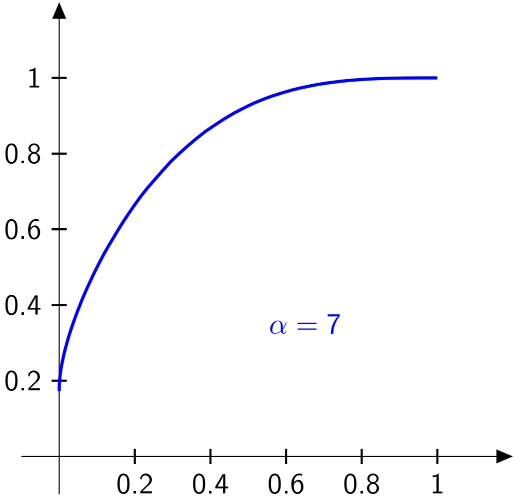

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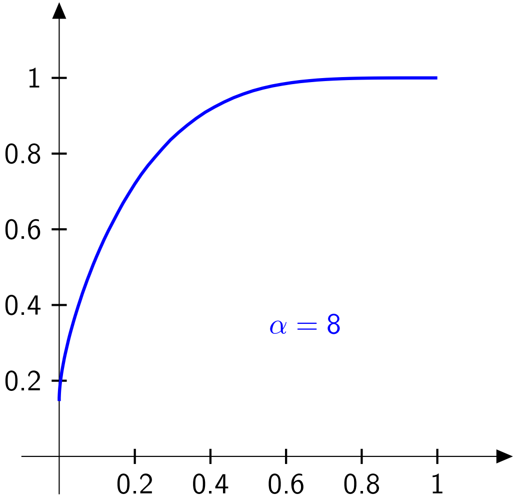

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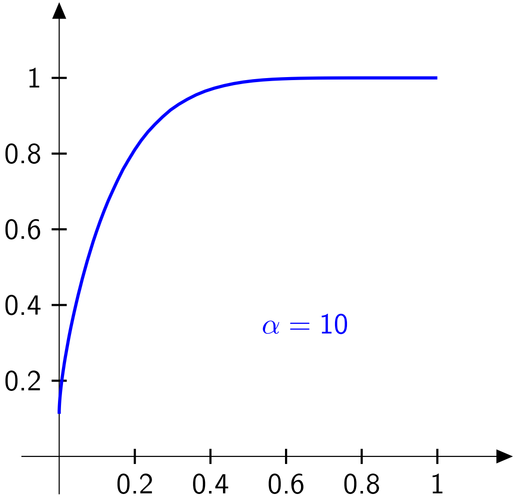

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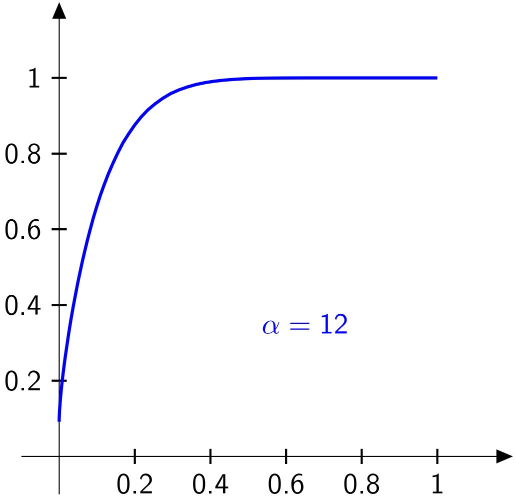

Numerical results