Weaves, webs and flows

Jan M. Swart (Czech Academy of Sciences)

joint with Nic Freeman

Stochastic flows

Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{X}$ be a probability space and a measurable space. By definition, a *stochastic flow* on $\mathcal{X}$ is a collection $(\mathbb{X}_{s,t})_{s \leq t}$ of random maps $\mathbb{X}_{s,t} : \mathcal{X} \to \mathcal{X}$, such that:

(i). $(s, t, \omega, x) \mapsto \mathbb{X}_{s,t}[\omega](x)$ is jointly measurable as a function on $\{(s, t) \in \mathbb{R}^2 : s \leq t\} \times \Omega \times \mathcal{X}$.

(ii). $\mathbb{X}_{s,s} = \mathrm{Id}$ and $\mathbb{X}_{t,u} \circ \mathbb{X}_{s,t} = \mathbb{X}_{s,u}$ $(s \leq t \leq u)$.

Sometimes (ii) is required only for deterministic $s \leq t \leq u$, i.e.,

(ii)'. $\mathbb{X}_{s,s} = \mathrm{Id}$ and $\mathbb{X}_{t,u} \circ \mathbb{X}_{s,t} = \mathbb{X}_{s,u}$ a.s. $(s \leq t \leq u)$.

A stochastic flow $(\mathbb{X}_{s,t})_{s \leq t}$ is *stationary* if:

(iii). $(\mathbb{X}_{s,t})_{s \leq t}$ is equal in law to $(\mathbb{X}_{s+r,t+r})_{s \leq t}$ for all $r \in \mathbb{R}$,

and we say that $(\mathbb{X}_{s,t})_{s \leq t}$ has *independent increments* if:

(iv). $X_{t_0,t_1}, \ldots, X_{t_{n-1},t_n}$ are independent for all $t_0 \leq \cdots \leq t_n$.

Stochastic flows

If $(\mathbb{X}_{s,t})_{s\leq t}$ is a stochastic flow with independent increments, $s\in\mathbb{R}$, and X_0 is an independent $\mathcal{X}$-valued random variable, then setting

$$X_t := \mathbb{X}_{s,s+t}(X_0) \qquad (t\geq 0)$$

defines a Markov process $(X_t)_{t\geq 0}$. If $(\mathbb{X}_{s,t})_{s\leq t}$ is stationary, then $(X_t)_{t\geq 0}$ is time-homogeneous.

Many Markov processes can be constructed from stochastic flows. Examples:

- Markov processes constructed from Poisson point processes.
- Strong solutions to stochastic differential equations relative to a fixed driving Brownian motion.

Stochastic flows

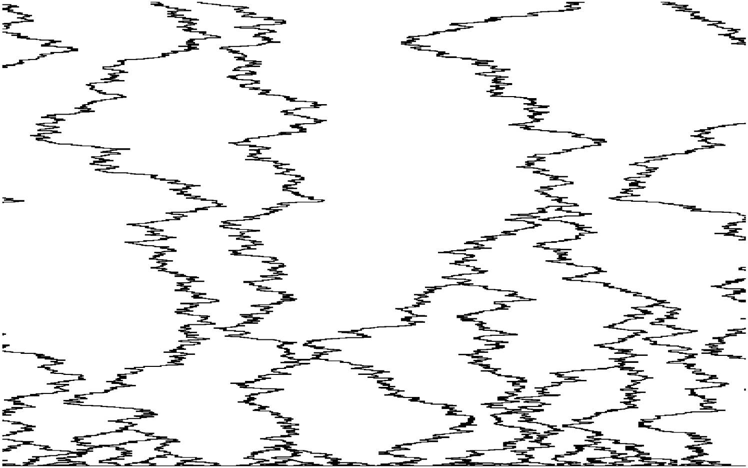
If $\vec{X}_0 = (X_0^1, \ldots, X_0^n)$ is an $\mathcal{X}^n$-valued random variable, independent of $(\mathbb{X}_{s,t})_{s\leq t}$, then setting

$$X_t^i := \mathbb{X}_{s,s+t}(X_0^i) \qquad (t \geq 0,\ 1 \leq i \leq n)$$

defines a stochastic process $(\vec{X}_t)_{t\geq 0}$. These *n-point motions* satisfy a natural consistency property.
Le Jan and Raimond (AoP 2004) have shown that each consistent family of Feller processes gives rise to a stationary stochastic flow (in the weak sense of (ii)') with independent increments.

Def a stochastic flow $(\mathbb{X}_{s,t})_{s\leq t}$ on $\mathbb{R}$ is *monotone* if for each $s \leq t$, the map $\mathbb{X}_{s,t} : \mathbb{R} \to \mathbb{R}$ is nondecreasing and right-continuous with $\lim_{x\to\pm\infty} \mathbb{X}_{s,t}(x) = \pm\infty$.

The Arratia flow



The n-point motions of the *Arratia flow* are *coalescing Brownian motions*.

Harris flows

By definition, a *correlation function* is a continuous function $\rho : \mathbb{R} \to [-1, 1]$ such that:

(i). ρ is continuous with $\rho(0) = 1$,

(ii). for each $x_1, \ldots, x_n \in \mathbb{R}$, setting $M_{ij} := \rho(x_i - x_j)$ defines a positive semidefinite matrix.

By Bochner's theorem, if μ is a symmetric probability measure on $\mathbb{R}$, then

$$\rho(x) := \int_{-\infty}^{\infty} \mu(\mathrm{d}y) e^{-2\pi ixy} \qquad (x \in \mathbb{R})$$

defines a correlation function, and each correlation function is of this form.

Harris flows

Harris (SPA 1984) proved that if ρ is a correlation function, then for each initial state in $\mathbb{R}^n$, there exists a unique solution to the martingale problem for the operator

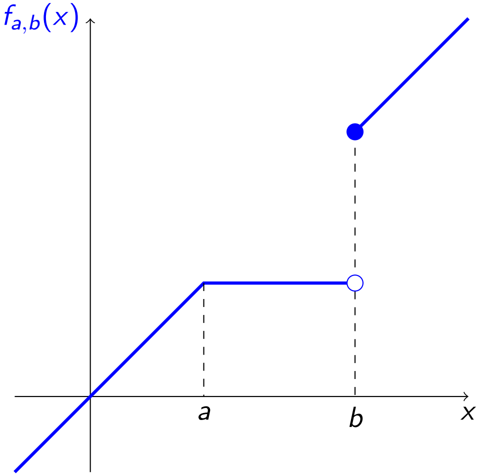
$$Gf(x_1,\ldots,x_n) := \tfrac{1}{2}\sum_{i,j=1}^{n} \rho(x_i - x_j)\tfrac{\partial^2}{\partial x_i \partial x_j} f(x),$$

with the additional condition that paths coalesce once they meet.[1] The solutions to this martingale problem are correlated Brownian motions that form the n-point motions for a *Harris flow* $(\mathbb{X}_{s,t})_{s\leq t}$.

$$\lim_{t\to 0} t^{-1}\mathbb{E}\big[(\mathbb{X}_{0,t}(x)-x)(\mathbb{X}_{0,t}(y)-y)\big] = \rho(x-y).$$

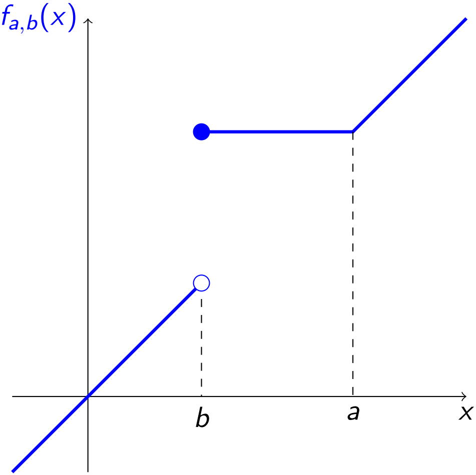
[1] Paths meet a.s. iff $\displaystyle\int_{0+}\frac{x}{1-\rho(x)}\mathrm{d}x < \infty$.

Monotone Lévy flows



$$f_{a,b}(x) := \begin{cases} a & \text{if } a < x < b \text{ or } b < x < a \\ x & \text{otherwise.} \end{cases}$$

Monotone Lévy flows



$$f_{a,b}(x) := \begin{cases} a & \text{if } a < x < b \text{ or } b < x < a \\ x & \text{otherwise.} \end{cases}$$

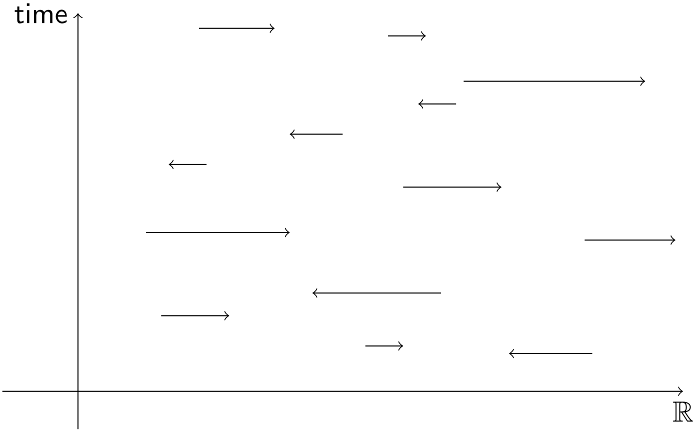
Monotone Lévy flows

Let ν be a locally finite measure on $\mathcal{A} := \mathbb{R}^2 \backslash \{(a,a) : a \in \mathbb{R}\}$ and let ℓ denote the Lebesgue measure on $\mathbb{R}$.
Let $\omega \subset \mathcal{A} \times \mathbb{R}$ be a Poisson point set with intensity $\nu \otimes \ell$.

Under suitable assumptions in ν, it should be possible to define a stochastic flow with the following informal description:

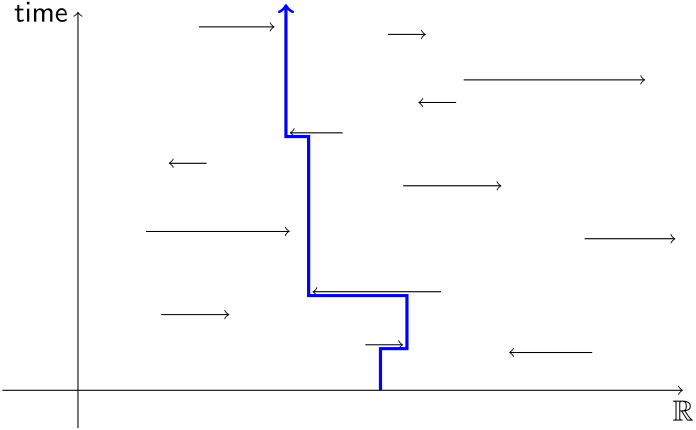
For each $(a,b,t) \in \omega$, we apply the map $f_{a,b}$ at time t.

Monotone Lévy flows



The n-point motions are then coalescing Lévy processes.

Monotone Lévy flows



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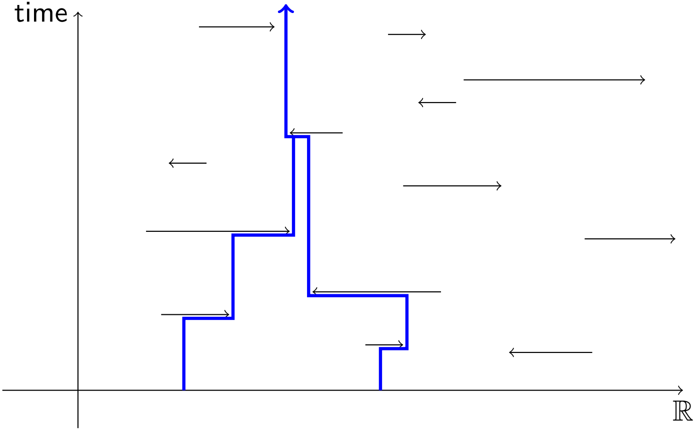
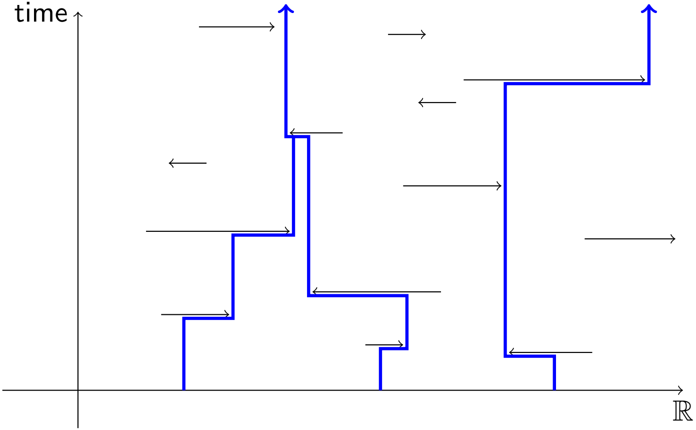
Monotone Lévy flows



The n-point motions are then coalescing Lévy processes.

Monotone Lévy flows



The n-point motions are then coalescing Lévy processes.

Dual stochastic flows

Let $\mathcal{M}$ be the space of nondecreasing and right-continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $\lim_{x\to\pm\infty} f(x) = \pm\infty$.
We define the *generalised inverse* of $f \in \mathcal{M}$ as

$$f^{-1}(y) := \inf \big\{ x \in \mathbb{R} : f(x) > y \big\}.$$

Let $(\mathbb{X}_{s,t})_{s\leq t}$ be a monotone stochastic flow on $\mathbb{R}$.

Then setting

$$\hat{\mathbb{X}}_{t,s} := \mathbb{X}^{-1}_{s,t} \qquad (s \leq t)$$

defines a *dual stochastic flow* in the sense that

$$\hat{\mathbb{X}}_{s,s} = \mathrm{Id} \quad \text{and} \quad \hat{\mathbb{X}}_{t,s} \circ \hat{\mathbb{X}}_{u,t} = \hat{\mathbb{X}}_{u,s} \qquad (u \geq t \geq s).$$

Dual stochastic flows

For the Arratia flow, the n-point motions of the dual stochastic flow are backward coalescing Brownian motions.

For Harris flows, the n-point motions of the dual stochastic flow are correlated backward coalescing Brownian motions.

For monotone Lévy flows, the n-point motions of the dual stochastic flow are coalescing Lévy processes.

The Hausdorff metric

Let $(\mathcal{X}, d)$ be a metric space. Let $d(x, A) := \inf\{d(x, y) : y \in A\}$.

Let $\mathcal{K}(\mathcal{X})$ be the set of all compact subsets of $\mathcal{X}$ and let $\mathcal{K}_+(\mathcal{X}) := \{K \in \mathcal{K}(\mathcal{X}) : K \neq \emptyset\}$.

The *Hausdorff metric* on $\mathcal{K}_+(\mathcal{X})$ is defined as

$$d_{\rm H}(K_1, K_2) := \sup_{x_1 \in K_1} d(x_1, K_2) \vee \sup_{x_2 \in K_2} d(x_2, K_1).$$

A *correspondence* between A_1, A_2 is a set $R \subset A_1 \times A_2$ such that:

$$\forall x_i \in A_i \ \exists x_j \in A_j \text{ s.t. } (x_i, x_j) \in R \qquad \big((i,j) = (1,2), (2,1)\big).$$

Let $\mathrm{Cor}(A_1, A_2)$ denote the set of all correspondences between A_1, A_2. Then

$$d_{\rm H}(K_1, K_2) = \inf_{R \in \mathrm{Cor}(K_1, K_2)} \sup_{(x_1, x_2) \in R} d(x_1, x_2).$$

The Hausdorff metric

Lemma If $(\mathcal{X}, d)$ is separable, then so is $(\mathcal{K}_+(\mathcal{X}), d_{\mathrm{H}})$.
If $(\mathcal{X}, d)$ is complete, then so is $(\mathcal{K}_+(\mathcal{X}), d_{\mathrm{H}})$.

Lemma A set $\mathcal{A} \subset \mathcal{K}_+(\mathcal{X})$ is compact iff there exists a compact $C \subset \mathcal{X}$ such that $K \subset C$ for all $K \in \mathcal{A}$.

Lemma Let $K_n \in \mathcal{K}_+(\mathcal{X})$ and let

$$\mathrm{Lim}\big((K_n)\big) := \big\{x \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } x_n \underset{n\to\infty}{\longrightarrow} x\big\},$$
$$\mathrm{Clus}\big((K_n)\big) := \big\{x \in \mathcal{X} : \exists\ _{n(k)\to\infty},\ x_{n(k)} \in K_{n(k)} \text{ s.t. } x_{n(k)} \underset{k\to\infty}{\longrightarrow} x\big\}.$$

Then $d_{\mathrm{H}}(K_n, K) \underset{n\to\infty}{\longrightarrow} 0$ iff

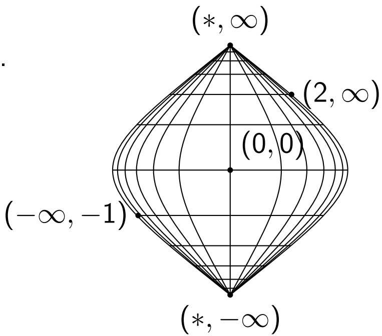
(i). $\exists C \subset \mathcal{K}_+(\mathcal{X})$ s.t. $K_n \subset C$ $\forall n$,

(ii). $\mathrm{Lim}(K) = K = \mathrm{Clus}(K)$.

The topology on $\mathcal{K}_+(\mathcal{X})$ does not depend on the choice of the metric on $\mathcal{X}$.

Squeezed space

Let $\overline{\mathbb{R}} := [-\infty, \infty]$.



It is possible to equip $\mathcal{R}(\overline{\mathbb{R}}) := \overline{\mathbb{R}} \times \mathbb{R} \cup \{(*, -\infty), (*, -\infty)\}$ with a metrisable topology such that $(x_n, t_n) \to (x, t)$ iff

(i). $t_n \to t$,

(ii). if $t \in \mathbb{R}$, then $x_n \to x$.

The space of continuous paths

Let $\Pi_{\rm c}$ denote the set of all π such that

(i). π is a compact subset of $\mathcal{R}(\overline{\mathbb{R}})$,

(ii). $(*, \pm\infty) \in \pi$,

(iii). $\big|\{x \in \overline{\mathbb{R}} : (x,t) \in \pi\}\big| \leq 1 \quad \forall t \in \mathbb{R}$.

For $\pi \in \Pi_{\rm c}$, we set

$$I_\pi := \big\{t \in \mathbb{R} : \exists x \in \overline{\mathbb{R}} \text{ s.t. } (x,t) \in \pi\big\},$$

and for $t \in I_\pi$, we let $\pi(t)$ denote the unique element of $\overline{\mathbb{R}}$ such that $\big(\pi(t), t\big) \in \pi$.

Then $I_\pi \subset \mathbb{R}$ is closed and $\pi : I_\pi \to \overline{\mathbb{R}}$ is continuous.

Conversely, for each continuous function $f : I \to \overline{\mathbb{R}}$ defined on a closed set $I \subset \mathbb{R}$, there exists a unique $\pi \in \Pi_{\rm c}$ such that $I_\pi = I$ and $p(t) = f(t)$ $(t \in I)$.

The space of continuous paths

Naturally $\Pi_{\rm c} \subset \mathcal{K}_+(\mathcal{R}(\overline{\mathbb{R}}))$.
We equip $\Pi_{\rm c}$ with the induced topology.
Informally, $\pi_n \to \pi$ iff $I_{\pi_n} \to I_\pi$ and the function $t \mapsto \pi_n(t)$ converges locally uniformly to $t \mapsto \pi(t)$.

We set

$$\Pi_{\rm c}^{|} := \big\{\pi \in \Pi_{\rm c} : I_\pi \text{ is an interval}\big\}.$$

For $\pi \in \Pi_{\rm c}^{|}$, we call $\sigma_\pi := \inf I_\pi$ the *starting time* and $\tau_\pi := \sup I_\pi$ the *final time*,[2] and we set

$$\Pi_{\rm c}^{\uparrow} := \big\{\pi \in \Pi_{\rm c}^{|} : \tau_\pi = \infty\big\},$$
$$\Pi_{\rm c}^{\downarrow} := \big\{\pi \in \Pi_{\rm c}^{|} : \sigma_\pi = -\infty\big\}.$$

Then $\Pi_{\rm c}^{|}, \Pi_{\rm c}^{\uparrow}$, and $\Pi_{\rm c}^{\downarrow}$ are closed subsets of $\Pi_{\rm c}$.

[2] By definition $\sigma_\pi := -\infty$ and $\tau_\pi := \infty$ if $I_\pi = \emptyset$.

The space of continuous paths

The *modulus of continuity* of a path $\pi \in \Pi_{\rm c}$ is defined as

$$m_{T,\delta}(\pi) := \sup\big\{|x_1 - x_2| : (x_1,t_1),(x_2,t_2) \in \pi \cap [-T,T]^2,\\ |t_1 - t_2| \leq \delta\big\}.$$

Recall that a set is called *precompact* if its closure is compact.

Compactness criterion A set $\mathcal{A} \subset \Pi_{\rm c}$ is precompact if and only if it is *equicontinuous*, i.e.,

$$\lim_{\delta\to 0} \sup_{\pi\in\mathcal{A}} m_{T,\delta}(\pi) = 0 \quad (T < \infty).$$

This generalises the classical Arzela-Ascoli theorem.

Theorem $\Pi_{\rm c}$ is a Polish space.

Continuous streams

Let $\Pi_c^{\updownarrow} := \Pi_c^{\uparrow} \cap \Pi_c^{\downarrow}$ denote the space of bi-infinite continuous paths. For $\pi_1, \pi_2 \in \Pi_c^{\updownarrow}$, define $\pi_1 \lhd \pi_2$ iff $\pi_1(t) \leq \pi_2(t)$ $(t \in \mathbb{R})$.

Def A *stream*[3] is a set $\mathcal{F} \subset \Pi_c^{\updownarrow}$ such that

- $\mathcal{F}$ is compact,
- $\mathcal{F}$ is *pervasive*, i.e., $\forall (x,t) \in \mathbb{R}^2$ $\exists \pi \in \mathcal{F}$ s.t. $(x,t) \in \pi$,
- $\mathcal{F}$ is *noncrossing*, i.e., $\forall \pi_1, \pi_2 \in \mathcal{F}$ either $\pi_1 \lhd \pi_2$ or $\pi_2 \lhd \pi_1$.

Given a random stream $\mathcal{F}$, we can define random maps $(\mathbb{X}_{s,t})_{s \leq t}$ and $(\hat{\mathbb{X}}_{t,s})_{t \geq s}$ on $\overline{\mathbb{R}}$ by

$$\left.\begin{aligned} \mathbb{X}_{s,t}(x) &:= \sup\big\{\pi(t) : \pi \in \mathcal{F},\ \pi(s) = x\big\}, \\ \hat{\mathbb{X}}_{t,s}(x) &:= \sup\big\{\pi(s) : \pi \in \mathcal{F},\ \pi(t) = x\big\}, \end{aligned}\right\} \quad (s \leq t,\ x \in \overline{\mathbb{R}}).$$

Many monotone stochastic flows and their duals can be obtained from a stream in this way.

[3] Or *flow of paths*.

Continuous streams

Random streams $\mathcal{F}$ and monotone stochastic flows $(\mathbb{X}_{s,t})_{s\leq t}$ are not in a one-to-one correspondence.

In general, the maps $(\mathbb{X}_{s,t})_{s\leq t}$ defined in terms of $\mathcal{F}$ may fail to satisfy the stochastic flow property $\mathbb{X}_{t,u}\circ\mathbb{X}_{s,t}=\mathbb{X}_{s,u}$.

Let $\mathcal{F}_{(x,s)}:=\{\pi\in\mathcal{F}:\pi(s)=x\}$.

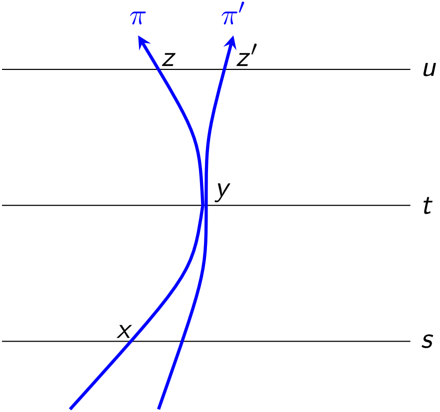
For each $x\in\overline{\mathbb{R}}$ and $s\in\mathbb{R}$, there exist $\pi^{\pm}_{(x,s)}\in\mathcal{F}_{(x,s)}$ such that

$$\pi^{-}_{(x,s)}\lhd\pi\lhd\pi^{+}_{(x,s)}\qquad\forall\pi\in\mathcal{F}_{(x,s)}.$$

One has

$$\mathbb{X}_{s,t}(x)=\pi^{+}_{(x,s)}(t).$$

Continuous streams



$\pi^+_{(x,s)} = \pi$ and $\pi(t) = y$ but $\pi^+_{(y,t)} = \pi' \neq \pi$.

As a result $\mathbb{X}_{t,u} \circ \mathbb{X}_{s,t}(x) = z' \neq z = \mathbb{X}_{s,u}(x)$.

Continuous streams

Recall that the stochastic flow property

(ii) $\mathbb{X}_{t,u} \circ \mathbb{X}_{s,t} = \mathbb{X}_{s,u}$ $(s \leq t \leq u)$ a.s.

sometimes has to be weakened to

(ii)' $\mathbb{X}_{t,u} \circ \mathbb{X}_{s,t} = \mathbb{X}_{s,u}$ a.s. $(s \leq t \leq u)$.

These difficulties stem from the need to make a choice (left- or right-continuous) at jumps of the function $x \mapsto \mathbb{X}_{s,t}(x)$.

In a stream $\mathcal{F}$, multiple paths can pass through a point (x, s).

> *In many ways, a stream $\mathcal{F}$ captures the intuitive idea of the "flow property" better than a stochastic flow $(\mathbb{X}_{s,t})_{s \leq t}$.*

The Brownian web

The Arratia flow $(\mathbb{X}_{s,t})_{s\leq t}$ and its dual $(\hat{\mathbb{X}}_{t,s})_{t\geq s}$ can be constructed in terms of a random stream $\mathcal{F}$.

This stream $\mathcal{F}$ is known as the *full Brownian web*, introduced in [Fontes & Newman '06].

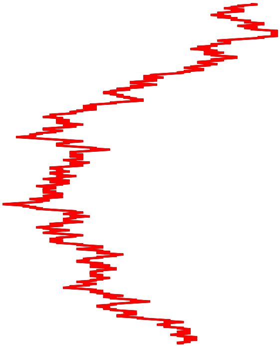
[Fontes, Isopi, Newman & Ravishankar (AoP 2004)]
There exists a random compact subset $\mathcal{W}\subset\Pi^{\uparrow}_{\rm c}$, called the *Brownian web*, such that:

(i). For deterministic $(x,s)\in\mathbb{R}^2$, there a.s. exists a unique $\pi_{(x,s)}\in\mathcal{W}$ with $\sigma_\pi=s$ and $\pi(s)=x$.

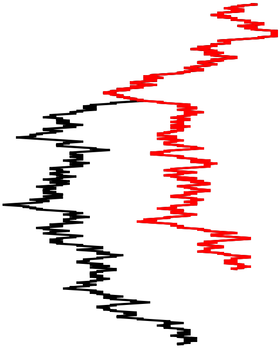
(ii). For deterministic $z_1,\ldots,z_n$, the paths $\pi_{z_1},\ldots,\pi_{z_n}$ are distributed as coalescing Brownian motions.

(iii). For deterministic dense countable $\mathcal{D}\subset\mathbb{R}^2$, one has $\mathcal{W}=\overline{\{\pi_z:z\in\mathcal{D}\}}$ a.s.

The Brownian web

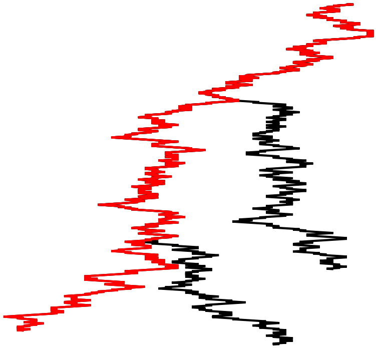


Coalescing Brownian motions.



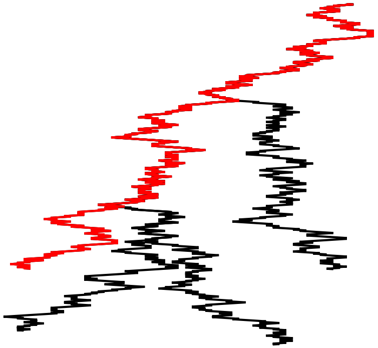
Coalescing Brownian motions.

The Brownian web



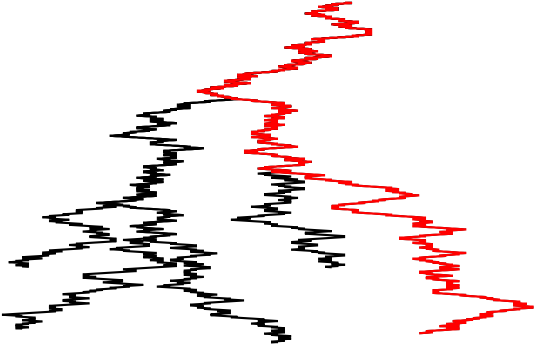
Coalescing Brownian motions.

The Brownian web



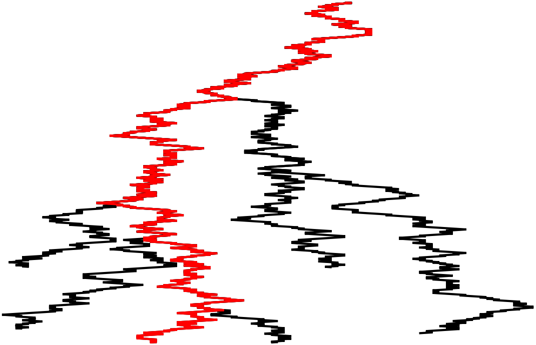
Coalescing Brownian motions.

The Brownian web



Coalescing Brownian motions.

The Brownian web



Coalescing Brownian motions.

The dual Brownian web

For a set $A \subset \mathbb{R}^2$, write $-A := \{-z : z \in A\}$.
In particular, $-\pi$ is the path π rotated over 180°.
Write $-\mathcal{W} := \{-\pi : \pi \in \mathcal{W}\}$.

Let $\mathcal{W}$ be a Brownian web. Then there exists an a.s. unique random set of paths $\hat{\mathcal{W}} \subset \Pi^{\downarrow}_{\rm c}$ such that

(i). $\hat{\mathcal{W}}$ is equally distributed with $-\mathcal{W}$,

(ii). paths in $\hat{\mathcal{W}}$ do not cross paths in $\mathcal{W}$.

We call $\hat{\mathcal{W}}$ the *dual Brownian web* associated with $\mathcal{W}$.

The full Brownian web

For deterministic $(x, s) \in \mathbb{R}^2$, define

$$\overline{\pi}_{(x,s)}(t) := \begin{cases} \pi_{(x,s)}(t) & \text{if } s \leq t, \\ \hat{\pi}_{(x,s)}(t) & \text{if } t \leq s, \end{cases}$$

where $\hat{\pi}_{(x,s)}$ is the a.s. unique path in $\hat{\mathcal{W}}$ starting in (x, s).

The *full Brownian web* is the random stream $\mathcal{F}$ defined as

$$\mathcal{F} := \overline{\{\overline{\pi}_{(x,s)} : (x, s) \in \mathcal{D}\}},$$

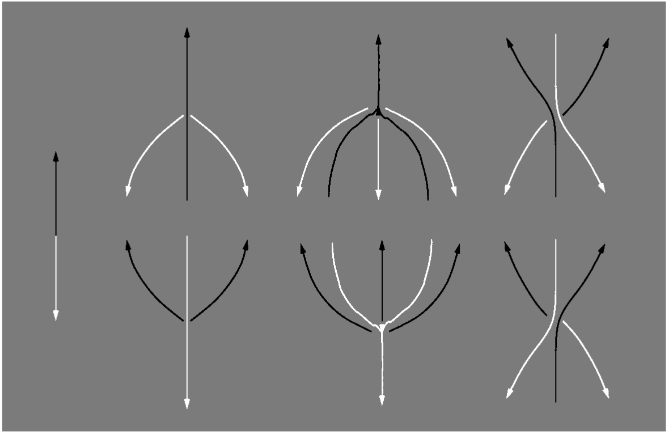
where $\mathcal{D}$ is any deterministic countable dense subset of $\mathbb{R}^2$.

The full Brownian web

The web / stream point of view has lead to a better understanding of the Arratia flow.

A natural concept of convergence. Webs and streams are random variables taking values in the spaces $\mathcal{K}_+(\Pi_{\mathrm{c}}^{\uparrow})$ and $\mathcal{K}_+(\Pi_{\mathrm{c}}^{\updownarrow})$, which are naturally equipped with the Hausdorff topology.

A better understanding of discontinuities. Points (x,t) where $\mathcal{F}_{(x,t)}$ contains more than one path correspond to discontinuities of the maps $x \mapsto \mathbb{X}_{t,u}(x)$ and $x \mapsto \hat{\mathbb{X}}_{t,s}(x)$ $(s < t < u)$.



Special points of the Brownian web.

Aim Develop a general theory of streams $\mathcal{F}$ and webs $\mathcal{W}$, which allows for monotone stochastic flows with discontinuous n-point motions.

On the following slide, we use the notation $[a, b] := [a \wedge b, a \vee b]$.

The space of cadlag paths

Let Π denote the set of all pairs $(\pi, \preceq)$ such that

(i). π is a compact subset of $\mathcal{R}(\overline{\mathbb{R}})$,

(ii). $(*, \pm\infty) \in \pi$,

(iii). $\{x \in \overline{\mathbb{R}} : (x,t) \in \pi\}$ is an interval $\forall t \in \mathbb{R}$,

(iv). $\preceq$ is a total order on π,

(v). $\pi^{\langle 2 \rangle} := \{(z, z') \in \pi^2 : z \preceq z'\}$ is a closed subset of $\mathcal{R}(\overline{\mathbb{R}})^2$,

(vi). $(x,s) \preceq (y,t)$ for all $(x,s),(y,t) \in \pi$ with $s < t$.

For $\pi \in \Pi$, we set

$$I_\pi := \big\{t \in \mathbb{R} : \exists x \in \overline{\mathbb{R}} \text{ s.t. } (x,t) \in \pi\big\},$$

and for $t \in I_\pi$, we define $\pi(t\pm)$ by

$$\{x \in \overline{\mathbb{R}} : (x,t) \in \pi\} =: [\pi(t-), \pi(t+)]$$
$$\text{with} \quad \big(\pi(t-), t\big) \preceq \big(\pi(t-), t\big).$$

The space of cadlag paths

For all $(\pi, \preceq) \in \Pi$,

- $I := I_\pi \subset \mathbb{R}$ is closed,
- $I \ni t \mapsto \pi(t-)$ is left-continuous,
- $I \ni t \mapsto \pi(t+)$ is right-continuous,
- if $t \in I$ can be approximated from the left, then $\pi(t-) = \lim_{I \ni s \uparrow t} \pi(s+)$,
- if $t \in I$ can be approximated from the right, then $\pi(t+) = \lim_{I \ni u \downarrow t} \pi(u-)$.

Conversely, each pair of functions $I \ni t \mapsto \pi(t-)$ and $I \ni t \mapsto \pi(t+)$ with these properties, defined on a closed subset $I \subset \mathbb{R}$, corrresponds to a path $(\pi, \preceq) \in \Pi$.

We can think of $t \mapsto \pi(t+)$ as a cadlag function whose left-continuous modification is $t \mapsto \pi(t-)$.

The space of cadlag paths

However, the functions $t \mapsto \pi(t+)$ and $t \mapsto \pi(t-)$ do not determine each other uniquely, since
$\pi(t-) = \lim_{I \ni s \uparrow t} \pi(s+)$ only if $t \in I$ can be approximated from the left, and
$\pi(t+) = \lim_{I \ni u \downarrow t} \pi(u-)$. only if $t \in I$ can be approximated from the right.

We allow for the case that $\pi(t-) \neq \pi(t+)$ at such points.

In particular, if $I_\pi = [s, \infty)$, then we allow for the case that $\pi(s-) \neq \pi(s+)$.

In this case $t \mapsto \pi(t+)$ is uniquely determined by $t \mapsto \pi(t-)$ but not vice versa.

The space of cadlag paths

Recall $\pi^{\langle 2\rangle} := \{(z,z')\in\pi^2 : z \preceq z'\} \subset \mathcal{R}(\overline{\mathbb{R}})^2$ is compact.
Let d be any metric generating the topology on $\mathcal{R}(\overline{\mathbb{R}})$.
Let d^2 be the metric on $\mathcal{R}(\overline{\mathbb{R}})^2$ defined as

$$d^2\big((z_1,z_1'),(z_2,z_2')\big) := d(z_1,z_2) \vee d(z_1',z_2'),$$

let d^2_{H} denote the associated Hausdorff metric on $\mathcal{K}_+\big(\mathcal{R}(\overline{\mathbb{R}})^2\big)$, and set

$$d_{\mathrm{part}}(\pi_1,\pi_2) := d^2_{\mathrm{H}}(\pi_1^{\langle 2\rangle},\pi_2^{\langle 2\rangle}) \qquad \big(\pi_1,\pi_2\in\Pi\big).$$

Let $\mathrm{Corr}_+(\pi_1,\pi_2)$ denote the set of all correspondences R between π_1 and π_2 that are *monotone* in the sense that

there are no $(z_1,z_2),(z_1',z_2')\in R$ such that $z_1 \prec_1 z_1'$ and $z_2' \prec_2 z_2$,

and set

$$d_{\mathrm{tot}}(\pi_1,\pi_2) := \inf_{R\in\mathrm{Cor}_+(\pi_1,\pi_2)} \sup_{(z_1,z_2)\in R} d(z_1,z_2) \qquad \big(\pi_1,\pi_2\in\Pi\big).$$

Theorem One has

$$d_{\mathrm{H}}(\pi_1,\pi_2) \le d_{\mathrm{part}}(\pi_1,\pi_2) \le d_{\mathrm{tot}}(\pi_1,\pi_2) \qquad (\pi_1,\pi_2 \in \Pi),$$

and d_{part} and d_{tot} generate the same topology on Π.

We can naturally view Π_{c} as a subset of Π. Then the topology on Π_{c} is the induced topology from Π.

Informally, $\pi_n \to \pi$ iff $I_{\pi_n} \to I_\pi$ and the function $t \mapsto \pi_n(t)$ converges in *Skorohod's M1-topology* to $t \mapsto \pi(t)$.

If we replace

3. $\{x \in \overline{\mathbb{R}} : (x,t) \in \pi\}$ is an interval $\forall t \in \mathbb{R}$,

by

3'. $\big|\{x \in \overline{\mathbb{R}} : (x,t) \in \pi\}\big| \le 2\ \forall t \in \mathbb{R}$,

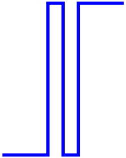
then we obtain *Skorohod's J1-topology*.

M2 and J2 topologies

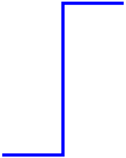
If we ignore the total order of paths and just measure their Hausdorff distance as sets, then we obtain *Skorohod's M2-topology* and *Skorohod's J2-topology* instead.

Problem of these topologies:

We do not want



to be close to



The space of cadlag paths

We define $\Pi^{|}, \Pi^{\uparrow}, \Pi^{\downarrow}$, and $\Pi^{\updownarrow}$ as before.
Paths in $\Pi^{\uparrow}$ can jump at their starting time.

We define the *M1-modulus of continuity* $m^{\mathrm{M1}}_{T,\delta}(\pi)$ of a path $\pi \in \Pi$ as

$$\begin{aligned}\sup\Big\{ d\big(x_2,[x_1,x_3]\big) :\ & (x_1,t_1),(x_2,t_2),(x_3,t_3) \in \pi \cap [-T,T]^2, \\ & (x_1,t_1) \preceq (x_2,t_2) \preceq (x_3,t_3),\ \ t_3 - t_1 \leq \delta \Big\}.\end{aligned}$$

Compactness criterion A set $\mathcal{A} \subset \Pi$ is precompact if and only if it is *M1-equicontinuous*, i.e.,

$$\lim_{\delta \to 0} \sup_{\pi \in \mathcal{A}} m^{\mathrm{M1}}_{T,\delta}(\pi) = 0 \quad (T < \infty).$$

Theorem Π is a Polish space and $\Pi^{|}, \Pi^{\uparrow}, \Pi^{\downarrow}$, and $\Pi^{\updownarrow}$ are closed subsets of Π.

Weaves

Recall that $\pi_1 \lhd \pi_2$ for $\pi_1, \pi_2 \in \Pi^{\updownarrow}$ is defined as $\pi_1(t\pm) \leq \pi_2(t\pm)$ $(t \in \mathbb{R})$.

For $\pi_1, \pi_2 \in \Pi^{\uparrow}$, we define $\pi_1 \lhd \pi_2$ iff

> there exist $\pi'_1, \pi'_2 \in \Pi^{\updownarrow}$ s.t. $\pi_i \subset \pi'_i$ $(i = 1, 2)$ and $\pi'_1 \lhd \pi'_2$.

Def A *weave* is a set $\mathcal{A} \subset \Pi^{\uparrow}$ such that

- $\mathcal{A}$ is compact,
- $\mathcal{A}$ is *pervasive*, i.e., $\forall (x, t) \in \mathbb{R}^2$ $\exists \pi \in \mathcal{A}$ s.t. $(x, t) \in \pi$,
- $\mathcal{A}$ is *noncrossing*, i.e., $\forall \pi_1, \pi_2 \in \mathcal{A}$ either $\pi_1 \lhd \pi_2$ or $\pi_2 \lhd \pi_1$.

A *stream* is a weave $\mathcal{F}$ such that $\mathcal{F} \subset \Pi^{\updownarrow}$.

Webs and streams

For any $\mathcal{A} \subset \Pi^{\uparrow}$, we define

$$\mathcal{A}_{\rm in} := \big\{\pi \in \Pi^{\uparrow} : \exists \pi' \in \mathcal{A} \text{ s.t. } \pi \subset \pi'\big\}.$$

We say that $\mathcal{A}$ is *inclusion-closed* if $\mathcal{A}_{\rm in} = \mathcal{A}$.

A *web* is a minimal inclusion-closed weave, i.e., a weave $\mathcal{W}$ such that

(i). $\mathcal{W}_{\rm in} = \mathcal{W}$,

(ii). if $\mathcal{A}$ is a weave such that $\mathcal{A}_{\rm in} = \mathcal{A}$ and $\mathcal{A} \subset \mathcal{W}$, then $\mathcal{A} = \mathcal{W}$.

Theorem For each weave $\mathcal{A}$, there exist a unique web $\mathcal{W} =: \mathrm{web}(\mathcal{A})$ and stream $\mathcal{F} =: \mathrm{stream}(\mathcal{A})$ such that $\mathcal{W} \subset \mathcal{A}_{\rm in}$ and $\mathcal{A} \subset \mathcal{F}_{\rm in}$.

Note that $\mathcal{A} \subset \mathcal{F}_{\rm in}$ implies that the paths of a weave can be extended to bi-infinite paths that do not cross!

Efficient covering of space

Let $\mathscr{W}$ be the space of weaves. Setting

$$\mathcal{A} \leq \mathcal{B} \quad \textit{iff} \quad \mathcal{A}_{\rm in} \cap \mathcal{B} \subset \mathcal{A} \subset \mathcal{B}$$

defines a partial order on $\mathscr{W}$ such that

- $\mathcal{A}$ is a web $\Leftrightarrow$ $\mathcal{A} = \mathcal{A}'$ for all $\mathcal{A}' \leq \mathcal{A}$,
- $\mathcal{A}$ is a stream $\Leftrightarrow$ $\mathcal{A} = \mathcal{A}'$ for all $\mathcal{A} \leq \mathcal{A}'$.

Edges

For $\pi \in \Pi^{\updownarrow}$ and $z \in \pi$, let

$$\pi^{\uparrow}(z) := \{z' \in \pi : z \prec z'\} \quad \text{and} \quad \pi^{\downarrow}(z) := \{z' \in \pi : z' \prec z\}.$$

A *separation point* of paths $\pi_1, \pi_2 \in \Pi^{\updownarrow}$ is a point $z \in \pi_1 \cap \pi_2$ such that

$$\pi_1^{\downarrow}(z) = \pi_2^{\downarrow}(z),$$

and no point $z' \in \pi_1 \cap \pi_2$ with $z \prec z'$ has this property.

For $\pi_1, \pi_2 \in \Pi^{\updownarrow}$ with $\pi_1 \lhd \pi_2$, we define $B = B(\pi_1, \pi_2)$ by

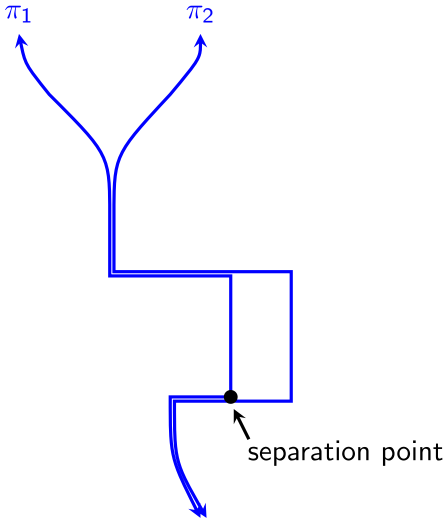
$$B := \big\{(x,t) \in \mathbb{R}^2 : \pi_1(t-) \wedge \pi_1(t+) \leq x \leq \pi_2(t-) \vee \pi_2(t+)\big\}.$$

If π_1, π_2 have a separation point z, then we define the *edge* created by π_1 and π_2 as

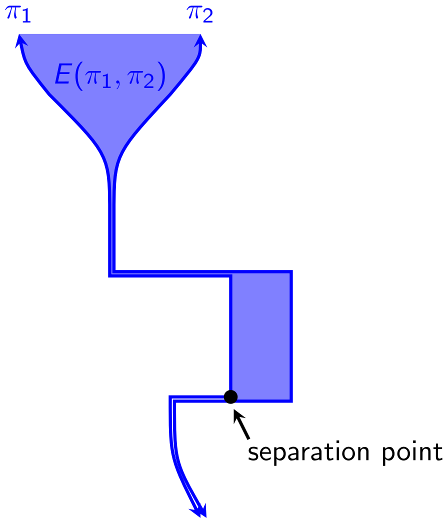
$$E(\pi_1, \pi_2) := B(\pi_1, \pi_2) \backslash \pi_1^{\downarrow}(z),$$

and we set $E(\pi_1, \pi_2) := B(\pi_1, \pi_2)$ otherwise.

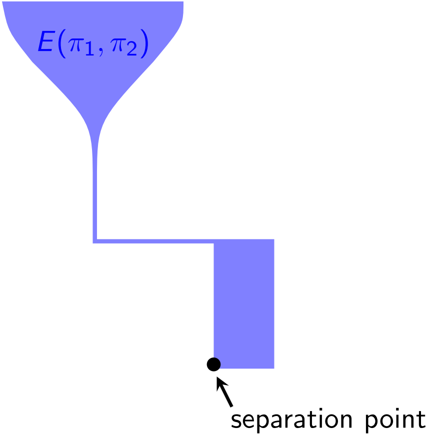
Edges



Edges



Edges



Webs and streams

We say that π_1 *crosses* π_2 if neither $\pi_1 \lhd \pi_2$ nor $\pi_2 \lhd \pi_1$.

A path π *enters* an edge E if $\exists\ z, z' \in \pi$
with $z \preceq z'$, $z \notin E$, $z' \in \mathring{E}$.

For a weave $\mathcal{A}$ and stream $\mathcal{F}$, one has

$$\begin{aligned}
\mathrm{stream}(\mathcal{A}) &= \big\{\pi \in \Pi^{\updownarrow} : \pi \text{ does not cross paths } \pi' \in \mathcal{A}\big\},\\
\mathrm{web}(\mathcal{F}) &= \big\{\pi \in \Pi^{\uparrow} : \pi \text{ does not enter } E(\pi_1, \pi_2)\\
&\qquad\qquad \text{for } \pi_1, \pi_2 \in \mathcal{F},\ \ \pi_1 \lhd \pi_2\big\}.
\end{aligned}$$

For the Brownian web $\mathcal{W}$ and full Browian web $\mathcal{F}$
one has $\mathcal{F} = \mathrm{stream}(\mathcal{W})$ and $\mathcal{W} = \mathrm{web}(\mathcal{F})$.

The space of weaves

We equip the space of weaves $\mathscr{W} \subset \mathcal{K}_+(\Pi)$ with the Hausdorff topology.

Theorem $\mathscr{W}$ is a Polish space.

We define the *antagonism modulus* $\mathfrak{a}_{T,\delta}(\mathcal{A})$ of a weave $\mathcal{A}$ as

$$\sup\big\{\,|z_2 - z_1| \vee |z_2' - z_1'| : z_i, z_i' \in [-T, T]^2,\ |z_i - z_i'| \leq \delta\ (i = 1, 2)$$
$$\exists \pi, \pi' \in \mathcal{A} \text{ s.t. } z_1, z_2 \in \pi,\ z_1 \preceq z_2,\ z_1', z_2' \in \pi',\ z_2' \preceq z_1'\big\}$$

Compactness criterion A set $\mathscr{A} \subset \mathscr{W}$ is precompact if and only if

$$\lim_{\delta \to 0} \sup_{\mathcal{A} \in \mathscr{A}} \mathfrak{a}_{T,\delta}(\mathcal{A}) = 0 \quad (T < \infty).$$

Ramification points

A *ramification point* of a stream $\mathcal{F}$ is a point $z \in \mathbb{R}^2$ such that $|\mathcal{F}_z| > 1$.

Theorem The set of ramification points of a stream has Lebesgue measure zero.

For a path π and $z' \in \mathbb{R}^2$ set $\pi + z' := \{z + z' : z \in \pi\}$.
For a set of paths $\mathcal{A}$ set $\mathcal{A} + z' := \{\pi + z' : \pi \in \mathcal{A}\}$.

A random stream $\mathcal{F}$ is *homogeneous* if $\mathcal{F} \stackrel{\mathrm{d}}{=} \mathcal{F} + z'$ $(z' \in \mathbb{R}^2)$.

Corollary If $\mathcal{F}$ is a homogeneous random stream, then for each deterministic $(x, s) \in \mathbb{R}^2$, there a.s. exists a unique $\pi_{(x,s)} := \pi \in \mathcal{F}$ such that $\pi(s) = x$.

Convergence of streams

Theorem Let $\mathcal{F}^n, \mathcal{F}$ be homogeneous streams. Then

$$\mathbb{P}\big[\mathcal{F}^n \in \cdot\,\big] \underset{n\to\infty}{\Longrightarrow} \mathbb{P}\big[\mathcal{F} \in \cdot\,\big]$$

in the sense of weak convergence of probability laws on $\mathscr{W}$ if and only if

$$\mathbb{P}\big[(\pi^n_{z_1}, \ldots, \pi^n_{z_m}) \in \cdot\,\big] \underset{n\to\infty}{\Longrightarrow} \mathbb{P}\big[(\pi_{z_1}, \ldots, \pi_{z_m}) \in \cdot\,\big]$$

in the sense of weak convergence of probability laws on $(\Pi^{\updownarrow})^n$ for all $z_1, \ldots, z_m \in \mathbb{R}^2$.

The map $\mathrm{stream}(\cdot) : \mathscr{W} \to \mathscr{W}$ is continuous but $\mathrm{web}(\cdot)$ is not.

Weak convergence in law of webs implies the same for streams, but the converse implication does not hold.

Stochastic flows

For a path π set $\pi \cap [s,u] := \big\{(x,t) \in \pi : t \in [s,u]\big\}$.
For a set of paths $\mathcal{A}$ set $\mathcal{A} \cap [s,u] := \big\{\pi \cap [s,u] : \pi \in \mathcal{A}\big\}$.

A random stream $\mathcal{F}$ has *independent increments* if

$$\mathcal{F} \cap [t_0, t_1], \ldots \mathcal{F} \cap [t_{n-1}, t_n] \quad \text{are independent } \forall t_0 < \cdots < t_n.$$

Theorem Let $\mathcal{F}$ be a homogeneous stream with independent increments. Then

$$\left.\begin{array}{l} \mathbb{X}_{s,t}(x) := \sup\big\{\pi(t) : \pi \in \mathcal{F},\ \pi(s) = x\big\}, \\ \hat{\mathbb{X}}_{t,s}(x) := \sup\big\{\pi(s) : \pi \in \mathcal{F},\ \pi(t) = x\big\}, \end{array}\right\} \quad (s \leq t,\ x \in \overline{\mathbb{R}}).$$

define a monotone stochastic flow (in the weak sense of (ii)') and its associated dual flow.