Sharpness of the phase transition for the contact process Guided Tour: Random Media, Eindhoven

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A Lattice e.g. $\Lambda = \mathbb{Z}^d$, more generally any infinite graph. Usually, with a translation-invariant structure, e.g. Cayley graph.

Definition The contact process $(\eta_t)_{t\geq 0}$ with infection rate λ is a Markov process taking values in the subets of Λ . Sites $i \in \eta_t$ are called *infected*.

- An infected site at *i* infects each neighboring healthy site *j* with rate λ.
- Infected sites recover with rate one.

Graphical representation

Draw recovery symbols – with Poisson rate 1. Draw an arrow from *i* to neighbor *j* with rate λ .



Open paths may follow arrows but must avoid recovery symbols.

Duality

For the dual process $\eta_t^{\dagger B}$ time runs backwards and all arrows are reversed.



 $\{\eta_t^A \cap B \neq \emptyset\} = \{\exists \text{ open path from } A \text{ to } B\} = \{A \cap \eta_t^{\dagger B} \neq \emptyset\}.$

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The percolation probability



$$heta(\lambda):=\mathbb{P}[(0,0)\leadsto\infty]=\mathbb{P}ig[\eta_t^{\{0\}}
eq\emptyset\;\forall t\geq 0ig]=\lim_{t
ightarrow\infty}\mathbb{P}ig[\eta_t^{\dagger\,\wedge}
eq 0ig].$$

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A simple subadditivity argument proves the existence of the limit

$$r(\lambda) = r := \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left[|\eta_t^{\{0\}}| \right].$$

For processes on $\Lambda = \mathbb{Z}^d$ and more generally on *Cayley graphs of subexponential growth*, one has $r \leq 0$.

On the other hand, on *nonamenable graphs*, it is known that $\theta(\lambda) > 0$ implies $r(\lambda) > 0$ [Swa09].

Sharpness of the phase transition



On general graphs, it is known that $r(\lambda) < 0$ iff $\lambda < \lambda_c$.

Sharpness of the phase transition.

Proof strategies:

- I Assume $\theta(\lambda_*) = 0$, conclude $r(\lambda) < 0$ for $\lambda < \lambda_*$.
- If Assume $r(\lambda_*) = 0$, conclude $\theta(\lambda) > 0$ for $\lambda > \lambda_*$.

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For unoriented percolation:

- Menshikov (1986) \approx Strategy I.
- Aizenman & Barsky (1987) Strategy II.
- Duminil-Copin & Tassion (2016) Strategy II.

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For oriented percolation & the contact process

- Bezuidenhout & Grimmett (1991) adapted the method of Aizenman & Barsky (1987).
- Method of Duminil-Copin & Tassion (2016) carries over without a change to oriented percolation; with some work also to the contact process.
- S. (2016) method based on harmonic functions & eigenmeasures.



The problem with Strategy I seems to be that it is hard to get universal upper bounds on $r(\lambda)$...



... whereas there seems to be hope to prove universal lower bounds on $\theta(\lambda)$. Indeed, all known proofs yield as a side result $\beta \leq 1$.

The method of Aizenman & Barsky (1987) requires the introduction of an *external field / spontaneous disease* and depends on *differential inequalities* involving the two parameters (infection and spontaneous disease) of the process.

The method of Duminil-Copin & Tassion (2016) does away with the external field and depends on a single differential inequality involving only the infection rate.

Define an equivalence relation on the set of all finite subsets of Λ by $A \sim B \iff A$ is a translation of B.

Let \tilde{A} denote the equivalence class containing A. Let $\tilde{\mathcal{P}}_{\mathrm{fin},+}$ denote the space of finite, nonzero subsets of Λ modulo translation.

[Sturm & S. '14] If r < 0, then the *contact process modulo* translations $(\tilde{\eta}_t)_{t\geq 0}$ has a unique quasi-invariant law μ . Moreover,

$$e^{-rt}\mathbb{P}\big[ilde{\eta}_t^{\{0\}} \in \cdot\,\big]\Big|_{ ilde{\mathcal{P}}_{\mathrm{fin},+}} \underset{t \to \infty}{\Longrightarrow} \mu.$$

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Eigenmeasures

A different way to view the previous result is as follows. Let \mathcal{P}_+ denote the space of all nonempty subsets of the lattice. Then

$$e^{-rt}\sum_{i\in\Lambda}\mathbb{P}[\eta_t^{\{i\}}\in\cdot]\Big|_{\mathcal{P}_+}\underset{t\to\infty}{\Longrightarrow}\nu,$$

where \Rightarrow denotes vague convergence and ν is a locally finite measure on \mathcal{P}_+ that evolves under the semigroup $(P_t)_{t\geq 0}$ of the contact process as

$$\nu P_t = e^{-rt}\nu \quad (t \ge 0),$$

i.e., ν is an eigenmeasure with eigenvalue r.

This different point of view is valid even if $r \ge 0$:

[S. '09] Each translation-invariant contact process defined on a countable group Λ has a translation-invariant eigenmeasure with eigenvalue *r*.

An eigenfunction

If two Markov processes X and Y are dual, then invariant laws of X give rise to harmonic functions of Y.

Similarly, an eigenmeasure ν^{\dagger} for the dual contact process η^{\dagger} gives rise to an eigenfunction for the generator G of the contact process η through the formula

$$h(A) := \int \nu^{\dagger}(\mathrm{d}B) \mathbb{1}_{\{A \cap B \neq \emptyset\}} \qquad (A \in \mathcal{P}_{\mathrm{fin}}).$$

This satisfies Gh = rh and moreover:

$$h(\emptyset) = 0$$

$$h(A) \le h(A) \ \forall A \subset B$$

$$h(A \cup B) \le h(A) + h(B)$$

$$h(\{0\}) = 1$$

$$h(i + A) = h(A)$$

monotone subadditive normalization translation invariance.

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Harmonic function

In particular, if r = 0, then Gh = 0, i.e., h is a harmonic function. This is good news for:

Strategy II Assume $r(\lambda_*) = 0$, conclude $\theta(\lambda) > 0$ for $\lambda > \lambda_*$. The harmonic function h for λ_* turns into a subharmonic function for $\lambda > \lambda_*$. Let G_{λ} denote the generator of the contact process with infection rate λ_* .

Lemma For each $\varepsilon > 0$, there exists a $\delta > 0$ such that $G_{\lambda_*}h = 0$ implies $G_{\lambda_*+\varepsilon}f_{\delta} \ge 0$, where

$$\boldsymbol{f}_{\delta} := \delta^{-1} (1 - e^{-\delta \boldsymbol{h}}).$$

Consequence:

$$\mathbb{P}[\eta_t^A \neq \emptyset \ \forall t \ge 0] \ge \delta f_{\delta}(A).$$

[Sturm & S. in progress] Both the method of Duminil-Copin & Tassion and the method with harmonic functions work more generally.

Additive particle systems can be constructed with a graphical representation involving infection arrows and recovery symbols. One can expect sharpness of the phase transition if, fixing all other parameters, the system goes through a phase transition at some critical recovery rate $\delta_c > 0$.

The method of Duminil-Copin & Tassion confirms this if connection probabilities inside and outside space-time boxes are positively correlated.

The method with harmonic functions confirms this provided there is only a single parameter describing the proportion of infection/recovery.

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Example of a result that can be proved using Duminil-Copin & Tassion but not using harmonic functions:

For a range-two contact process, sharpness as we increase the nearest-neighbor infection rate while keeping the infection rate at distance two constant.

Example of a result that can be proved using harmonic functions but not using Duminil-Copin & Tassion: Sharpness for a contact process where two neighboring sites always recover together.

The method using harmonic functions is technically easier in a continuous-time setting.

Open problem Monotone systems that are not additive.

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