Pathwise duality for monotone Markov processes.

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joint with A. Sturm and N. Latz (in progress)

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Outline

- Monotone probability kernels
- Poisson construction of Markov processes
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- Monotone systems duality

Probability kernels

For general sets S, T, let $\mathcal{F}(S, T)$ denote the set of all functions $f : S \to T$.

A *random mapping representation* of a probability kernel K from S to T is an $\mathcal{F}(S, T)$-valued random variable M such that

$$K(x, y) = \mathbb{P}[M(x) = y] \qquad (x \in S,\ y \in T).$$

We say that K is *representable* in $\mathcal{G} \subset \mathcal{F}(S, T)$ if M can be chosen so that it takes values in $\mathcal{G}$. We set

$$Kf(x) := \sum_{y \in T} K(x, y) f(y) = \mathbb{E}\big[f\big(M(x)\big)\big] \qquad \big(x \in S,\ f \in \mathcal{F}(T, \mathbb{R})\big).$$

Monotone probability kernels

For partially ordered sets S, T, let $\mathcal{F}_{\rm mon}(S,T)$ be the set of all monotone maps $m : S \to T$, i.e., those for which $x \leq x'$ implies $m(x) \leq m(x')$.

A probability kernel K is called *monotone* if

$$Kf \in \mathcal{F}_{\rm mon}(S,\mathbb{R}) \quad \forall f \in \mathcal{F}_{\rm mon}(T,\mathbb{R}),$$

and *monotonically representable* if K is representable in $\mathcal{F}_{\rm mon}(S,T)$.

Monotonical representability implies monotonicity:

$$\begin{aligned} & f \in \mathcal{F}_{\rm mon}(T,\mathbb{R}) \quad \text{and} \quad x \leq x' \quad \Rightarrow \\ & \quad Kf(x) = \mathbb{E}\big[f\big(M(x)\big)\big] \leq \mathbb{E}\big[f\big(M(x')\big)\big] = Kf(x'). \end{aligned}$$

Monotone probability kernels

J.A. Fill & M. Machida (AOP 2001) (and also D.A. Ross (unpublished)) discovered that the converse does not hold. There are counterexamples with $S = T = \{0,1\}^2$.

On the positive side, Kamae, Krengel & O'Brien (1977) and Fill & Machida (2001) have shown that:

(Sufficient conditions for monotone representability)
Let S, T be finite partially ordered sets and assume that at least one of the following conditions is satisfied:

(i) *S is totally ordered.*

(ii) *T is totally ordered.*

Then any monotone probability kernel from S to T is monotonically representable.

Stochastic order

In particular, setting $S = \{1, 2\}$, this proves that if μ_1, μ_2 are probability laws on T such that

$$\mu_1 f \leq \mu_2 f \quad \forall f \in \mathcal{F}_{\mathrm{mon}}(T, \mathbb{R}),$$

then it is possible to couple random variables M_1, M_2 with laws μ_1, μ_2 such that $M_1 \leq M_2$.

Markov semigroups

Let S be finite. By definition, a *Markov semigroup* is a collection of probability kernels $(P_t)_{t\geq 0}$ on S such that

$$P_0 = \lim_{t\downarrow 0} P_t = 1 \quad \text{and} \quad P_s P_t = P_{s+t}.$$

Each Markov semigroup is of the form

$$P_t := e^{tG} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n G^n \qquad (t \geq 0),$$

where the *generator* G satisfies

$$G(x,y) \geq 0 \quad (x \neq y) \quad \text{and} \quad \sum_{y\in S} G(x,y) = 0 \quad (x \in S).$$

We write

$$Gf(x) := \sum_{y\in T} G(x,y) f(y) \qquad \big(x \in S,\ f \in \mathcal{F}(T,\mathbb{R})\big).$$

Representability of semigroups

By definition, G is *representable in* $\mathcal{G} \subset \mathcal{F}(S, S)$ if G can be written as

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x) \big),$$

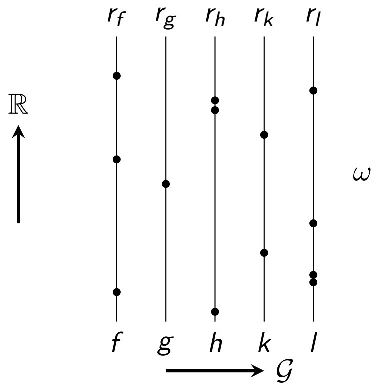
where $(r_m)_{m \in \mathcal{G}}$ are nonnegative constants (rates).

(Representability of semigroups)
Assume that $\mathcal{G}$ is closed under composition and contains the identity map. Then the following statements are equivalent:

(i) *G can be represented in $\mathcal{G}$.*

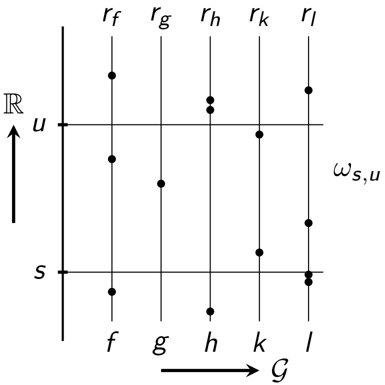
(ii) *P_t can be represented in $\mathcal{G}$ for all $t \geq 0$.*

Proof of (i)$\Rightarrow$(ii)



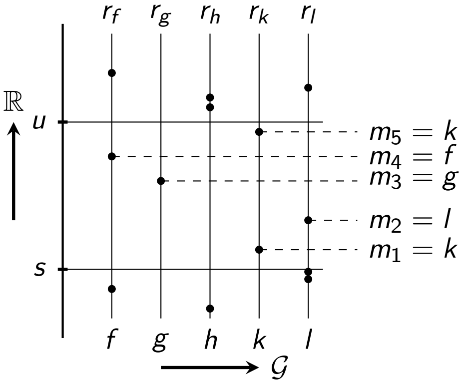
Let ρ be the measure on $\mathcal{G}$ defined by $\rho(\{m\}) := r_m$.
Let ℓ denote the Lebesgue measure on $\mathbb{R}$.
Let ω be a Poisson subset of $\mathcal{G} \times \mathbb{R}$ with intensity measure $\rho \otimes \ell$.

Proof of (i)⇒(ii)



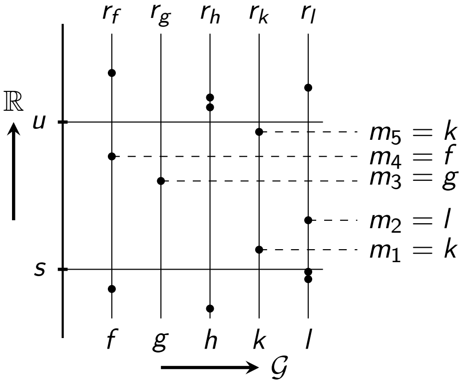
Let $\omega_{s,u} := \{(m,t) \in \omega : s < t \leq u\}$.

Proof of (i)⇒(ii)



Order the elements of $\omega_{s,u} := \{(m,t) \in \omega : s < t \leq u\}$ as $\omega_{s,u} = \big\{(m_1,t_1),\ldots,(m_n,t_n)\big\}$ with $t_1 < \cdots < t_n$.

Proof of (i)⇒(ii)



Define $\mathbb{X}_{s,u} := m_n \circ \cdots \circ m_1$.

Stochastic flows

The random maps $(\mathbb{X}_{s,u})_{s\leq u}$ form a *stochastic flow:*

$$\mathbb{X}_{s,s} = 1 \quad \text{and} \quad \mathbb{X}_{t,u} \circ \mathbb{X}_{s,t} = \mathbb{X}_{s,u} \quad (s \leq t \leq u),$$

with independent increments:

$$\mathbb{X}_{t_0,t_1}, \ldots, \mathbb{X}_{t_{n-1},t_n} \quad \text{independent for} \quad t_0 < \cdots < t_n.$$

If X_0 is independent of ω, then

$$X_t := \mathbb{X}_{0,t}(X_0) \qquad (t \geq 0)$$

defines a Markov process $(X_t)_{t\geq 0}$ with generator G, and

$$P_t(x,y) = \mathbb{P}[\mathbb{X}_{0,t}(x) = y]$$

gives the desired random mapping representation of the Markov semigroup $(P_t)_{t\geq 0}$ with generator G. ∎

Stochastic flows

We call the Poisson set ω a *graphical representation* of X.

Note: Since $\omega_{s,u} := \{(m,t) \in \omega : s < t \leq u\}$,
the stochastic flow $\mathbb{X}_{s,t}$ is right-continuous in s and t.
As a result, $(X_t)_{t\geq 0}$ has right-continuous sample paths.

Setting $\omega^-_{s,u} := \{(m,t) \in \omega : s \leq t < u\}$
yields a stochastic flow $\mathbb{X}^-_{s,t}$ with left-continuous sample paths.

Duality

Two Markov processes X and Y with state spaces S and R are *dual* with *duality function* $\psi : S \times R \to \mathbb{R}$ iff

$$\mathbb{E}\big[\psi(X_t, Y_0)\big] = \mathbb{E}\big[\psi(X_0, Y_t)\big] \qquad (*).$$

for all deterministic initial states X_0 and Y_0.
If $(*)$ holds for deterministic initial states, then also for random initial states, provided X_t is independent of Y_0 and X_0 is independent of Y_t.

In terms of semigroups $(P_t)_{t\geq 0}, (Q_t)_{t\geq 0}$ and generators G, H, duality says

$$\begin{aligned} & P_t\psi = \psi Q_t^\dagger \qquad (t \geq 0), \\ \Leftrightarrow \quad & G\psi = \psi H^\dagger, \end{aligned}$$

where $A^\dagger$ denotes the transpose of a matrix A.

Pathwise duality

Two maps $m : S \to S$ and $\hat{m} : R \to R$ are *dual* w.r.t. a duality function $\psi : S \times R \to T$ iff

$$\psi\big(m(x), y\big) = \psi\big(x, \hat{m}(y)\big) \qquad (x \in S,\ y \in R).$$

Assume that each $m \in \mathcal{G}$ has a dual map $\hat{m}$.
Let ω be a graphical representation for the process with generator

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x)\big).$$

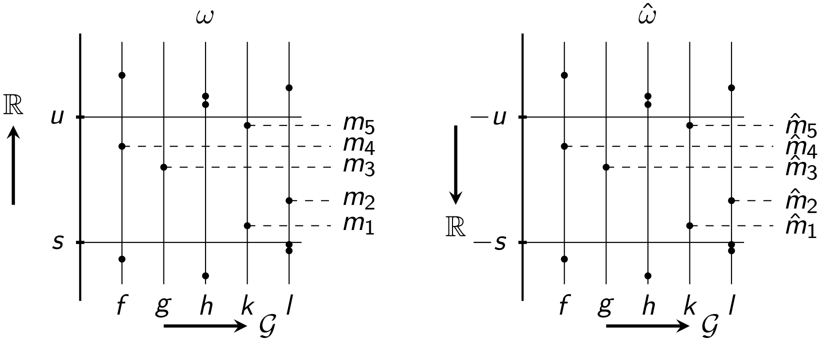
Then

$$\hat{\omega} := \big\{(\hat{m}, -t) : (m, t) \in \omega\big\}$$

is a graphical representation for the process with generator

$$Hf(y) = \sum_{m \in \mathcal{G}} r_m \big(f(\hat{m}(y)) - f(y)\big).$$

Pathwise duality



$$\mathbb{X}^{-}_{s,u} = m_1 \circ \cdots \circ m_1 \quad \text{and} \quad \mathbb{Y}_{-u,-s} = \hat{m}_1 \circ \cdots \hat{m}_5.$$

Pathwise duality

The stochastic flows $(\mathbb{X}_{s,u})_{s\leq u}$ constructed from ω and $(\mathbb{Y}_{s,u})_{s\leq u}$ constructed from $\hat{\omega}$ are dual:

$$\psi\big(\mathbb{X}^-_{s,u}(x),y\big)=\psi\big(x,\mathbb{Y}_{-u,-s}(y)\big)\qquad (x\in S,\ y\in R),$$

where $\mathbb{X}^-_{s,u}$ denotes the left-continuous modification of $\mathbb{X}_{s,u}$.

Proof

$$\begin{aligned}&\psi\big(m_5\circ m_4\circ m_3\circ m_2\circ m_1(x),y\big)\\ &\quad=\psi\big(m_4\circ m_3\circ m_2\circ m_1(x),\hat{m}_5(y)\big)\\ &\quad=\psi\big(m_3\circ m_2\circ m_1(x),\hat{m}_4\circ\hat{m}_5(y)\big)\\ &\quad=\cdots=\psi\big(x,\hat{m}_1\circ\hat{m}_2\circ\hat{m}_3\circ\hat{m}_4\circ\hat{m}_5(y)\big).\end{aligned}$$

∎

Pathwise duality

Two Markov processes X and Y are *pathwise dual* if they can be constructed from stochastic flows that are dual.

For duality functions $\psi : S \times R \to \mathbb{R}$, pathwise duality implies duality:

$$\begin{aligned}\mathbb{E}\big[\psi(X_t, Y_0)\big] &= \mathbb{E}\big[\psi\big(\mathbb{X}^-_{0,t}(X_0), Y_0\big)\big] \\ &= \mathbb{E}\big[\psi\big(X_0, \mathbb{Y}_{-t,0}(Y_0)\big)\big] = \mathbb{E}\big[\psi(X_0, Y_t)\big].\end{aligned}$$

Even though pathwise duality is much stronger than duality, lots of well-known dualities can be realized as pathwise dualities.

Pathwise duality

(Pathwise duality) *If the generators G and H of X and Y have random mapping representations of the form*

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x) \big),$$
$$Hf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(\hat{m}(y)) - f(y) \big),$$

where each map $\hat{m}$ is a dual of m, then X and Y are pathwise dual.

A formal dual

Let $\mathcal{F}(S,T)$ be the set of functions $f:S\to T$.
Let $(\mathbb{X}_{s,u})_{s\le u}$ be a stochastic flow on S. Then

$$\mathbb{F}_{s,u}(f):=f\circ\mathbb{X}^{-}_{-u,-s}\qquad\big(f\in\mathcal{F}(S,T)\big)$$

defines a stochastic flow $(\mathbb{F}_{s,u})_{s\le u}$ on $\mathcal{F}(S,T)$.
If F_0 is an $\mathcal{F}(S,T)$-valued random variable, independent of $(\mathbb{X}_{s,u})_{s\le u}$, then

$$F_t:=\mathbb{F}_{0,t}(F_0)=F_0\circ\mathbb{X}^{-}_{-t,0}\qquad(t\ge 0)$$

defines a Markov process $(F_t)_{t\ge 0}$ with values in $\mathcal{F}(S,T)$.
This Markov process is pathwise dual to X with duality function

$$\psi(x,f):=f(x)\qquad\big(x\in S,\ f\in\mathcal{F}(S,T)\big).$$

Indeed

$$\psi\big(\mathbb{X}^{-}_{s,u}(x),f\big)=f\circ\mathbb{X}^{-}_{s,u}(x)=\mathbb{F}_{-u,-s}(f)(x)=\psi\big(x,\mathbb{F}_{-u,-s}(f)\big).$$

Invariant subspaces

Def A subspace $\mathcal{H} \subset \mathcal{F}(S, T)$ is *invariant* under the action of $(\mathbb{F}_{s,u})_{s \leq u}$ if $f \in \mathcal{H}$ implies $\mathbb{F}_{s,u}(f) \in \mathcal{H}$. That means

$$f \in \mathcal{H} \quad \Rightarrow \quad f \circ \mathbb{X}_{s,u} \in \mathcal{H} \qquad (s \leq u).$$

Interesting pathwise duals are associated with *invariant subspaces* of $\mathcal{F}(S, T)$.

Assume that $\mathcal{H} \subset \mathcal{F}(S, T)$ is invariant and

$$R \ni y \mapsto \psi(\,\cdot\,, y) \in \mathcal{H}$$

is a bijection. Then setting

$$\mathbb{F}_{s,u}\big(\psi(\,\cdot\,, y)\big) =: \psi\big(\,\cdot\,, \mathbb{Y}_{s,u}(y)\big)$$

defines a stochastic flow $(\mathbb{Y}_{s,u})_{s \leq u}$ on R that is dual to $(\mathbb{X}_{s,u})_{s \leq u}$ with duality function ψ.

A bit of order theory

Let S be a finite partially ordered space. The "upset" and "downset" of $A \subset S$ are defined as

$$A^{\uparrow} := \{x \in S : x \geq a \text{ for some } a \in A\},$$
$$A^{\downarrow} := \{x \in S : x \leq a \text{ for some } a \in A\}.$$

A set $A \subset S$ is *increasing* (resp. *decreasing*) if $A^{\uparrow} = A$ (resp. $A^{\downarrow} = A$).

A *lattice* is a partially ordered set such that for every $x, y \in S$ there exist $x \vee y \in S$ and $x \wedge y \in S$, called the *supremum* or *join* and *infimum* or *meet* of x and y, respectively, such that

$$\{x\}^{\uparrow} \cap \{y\}^{\uparrow} = \{x \vee y\}^{\uparrow} \quad \text{and} \quad \{x\}^{\downarrow} \cap \{y\}^{\downarrow} = \{x \wedge y\}^{\downarrow}.$$

A bit of order theory

Let S be a partially ordered set. A map $m : S \to S$ is *monotone* if

$$x \leq y \quad \Rightarrow \quad m(x) \leq m(y) \qquad (x, y \in S).$$

We say that S is *bounded from below* resp. *above* if there exists an element 0 resp. 1 (necessarily unique) such that

$$0 \leq x \quad (x \in S) \quad \text{resp. } x \leq 1 \quad (x \in S).$$

Finite lattices are bounded from below and above.

A map $m : S \to S$ is *additive* if

$$m(0) = 0 \quad \text{and} \quad m(x \vee y) = m(x) \vee m(y) \qquad (x, y \in S).$$

Additive maps are monotone.

Monotone and additive duality

Let S be a partially ordered set (lattice), let $T := \{0,1\}$, and let

$$\mathcal{F}_{\rm mon}(S,T) := \big\{f \in \mathcal{F}(S,T) : f \text{ is monotone}\big\},$$
$$\mathcal{F}_{\rm add}(S,T) := \big\{f \in \mathcal{F}(S,T) : f \text{ is additive}\big\}.$$

Let X be a Markov process with state space S and generator

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m\big(f(m(x)) - f(x)\big).$$

If all maps $m \in \mathcal{G}$ are monotone (resp. additive), then $\mathbb{X}_{s,t}$ is monotone (resp. additive) and hence $f \circ \mathbb{X}_{s,t}$ is monotone (resp. additive) for all $f \in \mathcal{F}_{\rm mon}(S,T)$ (resp. $f \in \mathcal{F}_{\rm add}(S,T)$).

Thus $\mathcal{F}_{\rm mon}(S,T)$ (resp. $\mathcal{F}_{\rm add}(S,T)$) is invariant.

Dual spaces

Let S be a partially ordered set. A *dual* of S is a partially ordered set S' together with a bijection $S \ni x \mapsto x' \in S'$ such that

$$x \leq y \quad \text{if and only if} \quad x' \geq y'.$$

Example 1: For any partially ordered set S, we may take $S' := S$ but equipped with the reversed order, and $x \mapsto x'$ the identity map.

Example 2: If Λ is a set and $S \subset \mathcal{P}(\Lambda)$ is a set of subsets of Λ, equipped with the partial order of inclusion, then we may take for $x' := \Lambda \backslash x$ the complement of x and $S' := \{x' : x \in S\}$.

Additive duality

Let S be a finite lattice and let S' be its dual. Define $\psi : S \times S' \to T = \{0, 1\}$ by

$$\psi(x, y) = 1_{\{x \not\leq y'\}} = 1_{\{y \not\leq x'\}} \qquad (x \in S,\ y \in S').$$

(Additive functions) *For each* $f \in \mathcal{F}_{\mathrm{add}}(S, T)$*, there exists a unique* $y \in S'$ *such that*

$$f(x) = \psi(x, y) \qquad (x \in S).$$

(Additive duality) *A map* $m : S \to S$ *has a dual* $m' : S' \to S'$ *w.r.t.* ψ *if and only if* m *is additive. The dual map* m' *is unique and also an additive map.*

Siegmund's duality

Let $S = \{0, \ldots, n\}$ be totally ordered and let $S' := S$ equipped with the reversed order.
A map $m : S \to S$ is additive iff m is monotone and $m(0) = 0$.
Each such map has a dual $m' : S' \to S'$ that is monotone and satisfies $m'(n) = n$.

(Siegmund's dual) *Let X be a monotone Markov process in S such that 0 is a trap. Then X has a dual Y w.r.t. to the duality function $\psi(x, y) := 1_{\{x \not\leq y\}}$. The dual process is also monotone and has n as a trap. Moreover, the duality can be realized in a pathwise way.*

Additive particle systems

Let $S = \mathcal{P}(\Lambda)$ with Λ a finite set, and let $x \mapsto x' \in S' := \mathcal{P}(\Lambda)$ denote the complement map $x' := \Lambda \backslash x$.
Then $1_{\{x \not\subset y'\}} = 1_{\{x \cap y \neq \emptyset\}}$.

(Additive particle systems) *Let X be a Markov process in S whose generator can be represented in additive maps. Then X has a pathwise dual Y w.r.t. to the duality function $\psi(x, y) := 1_{\{x \cap y \neq \emptyset\}}$, and Y is also an additively representable Markov process.*

Examples: Voter model, contact process, exclusion process, systems of coalescing random walks.

Krone's duality

Steve Krone [AAP 1999] has studied a two-stage contact process, with state space of the form $S = \{0,1,2\}^\Lambda$.
He interprets $x(i) = 0, 1$, or 2 as an empty site, young, or adult organism, and defines maps

$$
\begin{array}{lll}
\text{grow up} & a_i & \cdots 1 \cdots\cdot \mapsto \cdots 2 \cdots\cdot \\
\text{give birth} & b_{ij} & \cdots 20 \cdots \mapsto \cdots 21 \cdots \\
\text{young dies} & c_i & \cdots 1 \cdots\cdot \mapsto \cdots 0 \cdots\cdot \\
\text{death} & d_i & \cdots 1 \cdots\cdot \mapsto \cdots 0 \cdots\cdot \\
\qquad\text{or} & & \cdots 2 \cdots\cdot \mapsto \cdots 0 \cdots\cdot \\
\text{grow younger} & e_i & \cdots 2 \cdots\cdot \mapsto \cdots 1 \cdots\cdot
\end{array}
$$

where in all cases not mentioned, the maps have no effect.

Krone's duality

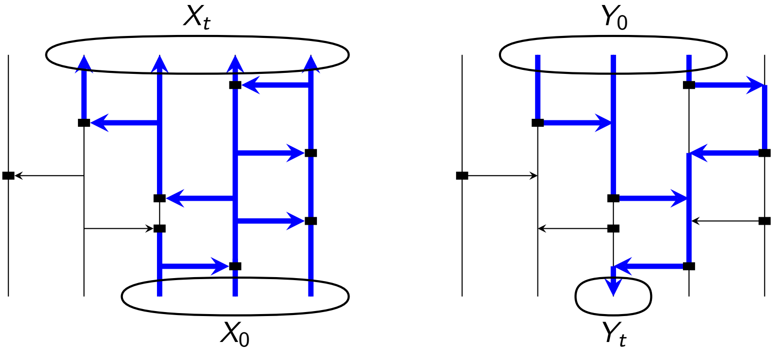
We set $S' := S$ and define $S \ni x \mapsto x' \in S'$ by $x'(i) := 2 - x(i)$. Then the duality function becomes

$$\psi(x, y) = 1_{\{x \not\leq y'\}} = 1_{\{\exists i \in \Lambda \text{ s.t. } x(i) + y(i) > 2\}}.$$

(Krone's dual) *The maps $a_i, b_{ij}, c_i, d_i, e_i$ are all additive and their duals are given by*

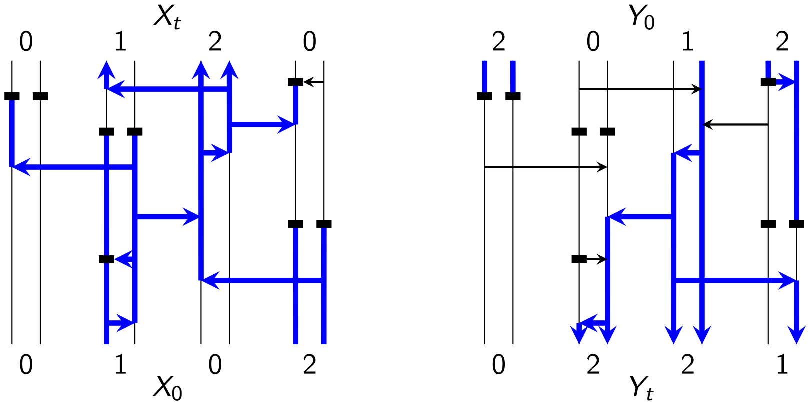
$$a'_i = a_i, \quad b'_{ij} = b_{ji}, \quad c'_i = e_i, \quad d'_i = d_i, \quad e'_i = c_i.$$

Percolation representations



Additive particle systems and their duals can be constructed in terms of open paths. In this example, X is a voter model and Y are coalescing random walks.

Percolation representations



We can also give a percolation representation of Krone's duality.

Percolation representations

By definition, a lattice S is *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad (x, y, z \in S).$$

If Λ is a partially ordered set, then $S := \mathcal{P}_{\rm dec}(\Lambda)$ with the order of set inclusion is a distributive lattice. *Birkhoff's representation theorem* says that every distributive lattice is of this form.

(Percolation representation) *An additive Markov process taking values in* $\mathcal{P}_{\rm dec}(\Lambda)$ *has a percolation representation together with its dual, which takes values in* $S' = \mathcal{P}_{\rm inc}(\Lambda)$*, with the duality function* $\psi(x, y) = 1_{\{x \cap y \neq \emptyset\}}$.

If Λ is equipped with the trivial order $x \not\leq y$ for all $x \neq y$, then $\mathcal{P}_{\rm dec}(\Lambda) = \mathcal{P}(\Lambda) = \mathcal{P}_{\rm inc}(\Lambda)$.

In Krone's example, $\Lambda = \{1, 2\}^{\Delta}$ with the product order.

Let Λ be countable and let $S := \{0,1\}^\Lambda$, equipped with the product topology. A map $m : S \to S$ is *local* if

(i) m is continuous,

(ii) $\big\{i \in \Lambda : \exists x \in S \text{ s.t. } m(x)(i) \neq x(i)\big\}$ is finite.

Note (i) is equivalent to:

(i)' for each $i \in \Lambda$, the function $x \mapsto m(x)(i)$ depends on finitely many coordinates.

Assume all $m \in \mathcal{G}$ are local. Under suitable assumptions, the interacting particle system X with generator

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x)\big)$$

can be constructed from a stochastic flow $(\mathbb{X}_{s,u})_{s \le u}$ based on a graphical representation ω.

Monotone systems duality

Notation:

$$\mathcal{C}(S,R) := \{f \in \mathcal{F}(S,R) : f \text{ is continuous}\},$$
$$\mathcal{L}(S,R) := \{f \in \mathcal{F}(S,R) : f \text{ is lower semi-continuous}\}.$$

Also: $\mathcal{C}_{\rm mon}(S,R) := \mathcal{C}(S,R) \cap \mathcal{F}_{\rm mon}(S,R)$ etc.
If all local maps $m \in \mathcal{G}$ are monotone, then a.s.

$$\mathbb{X}_{s,u} \in \mathcal{C}_{\rm mon}(S,S) \qquad (s \leq u).$$

Let $T := \{0,1\}$. Then

$$f \in \mathcal{C}_{\rm mon}(S,T) \quad \Rightarrow \quad f \circ \mathbb{X}_{s,u} \in \mathcal{C}_{\rm mon}(S,T),$$
$$f \in \mathcal{L}_{\rm mon}(S,T) \quad \Rightarrow \quad f \circ \mathbb{X}_{s,u} \in \mathcal{L}_{\rm mon}(S,T).$$

Thus, $\mathcal{L}_{\rm mon}(S,T)$ is preserved under $(\mathbb{F}_{s,u})_{s \leq u}$.

Monotone systems duality

Recall that $Y^{\uparrow} := \{z \in S : z \geq y \text{ for some } y \in Y\}$. One has

$$\mathcal{L}_{\rm mon}(S,T) = \{1_Y : Y \in \mathcal{I}\}$$
$$\text{with} \quad \mathcal{I} := \{Y \in \mathcal{P}(S) : Y \text{ is open and } Y^{\uparrow} = Y\}.$$

A convenient way to encode an open, increasing set is by writing down its minimal elements. A *minimal element* is an $y \in Y$ such that

$$z \in Y,\ z \leq y \text{ implies } z = y.$$

For each $Y \subset S$, let

$$Y^{\circ} := \{y \in Y : y \text{ is a minimal element of } Y\}.$$

It is easy to see that $(Y^{\uparrow})^{\uparrow} = Y^{\uparrow}$ and $(Y^{\circ})^{\circ} = Y^{\circ}$.

Monotone systems duality

$$\text{Let } S_{\rm fin} := \{y \in S : |y| < \infty\} \quad \text{with} \quad |y| := \sum_{i \in \Lambda} y(i),$$

$$\mathcal{H} := \{Y \in \mathcal{P}(S_{\rm fin}) : Y^\circ = Y\}.$$

(Encoding open increasing sets) *The map $Y \mapsto Y^\uparrow$ is a bijection from $\mathcal{H}$ to $\mathcal{I}$, and $Y \mapsto Y^\circ$ is its inverse.*

Define $\psi : S \times \mathcal{H} \to T$ by

$$\psi(x, Y) := 1_{\{\exists y \in Y \text{ s.t. } x \geq y\}} \qquad \big(x \in S,\ \ Y \in \mathcal{H}\big).$$

Then $\mathcal{H} \ni Y \mapsto \psi(\,\cdot\,, Y) \in \mathcal{L}_{\rm mon}(S, T)$ is a bijection.
There exists a unique stochastic flow $(\mathbb{Y}_{s,u})_{s \leq u}$ on $\mathcal{H}$ such that

$$\psi\big(\mathbb{X}^-_{s,u}(x), y\big) = \psi\big(x, \mathbb{Y}_{-u,-s}(Y)\big) \qquad (x \in S,\ \ Y \in \mathcal{H}).$$

Monotone systems duality

Define constant configurations $\underline{0}(i) := 0$, $\underline{1}(i) := 1$ $(i \in \Lambda)$. By monotonicity, the process X has an *upper invariant law*

$$\mathbb{P}^{\underline{1}}[X_t \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}.$$

By definition, the $\mathcal{H}$-valued process Y *survives* if

$$\mathbb{P}^{\{e_i\}}[Y_t \neq \emptyset\ \forall t \geq 0] > 0 \text{ for some } i \in \Lambda,$$

where $e_i(j) := 1_{\{i=j\}}$ $(i, j \in \Lambda)$.

(Nontrivial upper invariant law) One has $\overline{\nu} \neq \delta_{\underline{0}}$ if and only if Y survives. The law $\overline{\nu}$ is uniquely characterized by

$$\mathbb{E}\big[\psi(\overline{X}, \{y\})\big] = \mathbb{P}^{\{y\}}\big[Y_t \neq \emptyset\ \forall t \geq 0\big].$$

where $\overline{X}$ denotes a r.v. with law $\overline{\nu}$.

Proof

$$\mathbb{E}^{\underline{1}}\big[\psi(X_t,\{y\})\big] = \mathbb{E}^{\{y\}}\big[\psi(\underline{1},Y_t)\big] = \mathbb{E}^{\{y\}}\big[\exists y \in Y_t \text{ s.t. } \underline{1} \geq y\big]$$

$$= \mathbb{E}^{\{y\}}\big[Y_t \neq \emptyset\big] \underset{t\to\infty}{\longrightarrow} \mathbb{P}^{\{y\}}\big[Y_t \neq \emptyset \ \forall t \geq 0\big].$$

∎

Monotone systems duality

Equip the space $\mathcal{H}$ with an order such that

$$Y \leq Z \quad \Leftrightarrow \quad \psi(x, Y) \leq \psi(x, Z) \quad \forall x \in S,$$

and with a topology such that

$$Y_n \to Y \quad \Leftrightarrow \quad \psi(x, Y_n) \to \psi(x, Y) \quad \forall x \in S_{\rm fin}.$$

The minimal and maximal elements of $\mathcal{H}$ are $\emptyset$ and $\{\underline{0}\}$ since

$$\psi(x, \emptyset) = 0 \quad \psi(x, \{\underline{0}\}) = 1 \qquad (x \in S).$$

The *second largest* element of $\mathcal{H}$ is $Y_{\rm sec} := \{e_i : i \in \Lambda\}$, since

$$\psi(x, Y_{\rm sec}) = 1_{\{x \neq \underline{0}\}} \qquad (x \in S).$$

(Recall $e_i(j) := 1_{\{i=j\}}$ $(i, j \in \Lambda)$.)

Monotone systems duality

We say that X *survives* if

$$\mathbb{P}^{e_i}\big[X_t \neq \underline{0}\ \forall t \geq 0\big] > 0 \text{ for some } i \in \Lambda.$$

If $\underline{0}$ is a trap for $(X_t)_{t\geq 0}$, then $Y_0 \neq \{\underline{0}\}$ implies $Y_t \neq \{\underline{0}\}$ $(t \geq 0)$.

Work in progress We believe that

$$\mathbb{P}^{Y_{\rm sec}}[Y_t \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \overline{\mu}.$$

We call $\overline{\mu}$ the *upper invariant law* of $(Y_t)_{t\geq 0}$.
We believe that $\overline{\mu} \neq \delta_\emptyset$ if and only if X survives.
The law $\overline{\mu}$ is uniquely characterized by

$$\mathbb{E}\Big[\prod_{k=1}^{n}\psi(x_k,\overline{Y})\Big] = \mathbb{P}\big[\mathbb{X}_{0,t}(x_k) \neq \underline{0}\ \forall t \geq 0,\ k = 1,\ldots,n\big].$$

where $\overline{Y}$ denotes a r.v. with law $\overline{\mu}$ and $x_1,\ldots,x_n \in S_{\rm fin}$.

Some simulations

$i^{\uparrow}$	
i	$i^{\rightarrow}$

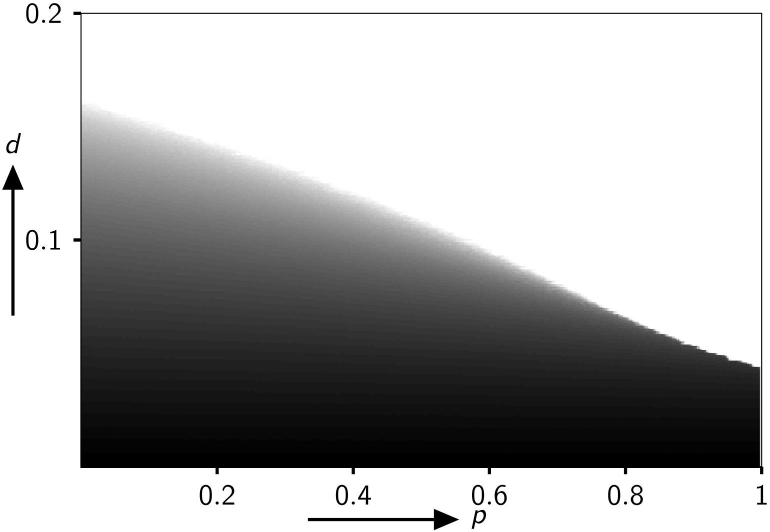
For each $i = (i_1, i_2) \in \mathbb{Z}^2$, let
$i^{\rightarrow} := (i_1 + 1, i_2)$ and $i^{\uparrow} := (i_1, i_2 + 1)$.

Let $p, d \in [0, 1]$ and let $X = (X_t)_{t \in \mathbb{N}}$ be a Markov chain with values in $\{0, 1\}^{\mathbb{Z}^2}$ such that independently for each i and t,

$$
\begin{array}{ll}
X_{t+1}(i) = X_t(i) \vee \big(X_t(i^{\rightarrow}) \wedge X_t(i^{\uparrow})\big) & \text{w. prob. } p(1-d), \\
X_{t+1}(i) = X_t(i) \vee X_t(i^{\rightarrow}) & \text{w. prob. } \tfrac{1}{2}(1-p)(1-d), \\
X_{t+1}(i) = X_t(i) \vee X_t(i^{\uparrow}) & \text{w. prob. } \tfrac{1}{2}(1-p)(1-d), \\
X_{t+1}(i) = 0 & \text{w. prob. } d.
\end{array}
$$

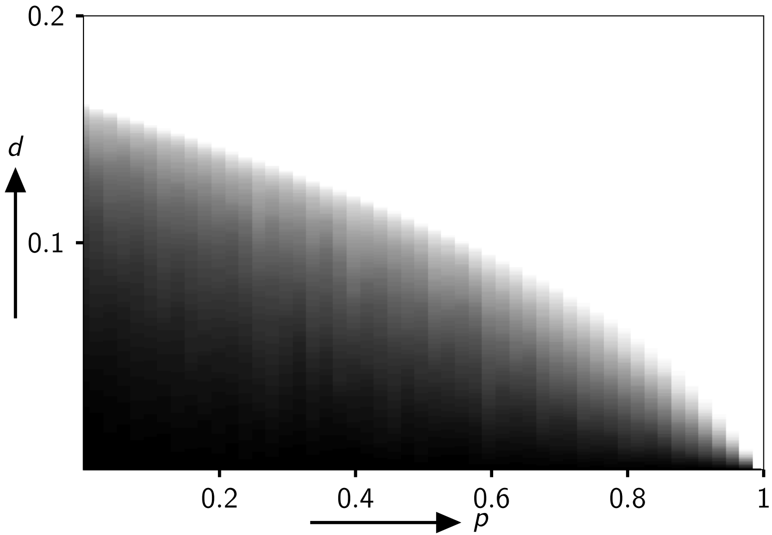
For $p = 0$ this model is additive.
For $p = 1$, it does not survive for any $d > 0$.

Some simulations



Density of the upper invariant law.

Some simulations



Survival probability started from a single one.