Intertwining of Markov processes

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Outline

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- The contact process on the hierarchical group

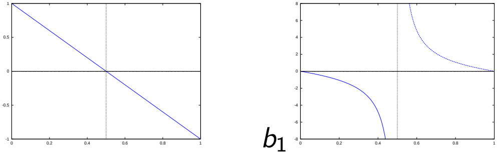
A change point problem

Let τ be exponentially distributed with mean one and let $Y_t := 1_{\{t\geq\tau\}}$. Let X be a diffusion in $[0,1]$ such that while $Y_t = y$, X_t evolves according to the generator

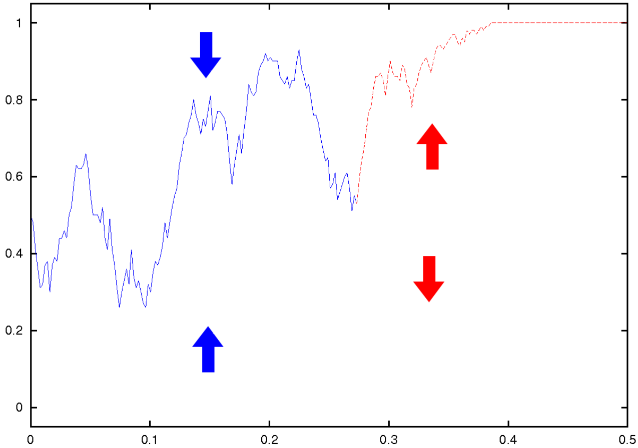
$$G_y f(x) := \tfrac{1}{2}x(1-x)\frac{\partial^2}{\partial x^2}f(x) + b_y(x)\frac{\partial}{\partial x}f(x) \qquad (y=0,1),$$

where

$$b_0(x) = 2(\tfrac{1}{2}-x), \quad b_1(x) = \frac{8x(1-x)(x-\frac{1}{2})}{1-4x(1-x)}.$$

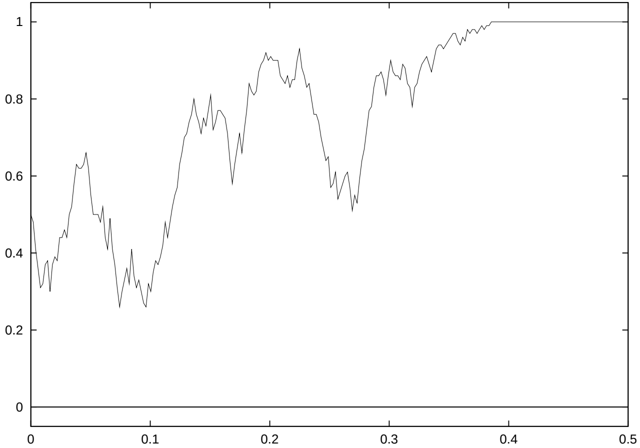
b_0  b_1

Wright-Fisher diffusion with drift



As long as $Y_t = 0$, X cannot reach the boundary. When $Y_t = 1$, X cannot cross the middle.

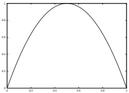
Wright-Fisher diffusion



If we forget about the colors, then X is just a Wright-Fished diffusion without drift!

Explanation

The process (X, Y) is Markov, but X is not autonomous, i.e., its dynamics depend on the state of Y. So how is it possible that X, on its own, is Markov?



In fact, one has

$$\mathbb{P}[Y_t = 0 \,|\, (X_s)_{0 \le s \le t}] = 4X_t(1 - X_t) \quad \text{a.s.}$$

In particular, this probability depends only on the endpoint of the path $(X_s)_{0 \le s \le t}$, and

$$\begin{aligned}&\mathbb{E}[b_{Y_t}(X_t) \,|\, X_t = x] = \\ &\quad 4x(1-x)b_0(x) + \big(1 - 4x(1-x)\big)\, b_1(x) = 0.\end{aligned}$$

General principle

[Rogers & Pitman '81] Let (X, Y) be a Markov process with state space $S \times T$ and generator $\hat{G}$, and let K be a probability kernel from S to T. Define $\hat{K} : \mathbb{R}^{S\times T} \to \mathbb{R}^S$ by

$$\hat{K} f(x) := \sum_{y\in S} K(x, y) f(x, y).$$

Let G be the generator of a Markov process in S and assume that

$$G\hat{K} = \hat{K}\hat{G}.$$

Then

$$\mathbb{P}[Y_0 = y \,|\, X_0] = K(X_0, y) \quad \text{a.s.}$$

implies

$$\mathbb{P}[Y_t = y \,|\, (X_s)_{0\leq s\leq t}] = K(X_t, y) \quad \text{a.s.} \qquad (t \geq 0).$$

and X, on its own, is a Markov process with generator G.

Adding structure to Markov processes

[Athreya & S. '10] Let X be a Markov processes with state space S and generator G, let K be a probability kernel from S to T and let $(G'_x)_{x\in S}$ be a collection of generators of T-valued Markov processes. Assume that

$$GK = \hat{K}\,\overline{G}$$

where $K:\mathbb{R}^T\to\mathbb{R}^S$ and $\overline{G}:\mathbb{R}^T\to\mathbb{R}^{S\times T}$ are defined by

$$Kf(x) := \sum_y K(x,y)f(y) \quad \text{and} \quad \overline{G}f(x,y) := G'_x f(y).$$

Then X can be coupled to a process Y such that (X,Y) is Markov, Y evolves according to the generator G'_x while X is in the state x, and

$$\mathbb{P}[Y_t = y \,|\, (X_s)_{0\le s\le t}] = K(X_t,y) \quad \text{a.s.} \qquad (t\ge 0).$$

We call Y an *added-on process on* X.

Intertwining of semigroups

[Fill '92] Let X and Y be Markov processes with state spaces S and T, semigroups $(P_t)_{t\geq 0}$ and $(P'_t)_{t\geq 0}$, and generators G and G'. Let K be a probability kernel from S to T and assume that

$$GK = KG'$$

Then one has the *intertwining relation*

$$P_t K = K P'_t \quad (t \geq 0)$$

and the processes X and Y can be coupled such that

$$\mathbb{P}[Y_t = y \,|\, (X_s)_{0\leq s\leq t}] = K(X_t, y) \quad \text{a.s.} \qquad (t \geq 0).$$

We call Y an *averaged Markov process on X*.

Example: Wright-Fisher diffusion

Generalization Let X be a Wright-Fisher diffusion without drift. Let Y be a process with state space $\{0,1,\ldots,\infty\}$ that jumps $k \mapsto k+1$ with rate $\frac{1}{2}(k+1)(2k+1)$ and gets absorbed in ∞ in finite time. Define a kernel $K:[0,1] \to \{0,1,\ldots,\infty\}$ by

$$K(x,y) = \begin{cases} 4x(1-x)\big(1-4x(1-x)\big)^y & (x \neq 0,1) \\ 1_{\{y=\infty\}} & (x = 0,1) \end{cases}$$

Then the processes started in $X_0 = \frac{1}{2}$ and $Y_0 = 0$ can be coupled such that

$$\mathbb{P}[Y_t = y \,|\, (X_s)_{0 \leq s \leq t}] = K(X_t, y) \quad \text{a.s.} \quad (t \geq 0)$$

and

$$\inf\big\{t \geq 0 : X_t \in \{0,1\}\big\} = \inf\{t \geq 0 : Y_t = \infty\}.$$

First passage times of birth and death processes

[Karlin & McGregor '59] Let Z be a Markov process with state space $\{0, 1, 2, \ldots\}$, started in $Z_0 = 0$, that jumps $k-1 \mapsto k$ with rate $b_k > 0$ and $k \mapsto k-1$ with rate $d_k > 0$ $(k \geq 1)$. Then

$$\tau_N := \inf\{t \geq 0 : Z_t = N\}$$

is distributed as a sum of independent exponentially distributed random variables whose parameters $\lambda_1 < \cdots < \lambda_N$ are the negatives of the eigenvalues of the generator of the process stopped in N.

Coupling of birth and death processes

[Diaconis & Miclos '09] Let $X_t := Z_{t \wedge \tau_N}$ be the stopped process and let $0 > -\lambda_1 > \cdots > -\lambda_N$ be its eigenvalues. Let X^+ be a pure birth process with birth rates $b_1, \ldots, b_N$ given by $\lambda_N, \ldots, \lambda_1$. Then it is possible to couple the processes X and X^+, both started in zero, in such a way that $X_t \leq X_t^+$ for all $t \geq 0$ and both processes arrive in N at the same time.

A probabilistic proof

Let G, G^+ be the generators of X, X^+. We claim that there exists a kernel K^+ such that

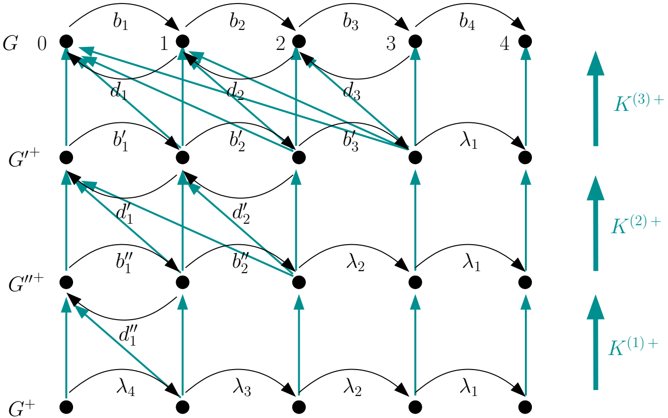
$$\begin{aligned} & K^+(x, \{0, \ldots, x\}) = 1 \qquad (0 \leq x \leq N), \\ & K^+(N, N) = 1, \end{aligned}$$

and moreover

$$K^+ G = G^+ K^+.$$

This can be proved by induction, using the Perron-Frobenius theorem in each step.

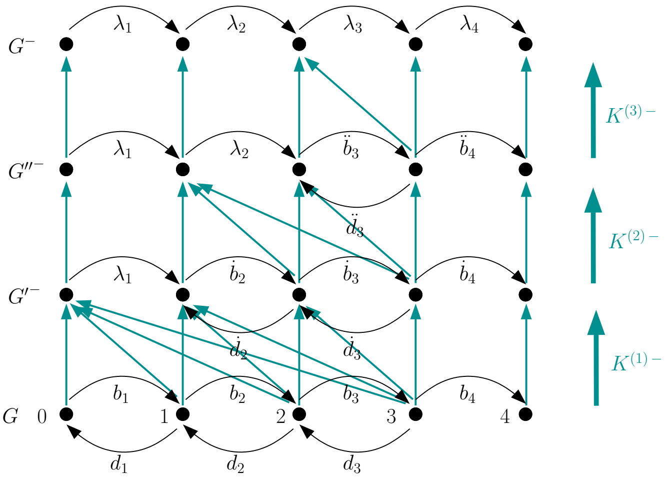
A probabilistic proof



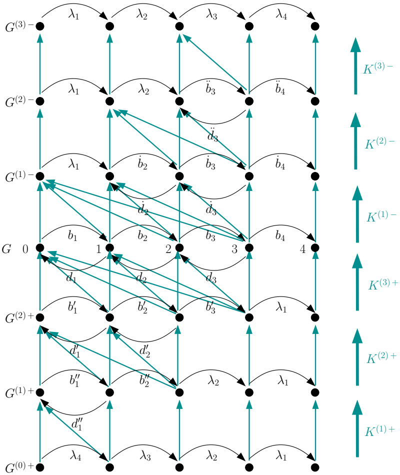
Coupling of birth and death processes

[S. '10] Let X_t and $\lambda_1, \ldots, \lambda_N$ be as before. Let X^- be a pure birth process with birth rates $b_1, \ldots, b_N$ given by $\lambda_1, \ldots, \lambda_N$. Then it is possible to couple the processes X and X^-, both started in zero, in such a way that $X^-_t \leq X_t$ for all $t \geq 0$ and both processes arrive in N at the same time.

A probabilistic proof



The complete figure

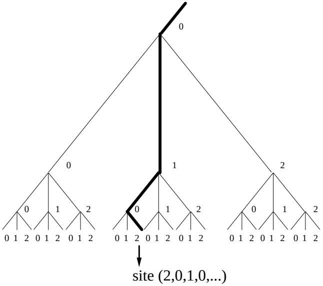


The hierarchical group

By definition, the *hierarchical group with freedom N* is the set

$$\Omega_N := \Big\{ i = (i_0, i_1, \ldots) : \begin{array}{l} i_k \in \{0, \ldots, N-1\}, \\ i_k \neq 0 \text{ for finitely many } k \end{array} \Big\},$$

equipped with componentwise addition modulo N. Think of sites $i \in \Omega_N$ as the leaves of an infinite tree. Then $i_0, i_1, i_2, \ldots$ are the labels of the branches on the unique path from i to the root of the tree.

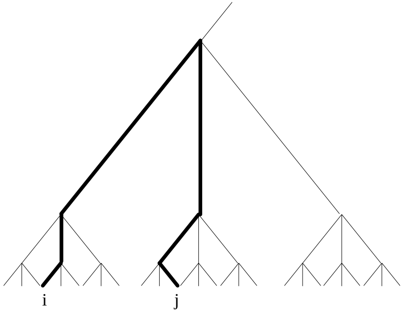


The hierarchical distance

Set

$$|i| := \inf\{k \geq 0 : i_m = 0 \ \forall m \geq k\} \qquad (i \in \Omega_N).$$

Then $|i - j|$ is the *hierarchical distance* between two elements $i, j \in \Omega_N$. In the tree picture, $|i - j|$ measures how high we must go up the tree to find the last common ancestor of i and j.



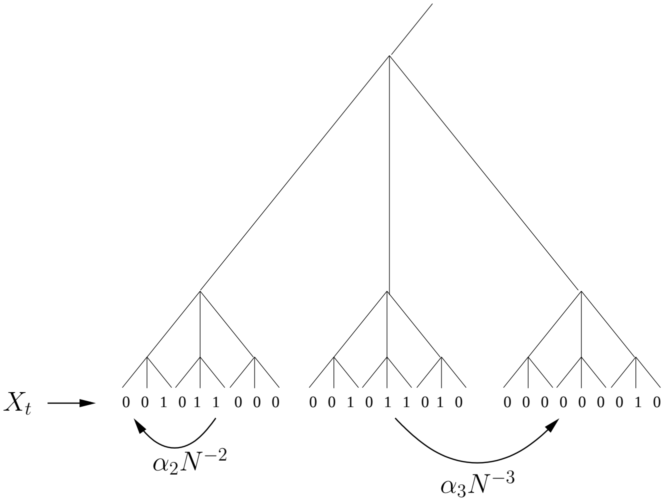
two sites at hierarchical distance 3

Hierarchical contact processes

Fix a *recovery rate* $\delta \geq 0$ and *infection rates* $\alpha_k \geq 0$ such that $\sum_{k=1}^{\infty} \alpha_k < \infty$. The *contact process* on Ω_N with these rates is the $\{0,1\}^{\Omega_N}$-valued Markov process $(X_t)_{t \geq 0}$ with the following description:

If $X_t(i) = 0$ (resp. $X_t(i) = 1$), then we say that the site $i \in \Omega_N$ is *healthy* (resp. *infected*) at time $t \geq 0$. An infected site i infects a healthy site j at hierarchical distance $k := |i - j|$ with rate $\alpha_k N^{-k}$, and infected sites become healthy with rate $\delta \geq 0$.

Hierarchical contact processes



Infection rates on the hierarchical group.

The critical recovery rate

We say that a contact process $(X_t)_{t\geq 0}$ on Ω_N with given recovery and infection rates *survives* if there is a positive probability that the process started with only one infected site never recovers completely, i.e., there are infected sites at any $t \geq 0$. For given infection rates, we let

$$\delta_{\rm c} := \sup\big\{\delta \geq 0 : \text{the contact process with infection rates } (\alpha_k)_{k\geq 1} \text{ and recovery rate } \delta \text{ survives}\big\}$$

denote the *critical recovery rate*. A simple monotone coupling argument shows that X survives for $\delta < \delta_{\rm c}$ and dies out for $\delta > \delta_{\rm c}$. It is not hard to show that $\delta_{\rm c} < \infty$. The question whether $\delta_{\rm c} > 0$ is more subtle.

(Non)triviality of the critical recovery rate

[Athreya & S. '10] Assume that $\alpha_k = e^{-\theta^k}$ $(k \geq 1)$. Then:

(a) If $N < \theta$, then $\delta_{\rm c} = 0$.

(b) If $1 < \theta < N$, then $\delta_{\rm c} > 0$.

More generally, we show that $\delta_{\rm c} = 0$ if

$$\liminf_{k\to\infty} N^{-k} \log(\beta_k) = -\infty, \quad \text{where} \quad \beta_k := \sum_{n=k}^{\infty} \alpha_n \quad (k \geq 1),$$

while $\delta_{\rm c} > 0$ if

$$\sum_{k=m}^{\infty} (N')^{-k} \log(\alpha_k) > -\infty,$$

for some $m \geq 1$ and $N' < N$.

Proof of extinction

Without loss of generality $\sum_{k=1}^{\infty} \alpha_k \leq 1$. Let $X^{(n)}$ be the process restricted to

$$\Omega_N^n := \big\{ i = \big(i_0, \ldots, i_{n-1} \big) : i_k \in \{0, \ldots, N-1\} \big\}.$$

We claim that

$$T := \mathbb{E}^{\delta_0} \big[\inf\{ t \geq 0 : X_t^{(n)} = \underline{0} \} \big] \leq N^{-n} (1 + \delta^{-1})^{N^n}. \qquad (1)$$

For $N < \theta$, this implies that sufficiently large blocks recover completely faster than they can infect other blocks of the same size, hence the result follows by comparison with subcritical branching.

Proof of extinction

To prove (1), we compare $X^{(n)}$ with a process $\tilde{X}^{(n)}$ where sites jump independently from each other from 0 to 1 with rate one and from 1 to 0 with rate δ. This process has a unique equilibrium law with

$$\mathbb{P}\big[\tilde{X}^{(n)}_t = \underline{0}\big] = \Big(\frac{\delta}{1+\delta}\Big)^{N^n}$$

Since $\tilde{X}^{(n)}$ stays on average a time N^{-n} in the state $\underline{0}$, we also have

$$\mathbb{P}\big[\tilde{X}^{(n)}_t = \underline{0}\big] = \frac{N^{-n}}{N^{-n} + \tilde{T}}$$

It follows that

$$T \leq \tilde{T} = N^{-n}\big((1+\delta^{-1})^{N^n} - 1\big) \leq N^{-n}(1+\delta^{-1})^{N^n}.$$

Proof of survival

We use added-on Markov processes to inductively derive bounds on the finite-time survival probability of finite systems. Let

$$\Omega_2^n := \{ i = (i_0, \ldots, i_{n-1}) : i_k \in \{0, 1\} \}$$

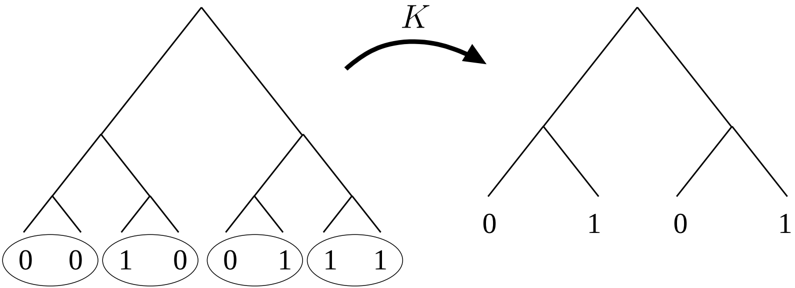
and let $S_n := \{0,1\}^{\Omega_2^n}$. We define a kernel from S_n to S_{n-1} by independently replacing blocks consisting of two spins by a single spin according to the stochastic rules:

$$00 \longrightarrow 0, \qquad 11 \longrightarrow 1,$$

$$\text{and} \quad 01 \text{ or } 10 \longrightarrow \begin{cases} 0 & \text{with probability } \xi, \\ 1 & \text{with probability } 1 - \xi, \end{cases}$$

where $\xi \in (0, \frac{1}{2}]$ is a constant, to be determined later.

Renormalization kernel



The probability of this transition is $1 \cdot (1 - \xi) \cdot \xi \cdot 1$.

An added-on process

Let X be a contact process on Ω_2^n with infection rates $\alpha_1, \ldots, \alpha_n$ and recover rate δ. Then X can be coupled to a process Y such that

$$\mathbb{P}[Y_t = y \,|\, (X_s)_{0 \le s \le t}] = K(X_t, y) \quad \text{a.s.} \qquad (t \ge 0),$$

where K is the kernel defined before, and

$$\xi := \gamma - \sqrt{\gamma^2 - \tfrac{1}{2}} \quad \text{with} \quad \gamma := \frac{1}{4}\Big(3 + \frac{\alpha_1}{2\delta}\Big).$$

Moreover, the process Y can be coupled to a finite contact process Y' on Ω_2^{n-1} with recovery rate $\delta' := 2\xi\delta$ and infection rates $\alpha'_1, \ldots, \alpha'_{n-1}$ given by $\alpha'_k := \frac{1}{2}\alpha_{k+1}$, in such a way that $Y'_t \le Y_t$ for all $t \ge 0$.

Renormalization

We may view the map $(\delta, \alpha_1, \ldots, \alpha_n) \mapsto (\delta', \alpha'_1, \ldots, \alpha'_{n-1})$ as an (approximate) renormalization transformation. By iterating this map n times, we get a sequence of recovery rates $\delta, \delta', \delta'', \ldots$, the last of which gives a upper bound on the spectral gap of the finite contact process X on Ω_2^n. Under suitable assumptions on the α_k's, we can show that this spectral gap tends to zero as $n \to \infty$, and in fact, we can derive explicit lower bounds on the probability that finite systems survive till some fixed time t.

A question

Question Can we find an *exact* renormalization map $(\delta, \alpha_1, \ldots, \alpha_n) \mapsto (\delta', \alpha'_1, \ldots, \alpha'_{n-1})$ for hierarchical contact processes, i.e., for each hierarchical contact process X on Ω_2^n, can we find an *averaged Markov process* Y that is itself a hierarchical contact process (or something similar) on Ω_2^{n-1}?