The contact process seen from a typical infected site

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Contact processes on groups

Λ: countable group with group action $(i,j) \mapsto ij$, inverse operation $i \mapsto i^{-1}$, and unit element (origin) 0.

a: function $a : \Lambda \times \Lambda \to [0,\infty)$ s.t.

$$\text{(i)} \quad a(i,j) = a(ki,kj) \qquad\qquad (i,j,k \in \Lambda),$$

$$\text{(ii)} \quad |a| := \sum_{i \in \Lambda} a(0,i) < \infty.$$

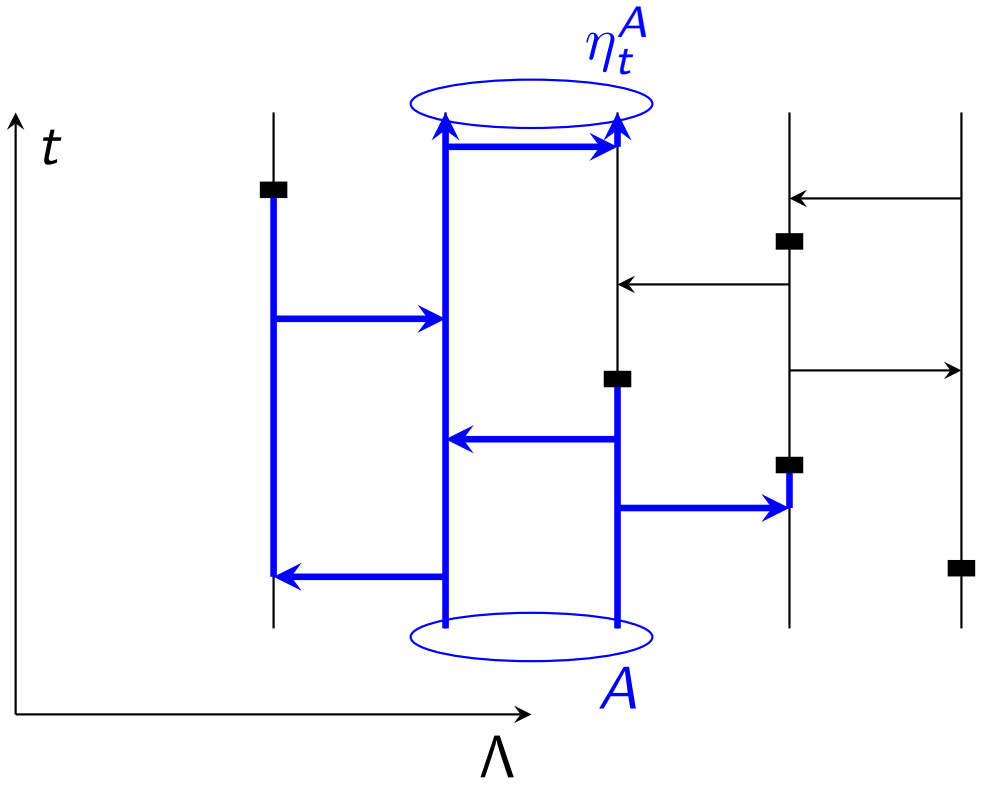
δ: nonnegative constant.

Definition The (Λ, a, δ)-*contact process* $(\eta_t)_{t \geq 0}$ is a Markov process taking values in the subets of Λ. Sites $i \in \eta_t$ are called *infected*.

- An infected site at i infects a healthy site at j with rate $a(i,j)$.
- Infected sites recover with rate δ.

Graphical representation

Draw recovery symbols ▬ with Poisson rate δ.
Draw an arrow from i to j with rate $a(i,j)$.



$$\eta_t^A = \{i \in \Lambda : (j,0) \rightsquigarrow (i,t) \text{ for some } j \in A\}.$$

Open paths may follow arrows but must avoid recovery symbols.

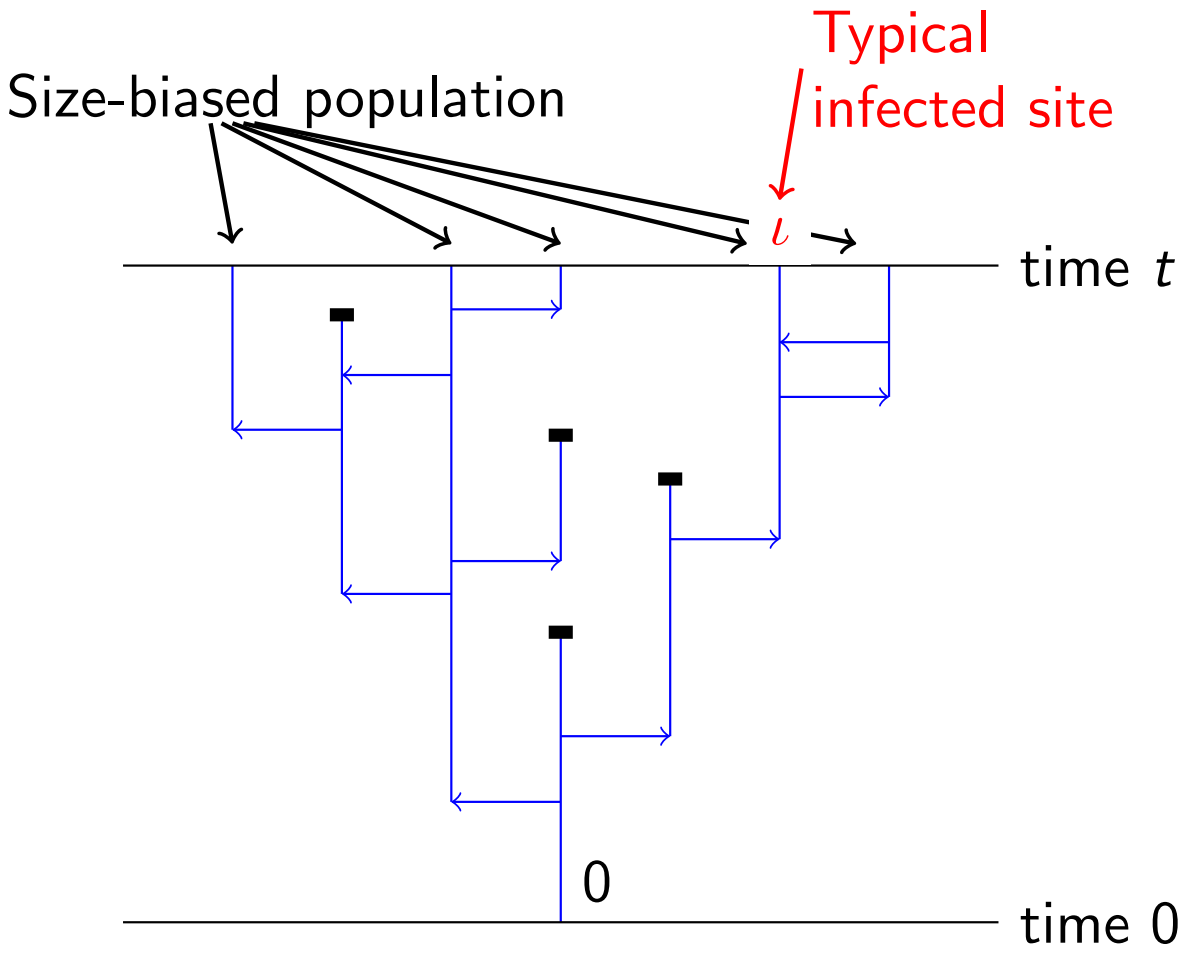
Campbell/Palm laws

Let ω be the Poisson processes of the graphical representation.
Let A be finite and nonempty.
Define a (normalized) *Campbell law*:

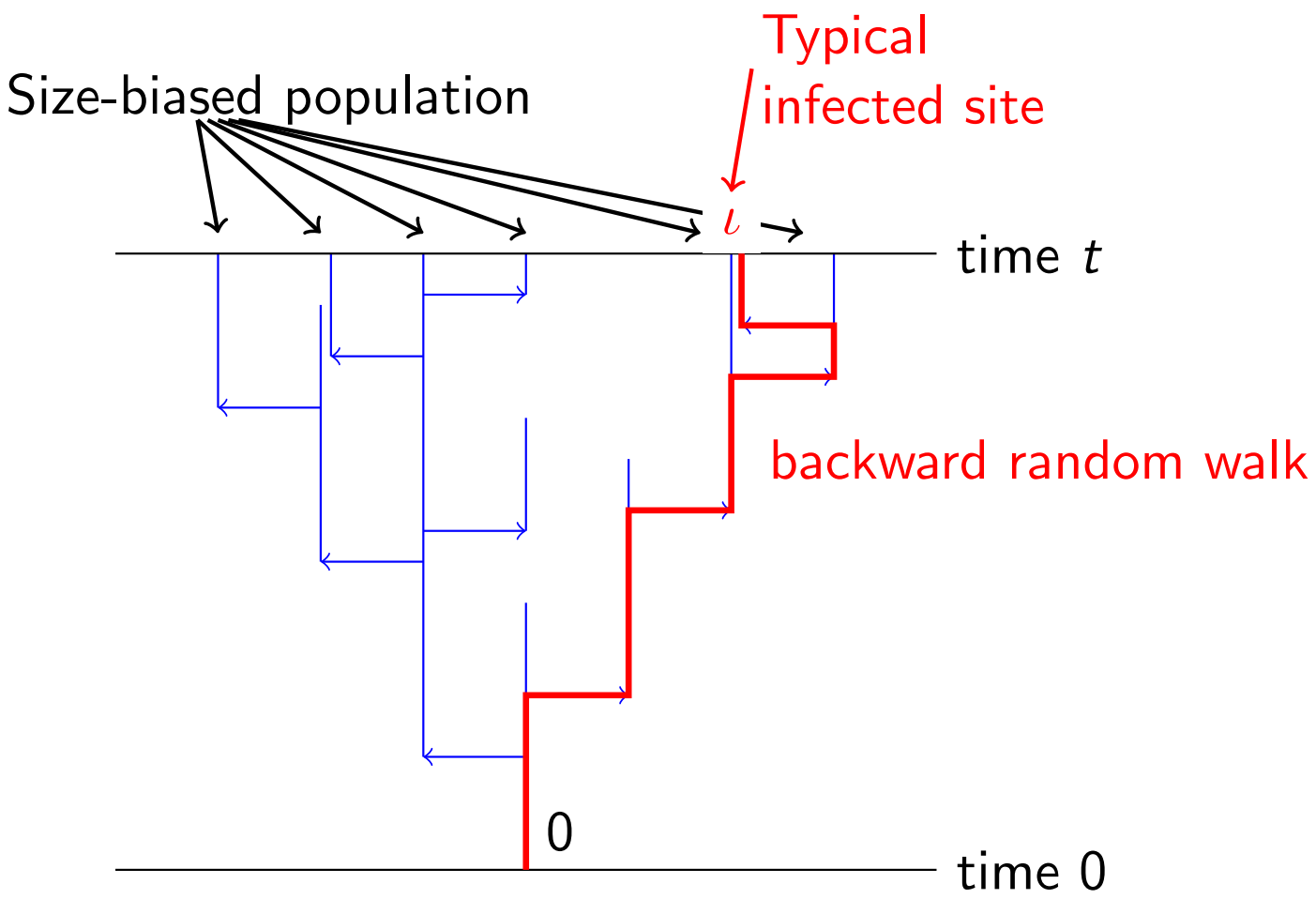
$$\hat{\mathbb{P}}_t^A[\omega \in \bullet,\ \iota = i] := \frac{\mathbb{P}[\omega \in \bullet,\ i \in \eta_t^A(\omega)]}{\mathbb{E}[|\eta_t^A|]}.$$

- $\hat{\mathbb{P}}_t^A[\omega \in \bullet]$ is the original law $\mathbb{P}$ size-biased on the number of infected sites $|\eta_t^A|$.
- Conditional on η_t^A, the site ι is chosen uniformly from all infected sites.

A typical infected site



Typical particles in branching processes



For branching processes, we have Kallenberg's *backward tree construction*.

The exponential growth rate

Why look at Campbell laws for a contact process?

Campbell laws are related to the quantity:

$$r(\Lambda, a, \delta) = r := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[|\eta_t^{\{0\}}|\big]$$

the *exponential growth rate* of the expected population size.

- $\delta \mapsto r(\delta)$ nonincreasing, Lipschitz continuous.
- $r < 0$ iff $\delta > \delta_{\mathrm{c}}$. [Menshikov '86, Aizenman & Barsky '87, Bezuidenhout & Grimmett '91, Aizenman & Jung '07].

Here $\delta_{\mathrm{c}} := \inf\{\delta > 0 :$ the (Λ, a, δ)-contact process dies out a.s.$\}$ critical point for survival.

Contact processes on nonamenable groups

Theorem (S. '09) The critical contact process on any nonamenable group dies out. Moreover, on nonamenable groups, survival implies $r > 0$.

Idea of the Proof The first statement follows from the second since $\delta \mapsto r(\delta)$ is continuous hence $\{\delta : r(\delta) > 0\}$ is open.

Define

$$\tilde{\mathbb{P}}_\lambda^{\{0\}}[\omega \in \bullet,\ \iota = i,\ \tau \in \mathrm{d}t] := \frac{\mathbb{P}[\omega \in \bullet,\ i \in \eta_t^{\{0\}}]e^{-\lambda t}\mathrm{d}t}{\int_0^t \mathbb{E}[|\eta_t^{\{0\}}|]e^{-\lambda t}\mathrm{d}t}.$$

Claim Assume $r = 0$ and the upper invariant law $\overline{\nu}$ is nontrivial. Then

$$\tilde{\mathbb{P}}_\lambda^{\{0\}}[\iota^{-1}\eta_\tau^{\{0\}} \in \bullet] \underset{\lambda \downarrow 0}{\Longrightarrow} \hat{\nu},$$

where $\hat{\nu}(\mathrm{d}A) = \overline{\nu}(\mathrm{d}A \,|\, 0 \in A)$ is the upper invariant law conditioned on the origin being infected.

Intermezzo: amenability

Let $AB := \{ij : i \in A,\ j \in B\}$ and let $A \triangle B := (A \backslash B) \cup (B \backslash A)$ denote the symmetric difference of A and B.
By definition, Λ is *amenable* if

> For every finite nonempty $\Delta \subset \Lambda$ and $\varepsilon > 0$, there exists a finite nonempty $A \subset \Lambda$ such that $|(A\Delta) \triangle A| \leq \varepsilon |A|$.

If Λ is finitely generated, then it suffices to check this for one finite symmetric generating set Δ. In this case, $(A\Delta) \triangle A$ is the boundary of A in the *Cayley graph* associated with Λ and Δ.

In nonamenable groups, the boundary of a finite set is never much smaller than its interior.

For example, $\mathbb{Z}^d$ is amenable, but regular trees are not.

Intermezzo: the upper invariant law

Set

$$\overline{\eta}_t := \{i \in \Lambda : -\infty \rightsquigarrow (i,t)\}$$

where $-\infty \rightsquigarrow$ indicates the presence of an open path in the graphical representation started at time $-\infty$.
Then $\overline{\nu} := \mathbb{P}[\overline{\eta}_t \in \bullet\,]$ is the *upper invariant law*.
Definition A measure μ on the set of subsets of Λ is *nontrivial* if $\mu(\{\emptyset\}) = 0$ and *homogeneous* if μ is invariant under translations $A \mapsto iA := \{ij : j \in A\}$.

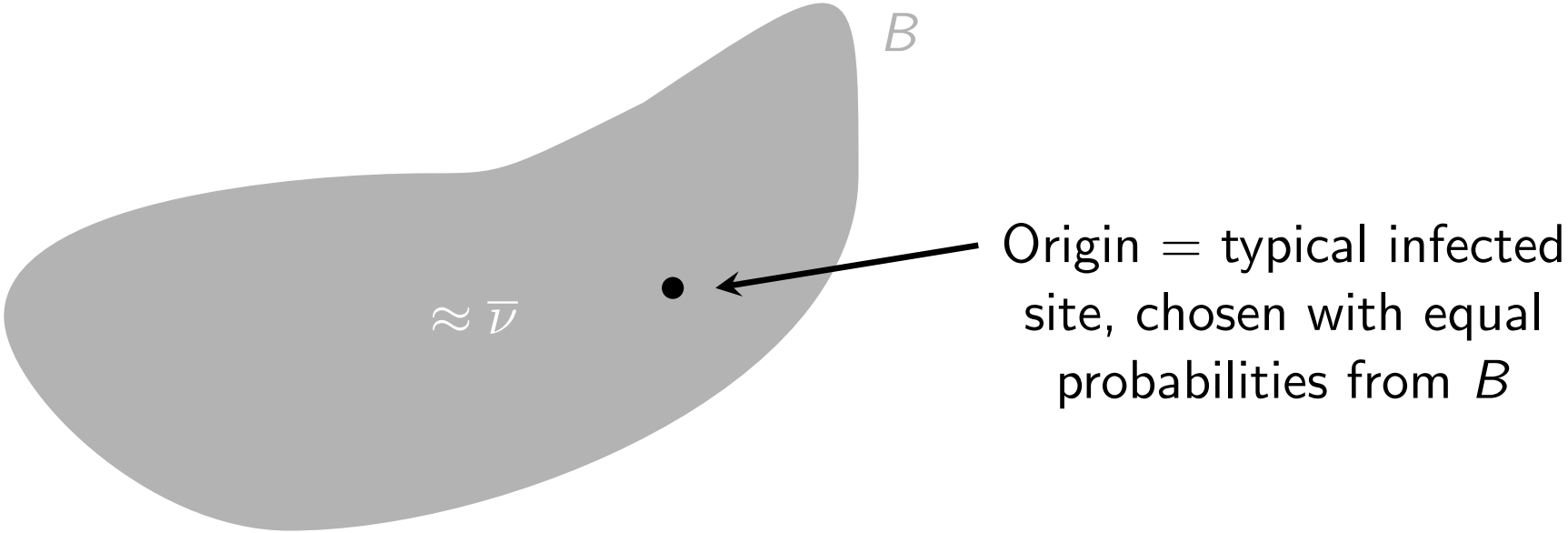
- $\overline{\nu}$ is a homogeneous invariant law.
- $\overline{\nu}$ is the *largest* invariant law (in the stochastic order).
- The only extremal homogeneous inv. laws are $\overline{\nu}$ and $\delta_\emptyset$.
- The process started in a homog. nontriv. law converges to $\overline{\nu}$. (Harris, 1974)

Idea of the proof (continued)

The fact that

$$\tilde{\mathbb{P}}_\lambda^{\{0\}}[\iota^{-1}\eta_\tau^{\{0\}} \in \bullet\,] \underset{\lambda\downarrow 0}{\Longrightarrow} \hat{\nu}$$

means that for small λ, the law $\tilde{\mathbb{P}}_\lambda^{\{0\}}[\iota^{-1}\eta_\tau^{\{0\}} \in \bullet\,]$ describes a random finite set B, that looks something like this:



Since the typical site lies far from the "boundary" of B, this contradicts nonamenability.

Derivative of the exponential growth rate

Theorem (Sturm & S. '11) For each δ in the *subcritical regime* $\delta > \delta_{\mathrm{c}}$, there exists a $\hat{\nu}_\delta$ concentrated on the finite subsets of Λ such that

$$\hat{\mathbb{P}}_t^A[\iota^{-1}\eta_t^A \in \bullet\,] \underset{t\to\infty}{\Longrightarrow} \hat{\nu}_\delta \qquad (A \subset \Lambda \text{ finite}).$$

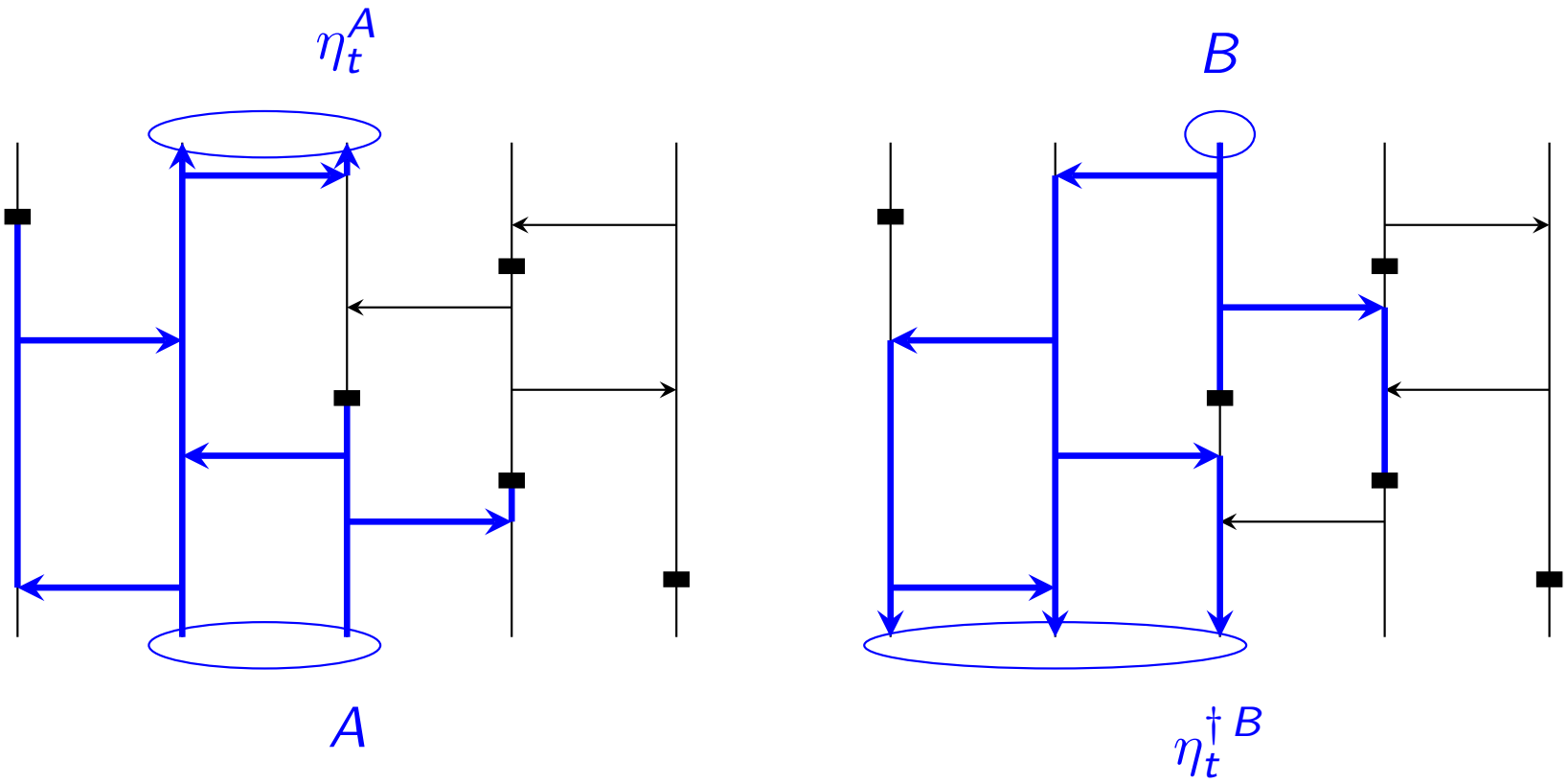
Theorem (Sturm & S. '11) The function $\delta \mapsto r(\Lambda, a, \delta)$ is continuously differentiable on $(\delta_{\mathrm{c}}, \infty)$ and satisfies

$$-\tfrac{\partial}{\partial\delta} r(\Lambda, a, \delta) = \frac{\mathbb{P}\big[\hat{\eta}^\delta \cap \hat{\eta}^{\dagger\,\delta} = \{0\}\big]}{\mathbb{E}\big[|\hat{\eta}^\delta \cap \hat{\eta}^{\dagger\,\delta}|^{-1}\big]} > 0 \qquad \big(\delta \in (\delta_{\mathrm{c}}, \infty)\big)$$

where $\hat{\eta}^\delta$ and $\hat{\eta}^{\dagger\,\delta}$ are independent random variables with laws $\hat{\nu}_\delta$ and $\hat{\nu}_\delta^\dagger$, respectively, and $\hat{\nu}_\delta^\dagger$ is the analogue of $\hat{\nu}_\delta$ for the dual $(\Lambda, a^\dagger, \delta)$-contact process.

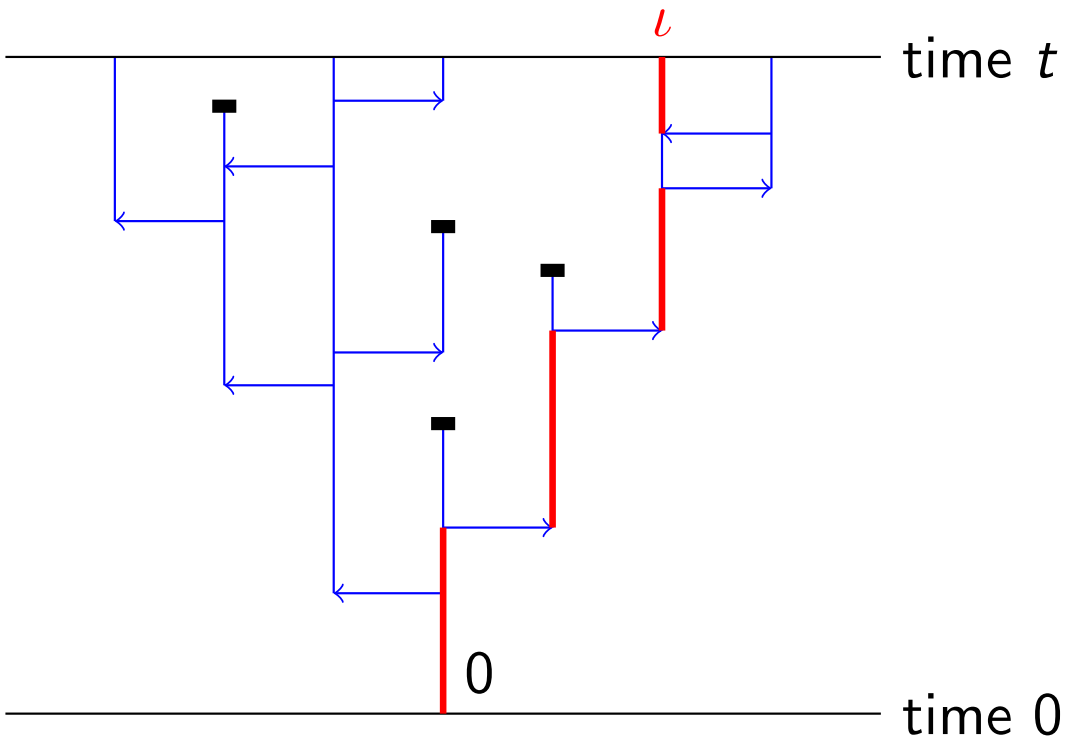
Intermezzo: duality

For the dual process $\eta_t^{\dagger\, B}$ time runs backwards and all arrows are reversed. Need to distinguish a from reversed $a^\dagger(i,j) := a(j,i)$.



$$\{\eta_t^A \cap B \neq \emptyset\} = \{\exists \text{ open path from } A \text{ to } B\} = \{A \cap \eta_t^{\dagger\, B} \neq \emptyset\}.$$

Russo's formula



Say that a point (i, s) is *pivotal* if all paths $(0,0) \rightsquigarrow (\iota, t)$ pass through it. Russo's formula implies:

$$-\frac{\partial}{\partial\delta} r(\Lambda, a, \delta) = \lim_{t\to\infty} \mathbb{E}\Big[\frac{1}{t}\int_0^t 1_{\{\exists \text{ pivotal at } s\}} \mathrm{d}s\Big].$$

Infinite starting measures

Let $\mathcal{P}_+ := \{B \subset \Lambda : B \neq \emptyset\}$ and

$$\mu_t^A := \sum_{i \in \Lambda} \mathbb{P}\big[\eta_t^{\{iA\}} \in \bullet\big]\big|_{\mathcal{P}_+}$$

be the "law" of the process started in the finite configuration A shifted to a "uniformly chosen" position in the lattice. This is an infinite measure but it can still be used to define conditional probabilities. In particular,

$$\mu_t^A\big(\ \cdot\ \big|\{B : 0 \in B\}\big) = \hat{\mathbb{P}}_t^A\big[\iota^{-1}\eta_t^A \in \bullet\big].$$

The advantage of this approach is that it preserves the translation invariance of the problem which is broken when we move the typical site to the origin.

Eigenmeasures

Equip $\mathcal{P} := \{A : A \subset \Lambda\} \cong \{0,1\}^\Lambda$ with the product topology.

Definition An *eigenmeasure* with *eigenvalue* λ is a nonzero, locally finite measure μ on $\mathcal{P}_+$ such that

$$\int \mu(\mathrm{d}A)\mathbb{P}[\eta_t^A \in \bullet]\big|_{\mathcal{P}_+} = e^{\lambda t}\mu \qquad (t \geq 0).$$

Let

$$E_t^A := \mathbb{E}\big[|\eta_t^A|\big] = \mu_t^A(\{B : 0 \in B\}).$$

Conjecture Each (Λ, a, δ)-contact process has an eigenmeasure $\mathring{\nu}$ with eigenvalue r such that

$$\frac{1}{E_t^A}\,\mu_t^A \underset{t\to\infty}{\Longrightarrow} \mathring{\nu}$$

for all finite nonempty A.

The upper invariant law revisited

Claim Set $\tilde{\mu}^A_\lambda := \int \mu^A_t e^{-\lambda t} \mathrm{d}t$ and $\tilde{E}^A_\lambda := \int E^A_t e^{-\lambda t} \mathrm{d}t$. Then there exist $\lambda_n \downarrow r$ such that

$$\frac{1}{\tilde{E}^A_{\lambda_n}} \tilde{\mu}^A_{\lambda_n} \underset{n\to\infty}{\Longrightarrow} \text{ an eigenmeasure with eigenvalue } r.$$

Theorem (S. '09) If $\overline{\nu}$ is nontrivial, then $\overline{\nu}$ is the only nontrivial spatially homogeneous eigenmeasure with eigenvalue 0.

Idea of proof Generalization of the proof that $\overline{\nu}$ is the only nontrivial spatially homogeneous invariant law.

The subcritical case

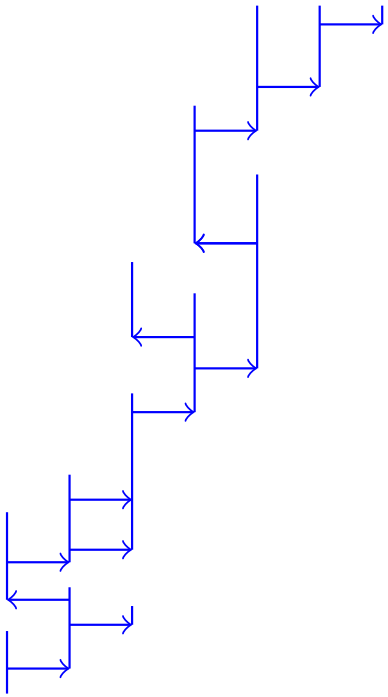
Theorem (Sturm & S. '11) Assume that $r < 0$. Then there exists, up to a multiplicative constant, a unique homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue r. For any nonzero, homogeneous, locally finite measure μ on $\mathcal{P}_+$, one has

$$e^{-rt} \int \mu(\mathrm{d}A)\, \mathbb{P}[\eta_t^A \in \bullet]\Big|_{\mathcal{P}_+(\Lambda)} \underset{t\to\infty}{\Longrightarrow} c\,\mathring{\nu},$$

where $\Rightarrow$ denotes vague convergence and $c > 0$.

Moreover, $\mathring{\nu}$ is concentrated on finite sets. There are no homogeneous eigenmeasures with eigenvalues other than r.

Backward path



The fact that $\mathring{\nu}$ is concentrated on finite sets means that there is almost a single path leading up to the typical particle.

Quasi-invariant laws

Define an equivalence relation on $\mathcal{P}_{\rm fin} := \{A \subset \Lambda : A \text{ is finite}\}$ by

$$A \sim B \quad \text{iff} \quad A = iB \text{ for some } i \in \Lambda,$$

and let $\tilde{A} := \{B : B \sim A\}$ denote the equivalence class containing A. Then $(\tilde{\eta}_t^A)_{t\geq 0}$ is the (Λ, a, δ)-contact process *modulo shifts*.

Let $\mathring{\nu}$ be an eigenmeasure with eigenvalue r that is concentrated on $\mathcal{P}_{\rm fin}$. Then there exists a random finite set Δ such that

$$\mathring{\nu} = \sum_i \mathbb{P}[i\Delta \in \cdot\,],$$

and $\mathbb{P}[\tilde{\Delta} \in \cdot\,]$ is a *quasi-invariant law* of the (Λ, a, δ)-contact process modulo shifts.

The subcritical case revisited

Theorem (Sturm & S. '11) In the subcritical regime $\delta > \delta_c$, the (Λ, a, δ)-contact process modulo shifts has a unique quasi-invariant law, which is the rescaled limit law starting from any finite initial state.

Proof uses Doob h-transform where h is defined in terms of the eigenmeasure of the dual process. In particular, the proof shows that subcritical (Λ, a, δ)-contact processes modulo shifts are *R-positive*. A similar discrete-time result has been derived earlier in [Ferrari, Kesten & Martínez '96].

Pivotal sites revisited

Define the *intersection measure* $\mu \circledast \nu$ of two measures μ and ν on $\mathcal{P}_+$ as the restriction to $\mathcal{P}_+$ of the image of the product measure $\mu \otimes \nu$ under the map $(A, B) \mapsto A \cap B$. Then

$$-\frac{\partial}{\partial\delta} r(\Lambda, a, \delta) = \frac{\int \mathring{\nu} \circledast \mathring{\nu}^\dagger(\mathrm{d}C) 1_{\{C=\{0\}\}}}{\int \mathring{\nu} \circledast \mathring{\nu}^\dagger(\mathrm{d}C) |C|^{-1} 1_{\{0\in C\}}}.$$

If Λ is finite, then we can normalize $\mathring{\nu} \circledast \mathring{\nu}^\dagger$ to a probability measure. Let ζ have this law and choose κ uniformly from ζ. Then

$$-\frac{\partial}{\partial\delta} r(\Lambda, a, \delta) = \frac{\mathbb{P}[\zeta = \{0\}]}{\mathbb{P}[\kappa = 0]} = \frac{|\Lambda|^{-1}\mathbb{P}[|\zeta| = 1]}{|\Lambda|^{-1}} = \mathbb{P}[|\zeta| = 1]$$

is the probability that ζ consists of a single point.

Outlook

Open Problem It can probably happen that $\mathring{\nu} \circledast \mathring{\nu}^\dagger$ is concentrated on finite sets but $\mathring{\nu}, \mathring{\nu}^\dagger$ on their own are not. Prove

$$-\frac{\partial}{\partial\delta} r(\Lambda, a, \delta) = \frac{\int \mathring{\nu} \circledast \mathring{\nu}^\dagger(\mathrm{d}C) 1_{\{C=\{0\}\}}}{\int \mathring{\nu} \circledast \mathring{\nu}^\dagger(\mathrm{d}C) |C|^{-1} 1_{\{0\in C\}}} > 0$$

assuming that there exists a pair of eigenmeasures $\mathring{\nu}, \mathring{\nu}^\dagger$ whose intersection measure $\mathring{\nu} \circledast \mathring{\nu}^\dagger$ is concentrated on finite sets.

In particular, is this always true in the regime $r > 0$?