Cooperative branching and pathwise duality for monotone systems

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Outline

- Pathwise duality for monotone systems

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- Cooperative branching

Probability kernels

For general sets S, T, let $\mathcal{F}(S,T)$ denote the set of all functions $f : S \to T$.

Let S, T be finite sets. A linear operator $A : \mathcal{F}(T,\mathbb{R}) \to \mathcal{F}(S,\mathbb{R})$ is uniquely characterized by its matrix $(A(x,y))_{x\in S,\ y\in T}$ through the formula

$$Af(x) := \sum_{y\in T} A(x,y)f(y) \qquad (x \in S).$$

A linear operator $K : \mathcal{F}(T,\mathbb{R}) \to \mathcal{F}(S,\mathbb{R})$ is a *probability kernel from* S *to* T if and only if

$$K(x,y) \geq 0 \quad \text{and} \quad \sum_{z\in T} K(x,z) = 1 \qquad (x \in S,\ \ y \in T).$$

Random mapping representations

Let K be a probability kernel from S to T.

A *random mapping representation* of K is an $\mathcal{F}(S,T)$-valued random variable M such that

$$K(x,y) = \mathbb{P}[M(x) = y] \qquad (x \in S,\ y \in T).$$

We say that K is *representable* in $\mathcal{G} \subset \mathcal{F}(S,T)$ if M can be chosen so that it takes values in $\mathcal{G}$.

Monotone probability kernels

For partially ordered sets S, T, let $\mathcal{F}_{\rm mon}(S, T)$ be the set of all monotone maps $m : S \to T$, i.e., those for which $x \leq x'$ implies $m(x) \leq m(x')$.

A probability kernel K is called *monotone* if

$$Kf \in \mathcal{F}_{\rm mon}(S,\mathbb{R}) \quad \forall f \in \mathcal{F}_{\rm mon}(T,\mathbb{R}),$$

and *monotonically representable* if K is representable in $\mathcal{F}_{\rm mon}(S, T)$.

Monotonical representability implies monotonicity:

$$\begin{aligned} & f \in \mathcal{F}_{\rm mon}(T,\mathbb{R}) \quad \text{and} \quad x \leq x' \quad \Rightarrow \\ & \quad Kf(x) = \mathbb{E}\big[f\big(M(x)\big)\big] \leq \mathbb{E}\big[f\big(M(x')\big)\big] = Kf(x'). \end{aligned}$$

Monotone probability kernels

J.A. Fill & M. Machida (AOP 2001) (and also D.A. Ross (unpublished)) discovered that the converse does not hold. There are counterexamples with $S = T = \{0, 1\}^2$.

On the positive side, Kamae, Krengel & O'Brien (1977) and Fill & Machida (2001) have shown that:

(Sufficient conditions for monotone representability)
Let S, T be finite partially ordered sets and assume that at least one of the following conditions is satisfied:

(i) *S is totally ordered.*

(ii) *T is totally ordered.*

Then any monotone probability kernel from S to T is monotonically representable.

Stochastic order

In particular, setting $S = \{1, 2\}$, this proves that if μ_1, μ_2 are probability laws on T such that

$$\mu_1 f \leq \mu_2 f \quad \forall f \in \mathcal{F}_{\mathrm{mon}}(T, \mathbb{R}),$$

then it is possible to couple random variables M_1, M_2 with laws μ_1, μ_2 such that $M_1 \leq M_2$.

Markov semigroups

Let S be finite. By definition, a *Markov semigroup* is a collection of probability kernels $(P_t)_{t\geq 0}$ on S such that

$$P_0 = \lim_{t\downarrow 0} P_t = 1 \quad \text{and} \quad P_s P_t = P_{s+t}.$$

Each Markov semigroup is of the form

$$P_t := e^{tG} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n G^n \qquad (t \geq 0),$$

where the *generator* G satisfies

$$G(x,y) \geq 0 \quad (x \neq y) \quad \text{and} \quad \sum_{y\in S} G(x,y) = 0 \quad (x \in S).$$

Representability of semigroups

By definition, G is *representable in* $\mathcal{G} \subset \mathcal{F}(S,S)$ if G can be written as

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x)\big),$$

where $(r_m)_{m \in \mathcal{G}}$ are nonnegative constants (rates).

(Representability of semigroups)
Assume that $\mathcal{G}$ is closed under composition and contains the identity map. Then the following statements are equivalent:

(i) *G can be represented in $\mathcal{G}$.*

(ii) *P_t can be represented in $\mathcal{G}$ for all $t \geq 0$.*

Stochastic flows

Proof of (i)⇒(ii) Let ω be a Poisson subset of $\mathcal{G} \times \mathbb{R}$ with local intensity $r_m \mathrm{d}t$ and let $\omega_{s,u} := \{(m,t) \in \omega : s < t \leq u\}$. Define random maps $(\mathbf{X}_{s,u})_{s \leq u}$ by composing the maps in $\omega_{s,u}$ in the order of the time at which they occur:

$$\mathbf{X}_{s,u} := m_n \circ \cdots \circ m_1$$

$$\text{with} \quad \omega_{s,u} = \big\{(m_1,t_1),\ldots,(m_n,t_n)\big\}, \quad t_1 < \cdots < t_n.$$

The $(\mathbf{X}_{s,u})_{s \leq u}$ form a *stochastic flow:*

$$\mathbf{X}_{s,s} = 1 \quad \text{and} \quad \mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u} \quad (s \leq t \leq u),$$

with independent increments:

$$\mathbf{X}_{t_0,t_1},\ldots,\mathbf{X}_{t_{n-1},t_n} \quad \text{independent for} \quad t_0 < \cdots < t_n.$$

Stochastic flows

If X_0 is independent of ω, then

$$X_t := \mathbf{X}_{0,t}(X_0) \qquad (t \geq 0)$$

defines a Markov process $(X_t)_{t\geq 0}$ with generator G, and

$$P_t(x,y) = \mathbb{P}[\mathbf{X}_{0,t}(x) = y]$$

gives the desired random mapping representation of the Markov semigroup $(P_t)_{t\geq 0}$ with generator G. ∎

We call the Poisson set ω a *graphical representation* of X.

Note: We have defined $\mathbf{X}_{s,t}$ right-continuous in s and t. As a result, $(X_t)_{t\geq 0}$ has right-continuous sample paths.

Duality

Two Markov processes X and Y with state spaces S and T are *dual* with *duality function* $\psi : S \times T \to \mathbb{R}$ iff

$$\mathbb{E}\big[\psi(X_t, Y_0)\big] = \mathbb{E}\big[\psi(X_0, Y_t)\big] \qquad (*).$$

for all deterministic initial states X_0 and Y_0.
If $(*)$ holds for deterministic initial states, then also for random initial states, provided X_t is independent of Y_0 and X_0 is independent of Y_t.

In terms of semigroups $(P_t)_{t\geq 0}, (Q_t)_{t\geq 0}$ and generators G, H, duality says

$$\begin{aligned} & P_t\psi = \psi Q_t^\dagger \qquad (t \geq 0), \\ \Leftrightarrow \quad & G\psi = \psi H^\dagger, \end{aligned}$$

where $A^\dagger$ denotes the adjoint of a matrix A.

Pathwise duality

Two maps $m : S \to S$ and $\hat{m} : T \to T$ are *dual* w.r.t. the duality function ψ iff

$$\psi\big(m(x), y\big) = \psi\big(x, \hat{m}(y)\big) \qquad (x \in S,\ y \in T).$$

Two stochastic flows $(\mathbf{X}_{s,t})_{s \leq t}$ and $(\mathbf{Y}_{s,t})_{s \leq t}$ with independent increments are *dual* w.r.t. the duality function ψ if:

(i) A.s. $\forall\ s \leq t$, the maps $\mathbf{X}^{-}_{s,t}$ and $\mathbf{Y}_{-t,-s}$ are dual w.r.t. ψ.

(ii) $(\mathbf{X}^{-}_{t_0,t_1}, \mathbf{Y}_{-t_1,-t_0}), \ldots, (\mathbf{X}^{-}_{t_{n-1},t_n}, \mathbf{Y}_{-t_n,-t_{n-1}})$ are independent for $t_0 < \cdots < t_n$.

To get a sensible definition, we have to take the left-continuous modification $\mathbf{X}^{-}_{s,t} := \mathbf{X}_{s-,t-}$ (if $\mathbf{Y}_{s,t}$ is right-continuous as usual).

Pathwise duality

Two Markov processes X and Y are *pathwise dual* if they can be constructed from stochastic flows that are dual.
Pathwise duality implies duality:

$$\begin{aligned}\mathbb{E}\big[\psi(X_t, Y_0)\big] &= \mathbb{E}\big[\psi\big(\mathbf{X}^-_{0,t}(X_0), Y_0\big)\big]\\ &= \mathbb{E}\big[\psi\big(X_0, \mathbf{Y}_{-t,0}(Y_0)\big)\big] = \mathbb{E}\big[\psi(X_0, Y_t)\big].\end{aligned}$$

Even though pathwise duality is much stronger than duality, lots of well-known dualities can be realized as pathwise dualities.

Pathwise duality

(Pathwise duality) *If the generators G and H of X and Y have random mapping representations of the form*

$$
\begin{aligned}
Gf(x) &= \sum_{m\in\mathcal{G}} r_m\big(f(m(x)) - f(x)\big),\\
Hf(x) &= \sum_{m\in\mathcal{G}} r_m\big(f(\hat{m}(y)) - f(y)\big),
\end{aligned}
$$

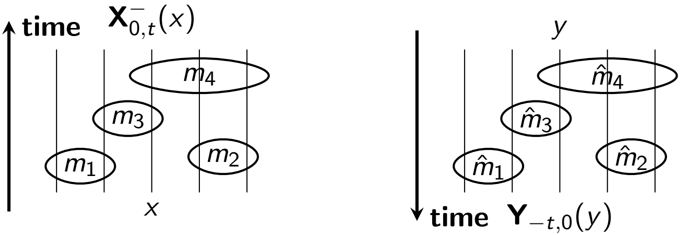
where each map $\hat{m}$ is a dual of m, then X and Y are pathwise dual.

Proof Given a graphical representation ω of X, we can define a graphical representation $\hat{\omega}$ for Y by

$$\hat{\omega} := \big\{(\hat{m}, -t) : (m, t) \in \omega\big\}.$$

Then the stochastic flows $(\mathbf{X}_{s,t})_{s\leq t}$ and $(\mathbf{Y}_{s,t})_{s\leq t}$ associated with ω and $\hat{\omega}$ are dual. ∎

Pathwise duality



In this picture

$$\mathbf{X}^-_{0,t} = m_4 \circ \cdots \circ m_1 \quad \text{is dual to} \quad \mathbf{Y}_{-t,0} = \hat m_1 \circ \cdots \circ \hat m_4.$$

Invariant subspaces

Let $\mathcal{P}(S)$ be the set of all subsets of S.
Let $m^{-1} : \mathcal{P}(S) \to \mathcal{P}(S)$ denote the *inverse image map*

$$m^{-1}(A) := \{x \in S : m(x) \in A\}.$$

Observation m^{-1} is dual to m w.r.t. to the duality function

$$\psi(x, A) := 1_{\{x \in A\}}.$$

Consequence Each Markov process X with state space S (and given random mapping representation) has a pathwise dual Y with state space $\mathcal{P}(S)$ and generator

$$Hf(A) := \sum_{m \in \mathcal{G}} r_m \big(f(m^{-1}(A)) - f(A) \big)$$

In practice, this dual is not very useful since the space $\mathcal{P}(S)$ is very big. *Useful* duals are associated with *invariant subspaces* of $\mathcal{P}(S)$.

A bit of order theory

Let S be a finite partially ordered space. The "upset" and "downset" of $A \subset S$ are defined as

$$A^{\uparrow} := \{x \in S : x \geq a \text{ for some } a \in A\},$$
$$A^{\downarrow} := \{x \in S : x \leq a \text{ for some } a \in A\}.$$

A set $A \subset S$ is *increasing* (resp. *decreasing*) if $A^{\uparrow} = A$ (resp. $A^{\downarrow} = A$) and a *principal filter* (resp. *principal ideal*) if A is of the form $A = \{a\}^{\uparrow}$ (resp. $A = \{a\}^{\downarrow}$) for some $a \in S$. We let

$$\mathcal{P}_{\rm inc}(S) := \{A \subset S : A \text{ is increasing}\},$$
$$\mathcal{P}_{\rm !inc}(S) := \{A \subset S : A \text{ is a principal filter}\},$$
$$\mathcal{P}_{\rm dec}(S) := \{A \subset S : A \text{ is decreasing}\},$$
$$\mathcal{P}_{\rm !dec}(S) := \{A \subset S : A \text{ is a principal ideal}\}.$$

A bit of order theory

A partially ordered set S is *bounded from below* resp. *above* if there exists an element 0 resp. 1 such that

$$0 \leq x \quad (x \in S) \quad \text{resp. } x \leq 1 \quad (x \in S).$$

A *lattice* is a partially ordered set such that for every $x, y \in S$ there exist $x \vee y \in S$ and $x \wedge y \in S$ called the *supremum* or *join* and *infimum* or *meet* of x and y, respectively, such that

$$\{x\}^{\uparrow} \cap \{y\}^{\uparrow} = \{x \vee y\}^{\uparrow} \quad \text{and} \quad \{x\}^{\downarrow} \cap \{y\}^{\downarrow} = \{x \wedge y\}^{\downarrow}.$$

Finite lattices are bounded from below and above.

A map $m : S \to S$ is *additive* if

$$m(0) = 0 \quad \text{and} \quad m(x \vee y) = m(x) \vee m(y) \qquad (x, y \in S).$$

Monotone and additive maps

(Monotone and additive maps)
(i) *Let S and T be partially ordered sets and let $m : S \to T$ be a map. Then m is monotone if and only if*

$$m^{-1}(A) \in \mathcal{P}_{\mathrm{dec}}(S) \text{ for all } A \in \mathcal{P}_{\mathrm{dec}}(T).$$

(ii) *If S and T are finite lattices, then m is additive if and only if*

$$m^{-1}(A) \in \mathcal{P}_{!\mathrm{dec}}(S) \text{ for all } A \in \mathcal{P}_{!\mathrm{dec}}(S).$$

Dual spaces

Let S be a partially ordered set. A *dual* of S is a partially ordered set S' together with a bijection $S \ni x \mapsto x' \in S'$ such that

$$x \leq y \quad \text{if and only if} \quad x' \geq y'.$$

Example 1: For any partially ordered set S, we may take $S' := S$ but equipped with the reversed order, and $x \mapsto x'$ the identity map.

Example 2: If Λ is a set and $S \subset \mathcal{P}(\Lambda)$ is a set of subsets of Λ, equipped with the partial order of inclusion, then we may take for $x' := \Lambda \backslash x$ the complement of x and $S' := \{x' : x \in S\}$.

Additive systems duality

Let X be a Markov process in a finite lattice S.
Assume that the generator of X is representable in additive maps.
Then X has a pathwise dual that takes values in the invariant subspace $\mathcal{P}_{\mathrm{!dec}}(S) \subset \mathcal{P}(S)$.
A convenient way to encode an element $A \in \mathcal{P}_{\mathrm{!dec}}(S)$ is to write

$$A = \{y'\}^{\downarrow} \quad \text{with} \quad y \in S'.$$

Identifying $\mathcal{P}_{\mathrm{!dec}}(S) \cong S'$, the duality function becomes

$$\psi(x, y) = 1_{\{x \leq y'\}} = 1_{\{y \leq x'\}} \qquad (x \in S, \; y \in S').$$

(Additive duality) *A map $m : S \to S$ has a dual $m' : S' \to S'$ w.r.t. ψ if and only if m is additive. The dual map m' is unique and also an additive map.*

Siegmund's duality

Let $S = \{0, \ldots, n\}$ be totally ordered and let $S' := S$ equipped with the reversed order.
A map $m : S \to S$ is additive iff m is monotone and $m(0) = 0$.
Each such map has a dual $m' : S' \to S'$ that is monotone and satisfies $m'(n) = n$.

(Siegmund's dual) *Let X be a monotone Markov process in S such that 0 is a trap. Then X has a dual Y w.r.t. to the duality function $\psi(x, y) := 1_{\{x \leq y\}}$. The dual process is also monotone and has n as a trap. Moreover, the duality can be realized in a pathwise way.*

Additive particle systems

Let $S = \mathcal{P}(\Lambda)$ with Λ a finite set, and let $x \mapsto x' \in S' := \mathcal{P}(\Lambda)$ denote the complement map $x' := \Lambda \backslash x$.

(Additive particle systems) *Let X be a Markov process in S whose generator can be represented in additive maps. Then X has a pathwise dual Y w.r.t. to the duality function $\psi(x, y) := 1_{\{x \cap y = \emptyset\}}$, and Y is also an additively representable Markov process.*

Examples: Voter model, contact process, exclusion process, systems of coalescing random walks.

Krone's duality

Steve Krone [AAP 1999] has studied a two-stage contact process, with state space of the form $S = \{0, 1, 2\}^\Lambda$.
He interprets $x(i) = 0, 1$, or 2 as an empty site, young, or adult organism, and defines maps

grow up	a_i	$\cdots 1 \cdots\cdot \mapsto \cdots 2 \cdots\cdot$
give birth	b_{ij}	$\cdots 20 \cdots \mapsto \cdots 21 \cdots$
young dies	c_i	$\cdots 1 \cdots\cdot \mapsto \cdots 0 \cdots\cdot$
death	d_i	$\cdots 1 \cdots\cdot \mapsto \cdots 0 \cdots\cdot$
or		$\cdots 2 \cdots\cdot \mapsto \cdots 0 \cdots\cdot$
grow younger	e_i	$\cdots 2 \cdots\cdot \mapsto \cdots 1 \cdots\cdot$

where in all cases not mentioned, the maps have no effect.

Krone's duality

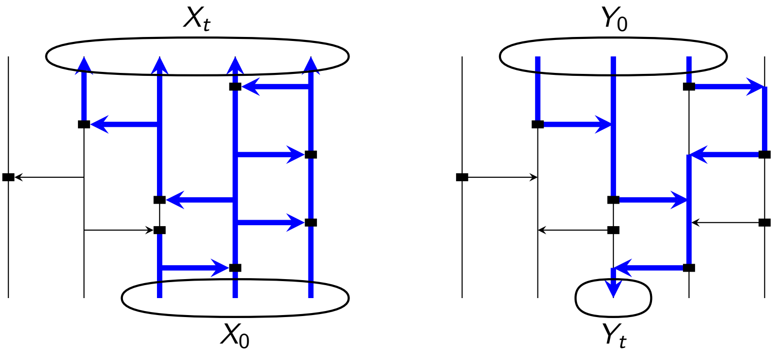
We set $S' := S$ and define $S \ni x \mapsto x' \in S'$ by $x'(i) := 2 - x(i)$. Then the duality function becomes

$$\psi(x,y) = 1_{\{x \leq y'\}} = 1_{\{x(i) + y(i) \leq 2 \ \forall \ i \in \Lambda\}}.$$

(Krone's dual) *The maps $a_i, b_{ij}, c_i, d_i, e_i$ are all additive and their duals are given by*

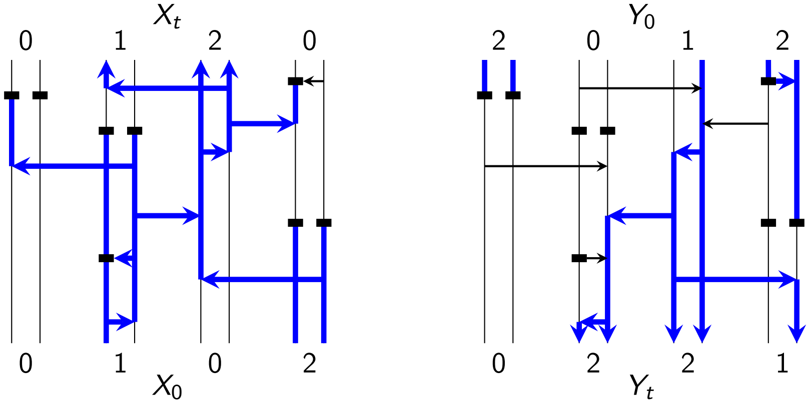
$$a'_i = a_i, \quad b'_{ij} = b_{ji}, \quad c'_i = e_i, \quad d'_i = d_i, \quad e'_i = c_i.$$

Percolation representations



Additive particle systems and their duals can be constructed in terms of open paths. In this example, X is a voter model and Y are coalescing random walks.

Percolation representations



We can also give a percolation representation of Krone's duality.

Percolation representations

By definition, a lattice S is *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad (x, y, z \in S).$$

If Λ is a partially ordered set, then $S := \mathcal{P}_{\rm dec}(\Lambda)$ with the order of set inclusion is a distributive lattice. *Birkhoff's representation theorem* says that every distributive lattice is of this form.

(Percolation representation) *An additive Markov process taking values in* $\mathcal{P}_{\rm dec}(\Lambda)$ *has a percolation representation together with its dual, which takes values in* $S' = \mathcal{P}_{\rm inc}(\Lambda)$, *with the duality function* $\psi(x, y) = 1_{\{x \cap y \neq \emptyset\}}$.

If Λ is equipped with the trivial order $x \not\leq y$ for all $x \neq y$, then $\mathcal{P}_{\rm dec}(\Lambda) = \mathcal{P}(\Lambda) = \mathcal{P}_{\rm inc}(\Lambda)$.

In Krone's example, $\Lambda = \{1, 2\}^{\Delta}$ with the product order.

Monotone systems duality

Let S be a finite lattice and let $m : S \to S$ be monotone. Then m is automatically *superadditive:*

$$m(x \vee y) \geq m(x) \vee m(y)$$

For monotone maps that are not additive, this inequality is strict. A good example is the *cooperative branching map*

$$\begin{aligned} 110 &\mapsto 111, \\ 100 &\mapsto 100, \\ 010 &\mapsto 010, \end{aligned}$$

which can be interpreted as two individuals cooperating to give birth to a third one.

Monotone systems duality

Let X be a Markov process in a finite partially ordered set S. Assume that the generator of X is representable in monotone maps.
Let $(\mathbf{X}_{s,t})_{s\leq t}$ be the associated stochastic flow. The maps $\mathbf{X}_{s,t}$ are now monotone, but in general not additive. It follows that

$$\mathbf{X}_{s,t}^{-1}(A) \in \mathcal{P}_{\rm dec}(S) \text{ for all } A \in \mathcal{P}_{\rm dec}(S),$$
$$\mathbf{X}_{s,t}^{-1}(A) \in \mathcal{P}_{\rm inc}(S) \text{ for all } A \in \mathcal{P}_{\rm inc}(S).$$

Setting $\mathbf{Z}_{s,t}(A) := \mathbf{X}_{-t,-s}^{-1}(A)$ defines a dual stochastic flow with values in $\mathcal{P}_{\rm dec}(S)$ or $\mathcal{P}_{\rm inc}(S)$. This yields two distinct pathwise duals that are related by taking complements.

If X is not additive, then $\mathbf{Z}_{s,t}$ sometimes maps elements of $\mathcal{P}_{\rm !dec}(S)$ into sets that have more than one maximal element.

Monotone systems duality

A convenient way to encode an element $A \in \mathcal{P}_{\rm dec}(S)$ is to write down its maximal elements. By definition, $x \in A$ is a *maximal element* of A if

$$w \in A,\ w \geq x \quad \text{implies} \quad w = x.$$

Setting

$$Y_t := \{y \in S' : y' \text{ is a maximal element of } \mathbf{Z}_{0,t}(A)\} \qquad (t \geq 0)$$

yields a Markov process taking values in the finite subsets of S' that is dual to $(X_t)_{t\geq 0}$ w.r.t. the duality function

$$\psi(x, Y) = 1_{\{x \leq y' \text{ for some } y \in Y\}}.$$

In the special case that $(X_t)_{t\geq 0}$ is additive, $(Y_t)_{t\geq 0}$ has the property that

$$Y_0 = \{y_0\} \quad \text{implies} \quad Y_t = \{y_t\} \quad (t \geq 0),$$

where $(y_t)_{t\geq 0}$ is the additive dual of $(X_t)_{t\geq 0}$.

Monotone systems duality

Alternatively, encode *increasing* sets by their *minimal* elements. Let Λ be countable and equip $S = \{0,1\}^\Lambda$ with the product order and topology. For each $Y \subset S$, let

$$Y^\uparrow := \{z \in S : z \geq y \text{ for some } y \in Y\},$$
$$Y^\circ := \{y \in Y : y \text{ is a minimal element of } Y\}.$$

It is easy to see that $(Y^\uparrow)^\uparrow = Y^\uparrow$ and $(Y^\circ)^\circ = Y^\circ$. Set $S_{\rm fin} := \{y \in S : |y| < \infty\}$ with $|y| := \sum_i y(i)$ and

$$\mathcal{I}(\Lambda) := \{Y : Y \text{ is open and } Y^\uparrow = Y\},$$
$$\mathcal{H}(\Lambda) := \{Y : Y \subset S_{\rm fin} \text{ and } Y^\circ = Y\}.$$

(Encoding open increasing sets) *The map $Y \mapsto Y^\uparrow$ is a bijection from $\mathcal{H}(\Lambda)$ to $\mathcal{I}(\Lambda)$, and $Y \mapsto Y^\circ$ is its inverse.*

Monotone systems duality

Equip $\mathcal{I}(\Lambda)$ with a topology such that $Y^{(n)} \to Y$ if and only if their complements converge in the Hausdorff topology. Then each monotonely representable interacting particle system with values in $S = \{0,1\}^\Lambda$ has a pathwise dual with values in $\mathcal{I}(\Lambda)$, or alternatively $\mathcal{H}(\Lambda)$.

If we take $\mathcal{H}(\Lambda)$ as the state space of the dual, then the duality function becomes

$$\psi(x, Y) = 1_{\{x \geq y \text{ for some } y \in Y\}} \qquad \big(x \in S,\ \ Y \in \mathcal{H}(\Lambda)\big).$$

Monotone systems duality

In the special case that $(X_t)_{t\geq 0}$ is additive, the $\mathcal{H}(\Lambda)$-valued dual process preserves the subspace of all Y_t of the form

$$Y_t = \{\delta_i : i \in \Delta_t\} \quad \text{with} \quad \Delta_t \subset \Lambda.$$

Now the process $(\Delta_t)_{t\geq 0}$ is the additive dual of $(X_t)_{t\geq 0}$.

In general, Y_t is a set whose elements $y \in \{0,1\}^\Lambda$ satisfy $|y| := \sum_{i\in\Lambda} y(i) < \infty$.
For example, if $Y_0 = \delta_k$ and Y_t contains an element $Y_t \ni y = \delta_i + \delta_j$, this may express that k contains a particle at time 0 provided both its parents i and j are alive at time $-t$.

Monotone systems duality

By monotonicity, the process X has an *upper invariant law*

$$\mathbb{P}^{\underline{1}}[X_t \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}.$$

By definition, Y *survives* if

$$\mathbb{P}^{\{\delta_i\}}[Y_t \neq \emptyset \ \forall t \geq 0] > 0$$

for some $i \in \Lambda$.

(Nontrivial upper invariant law) One has $\overline{\nu} \neq \delta_{\underline{0}}$ if and only if Y survives. The law $\overline{\nu}$ is uniquely characterized by

$$\mathbb{E}\big[\psi(\overline{X}, \{y\})\big] = \mathbb{P}^{\{y\}}\big[Y_t \neq \emptyset \ \forall t \geq 0\big].$$

where $\overline{X}$ denotes a r.v. with law $\overline{\nu}$.

Proof

$$\mathbb{E}^{\underline{1}}\big[\psi(X_t,\{y\})\big] = \mathbb{E}^{\{y\}}\big[\psi(\underline{1},Y_t)\big] = \mathbb{E}^{\{y\}}\big[\exists y \in Y_t \text{ s.t. } \underline{1} \geq y\big]$$
$$= \mathbb{E}^{\{y\}}\big[Y_t \neq \emptyset\big] \underset{t\to\infty}{\longrightarrow} \mathbb{P}^{\{y\}}\big[Y_t \neq \emptyset \ \forall t \geq 0\big].$$

∎

Monotone systems duality

We equip $\mathcal{H}(\Lambda)$ with a partial order by setting

$$Y \leq Z \quad \text{iff} \quad Y^{\uparrow} \subset Z^{\uparrow}.$$

The largest element of $\mathcal{H}(\Lambda)$ is

$$\{\underline{0}\} \quad \text{with} \quad \{\underline{0}\}^{\uparrow} = \{0,1\}^{\Lambda}.$$

The second largest element of $\mathcal{H}(\Lambda)$ is

$$Y_* := \{\delta_i : i \in \Lambda\} \quad \text{with} \quad Y_*^{\uparrow} = \{x : x \neq \underline{0}\}.$$

If $\underline{0}$ is a trap for $(X_t)_{t\geq 0}$, then $Y_0 \neq \{\underline{0}\}$ implies $Y_t \neq \{\underline{0}\}$ $(t \geq 0)$. Now, by monotonicity,

$$\mathbb{P}^{Y_*}[Y_t \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \overline{\mu}.$$

We call $\overline{\mu}$ the *upper invariant law* of $(Y_t)_{t\geq 0}$.

Monotone systems duality

We say that X *survives* if

$$\exists i \in \Lambda \quad \text{s.t.} \quad \mathbb{P}^{\delta_i}\big[X_t \neq \underline{0}\ \forall t \geq 0\big] > 0.$$

(Nontrivial upper invariant law) One has $\overline{\mu} \neq \delta_\emptyset$ if and only if X survives. The law $\overline{\mu}$ is uniquely characterized by

$$\mathbb{E}\Big[\prod_{k=1}^{n} \psi(x_k, \overline{Y})\Big] = \mathbb{P}\big[\mathbf{X}_{0,t}(x_k) \neq \underline{0}\ \forall t \geq 0,\ \ k = 1, \ldots, n\big].$$

where $\overline{Y}$ denotes a r.v. with law $\overline{\mu}$.

Proof

$$\begin{aligned}\mathbb{E}^{Y_*}\Big[\prod_{k=1}^{n} \psi(x_k, Y_t)\Big] &= \mathbb{E}\Big[\prod_{k=1}^{n} \psi(\mathbf{X}_{0,t}(x_k), Y_*)\Big] \\ &= \mathbb{P}\big[\mathbf{X}_{0,t}(x_k) \neq \underline{0}\ \forall k = 1, \ldots, n\big].\end{aligned}$$

The sexual reproduction process

DeMasi, Ferrari & Lebowitz [JSP 1986], *C. Noble* [AOP 1992], *R. Durrett* [JAP 1992], and *C. Neuhauser and S.W. Pacala* [AAP 1999] consider a *sexual reproduction process* $(X_t)_{t\geq 0}$ taking values in the space of all configurations $\ldots 101101001001\ldots$, that evolves as:

$$\begin{array}{rrcllr}
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\text{(death)} & 1 & \mapsto & 0 & \text{with rate} & 1,
\end{array}$$

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Interpretation:

- 'Sexual' reproduction.

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Interpretation:

- 'Sexual' reproduction.
- Competition for limited space.

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\text{(death)} & 1 & \mapsto & 0 & \text{with rate} & 1,
\end{array}$$

Interpretation:

- ‘Sexual’ reproduction.
- Competition for limited space.
- Death.

The sexual reproduction process

Consider the maps

$$\begin{aligned}
\mathtt{death}_i(x) &:= x - 1_{\{x(i)=1\}}\delta_i,\\
\mathtt{coop}_{ijk}(x) &:= \big[x + 1_{\{x(i)=1,\ x(j)=1\}}\delta_k\big] \wedge 1,\\
\mathtt{bran}_{kij}(x) &:= \big[x + 1_{\{x(k)=1\}}(\delta_i + \delta_j - \delta_k)\big] \wedge 1,
\end{aligned}$$

i.e.,

$$\begin{array}{ll}
\text{death} & \cdots 1 \cdots \mapsto \cdots 0 \cdots\ , \\
\text{coop} & \cdots 110 \cdots \mapsto \cdots 111 \cdots\ , \\
\text{bran} & \cdots 001 \cdots \mapsto \cdots 110 \cdots\ , \\
 & \cdots 011 \cdots \mapsto \cdots 110 \cdots\ , \\
 & \cdots 101 \cdots \mapsto \cdots 110 \cdots\ , \ldots \text{etc.}
\end{array}$$

The sexual reproduction process

Then the maps

$$\mathtt{death}^\bullet_i(Y) := \{y \in Y : y(i) \neq 1\},$$
$$\mathtt{coop}^\bullet_{ijk}(x) := Y \cup \mathtt{bran}_{kij}(Y)$$

are dual to $\mathtt{death}_i$ and $\mathtt{coop}_{ijk}$ w.r.t. the duality function

$$\psi(x, Y) = 1_{\{x \geq y \text{ for some } y \in Y\}} \qquad \big(x \in S,\ \ Y \in \mathcal{H}(\Lambda)\big).$$

If $(Y_t)_{t \geq 0}$ is the Markov process with generator

$$G_\bullet(Y) := \sum_i \big\{f\big(\mathtt{death}^\bullet_i(Y)\big) - f\big(Y\big)\big\}$$
$$\tfrac{1}{2}\lambda \sum_{ijk} \big\{f\big(\mathtt{coop}^\bullet_{ijk}(Y)\big) - f\big(Y\big)\big\},$$

then $(Y^\circ_t)_{t \geq 0}$ is the $\mathcal{H}(\Lambda)$-valued dual process w.r.t. the duality function ψ.

Gray's duality

Gray [AOP 1986] introduced a dual for monotone spin systems that is essentially the Markov process $(Y_t)_{t\geq 0}$ of the previous slide, started in an initial state of the form $Y_0 = \{y\}$ for some $y \in S$.

In particular, the associated process $(Y^\circ_t)_{t\geq 0}$ with $Y^\circ := \{y \in Y : y \text{ is a minimal element of } Y\}$ is our $\mathcal{H}(\Lambda)$-valued dual.

The sexual reproduction process

Recall that:

- $(X_t)_{t\geq 0}$ has a nontrivial invariant law iff $(Y^\circ_t)_{t\geq 0}$ survives.
- $(Y^\circ_t)_{t\geq 0}$ has a nontrivial invariant law iff $(X_t)_{t\geq 0}$ survives.

Let

$$\lambda_{\rm c} := \inf\{\lambda \geq 0 : (X_t)_{t\geq 0} \text{ survives}\},$$
$$\lambda'_{\rm c} := \inf\{\lambda \geq 0 : (X_t)_{t\geq 0} \text{ has a nontrivial invariant law}\}$$

Conjecture $\lambda'_{\rm c} \leq \lambda_{\rm c}$ with equality on $\mathbb{Z}^d$.

Theorem On trees of sufficiently high degree, $\lambda'_{\rm c} < \lambda_{\rm c}$.

Proof $(Y^\circ_t)_{t\geq 0}$ survives while $(X_t)_{t\geq 0}$ dies out.

The sexual reproduction process

It seems quite plausible that

$$X \text{ survives} \quad \Rightarrow \quad \overline{\nu} \text{ nontrivial.}$$

(*Warning:* Not true for coalescing random walks.)
However, it is not clear why this should hold for Y since it may happen that Y survives but

$$\inf\{|y| : y \in Y_t\} \underset{t\to\infty}{\longrightarrow} \infty.$$

In this case $Y_t^{\uparrow} \downarrow \emptyset$ as $t \to \infty$.

Durrett and Gray [1985] gave an example of a model with cooperative branching on $\mathbb{Z}^2$ that cannot escape a bounding rectangle and hence does not survive, yet has a nontrivial upper invariant law.

Fast stirring

Let $(X_t)_{t\geq 0}$ evolve as:

$$\begin{array}{rrcllc}
\text{(coop. bra.)} & 110 & \mapsto & 111 & \text{with rate} & \tfrac{1}{2}\lambda, \\
\text{(coop. bra.)} & 011 & \mapsto & 111 & \text{with rate} & \tfrac{1}{2}\lambda, \\
\text{(death)} & 1 & \mapsto & 0 & \text{with rate} & 1, \\
\text{(stirring)} & 10 & \mapsto & 01 & \text{with rate} & \varepsilon^{-1}, \\
\text{(stirring)} & 01 & \mapsto & 10 & \text{with rate} & \varepsilon^{-1}.
\end{array}$$

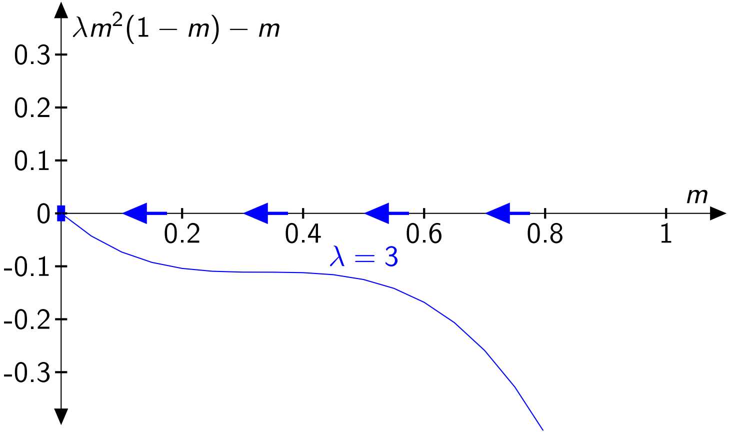
Set

$$m_\varepsilon(x,t) := \mathbb{P}[X_{\varepsilon^{-2}t}(\lfloor \varepsilon^{-1}x \rfloor) \qquad (x \in R,\ t \geq 0).$$

[DeMasi, Ferrari & Lebowitz '86] In the fast stirring limit $\varepsilon \downarrow 0$, the particle density m_ε converges to a solution of

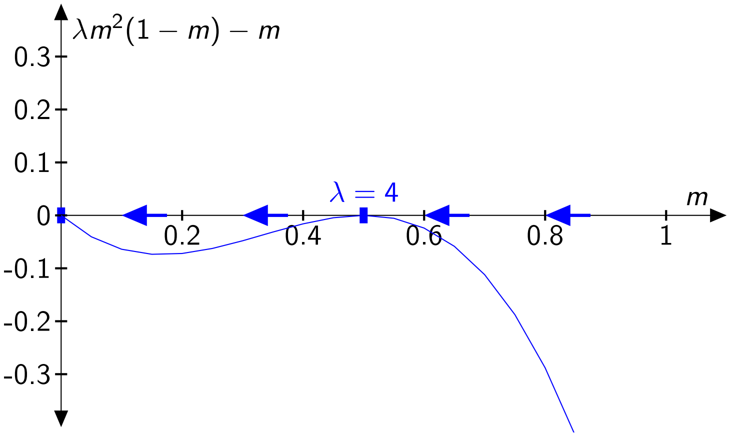
$$\frac{\partial}{\partial t} m = \frac{\partial^2}{\partial x^2} m + \lambda m^2 (1-m) - m.$$

Constant densities



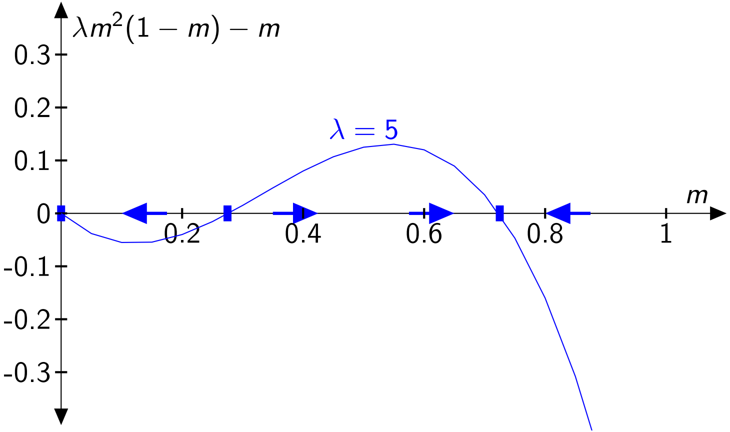
For $\lambda < 4$, the equation $\frac{\partial}{\partial t} m = \lambda m^2(1-m) - m$ has only one stable fixed point $m = 0$.

Constant densities



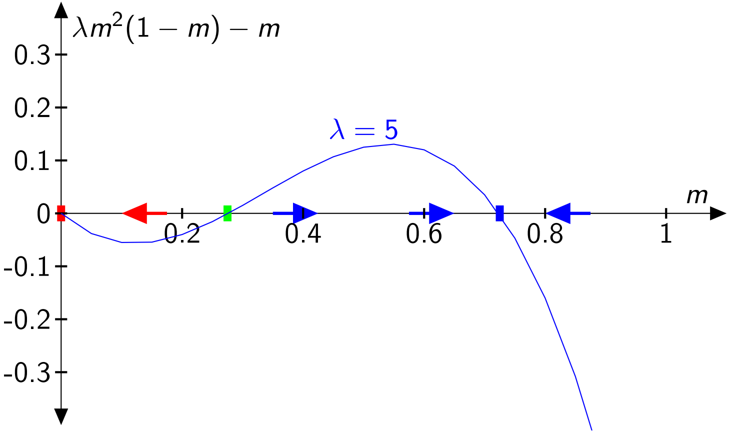
For $\lambda = 4$, the equation $\frac{\partial}{\partial t}m = \lambda m^2(1-m) - m$ obtains a second fixed point at $m = 0.5$.

Constant densities



For $\lambda > 4$, the equation $\frac{\partial}{\partial t} m = \lambda m^2(1-m) - m$ has one unstable and two stable fixed points.

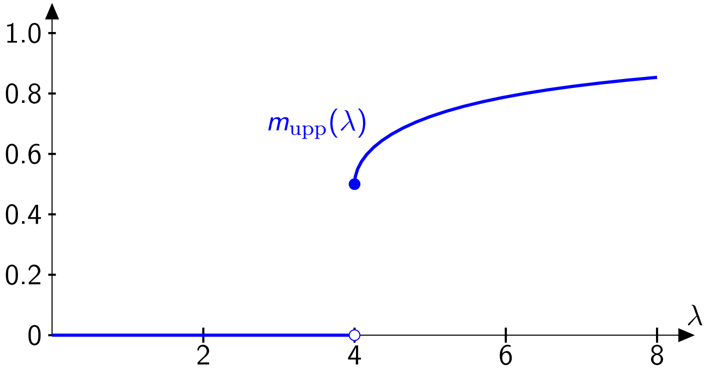
Constant densities



The unstable fixed point represents a critical density below which the population is doomed to die out.

Constant densities

Starting with density $m(x,0) \equiv 1$, the hydrodynamic limit converges to the upper fixed point $\lim_{t\to\infty} m(x,t) = m_{\rm upp}$.



We observe a first-order phase transition.

The stochastic model

Define

- The process *survives* if $\mathbb{P}^x[X_t \neq \underline{0} \ \forall t \geq 0] > 0$ for some, and hence for all initial states with $1 < |x| < \infty$.
- The process is *stable* if the upper invariant law is nontrivial.

Monotonicity implies that there exist $\lambda_{\rm c}, \lambda'_{\rm c}$ such that

- The process survives for $\lambda > \lambda_{\rm c}$ and dies out for $\lambda < \lambda_{\rm c}$.
- The process is stable for $\lambda > \lambda'_{\rm c}$ and unstable for $\lambda < \lambda'_{\rm c}$.

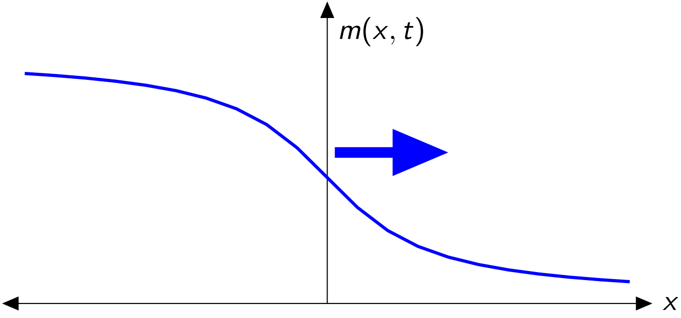
Open problem: Prove that $\lambda_{\rm c} = \lambda'_{\rm c}$.

[Noble '92] $2 \leq \lambda'_{\rm c}(\varepsilon)$ for all $\varepsilon > 0$ and $\limsup_{\varepsilon \downarrow 0} \lambda'_{\rm c}(\varepsilon) \leq 4.5$.

Conjecture: $\lim_{\varepsilon \downarrow 0} \lambda'_{\rm c}(\varepsilon) = 4.5$.

Travelling waves

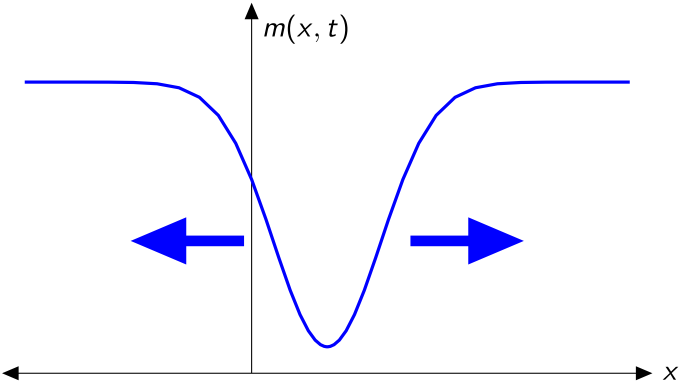
For $\lambda > 4$, the equation $\frac{\partial}{\partial t} m = \frac{\partial^2}{\partial x^2} m + \lambda m^2(1 - m) - m$ has travelling wave solutions.



[DeMasi, Ianiro, Pellegrinotti, & Presutti '84] The propagation speed is positive for $\lambda > 4.5$, and negative for $4 < \lambda < 4.5$.

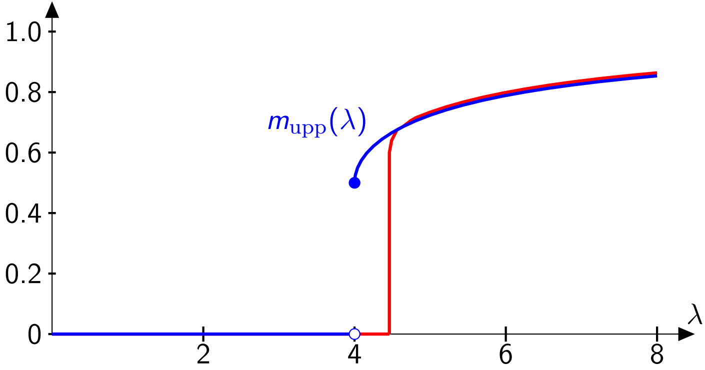
Metastability

For $4 < \lambda < 4.5$ and ε small, rare random events bring the local particle density below a critical value.



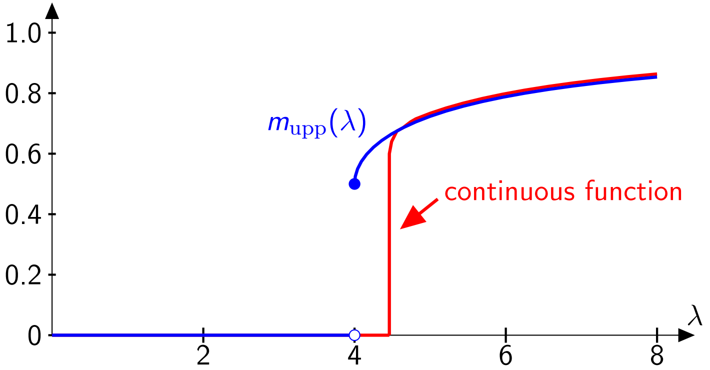
The interval of low population density spreads in both directions.

The upper invariant law



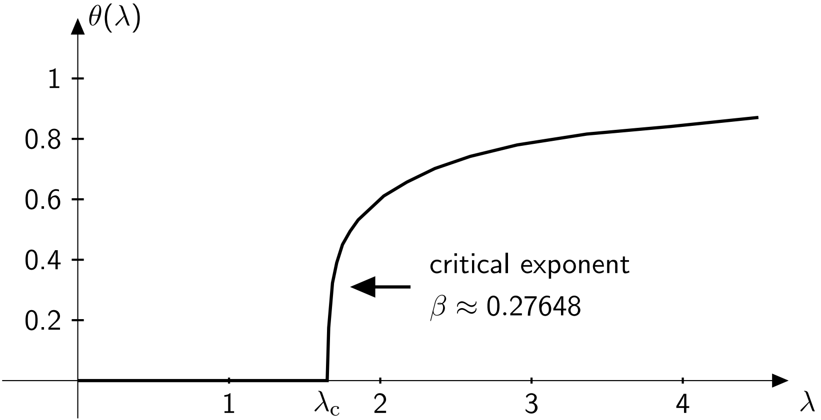
[Noble '92] For small $\varepsilon > 0$, the density of the upper invariant law is at least $m_{\mathrm{upp}}(\lambda)$ for $\lambda > 4.5$ and close to zero for $\lambda < 4.5$.

The upper invariant law



Conjecture For fixed $\varepsilon > 0$, the phase transition is second order and in the same universality class as the contact process.

The upper invariant law



Density of the upper invariant law of the 1D contact process.
$\theta(\lambda) \propto (\lambda - \lambda_c)^\beta$ as $\lambda \downarrow \lambda_c$

Equality of the critical points

Recall that $\lambda_{\rm c}$ and $\lambda'_{\rm c}$ are the critical points for survival of finite systems resp. for the density of the upper invariant law.

For the contact process, $\lambda_{\rm c} = \lambda'_{\rm c}$ by self-duality.

The sexual reproduction process without stirring is an attractive spin system.
For such systems, Bezuidenhout and Gray (1994) prove that survival implies a lower bound in terms of supercritical oriented percolation and hence nontriviality of the upper invariant law.
It follows that $\lambda'_{\rm c} \leq \lambda_{\rm c}$ (without stirring).

Conversely, nontriviality of the upper invariant law seems to imply a positive propagation speed and hence survival. Proof?

A cooperative branching-coalescent

Let $(X_t)_{t\geq 0}$ with $X_t = (X_t(i))_{i\in\mathbb{Z}}$ take values in the space of all configurations $\ldots 101101001001\ldots$ and evolve as:

$$
\begin{array}{lrcllc}
\text{(coop. bra.)} & 110 & \mapsto & 111 & \text{with rate} & \frac{1}{2}\lambda, \\
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\text{(coal. RW)} & 10 & \mapsto & 01 & \text{with rate} & \frac{1}{2}, \\
\text{(coal. RW)} & 01 & \mapsto & 10 & \text{with rate} & \frac{1}{2}, \\
\text{(coal. RW)} & 11 & \mapsto & 01 & \text{with rate} & \frac{1}{2}, \\
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\end{array}$$

Interpretation:

- Cooperative reproduction.

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Interpretation:

- Cooperative reproduction.
- Competition for limited space.

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\end{array}$$

Interpretation:

- Cooperative reproduction.
- Competition for limited space.
- Migration.

A cooperative branching-coalescent

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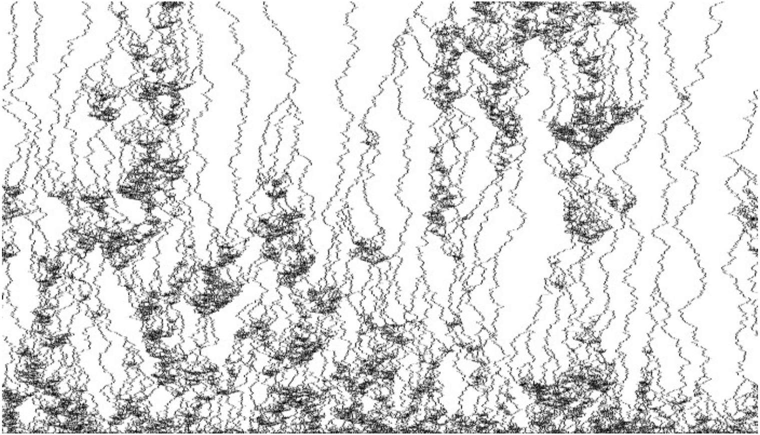
\end{array}
$$

Interpretation:

- Cooperative reproduction.
- Competition for limited space.
- Migration.
- No spontaneous deaths!

A cooperative branching-coalescent



Time = upwards, black = a particle, $\lambda = 2.333$.

Critical points

Define

- The process *survives* if $\mathbb{P}^x\big[|X_t| > 1 \ \forall t \geq 0\big] > 0$ for some, and hence for all initial states with $1 < |x| < \infty$ particles. Note: a single particle can neither die nor reproduce!
- The process is *stable* if there exists an invariant law that is concentrated on nonzero states.

Monotonicity implies that there exist $\lambda_{\rm c}, \lambda'_{\rm c}$ such that

- The process survives for $\lambda > \lambda_{\rm c}$ and dies out for $\lambda < \lambda_{\rm c}$.
- The process is stable for $\lambda > \lambda'_{\rm c}$ and unstable for $\lambda < \lambda'_{\rm c}$.

[Sturm & S. '14] $1 \leq \lambda_{\rm c}, \lambda'_{\rm c} < \infty$.

Numerically: $\lambda_{\rm c} \approx \lambda'_{\rm c} \approx 2.47 \pm 0.02$.

Open problem: Prove that $\lambda_{\rm c} = \lambda'_{\rm c}$.

Proof of the phase transition

Note: If we combine normal branching:

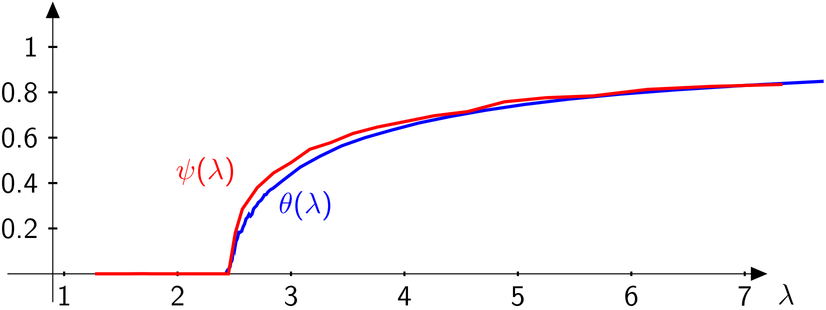
$$01 \mapsto 11 \text{ and } 10 \mapsto 11 \text{ at rate } \tfrac{1}{2}\lambda \text{ each,}$$

with coalescence, then the process converges to an invariant law that is product measure with intensity $\lambda/(1+\lambda)$
-no phase transition!

For the *cooperative* branching-coalescent, particles die at a rate proportional to the number of neighboring pairs 11, and particles are born at a rate less than λ times that number
-no survival and no nontrivial invariant law for $\lambda \leq 1$.

For large λ, survival and existence of a nontrivial invariant law follow from comparison with oriented percolation.

Critical points



$\psi(\lambda) := \mathbb{P}\big[|X_t| > 1\ \forall t \geq 0\big]$ starting with two particles on neighboring sites.

$\theta(\lambda) := \mathbb{P}[X_\infty(0) = 1]$ where X_∞ distributed according to the upper invariant law.

Critical exponent

Numerically, the density of the upper invariant law satisfies

$$\theta(\lambda) \propto (\lambda - \lambda_{\rm c})^{\beta} \qquad \text{as } \lambda \downarrow \lambda_{\rm c},$$

with

$$\beta \approx 0.5 \pm 0.1,$$

which differs from the $\beta \approx 0.27648$ of the contact process.

The subcritical regime

Consider

$$\mathbb{P}\big[|X_t| > 1\big] \qquad \text{with} \quad X_0 = \delta_0 + \delta_1 \quad \text{(two particles)},$$
$$\mathbb{P}\big[X_t(0) = 1\big] \qquad \text{with} \quad X_0 = \underline{1} \quad \text{(fully occupied)}.$$

[Bezuidenhout & Grimmett '91] For the contact process, in the subcritical regime $\lambda < \lambda_c$, both quantities decay exponentially fast to zero.

[Sturm & S. '14] For the cooperative branching-coalescent, both quantities decay not faster than as $t^{-1/2}$. For $\lambda \leq \frac{1}{2}$, this is the exact rate of convergence.

Proof of the lower bound: By monotonicity, we can estimate the cooperative branching-coalescent by a pure coalescent, for which both quantities decay like $t^{-1/2}$.

Proof of the upper bound: Write $x(i,j,k) := (x(i), x(j), x(k))$. Since

$$\frac{\partial}{\partial t}\mathbb{P}[X_t(0) = 1] = (\lambda - 1)\mathbb{P}[X_t(0,1) = 11] - \lambda\mathbb{P}[X_t(0,1,2) = 111]$$

it suffices to prove

$$\mathbb{P}[X_t(0,1) = 11] \leq Ct^{-3/2}.$$

We use the duality function

$$\psi(x, Y) = 1_{\{x \leq y' \text{ for some } y \in Y\}},$$

or equivalently

$$\phi(x, Y) := 1 - \psi(x, Y) = 1_{\{x \wedge y \neq 0 \text{ for all } y \in Y\}}.$$

Our quantity of interest is

$$\mathbb{P}[X_t(0,1) = 11] = \mathbb{E}[\phi(X_t, Y_0)] = \mathbb{E}[\phi(X_0, Y_t)],$$

where $Y_0 = \{\delta_0, \delta_1\}$.

The subcritical regime

We need to show that

$$\mathbb{P}[\underline{0} \notin Y_t] \leq Ct^{-3/2},$$

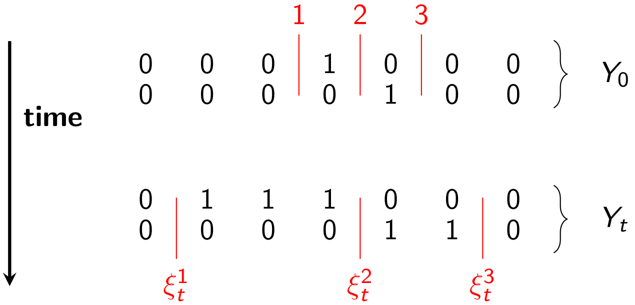
since $\underline{0} \in Y$ implies

$$\phi(x, Y) = 1_{\{x \wedge y \neq 0 \text{ for all } y \in Y\}} = 0 \quad \forall x.$$

In the absence of cooperative branching, when there is only coalescing random walk evolution, the dual process $(Y_t)_{t \geq 0}$ evolves as a collection of coupled voter models.

The subcritical regime

If the cooperative branching rate λ is zero, then the first time that $\underline{0} \in Y_t$ is the first time that two out of three walkers meet.



Coalescing random walks

Let $(\xi_t^i)_{t\geq 0}^{i\in\mathbb{Z}}$ be coalescing random walks, started from every site $i \in \mathbb{Z}$.

Let $\tau_{ij} := \inf\{t \geq 0 : \xi_t^i = \xi_t^j\}$.

Facts:

$$\mathbb{P}[\tau^{12} \wedge \tau^{23} > t] \sim \frac{1}{2\sqrt{\pi}} t^{-3/2},$$

$$\mathbb{E}[\tau^{ij} \wedge \tau^{jk}] = (j-i)(k-j) \qquad (i < j < k).$$

The case with branching

If a cooperative branching event occurs, then we use *subduality:* it suffices to show that both Y'_{t+s} and Y''_{t+s} die out.

time

$$\left.\begin{array}{c|ccc|cc|c} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right\} \; Y_t$$

$$\left.\begin{array}{ccccccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right\} \; Y_{t+s}$$

$$\left.\begin{array}{c|ccc|cc|c} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right\} \; Y'_{t+s}$$

$$\left.\begin{array}{cc|c|c|ccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right\} \; Y''_{t+s}$$

The case with branching

This leads to a (dependent) branching process where triples of random walks die as soon as two out of the three meet, but before it dies, with rate 2λ, a triple can give birth to a new triple of random walks, started on neighboring positions. As long as $\lambda < \frac{1}{2}$, it can be shown that this branching process dies out and the probability to be alive at time t decays as $t^{-3/2}$.