The Brownian net

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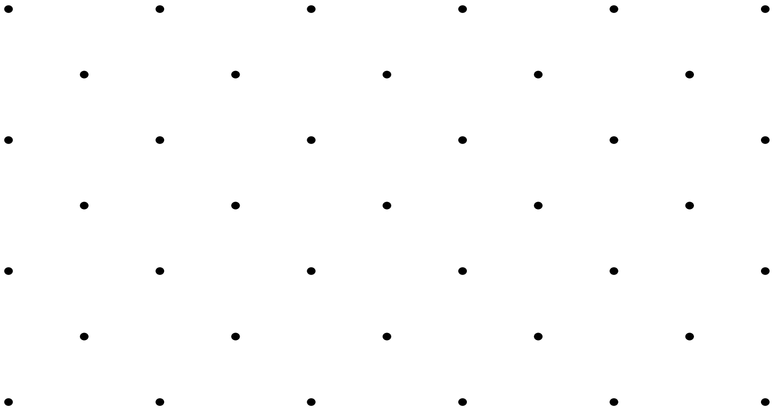
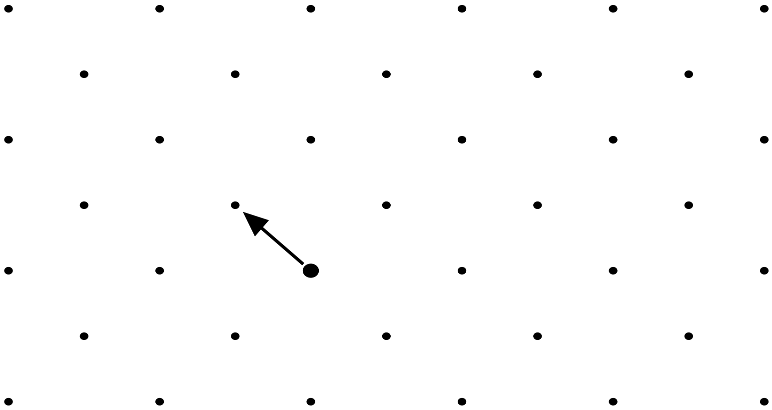
Erlangen, 20 November, 2014
based on work of C. Newman, K. Ravishankar, R. Sun,
E. Schertzer, & me.

Arrow configurations



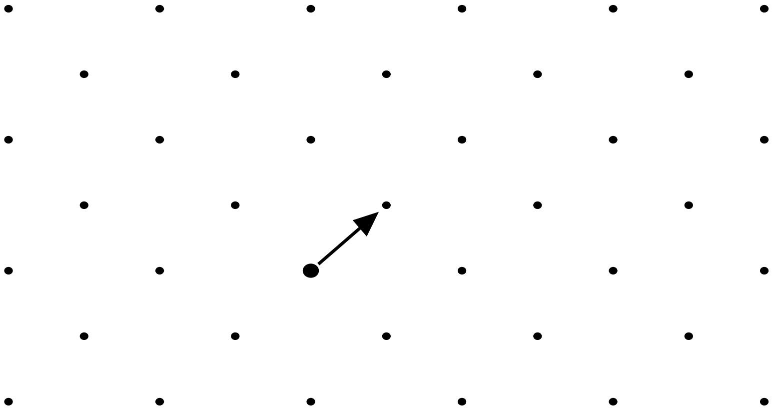
$$\mathbb{Z}^2_{\mathrm{even}} := \{(x,t) \in \mathbb{Z}^2 : x + t \text{ is even}\}.$$

Arrow configurations



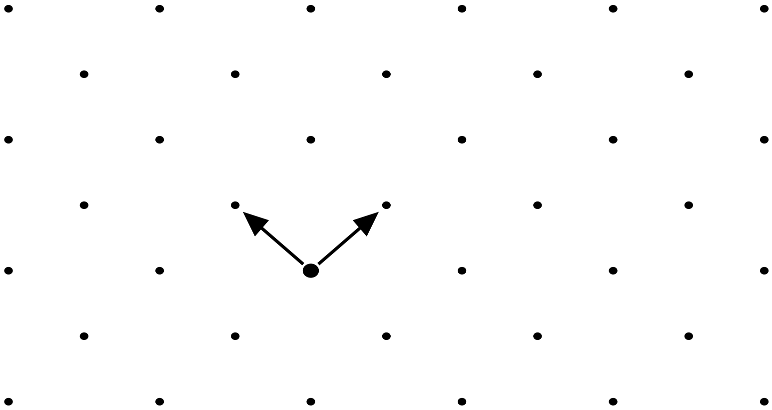
With probability p_l we draw an arrow to the left.

Arrow configurations



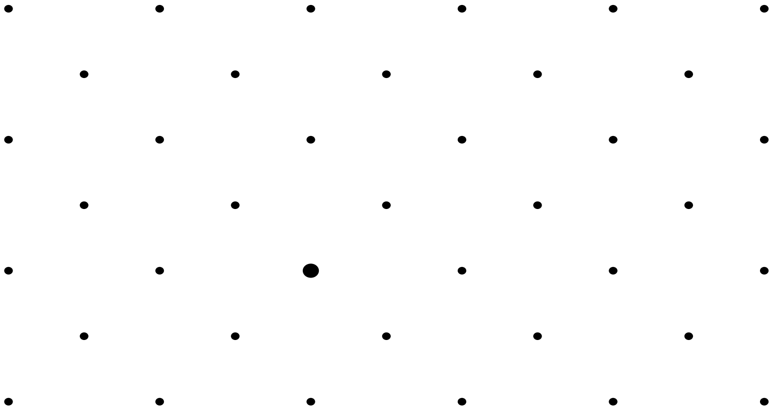
With probability p_{r} we draw an arrow to the right.

Arrow configurations



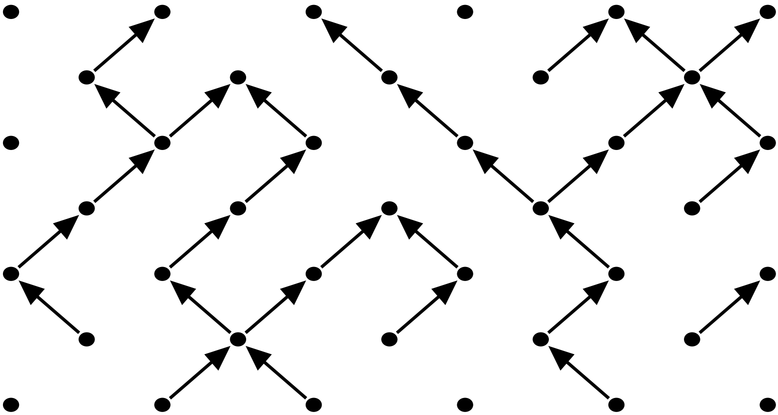
With probability p_{b} we draw two arrows.

Arrow configurations



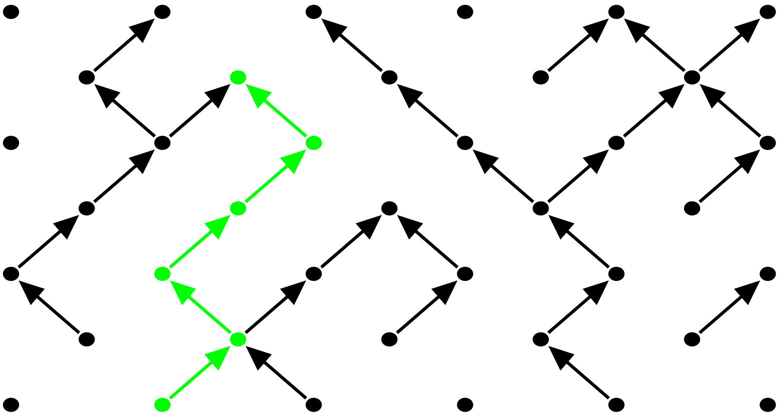
And with probability p_k we draw no arrows at all.

Arrow configurations



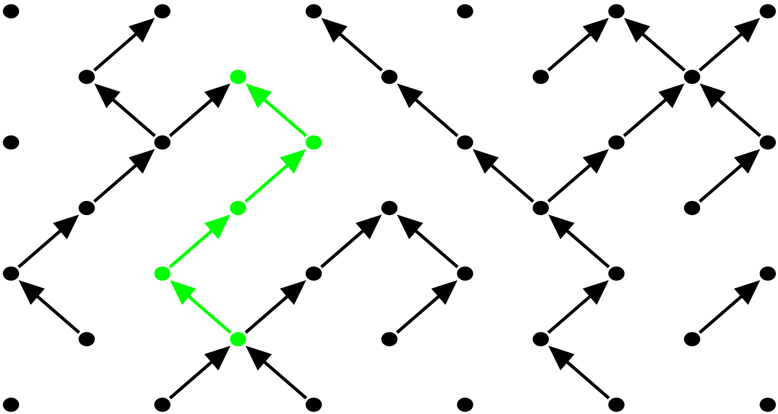
We do this independently for each point.

Arrow configurations



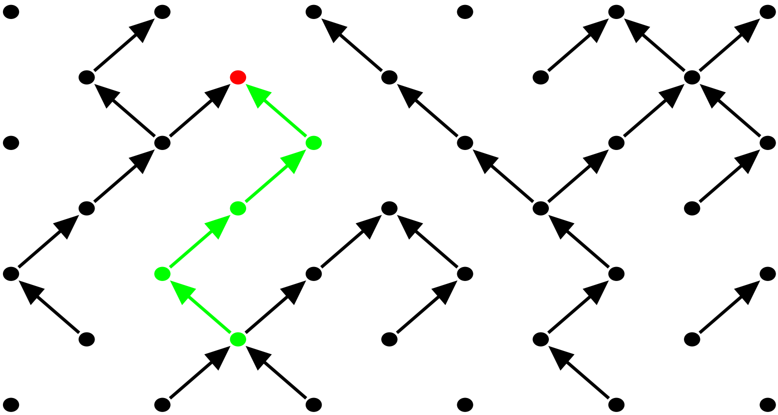
We are interested in open paths.

Arrow configurations



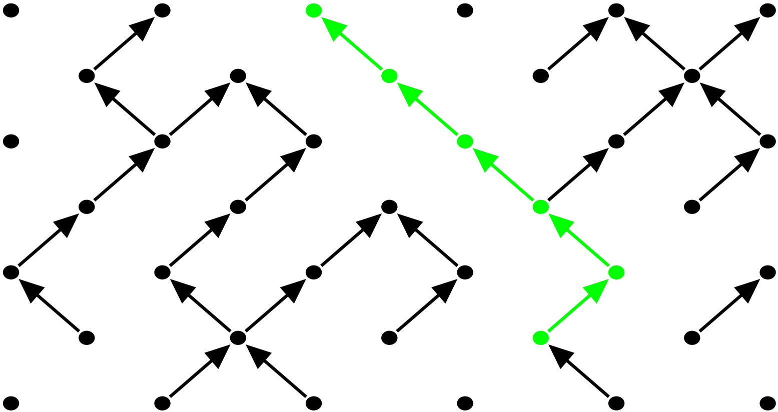
Paths can start at any point in $\mathbb{Z}^2_{\mathrm{even}}$.

Arrow configurations



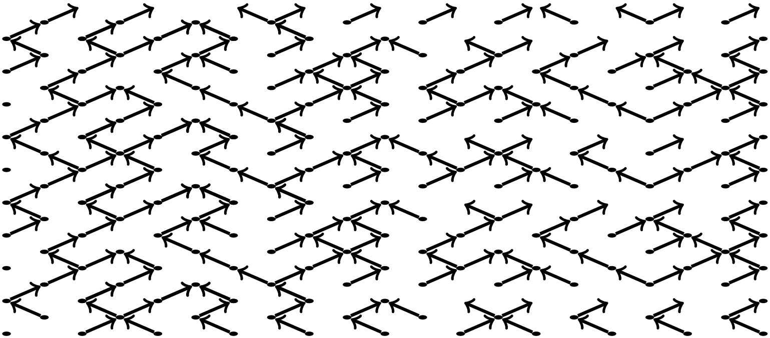
Paths either end at killing points. . .

Arrow configurations



. . . or carry on forever.

Scaling limit



We rescale diffusively, multiplying all spatial distances with ε and all temporal distances with ε^2.

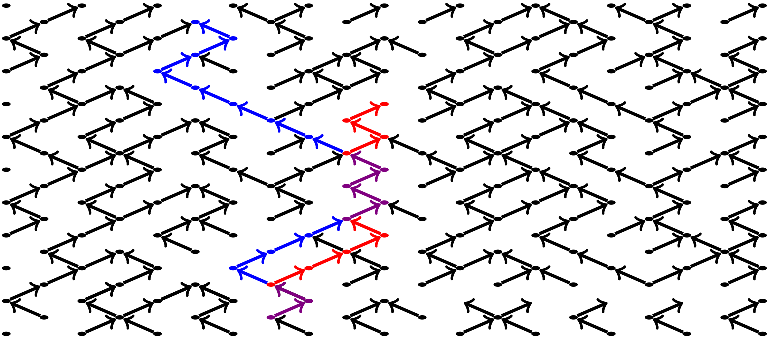
Scaling limit

Claim Assume that

$$\begin{aligned} \varepsilon^{-1}(p_{\mathrm{r}} - p_{\mathrm{l}} - p_{\mathrm{b}}) &\to \beta_-, \\ \varepsilon^{-1}(p_{\mathrm{r}} - p_{\mathrm{l}} + p_{\mathrm{b}}) &\to \beta_+, \\ \varepsilon^{-2} p_{\mathrm{k}} &\to \delta. \end{aligned}$$

Then the collection $\mathcal{U}$ converges to a diffusive scaling limit $\mathcal{N}^{\delta}_{\beta_-,\beta_+}$.

Scaling limit



At each point $z \in \mathbb{Z}^2_{\text{even}}$ there starts an a.s. unique left-most path l_z and right-most path r_z.

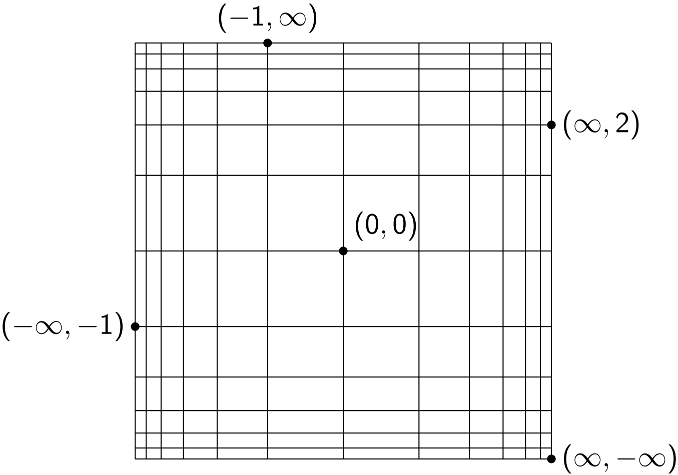
Scaling limit

Under the assumptions

$$\begin{aligned} \varepsilon^{-1}(p_{\rm r} - p_{\rm l} - p_{\rm b}) &\to \beta_-, \\ \varepsilon^{-1}(p_{\rm r} - p_{\rm l} + p_{\rm b}) &\to \beta_+, \\ \varepsilon^{-2} p_{\rm k} &\to \delta, \end{aligned}$$

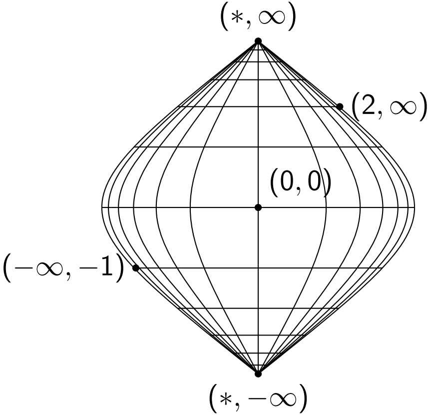
left- and right-most paths converge to Brownian motions with drift β_- and β_+, respectively, and exponential lifetimes with mean $1/\delta$.

Topological matters



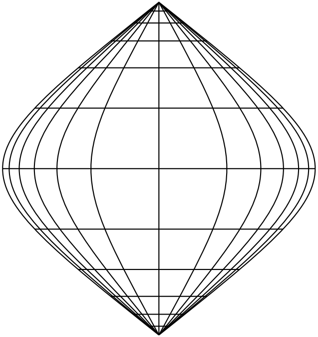
We first compactify $\mathbb{R}^2$ to $[-\infty, \infty]^2$...

Topological matters



...and then contract $[-\infty, \infty] \times \{-\infty\}$ and $[-\infty, \infty] \times \{\infty\}$ to single points.

Topological matters

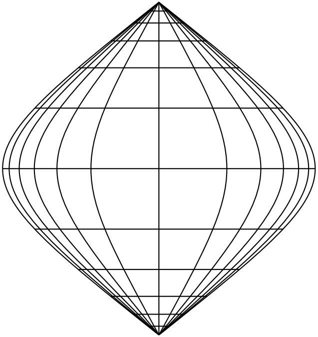


Alternatively, map $\mathbb{R}^2$ into itself with the map

$$\Theta(x,t) := \Big(\frac{\tanh(x)}{1+|t|}, \tanh(t)\Big),$$

and take the closure.

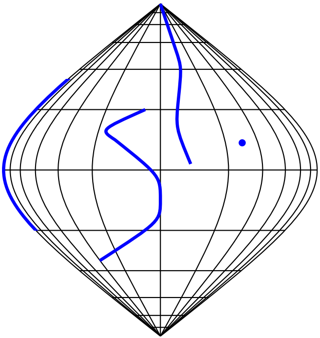
Topological matters



Another equivalent formulation is: take the completion of $\mathbb{R}^2$ w.r.t. the metric

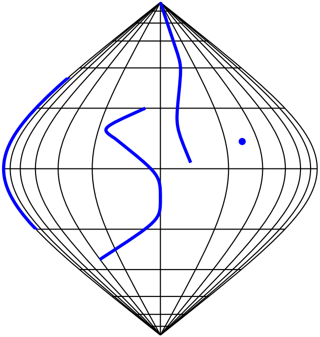
$$d(z, z') := \big|\Theta(z) - \Theta(z')\big|.$$

Topological matters



A *path* is a continuous function $\pi : [\sigma_\pi, \tau_\pi] \to [-\infty, \infty]$,
with $-\infty \leq \sigma_\pi \leq \tau_\pi \leq \infty$.

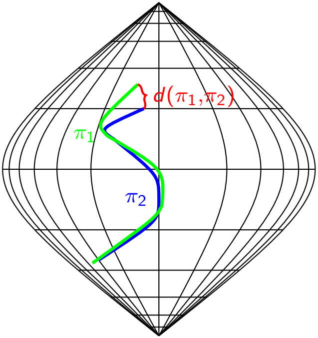
Topological matters



We identify a path with its graph

$$\{(\pi(t), t) : t \in [\sigma_\pi, \tau_\pi]\}.$$

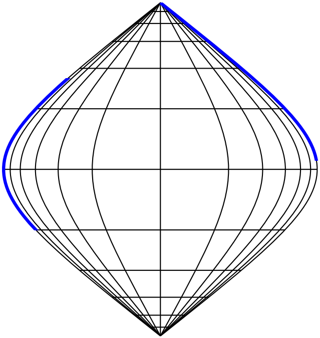
Topological matters



We equip the space Π of all paths with the Hausdorff metric

$$d(\pi_1,\pi_2) = \sup_{z_1\in\pi_1} \inf_{z_2\in\pi_2} d(z_1,z_2) \vee \sup_{z_2\in\pi_2} \inf_{z_1\in\pi_1} d(z_1,z_2).$$

Topological matters



By adding trivial paths that are constantly $-\infty$ or $+\infty$, we can make the set $\mathcal{U}$ of open paths into a compact subset of Π.

Topological matters

We equip the space $\mathcal{K}(\Pi)$ of all compact subsets of the space of paths Π with the Hausdorff metric

$$d(\mathcal{U}_1,\mathcal{U}_2) = \sup_{\pi_1\in\mathcal{U}_1} \inf_{\pi_2\in\mathcal{U}_2} d(\pi_1,\pi_2) \vee \sup_{\pi_2\in\mathcal{U}_2} \inf_{\pi_1\in\mathcal{U}_1} d(\pi_1,\pi_2).$$

We define a diffusive scaling map S_ε by

$$S_\varepsilon(x,t) := (\varepsilon x, \varepsilon^2 t).$$

Topological matters

Theorem Let $\varepsilon_n \downarrow 0$ and let $\mathcal{U}_n$ be the sets of open paths in arrow configurations with parameters satisfying

$$\begin{aligned}
\varepsilon_n^{-1}\big(p_{\rm r}(n) - p_{\rm l}(n) - p_{\rm b}(n)\big) &\to \beta_-,\\
\varepsilon_n^{-1}\big(p_{\rm r}(n) - p_{\rm l}(n) + p_{\rm b}(n)\big) &\to \beta_+,\\
\varepsilon_n^{-2} p_{\rm k}(n) &\to \delta.
\end{aligned}$$

Then

$$\mathbb{P}[S_{\varepsilon_n}(\mathcal{U}_n) \in \cdot\,] \underset{n\to\infty}{\Longrightarrow} \mathbb{P}[\mathcal{N}^{\delta}_{\beta_-,\beta_+} \in \cdot\,],$$

where $\Rightarrow$ denotes weak convergence of probability laws on $\mathcal{K}(\Pi)$. The limiting object is a *Brownian net with killing*.

If $\beta = \beta_- = \beta_+$ and $\delta = 0$, then the limiting object $\mathcal{W}_\beta := \mathcal{N}^0_{\beta,\beta}$ is a *Brownian web* with drift β. In particular, $\mathcal{W} := \mathcal{W}_0$ is the *standard Brownian web.*

- For each deterministic $z \in \mathbb{R}^2$, almost surely there is a unique path $p_z \in \mathcal{W}$.

The Brownian web

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- For each deterministic $z \in \mathbb{R}^2$, almost surely there is a unique path $p_z \in \mathcal{W}$.
- For any deterministic finite set of points $z_1, \ldots, z_k \in \mathbb{R}^2$, the collection $(p_{z_1}, \ldots, p_{z_k})$ is distributed as coalescing Brownian motions.

The Brownian web

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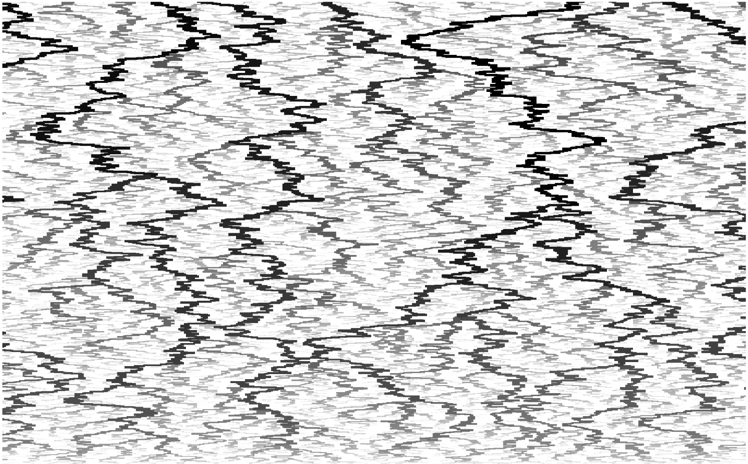
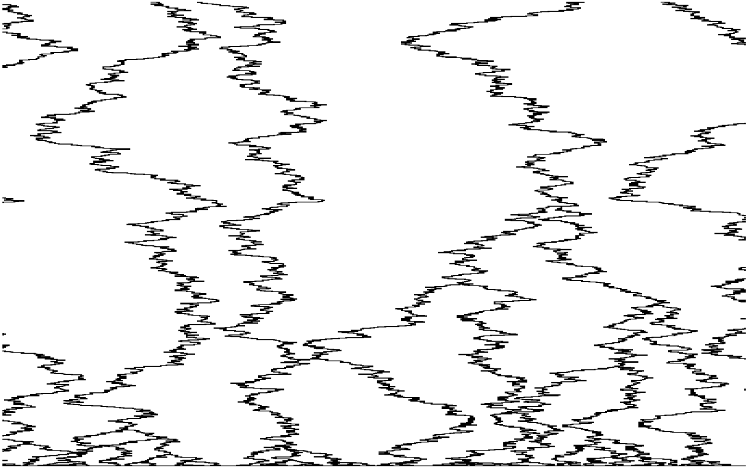
- For each deterministic $z \in \mathbb{R}^2$, almost surely there is a unique path $p_z \in \mathcal{W}$.
- For any deterministic finite set of points $z_1, \ldots, z_k \in \mathbb{R}^2$, the collection $(p_{z_1}, \ldots, p_{z_k})$ is distributed as coalescing Brownian motions.
- For any deterministic countable dense subset $\mathcal{D} \subset \mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure of $\{p_z : z \in \mathcal{D}\}$.



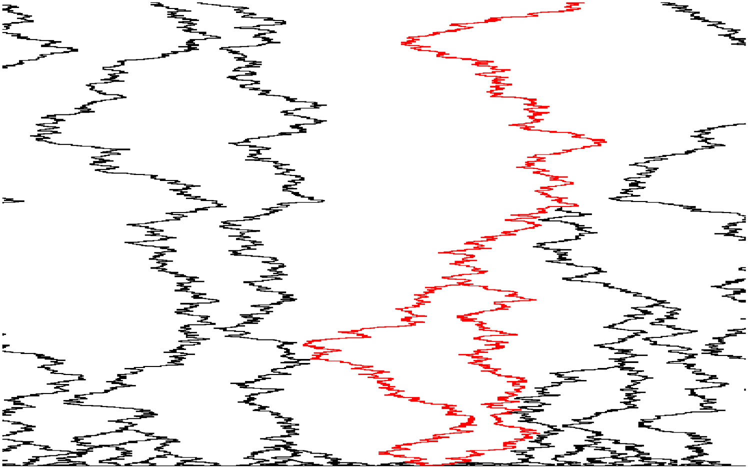
Artist's impression of the Brownian web.

The Brownian web



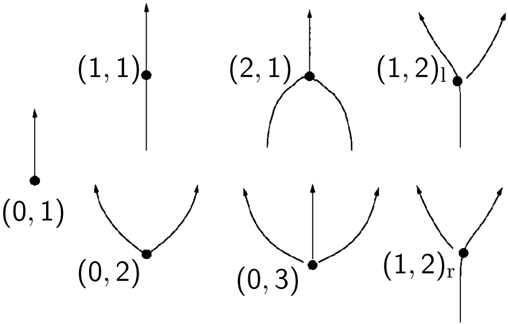
Paths started at time zero.

The Brownian web



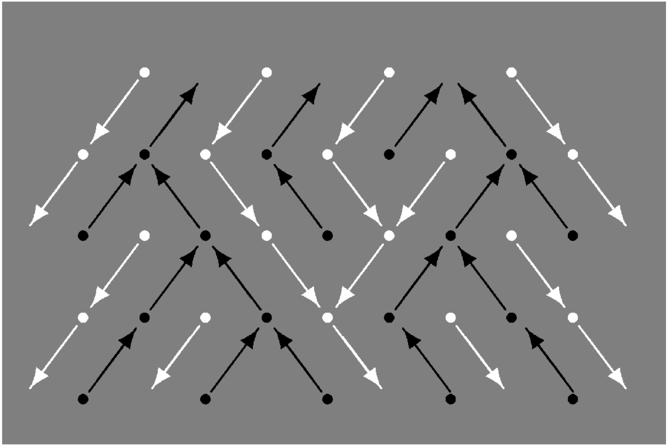
There exists random points where two paths start.

Special points



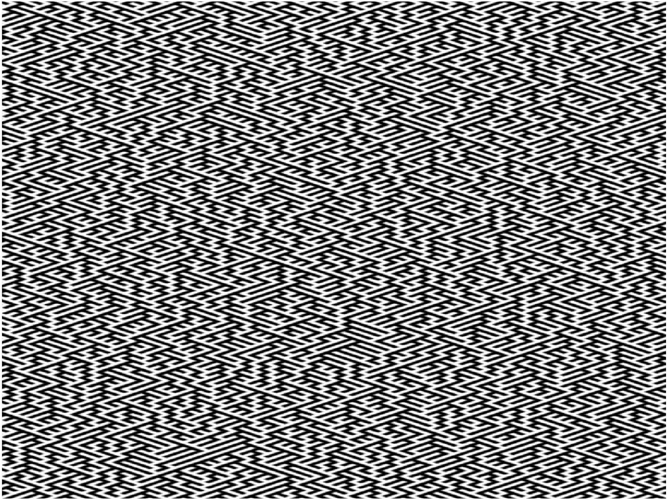
Special points are classified according to the number of incoming and outgoing paths. There exists 7 types of special points.

Dual arrows



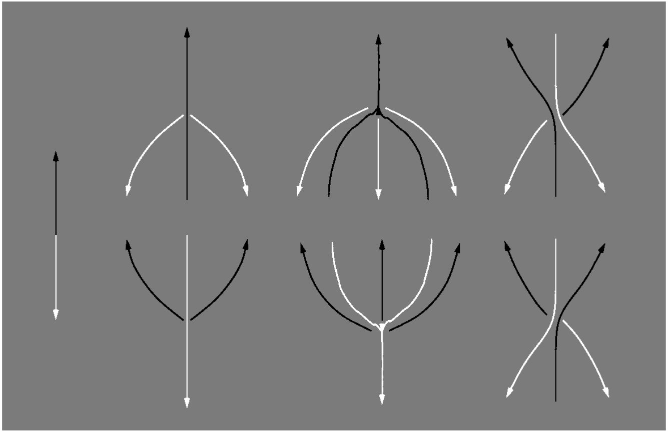
Forward and dual arrows.

Dual Brownian web



Approximation of the forward and dual Brownian web.

Special points revisited

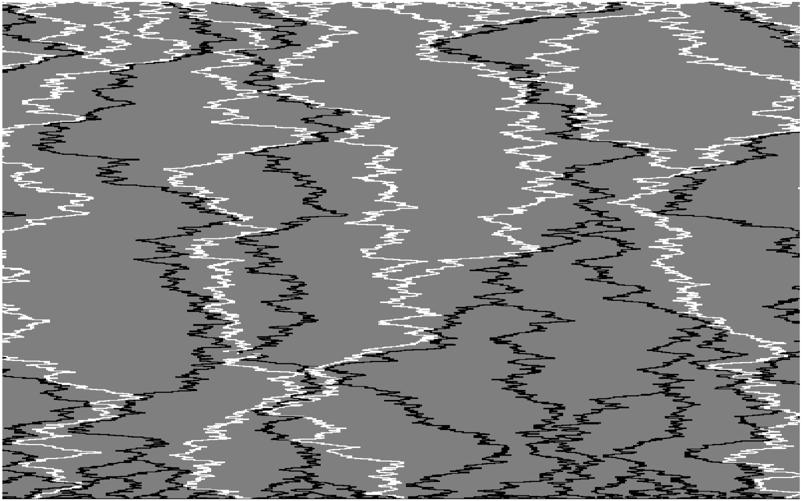


Structure of dual paths at special points.

To each Brownian web $\mathcal{W}$, we can associate an a.s. unique *dual web* $\hat{\mathcal{W}}$ that is equally distributed with $\mathcal{W}$ except for a rotation over 180°.

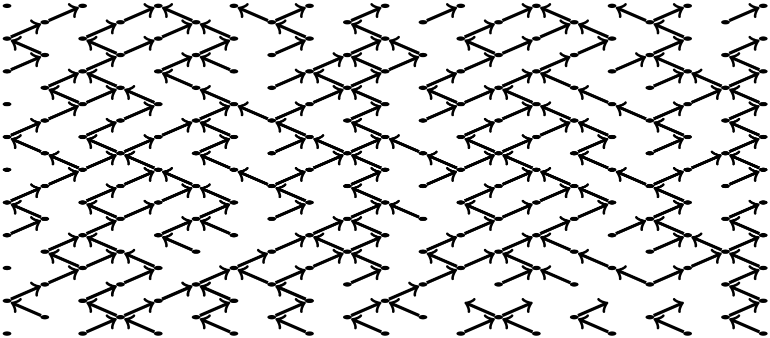
Fix a deterministic finite set of starting points and condition on the forward paths starting at these points.
Then paths of the dual web are Brownian motions with immediate reflection off the fixed forward paths.

Dual Brownian web



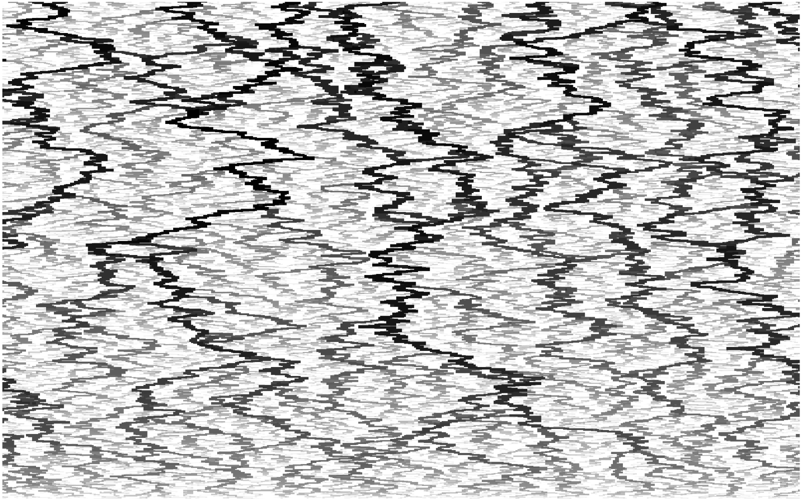
Forward and dual paths started from fixed times.

Left- and right-most paths



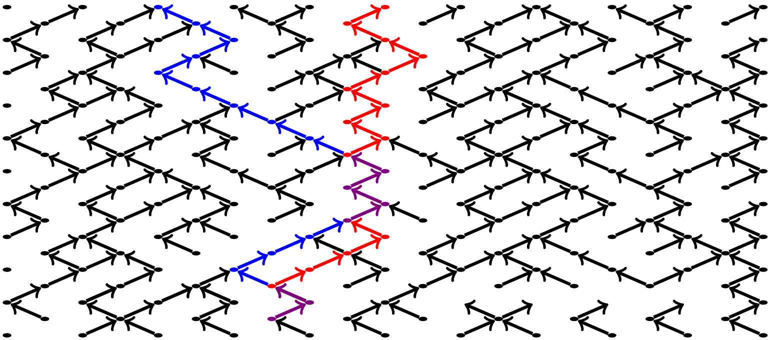
Consider an arrow configuration with
branching probability $p_{\mathrm{b}} > 0$ but killing probability $p_{\mathrm{k}} = 0$.

Left- and right-most paths



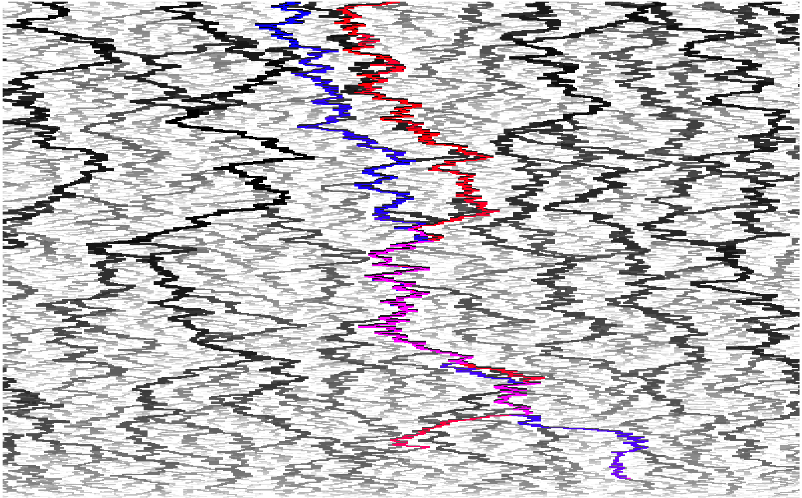
Artist's impression of the Brownian net.

Left- and right-most paths



Left- and right-most paths interact with a form of sticky interaction.

Left- and right-most paths



In the limit, left- and right-most paths are Brownian motions with drift $\beta_- < \beta_+$.

Left- and right-most paths

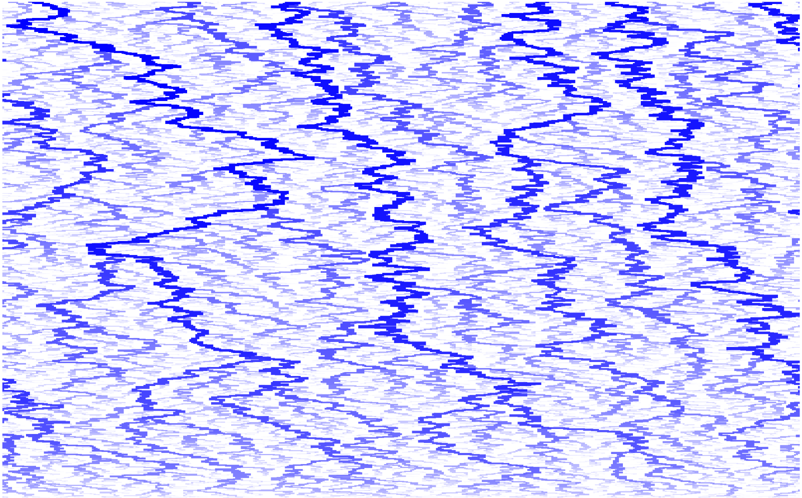
The interaction between left-most and right-most paths is described by the stochastic differential equation (SDE):

$$\begin{aligned} \mathrm{d}L_t &= 1_{\{L_t \neq R_t\}} \mathrm{d}B^{\mathrm{l}}_t + 1_{\{L_t = R_t\}} \mathrm{d}B^{\mathrm{s}}_t + \beta_- \mathrm{d}t, \\ \mathrm{d}R_t &= 1_{\{L_t \neq R_t\}} \mathrm{d}B^{\mathrm{r}}_t + 1_{\{L_t = R_t\}} \mathrm{d}B^{\mathrm{s}}_t + \beta_+ \mathrm{d}t, \end{aligned}$$

where $B^{\mathrm{l}}_t, B^{\mathrm{r}}_t, B^{\mathrm{s}}_t$ are independent Brownian motions, and L_t and R_t are subject to the constraint that $L_t \leq R_t$ for all $t \geq \tau := \inf\{u \geq 0 : L_u = R_u\}$.

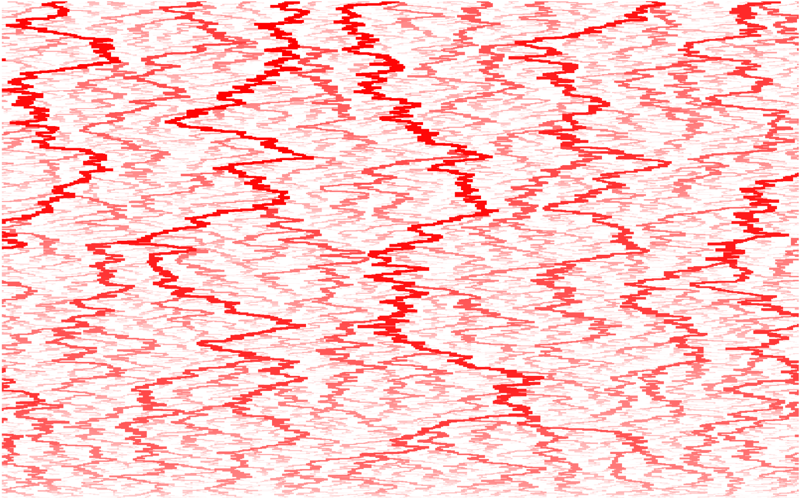
The set $\{t : L_t = R_u\}$ is nowhere dense and has positive Lebesgue measure whenever it is nonempty.

The left Brownian web



The left-most paths converge to a left Brownian web. . .

The right Brownian web



...and the right-most paths to a right Brownian web.

Hopping construction of the Brownian net

By definition, an *intersection time* of two paths π_1, π_2 is a time $t > \sigma_{\pi_1} \vee \sigma_{\pi_2}$ such that $\pi_1(t) = \pi_2(t)$.
We may concatenate two paths at an intersection time by putting

$$\pi(s) := \begin{cases} \pi_1(s) & \big(s \in [\sigma_{\pi_1}, t]\big), \\ \pi_2(s) & \big(s \in [t, \infty]\big). \end{cases}$$

Let $(\mathcal{W}^{\rm l}, \mathcal{W}^{\rm r})$, be a *left-right Brownian web*.
Let $\mathcal{D} \subset \mathbb{R}^2$ be deterministic, countable, and dense and let $\mathcal{W}^{\rm l}(\mathcal{D})$ and $\mathcal{W}^{\rm r}(\mathcal{D})$ denote the left- and right-most paths started from $\mathcal{D}$.

Let $\mathtt{Hop}\big(\mathcal{W}^{\rm l}(\mathcal{D}) \cup \mathcal{W}^{\rm r}(\mathcal{D})\big)$ denote the smallest set containing $\mathcal{W}^{\rm l}(\mathcal{D}) \cup \mathcal{W}^{\rm r}(\mathcal{D})$ that is closed under concatenation of paths at intersection times.

Hopping construction $\mathcal{N}^0_{\beta_-,\beta_+} = \overline{\mathtt{Hop}\big(\mathcal{W}^{\rm l}(\mathcal{D}) \cup \mathcal{W}^{\rm r}(\mathcal{D})\big)}$.

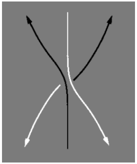
Hopping construction of the Brownian net

It is not allowed to "hop" from π_1 onto π_2 at the starting time σ_{π_2} of π_2.

In view of the special points of type $(1,2)$, if we would allow this sort of hopping, then we would obtain too many paths.

(In fact, after taking the closure, we'd obtain the whole space Π.)

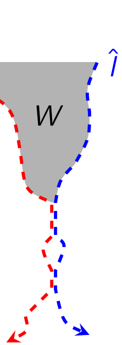
We will later see, however, that we *do* get a sensible limit if we allow hopping only at a cleverly chosen Poisson subset of the points of type $(1,2)$...



Wedge construction of the Brownian net

Let $(\mathcal{W}^{\rm l}, \mathcal{W}^{\rm r}, \hat{\mathcal{W}}^{\rm l}, \hat{\mathcal{W}}^{\rm r})$, be a left-right Brownian web together with its dual left and right webs.

This object is symmetric w.r.t. rotation over 180°.



A dual left and right path together define a *wedge* W.

Wedge construction of the Brownian net

Given an open set $A \subset \mathbb{R}^2$ and a path $\pi \in \Pi$, we say π *enters* A if there exist $\sigma_\pi < s < t$ such that $\pi(s) \notin A$ and $\pi(t) \in A$. We say π *enters* A *from outside* if there exists $\sigma_\pi < s < t$ such that $\pi(s) \notin \overline{A}$ and $\pi(t) \in A$.

A *wedge* is an open set of the form:

$$W(\hat{r}, \hat{l}) := \{(x,t) \in \mathbb{R}^2 : \hat{\tau}_{\hat{r},\hat{l}} < t < \hat{\sigma}_{\hat{l}} \wedge \hat{\sigma}_{\hat{r}},\ \hat{r}(t) < x < \hat{l}(t)\},$$

where $\hat{\tau}_{\hat{r},\hat{l}}$ is the first meeting time of $\hat{r}$ and $\hat{l}$.

Wedge construction

$$\mathcal{N}^0_{\beta_-,\beta_+}$$
$$= \{\pi \in \Pi : \pi \text{ does not enter wedges of } (\hat{\mathcal{W}}^{\mathrm{l}}, \hat{\mathcal{W}}^{\mathrm{r}}) \text{ from outside}\}.$$

The wedge construction shows that paths of $\mathcal{N}^0_{\beta_-,\beta_+}$ cannot cross dual left- or right-most in the wrong direction.
But this condition alone is not enough to guarantee that a path lies in $\mathcal{N}^0_{\beta_-,\beta_+}$.

In the special case $\beta_- = \beta_+$, the left and right webs coincide.
In this case, wedges can still be defined and the wedges give an a.s. construction of the Brownian web in terms of its dual.

Also for the Brownian web $\mathcal{W}$, the condition that a path π does not cross any dual path $\hat{\pi} \in \hat{\mathcal{W}}$ is not enough to guarantee that $\pi \in \mathcal{W}$.

(A counterexample can be constructed by concatenating a piece of a dual path with a forward path.)

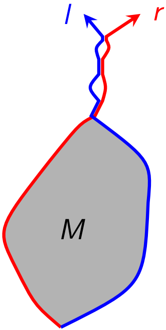
Wedge construction of the Brownian net

The equivalence of the hopping and wedge constructions can be used to prove convergence of diffusively rescaled discrete nets to the Brownian net.
Tightness comes for free from the tightness of the left- and right webs while any limit point $\mathcal{N}_*$ satisfies

$$\mathcal{N}_{\text{hop}} \subset \mathcal{N}_* \subset \mathcal{N}_{\text{wedge}}.$$

However, this proof works only because the discrete nets are nearest-neighbor and (hence) the associated discrete left and right webs have duals.

Mesh construction of the Brownian net

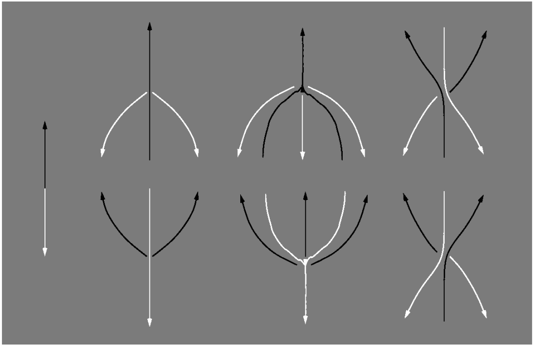


A *mesh* M is the open area enclosed by a right- and left-most path, starting from the same point, that are initially ordered the "wrong" way.

Mesh construction

$\mathcal{N}^0_{\beta_-,\beta_+} = \{\pi \in \Pi : \pi$ does not enter meshes of $(\mathcal{W}^{\mathrm{l}}, \mathcal{W}^{\mathrm{r}})\}$.

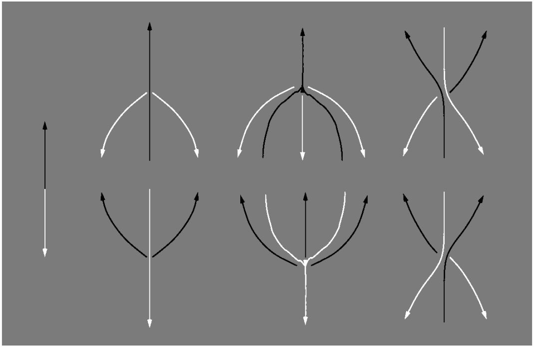
Marking constructions



Recall that points of the Brownian web are classified according to the number of incoming and outgoing paths $(m_{\rm in}, m_{\rm out})$.

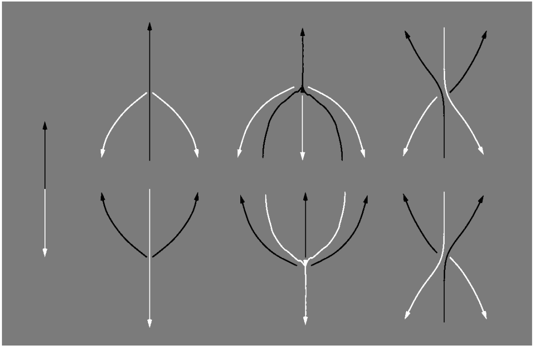
With respect to Lebesgue measure, a.e. point is of type $(0, 1)$.

Marking constructions



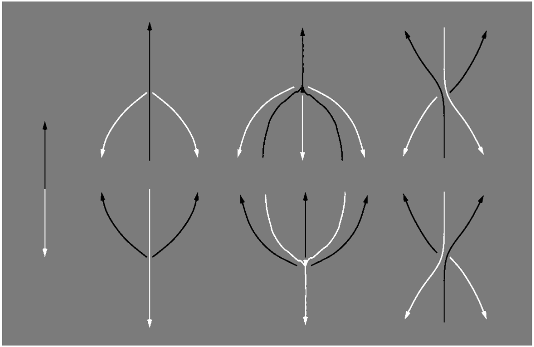
The sets of points of types $(2, 1)$ and $(0, 3)$ are countable.

Marking constructions



With respect to the *length measure* $\mu_{\rm length}$ of the forward web, a.e. point is of type $(1, 1)$.

Marking constructions



With respect to the *intersection local measure* μ_{int} of the forward and dual webs, a.e. point is of type $(1,2)$.

Marking constructions

The *length measure* $\mu_{\rm length}$ is a measure on $\mathbb{R}^2$ that is concentrated on points of type $(1,1)$ such that for every path $\pi\in\mathcal{W}$ and $\sigma_\pi\leq s\leq u<\infty$,

$$\mu_{\rm length}\big(\{(\pi(t),t):t\in[s,u]\}\big)=u-s.$$

The *intersection local measure* $\mu_{\rm int}$ is a measure on $\mathbb{R}^2$ that is concentrated on points of type $(1,2)$ such that for every two paths $\pi\in\mathcal{W}$ and $\hat\pi\in\hat{\mathcal{W}}$,

$$\begin{aligned}&\mu_{\rm int}\Big(\big\{(x,t)\in\mathbb{R}^2:\sigma_\pi<t<\hat\sigma_{\hat\pi},\ \pi(t)=x=\hat\pi(t)\big\}\Big)\\&\quad=\lim_{\varepsilon\downarrow0}\varepsilon^{-1}\Big|\big\{t\in\mathbb{R}:\sigma_\pi<t<\hat\sigma_{\hat\pi},\ |\pi(t)-\hat\pi(t)|\leq\varepsilon\big\}\Big|.\end{aligned}$$

These measures are σ-finite, but not locally finite; they give infinite measure to any nonempty open subset of $\mathbb{R}^2$.

Marking constructions

Let $\mu^{\mathrm{l}}_{\mathrm{int}}$ and $\mu^{\mathrm{r}}_{\mathrm{int}}$ be the restrictions of μ_{int} to the set of points of type $(1,2)_{\mathrm{l}}$ and $(1,2)_{\mathrm{r}}$, respectively.

Modified web Let $\mathcal{W}$ be a Brownian web with drift β and let S be a Poisson set with intensity $c_{\mathrm{l}}\mu^{\mathrm{l}}_{\mathrm{int}} + c_{\mathrm{r}}\mu^{\mathrm{r}}_{\mathrm{int}}$. Then, for any finite $\Delta_n \uparrow S$, the limit

$$\mathcal{W}' := \lim_{\Delta_n \uparrow S} \mathtt{switch}_{\Delta_n}(\mathcal{W})$$

exists and is a Brownian web with drift $\beta' = \beta + c_{\mathrm{l}} - c_{\mathrm{r}}$.

In particular, if $c_{\mathrm{r}} = 0$, then $(\mathcal{W}, \mathcal{W}')$ is a left-right Brownian web.

Marking constructions

Let $\mathcal{W}$ be a "reference" Brownian web with drift β.
Let S_{12} be a Poisson set with intensity $c_{\rm l}\mu^{\rm l}_{\rm int} + c_{\rm r}\mu^{\rm r}_{\rm int}$.
Let S_{11} be a Poisson set with intensity $\delta\mu_{\rm length}$.
Marking construction For any finite $\Delta_n \uparrow S_{12}$, the limit

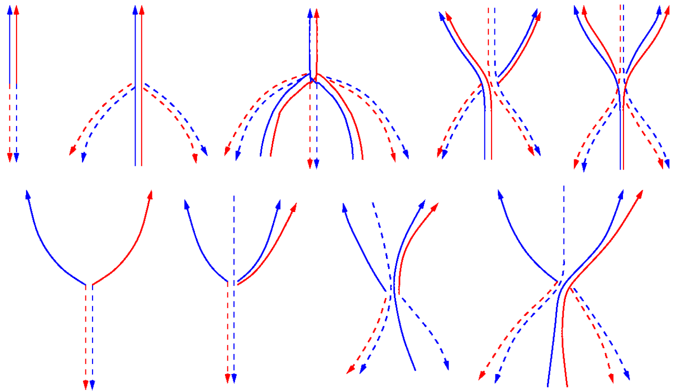
$$\mathcal{N} := \lim_{\Delta_n \uparrow S_{12}} \mathtt{hop}_{\Delta_n}(\mathcal{W})$$

exists and is a Brownian net (without killing) with left and right drifts

$$\beta_- = \beta - c_{\rm r} \quad \text{and} \quad \beta_+ = \beta + c_{\rm l}.$$

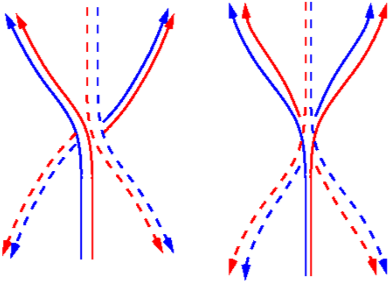
Moreover, the set of all paths in $\mathcal{N}$ stopped at the first time they hit a point in S_{11} is a Brownian net with left and right drifts β_-, β_+ and killing rate δ.

Marking constructions



Modulo symmetry, there exist 9 types of special points of a left-right Brownian web, or equivalently, a Brownian net.

Marking constructions



Separation points (on the right) are of type $(1, 2)$ in both the left and right web, but differently oriented.

There are countably many separation points.
In a marking construction starting from only the left web, these are the Poisson points where we change the reference web.

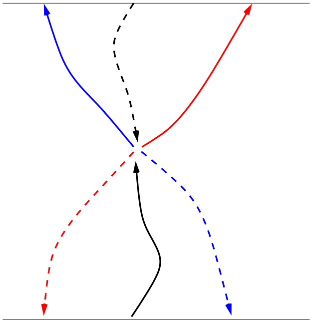
Marking constructions

Web inside net Let S be the set of separation points of a Brownian net $\mathcal{N}^0_{\beta_-,\beta_+}$. Conditional on $\mathcal{N}$, let $(\alpha_z)_{z\in S}$ be i.i.d. $\{-1,+1\}$-valued random variables with $\mathbb{P}[\alpha_z = +1] = r$. Then

$$\mathcal{W} := \{\pi \in \Pi : \pi \text{ leaves each } z \in S \text{ in the direction } \alpha_z\}$$

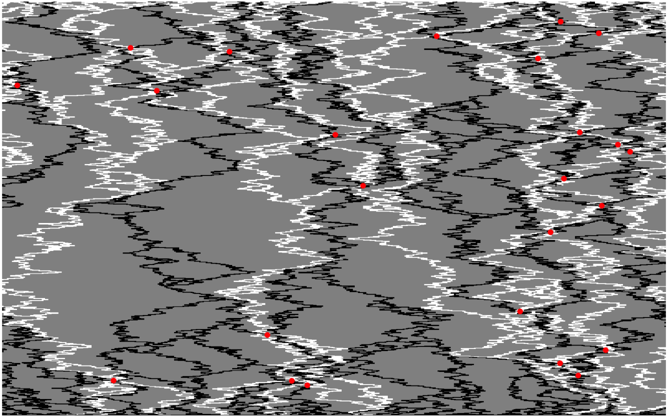
is a Brownian web with drift $(1-r)\beta_- + r\beta_+$.

Relevant separation points



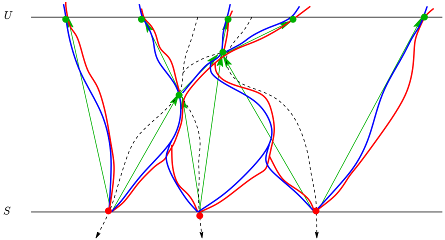
By definition, a separation point $z = (x, t)$ with $S < t < U$ is S, U-*relevant* if there is a path $\pi \in \mathcal{N}$ entering z starting at time S, and there are $l \in \mathcal{W}^{\mathrm{l}}(z)$, $r \in \mathcal{W}^{\mathrm{r}}(z)$ such that $l < r$ on (t, U).

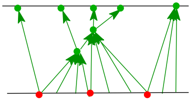
Relevant separation points



'Relevant' separation points, where the forward Brownian net crosses its dual, are locally finite (with an explicit density).

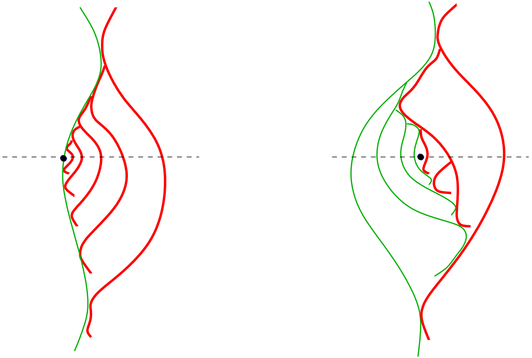
Relevant separation points





The finite graph representation says that we only need to know the orientation of relevant separation points to decide how a path moves between *deterministic* times.
This construction is closely linked to the construction of the Brownian net using meshes.

Fractal structure



(C_l) (C_n)

The Brownian net has a fractal structure. There are random times where infinitely many choices are needed to pass through certain points.

Historical notes

- R. Arratia ('79,'81), motivated by scaling limits of the 1D voter model, studies coalescing Brownian motions started from each point in space and time.
- B. Tóth and W. Werner ('98) arrive at the same object by studying the true self-repellent motion. They classify special points and use right-continuity to choose a unique path at points of multiplicity.
- F. Soucaliuc, B. Tóth, and W. Werner ('00) prove that paths in the dual web are reflected off forward paths.
- L. Fontes, M. Isopi, C. Newman, and K. Ravishankar ('04) invent the name "Brownian web", viewed this as a compact set of paths, and prove weak convergence w.r.t. to the Hausdorff topology.
- C. Newman, K. Ravishankar, and R. Sun ('05) prove convergence of coalescing non-nearest neighbor random walks to the Brownian web.

Historical notes

- R. Sun and J.S. ('08) invent the name Brownian net and the hopping, wedge, and mesh constructions, which are all based on the left-right SDE.
- E. Schertzer, R. Sun and J.S. ('09) classify special points of the Brownian net.
- C. Howitt and J. Warren ('09) construct sticky pairs of Brownian webs by means of a martingale problem.
- C. Newman, K. Ravishankar, and E. Schertzer ('10) publish the marking construction of the Brownian net, conceived around '05.
- C. Newman, K. Ravishankar, and E. Schertzer ('13) construct the Brownian net with killing.
- E. Schertzer, R. Sun and J.S. ('14) study stochastic flows using marked webs.
- C. Newman, K. Ravishankar, and E. Schertzer (announced) study voter model perturbations.

Branching-coalescing point set

For any closed subset $A \subset \mathbb{R}$,

$$\xi_t := \big\{\pi(t) : \exists \pi \in \mathcal{N}^{\delta}_{\beta_-,\beta_+} \text{ s.t. } \sigma_\pi = 0,\ \pi(0) \in A\big\}$$

defines a Feller process taking values in the closed subsets of $\mathbb{R}$.

(i) Reversible invariant law: the law of a Poisson point set with intensity $\beta_+ - \beta_-$.

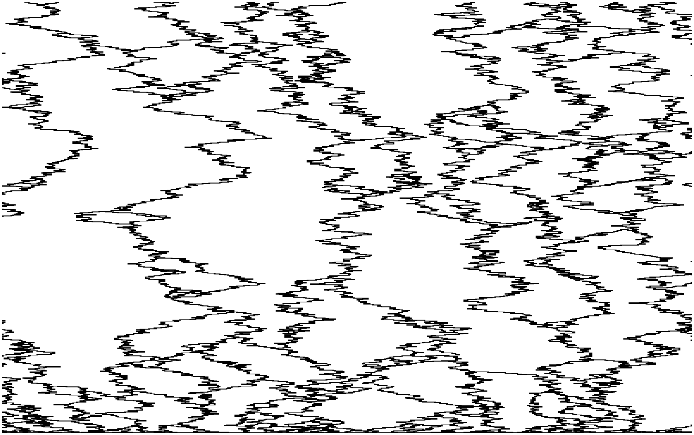
(ii) For deterministic $t > 0$, a.s. ξ_t is a locally finite subset of $\mathbb{R}$.

(iii) There exists a dense set of random times $\tau > 0$ such that ξ_τ has no isolated points.

Open problem: generator characterization!

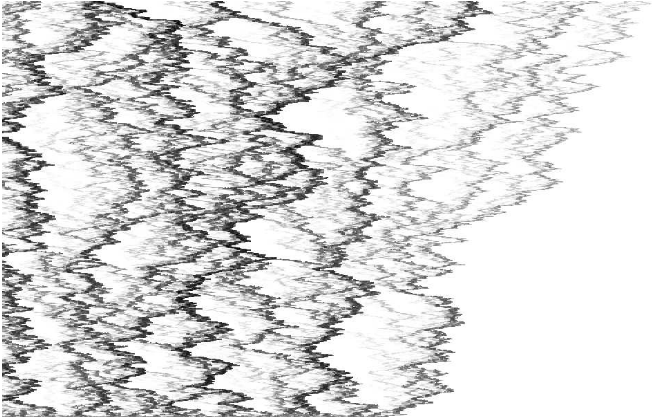
Thm Phase transition between survival and extinction at some δ_{c}.

The branching-coalescing point set



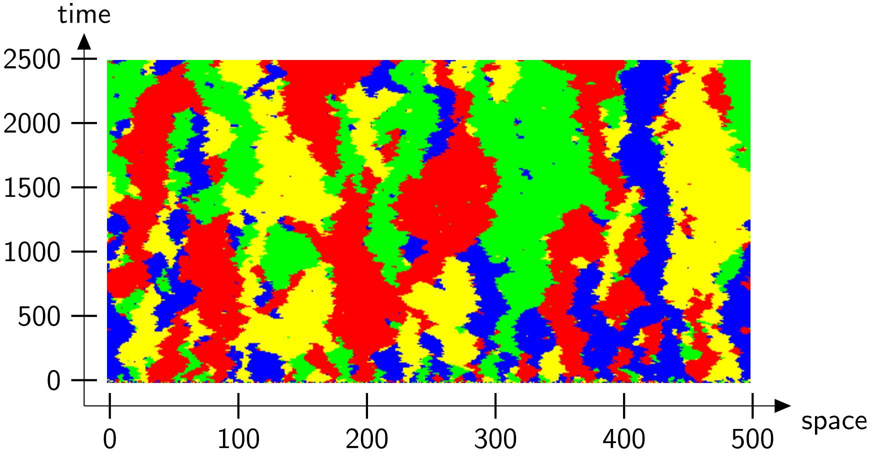
The branching-coalescing point set with
$\beta_- = -1$, $\beta_+ = 1$, $\delta = 0$ started in $\xi_0 = \mathbb{R}$.

Howitt-Warren flows



A one-sided erosion flow.

A one-dimensional Potts model



A low-temperature one-dimensional Potts model.