Cooperative branching

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Cooperative branching

Cooperative branching is a type of dynamics for interacting particle systems, where *two* particles together produce a third particle.

In physics notation for reaction-diffusion models: $2A \mapsto 3A$.

This sort of dynamics, together with $3A \mapsto 2A$, was already considered by *F. Schlögl* [Z. Phys. 1972].

Lebowiz, Presutti and Spohn [JSP 1988] call this *binary reproduction*.

C. Noble [AOP 1992], *R. Durrett* [JAP 1992], and *C. Neuhauser and S.W. Pacala* [AAP 1999] consider a model with $2A \mapsto 3A$ (cooperative branching) and $A \mapsto \emptyset$ (deaths). They call this the *sexual reproduction process*.

J. Blath and N. Kurt [ECP 2011] considered a *cooperative caring double-branching annihilating random walk*, and *A. Sturm and J.S.* [AAP 2014] studied a *cooperative branching-coalescent*.

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The sexual reproduction process

Let $(X_t)_{t\geq 0}$ with $X_t = (X_t(i))_{i\in\mathbb{Z}}$ take values in the space of all configurations $\ldots 101101001001\ldots$ and evolve as:

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Interpretation:

- 'Sexual' reproduction.

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- ‘Sexual’ reproduction.
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Fast stirring

Let $(X_t)_{t\geq 0}$ with $X_t = (X_t(i))_{i\in\mathbb{Z}}$ take values in the space of all configurations $\ldots 101101001001\ldots$ and evolve as:

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$$

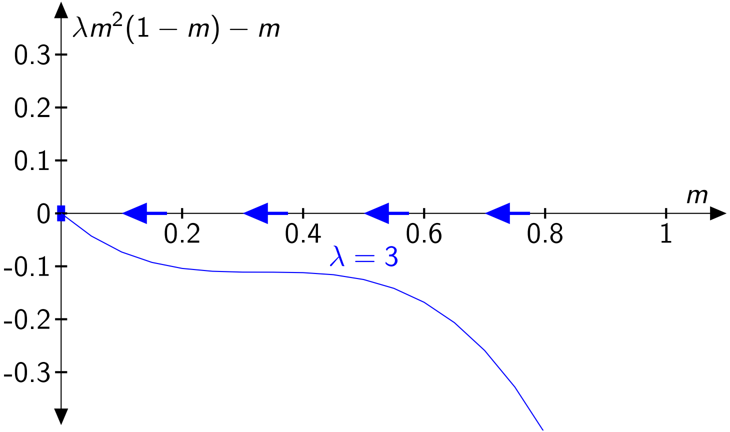
Set

$$m_\varepsilon(x,t) := \mathbb{P}[X_{\varepsilon^{-2}t}(\lfloor \varepsilon x \rfloor) \qquad (x \in R,\ t \geq 0).$$

[DeMasi, Ferrari & Lebowitz '86] In the fast stirring limit $\varepsilon \downarrow 0$, the particle density m_ε converges to a solution of

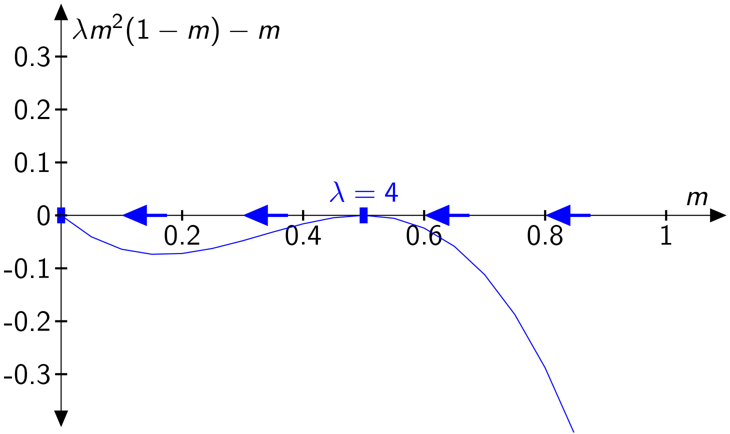
$$\frac{\partial}{\partial t} m = \frac{\partial^2}{\partial x^2} m + \lambda m^2(1-m) - m.$$

Constant densities



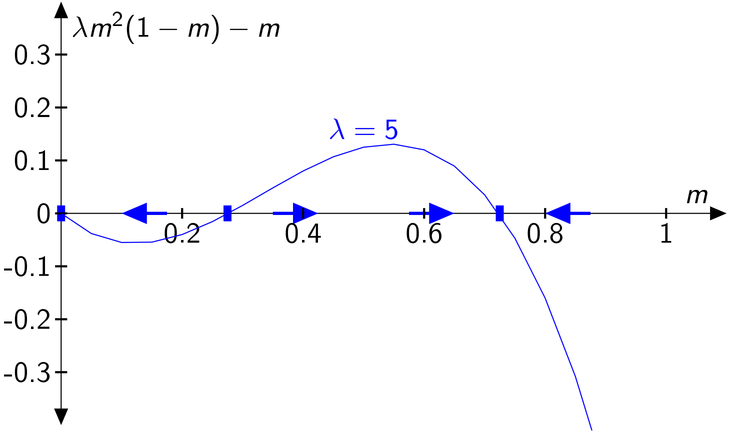
For $\lambda < 4$, the equation $\frac{\partial}{\partial t}m = \lambda m^2(1-m) - m$ has only the stable fixed point $m = 0$.

Constant densities



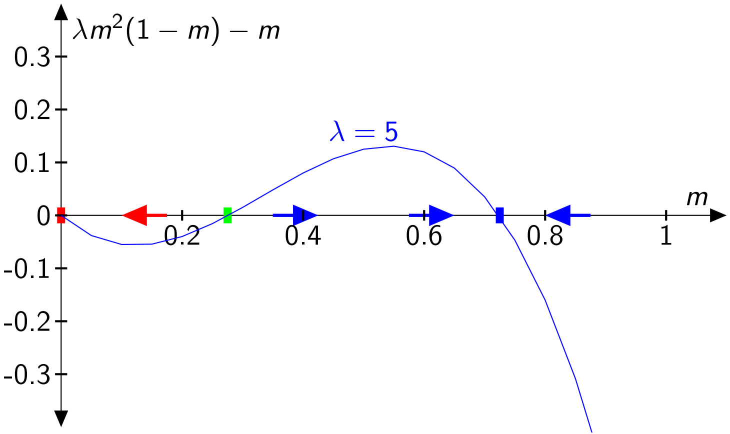
For $\lambda = 4$, the equation $\frac{\partial}{\partial t} m = \lambda m^2(1-m) - m$ obtains a second fixed point at $m = 0.5$.

Constant densities



For $\lambda > 4$, the equation $\frac{\partial}{\partial t} m = \lambda m^2(1-m) - m$ has one unstable and two stable fixed points.

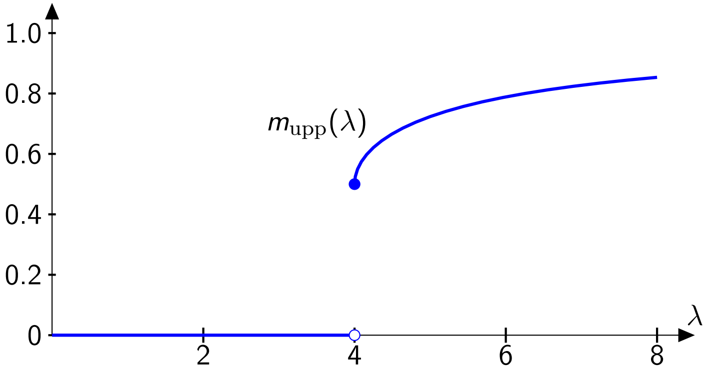
Constant densities



The unstable fixed point represents a critical density below which the population is doomed to die out.

Constant densities

Starting with density $m(x,0) \equiv 1$, the hydrodynamic limit converges to the upper fixed point $\lim_{t\to\infty} m(x,t) = m_{\mathrm{upp}}$.



We observe a first-order phase transition.

The stochastic model

Define

- The process *survives* if $\mathbb{P}^x[X_t \neq \underline{0}\ \forall t \geq 0] > 0$ for some, and hence for all finite nonzero initial states x.
- The process is *stable* if there exists an invariant law that is concentrated on nonzero states.

Monotonicity implies that there exist $\lambda_{\rm c}, \lambda'_{\rm c}$ such that

- The process survives for $\lambda > \lambda_{\rm c}$ and dies out for $\lambda < \lambda_{\rm c}$.
- The process is stable for $\lambda > \lambda'_{\rm c}$ and unstable for $\lambda < \lambda'_{\rm c}$.

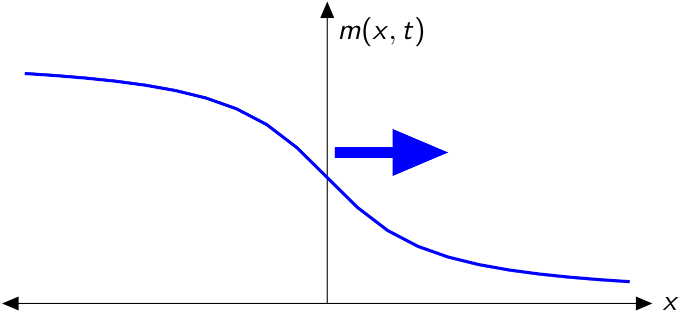
Open problem: Prove that $\lambda_{\rm c} = \lambda'_{\rm c}$.

[Noble '92] $2 \leq \lambda'_{\rm c}(\varepsilon)$ for all $\varepsilon > 0$ and $\limsup_{\varepsilon \downarrow 0} \lambda'_{\rm c}(\varepsilon) \leq 4.5$.

Conjecture: $\lim_{\varepsilon \downarrow 0} \lambda'_{\rm c}(\varepsilon) = 4.5$.

Travelling waves

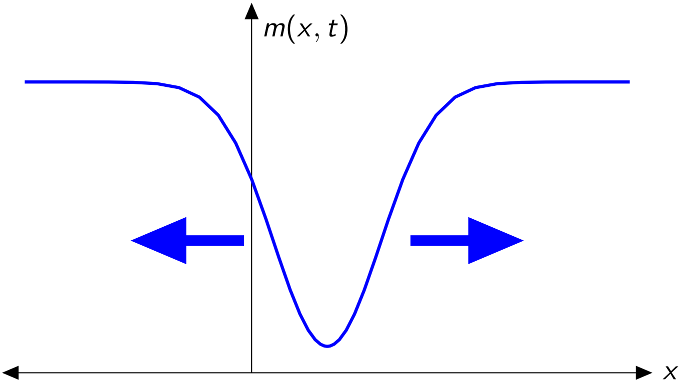
For $\lambda > 4$, the equation $\frac{\partial}{\partial t} m = \frac{\partial^2}{\partial x^2} m + \lambda m^2 (1-m) - m$ has travelling wave solutions.



[DeMasi, Ianiro, Pellegrinotti, & Presutti '84] The propagation speed is positive for $\lambda > 4.5$, and negative for $4 < \lambda < 4.5$.

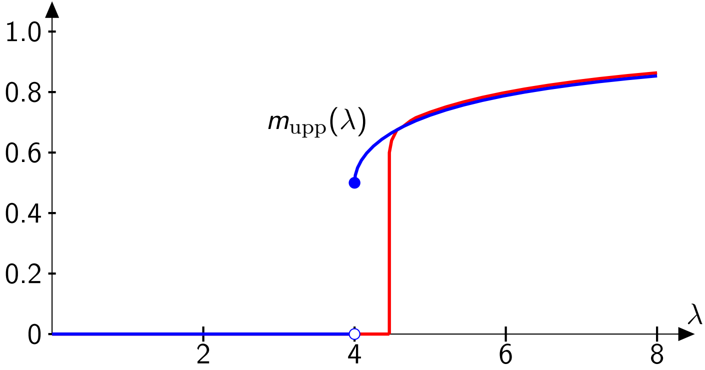
Metastability

For $4 < \lambda < 4.5$ and ε small, rare random events bring the local particle density below a critical value.



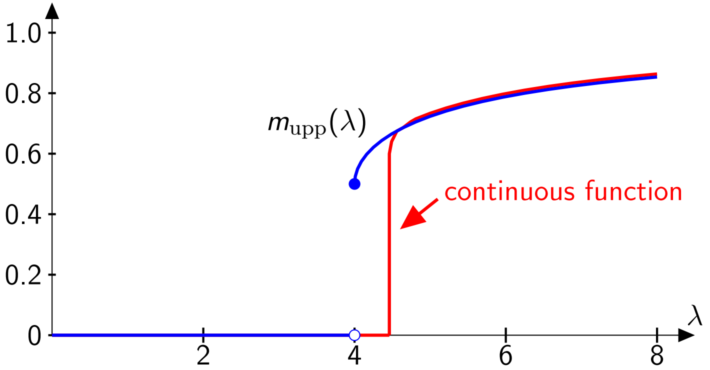
The interval of low population density spreads in both directions.

The upper invariant law



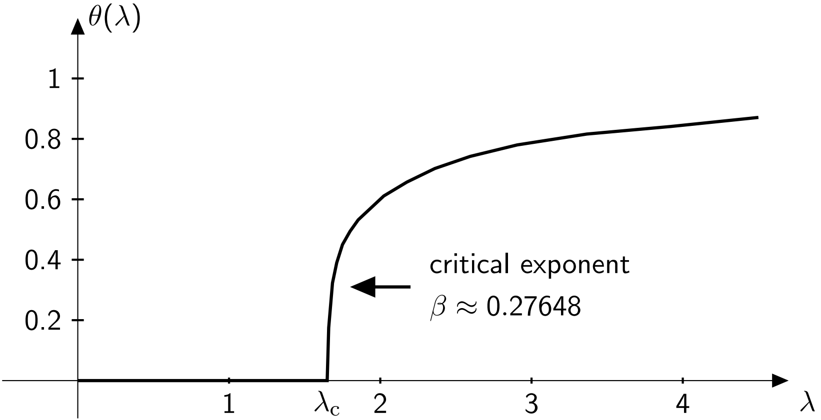
[Noble '92] For small $\varepsilon > 0$, the density of the upper invariant law is at least $m_{\text{upp}}(\lambda)$ for $\lambda > 4.5$ and close to zero for $\lambda < 4.5$.

The upper invariant law



Conjecture For fixed $\varepsilon > 0$, the phase transition is second order and in the same universality class as the contact process.

The upper invariant law



Density of the upper invariant law of the 1D contact process.
$\theta(\lambda) \propto (\lambda - \lambda_c)^\beta$ as $\lambda \downarrow \lambda_c$

Equality of the critical points

Recall that $\lambda_{\rm c}$ and $\lambda'_{\rm c}$ are the critical points for survival of finite systems resp. for the density of the upper invariant law.

For the contact process, $\lambda_{\rm c} = \lambda'_{\rm c}$ by self-duality.

The sexual reproduction process without stirring is an attractive spin system.
For such systems, Bezuidenhout and Gray (1994) prove that survival implies a lower bound in terms of supercritical oriented percolation and hence nontriviality of the upper invariant law.
It follows that $\lambda'_{\rm c} \leq \lambda_{\rm c}$ (without stirring).

Conversely, nontriviality of the upper invariant law seems to imply a positive propagation speed and hence survival. Proof?

A cooperative branching-coalescent

Let $(X_t)_{t\geq 0}$ with $X_t = (X_t(i))_{i\in\mathbb{Z}}$ take values in the space of all configurations $\ldots 101101001001 \ldots$ and evolve as:

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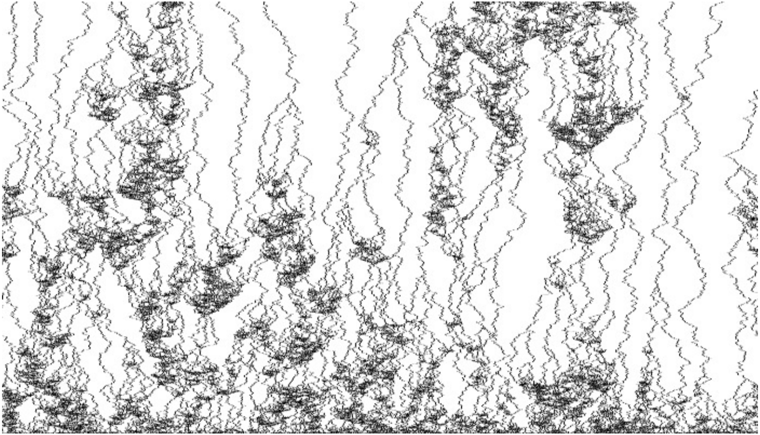
- Migration.
- No spontaneous deaths!

A cooperative branching-coalescent



Time = upwards, black = a particle, $\lambda = 2.333$.

Critical points

Define

- The process *survives* if $\mathbb{P}^x\big[|X_t| > 1\ \forall t \geq 0\big] > 0$ for some, and hence for all initial states with $1 < |x| < \infty$ particles. Note: a single particle can neither die nor reproduce!
- The process is *stable* if there exists an invariant law that is concentrated on nonzero states.

Monotonicity implies that there exist $\lambda_{\rm c}, \lambda'_{\rm c}$ such that

- The process survives for $\lambda > \lambda_{\rm c}$ and dies out for $\lambda < \lambda_{\rm c}$.
- The process is stable for $\lambda > \lambda'_{\rm c}$ and unstable for $\lambda < \lambda'_{\rm c}$.

[Sturm & S. '14] $1 \leq \lambda_{\rm c}, \lambda'_{\rm c} < \infty$.

Numerically: $\lambda_{\rm c} \approx \lambda'_{\rm c} \approx 2.47 \pm 0.02$.

Open problem: Prove that $\lambda_{\rm c} = \lambda'_{\rm c}$.

Proof of the phase transition

Note: If we combine normal branching:

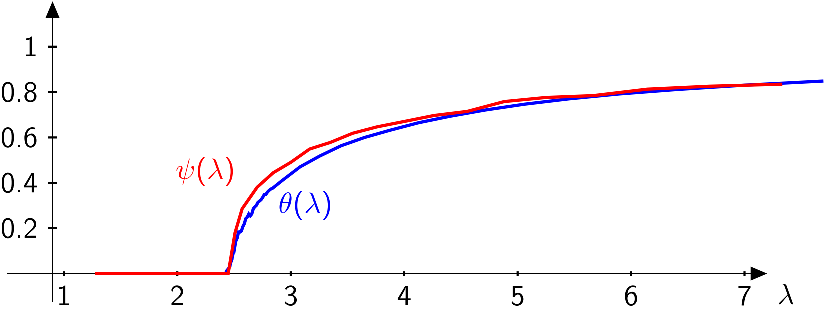
$$01 \mapsto 11 \text{ and } 10 \mapsto 11 \text{ at rate } \tfrac{1}{2}\lambda \text{ each},$$

with coalescence, then the process converges to an invariant law that is product measure with intensity $\lambda/(1+\lambda)$
-no phase transition!

For the *cooperative* branching-coalescent, particles die at a rate proportional to the number of neighboring pairs 11, and particles are born at a rate less than λ times that number
-no survival and no nontrivial invariant law for $\lambda \leq 1$.

For large λ, survival and existence of a nontrivial invariant law follow from comparison with oriented percolation.

Critical points



$\psi(\lambda) := \mathbb{P}\big[|X_t| > 1\ \forall t \geq 0\big]$ starting with two particles on neighboring sites.

$\theta(\lambda) := \mathbb{P}[X_\infty(0) = 1]$ where X_∞ distributed according to the upper invariant law.

Critical exponent

Numerically, the density of the upper invariant law satisfies

$$\theta(\lambda) \propto (\lambda - \lambda_{\rm c})^{\beta} \qquad \text{as } \lambda \downarrow \lambda_{\rm c},$$

with

$$\beta \approx 0.5 \pm 0.1,$$

which differs from the $\beta \approx 0.27648$ of the contact process.

The subcritical regime

Consider

$$\mathbb{P}\big[|X_t| > 1\big] \qquad \text{with} \quad X_0 = \delta_0 + \delta_1 \quad \text{(two particles)},$$
$$\mathbb{P}\big[X_t(0) = 1\big] \qquad \text{with} \quad X_0 = \underline{1} \quad \text{(fully occupied)}.$$

[Bezuidenhout & Grimmett '91] For the contact process, in the subcritical regime $\lambda < \lambda_{\rm c}$, both quantities decay exponentially fast to zero.

[Sturm & S. '14] For the cooperative branching-coalescent, both quantities decay not faster than as $t^{-1/2}$. For $\lambda \leq \frac{1}{2}$, this is the exact rate of convergence.

Proof of the lower bound: By monotonicity, we can estimate the cooperative branching-coalescent by a pure coalescent, for which both quantities decay like $t^{-1/2}$.

Graphical representations

-To prepare for the upper bound, we need a bit of theory.-

Let $(X_t)_{t\geq 0}$ be a Markov process with state space of the form $S = \{0,1\}^\Lambda$, where Λ is a countable set, and generator of the form

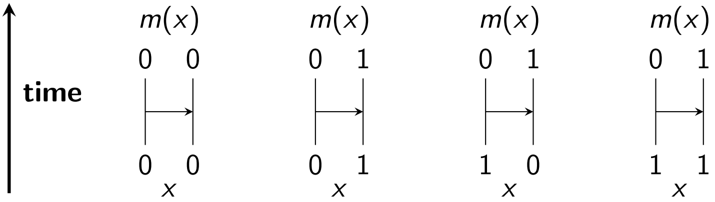
$$Gf(x) = \sum_{m\in\mathcal{M}} r_m\big(f(m(x)) - f(x)\big),$$

where $r_m \geq 0$ are rates and $m \in \mathcal{M}$ are *local maps* $m : S \to S$.

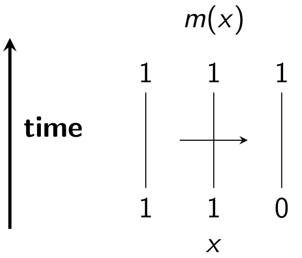
Then $(X_t)_{t\geq 0}$ can be constructed with a *graphical representation* where each local map m is applied at the times of an independent Poisson process with rate r_m.

Local maps

Example: coalescing random walk jump.



Example: cooperative branching.

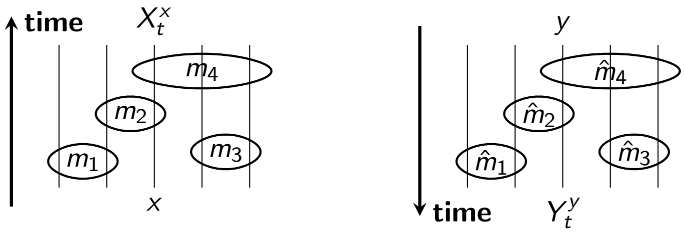


in all other cases nothing happens

Dual maps

Two local maps $m : S \to S$ and $\hat{m} : \hat{S} \to \hat{S}$ are *dual* with respect to a duality function $\psi : S \times \hat{S} \to \mathbb{R}$ if

$$\psi\big(m(x), y\big) = \psi\big(x, \hat{m}(y)\big) \qquad (x \in S,\ \ y \in \hat{S}).$$



[JK] *pathwise duality* $\qquad \psi\big(X^x_t, y\big) = \psi\big(x, Y^y_t\big) \quad \text{a.s.}$

$[0, t] \ni s \mapsto \psi\big(X^x_{s-}, Y^y_{t-s}\big)$ a.s. constant, $\quad X^x_{s-}$ indep. of Y^y_{t-s}.

A formal dual

Let $m : S \to S$ and let $m^{-1} : \mathcal{P}(S) \to \mathcal{P}(S)$ be the *inverse image map*, where $\mathcal{P}(S) := \{A : A \subset S\}$.

Then m^{-1} is dual to m w.r.t. the function $\psi(x, A) := 1_{\{x \in A\}}$.

$$\psi\big(m(x), A\big) = 1_{\{m(x) \in A\}} = 1_{\{x \in m^{-1}(A)\}} = \psi\big(x, m^{-1}(A)\big).$$

Every Markov process $(X_t)_{t \geq 0}$ in S has a pathwise dual $(A_t)_{t \geq 0}$ taking values in $\mathcal{P}(S)$.

Idea: Look for invariant subspaces of $\mathcal{P}(S)$.

Monotone maps

Def m *montone* iff $\quad x \leq y \ \Rightarrow\ m(x) \leq m(y).$

Def $B^{\downarrow} := \{x \in S : x \leq y \text{ for some } y \in B\}.$

Def A *decreasing* iff $\quad x \leq y \in A \quad \Rightarrow x \in A$

$$\Leftrightarrow \quad A = B^{\downarrow} \text{ for some } B.$$

Lemma m monotone iff m^{-1} maps decreasing sets into decreasing sets

$$\Leftrightarrow \quad \exists \text{ dual } \hat{m} \text{ w.r.t. } \psi(x, B) = 1_{\{x \in B^{\downarrow}\}}.$$

Gray's (1986) dual $(B_t)_{t \geq 0}$ for general monotone (spin) systems.

Additive maps

Def m *additive* iff

- $m(\underline{0}) = \underline{0}$
- $m(x \vee y) = m(x) \vee m(y)$

Def A *ideal* iff $A = \{y\}^{\downarrow}$ for some $y \in S$.

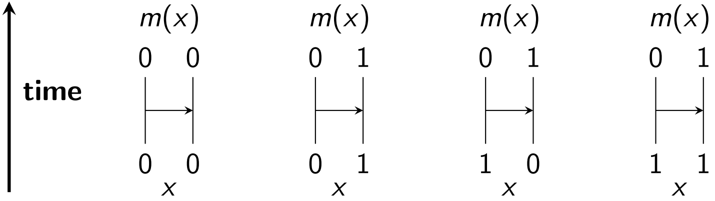
Lemma m additive iff m^{-1} maps ideals into ideals

$\Leftrightarrow$ $\exists$ dual $\hat{m}$ w.r.t. $\psi(x, y) = 1_{\{x \leq y\}}$.

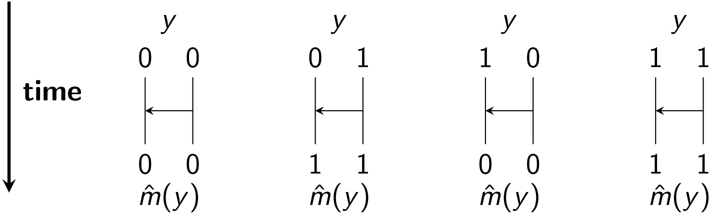
Additive systems duality **[Griffeath 1979]**.

Coalescing random walk duality

Example: The coalescing random walk map is additive.

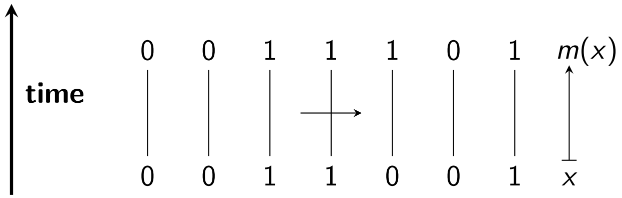


Its dual w.r.t. $\psi(x, y) = 1_{\{x \leq y\}}$ is the voter model map.

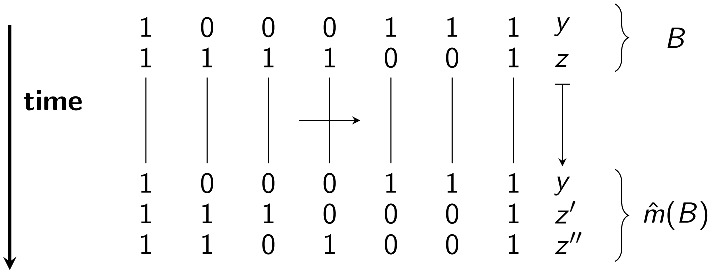


Cooperative branching duality

Example: The cooperative branching map is monotone, but only *superadditive*: $m(x \vee y) \geq m(x) \vee m(y)$.



Cooperative branching duality



$$m(x) \in B^{\downarrow} \quad \text{iff} \quad x \in \big(\hat{m}(B)\big)^{\downarrow}.$$

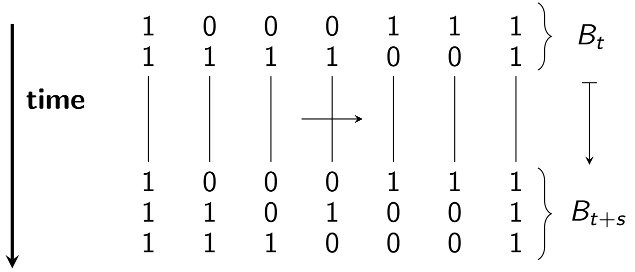
$$m(x) \leq y \text{ for some } y \in B \quad \text{iff} \quad x \leq y' \text{ for some } y \in \hat{m}(B).$$

Cooperative branching duality

The dual process $(B_t)_{t\geq 0}$ of the cooperative branching-coalescent takes values in the space $\mathcal{P}\big(\{0,1\}^{\mathbb{Z}}\big)$.

Each element of B_t evolves as a voter model.

At a cooperative branching event, some elements of B_t split into two new elements.



Extinction

Start $X_0 = \underline{1} = \ldots 1111111 \ldots$

$$B_0 = \{y_0\} \text{ with } y_0 = \ldots 111\underset{\substack{\uparrow \\ \text{origin}}}{0}111 \ldots$$

$\mathbb{P}[X_t(0) = 0] = \mathbb{P}[X_t \leq y_0] = \mathbb{P}[\underline{1} \leq y \text{ for some } y \in B_t] = \mathbb{P}[\underline{1} \in B_t]$.

Lemma The cooperative branching-coalescent is stable if and only if the dual "survives" in the sense that

$$\mathbb{P}[\underline{1} \notin B_t \ \forall t \geq 0] > 0.$$

(Alternatively, let $B^{\mathrm{c}} := \{1 - y : y \in B\}$.
Then $x \in B^{\downarrow}$ if and only if "$x \cap y \neq \emptyset$" for some $y \in B^{\mathrm{c}}$.
Extinction now means that $\underline{0} \in B^{\mathrm{c}}_t$ for some $t \geq 0$.)

Extinction

Start $X_0 = \underline{1} = \ldots 1111111 \ldots$

$$B_0 = \{y_0\} \text{ with } y_0 = \ldots 111\underset{\substack{\uparrow \\ \text{origin}}}{0}111 \ldots$$

Claim

$$\mathbb{P}[X_t(0) = 1] = \mathbb{P}[\underline{1} \notin B_t] \leq Ct^{-1/2}.$$

Proof

$$\frac{\partial}{\partial t}\mathbb{P}[X_t(0) = 1] = (\lambda - 1)\mathbb{P}[X_t(0:1) = 11] - \lambda\mathbb{P}[X_t(0:2) = 111]$$

Suffices to prove:

$$\mathbb{P}[X_t(0:1) = 11] \leq Ct^{-3/2}.$$

Extinction

Start $X_0 = \underline{1} = \ldots 1111111 \ldots$

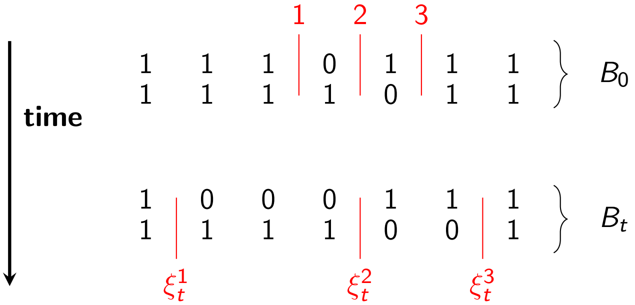
$$B_0 = \{y_0, y_1\} \text{ with } \begin{aligned} y_0 &= \ldots 1110111 \ldots \\ y_1 &= \ldots 1111011 \ldots \end{aligned}$$

Claim

$$\mathbb{P}[X_t(0:1) = 11] = \mathbb{P}[\underline{1} \notin B_t] \leq Ct^{-3/2}.$$

The case without branching

If the cooperative branching rate λ is zero, then the first time that $\underline{1} \in B_t$ is the first time that two out of three walkers meet.



Coalescing random walks

Let $(\xi_t^i)_{t \geq 0}^{i \in \mathbb{Z}}$ be coalescing random walks, started from every site $i \in \mathbb{Z}$.

Let $\tau_{ij} := \inf\{t \geq 0 : \xi_t^i = \xi_t^j\}$.

Facts:

$$\mathbb{P}[\tau^{12} \wedge \tau^{23} > t] \sim \frac{1}{2\sqrt{\pi}} t^{-3/2},$$

$$\mathbb{E}[\tau^{ij} \wedge \tau^{jk}] = (j-i)(k-j) \qquad (i < j < k).$$

The case with branching

If a cooperative branching event occurs, then we use *subduality:* it suffices to show that both B'_{t+s} and B''_{t+s} die out.

time

$$\left.\begin{array}{c|ccc|cc|c} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right\} B_t$$

$$\left.\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right\} B_{t+s}$$

$$\left.\begin{array}{c|ccc|cc|c} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right\} B'_{t+s}$$

$$\left.\begin{array}{cc|c|c|ccc} 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{array}\right\} B''_{t+s}$$

The case with branching

This leads to a (dependent) branching process where triples of random walks die as soon as two out of the three meet, but before it dies, with rate 2λ, a triple can give birth to a new triple of random walks, started on neighboring positions. As long as $\lambda < \frac{1}{2}$, it can be shown that this branching process dies out and the probability to be alive at time t decays as $t^{-3/2}$.