The contact process seen from a typical infected site

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Contact processes on groups

Λ: countable group with group action $(i,j) \mapsto ij$, inverse operation $i \mapsto i^{-1}$, and unit element (origin) 0.

a: function $a : \Lambda \times \Lambda \to [0,\infty)$ s.t.

$$\begin{aligned} &\text{(i)} \quad a(i,j) = a(ki,kj) \qquad\qquad (i,j,k \in \Lambda),\\ &\text{(ii)} \quad |a| := \sum_{i \in \Lambda} a(0,i) < \infty. \end{aligned}$$

δ: nonnegative constant.

Definition The (Λ, a, δ)-*contact process* $(\eta_t)_{t \geq 0}$ is a Markov process taking values in the subets of Λ. Sites $i \in \eta_t$ are called *infected*.

- An infected site at i infects a healthy site at j with rate $a(i,j)$.
- Infected sites recover with rate δ.

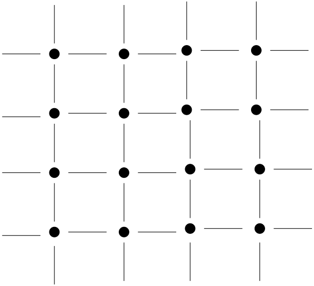
Cayley graphs

Let $\Delta \subset \Lambda$ be a finite, symmetric generating set for Λ.

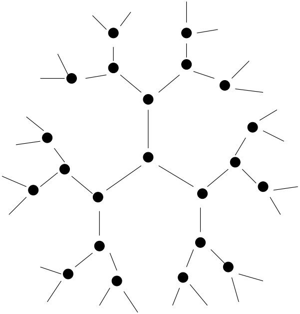
Definition The (left) *Cayley graph* $\mathcal{G}(\Lambda, \Delta)$ is defined as

- Vertex set Λ.
- An edge connects $i, j \in \Lambda$ iff there is a $k \in \Delta$ such that $j = ki$.

Examples $\mathbb{Z}^d$ with nearest-neighbor bonds, regular tree $\mathbb{T}_d$.



$\mathbb{Z}^2$



$\mathbb{T}_2$

Nearest neighbor processes

Let $\Delta \subset \Lambda$ be a finite, symmetric generating set for Λ.

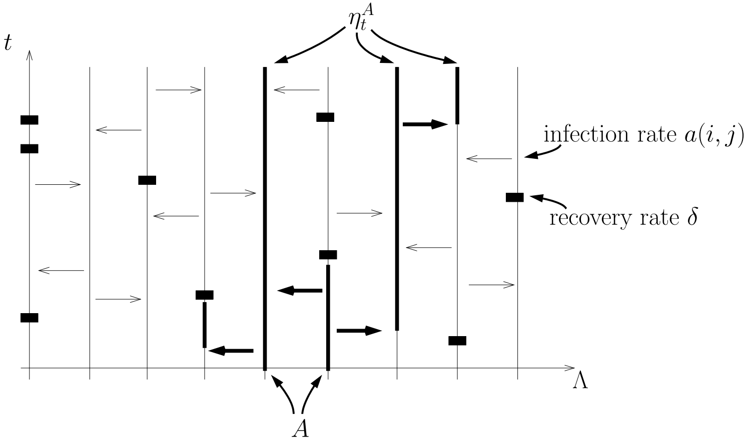
Let $a(i,j) := \lambda 1_{\{i \in \Delta j\}}$, i.e., $a(i,j) = \lambda$ if i,j are connected by an edge in $\mathcal{G}(\Lambda, \Delta)$ and zero otherwise.

Then the (Λ, a, δ)-contact process is the *nearest-neighbor* contact process on $\mathcal{G}(\Lambda, \Delta)$ with infection rate λ and recovery rate δ.

Remark There are interesting contact processes on groups that are not finitely generated, e.g. the hierarchical group. *Graphs are not all that matters.*

Remark Asymmetric processes (with $a(i,j) \neq a(j,i)$) are also interesting.

Graphical representation



Duality

Set $a^\dagger(i,j) := a(j,i)$.
Let $(\eta_t^A)_{t\geq 0}$ denote the (Λ, a, δ)-contact process started in $\eta_0^A = A$.
Let $(\eta_t^{\dagger\, B})_{t\geq 0}$ denote the $(\Lambda, a^\dagger, \delta)$-contact process started in $\eta_0^{\dagger\, B} = B$. Then

$$\mathbb{P}\big[\eta_t^A \cap B \neq \emptyset\big] = \mathbb{P}\big[A \cap \eta_t^{\dagger\, B} \neq \emptyset\big] \qquad (t \geq 0).$$

Remark The (Λ, a, δ)- and $(\Lambda, a^\dagger, \delta)$-contact processes are in general different and need to be distinguished.

Survival

Definition A (Λ, a, δ)-contact process *survives* if

$$\theta = \theta(\Lambda, a, \delta) := \mathbb{P}\big[\eta_t^{\{0\}} \neq \emptyset \ \forall t \geq 0\big] > 0.$$

It *survives locally* if

$$\liminf_{t\to\infty} \mathbb{P}\big[0 \in \eta_t^{\{0\}}\big] > 0.$$

$$\delta_{\rm c}(\Lambda, a) := \sup\{\delta \geq 0 : \text{the } (\Lambda, a, \delta)\text{-contact process survives}\},$$
$$\delta'_{\rm c}(\Lambda, a) := \sup\{\delta \geq 0 : \text{the } (\Lambda, a, \delta)\text{-contact process survives locally}\}.$$

The upper invariant law

Definition A measure μ on the set of subsets of Λ is *nontrivial* if $\mu(\{\emptyset\}) = 0$ and *homogeneous* if μ is invariant under translations $A \mapsto iA := \{ij : j \in A\}$.
Set

$$\overline{\eta}_t := \{i \in \Lambda : -\infty \rightsquigarrow (i, t)\}$$

where $-\infty \rightsquigarrow$ indicates the presence of an open path in the graphical representation started at time $-\infty$.
Then $\overline{\nu} := \mathbb{P}[\overline{\eta}_t \in \cdot\,]$ is the *upper invariant law*.

- $\overline{\nu}$ is a homogeneous invariant law.
- $\overline{\nu}$ is the *largest* invariant law (in the stochastic order).
- $\overline{\nu}$ is nontrivial iff the $(\Lambda, a^\dagger, \delta)$-contact process survives.
- All homog. invar. laws are convex combinations of $\overline{\nu}$ and $\delta_\emptyset$.
- The process started in a homog. nontriv. law converges to $\overline{\nu}$. (Harris, 1974)

The process on $\mathbb{Z}^d$

For *symmetric, finite-range* processes on $\mathbb{Z}^d$ much more is known: (Bezuidenhout & Grimmett 1990 and references therein)

- Survival implies local survival, $\delta_{\rm c} = \delta'_{\rm c}$.
- The critical process dies out: no survival at $\delta = \delta_{\rm c}$.
- Complete convergence:

$$\mathbb{P}[\eta_t^{\{0\}} \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \theta\overline{\nu} + (1-\theta)\delta_\emptyset.$$

- *All* invar. laws are convex combinations of $\overline{\nu}$ and $\delta_\emptyset$.
- Shape theorem.

For *asymmetric* processes much less has been proved and statements need modifications!

The process on trees

For nearest-neighbor processes on regular trees $\mathbb{T}_d$, the situation is quite different: (See Liggett's 1999 book)

- $\delta'_{\rm c} < \delta_{\rm c}$.
- Local survival on $[0, \delta'_{\rm c})$, global but not local survival on $[\delta'_{\rm c}, \delta_{\rm c})$.
- No complete convergence on $[\delta'_{\rm c}, \delta_{\rm c})$.
- Many extremal invariant measures on $[\delta'_{\rm c}, \delta_{\rm c})$.

Question Which properties of $\mathbb{Z}^d$ resp. $\mathbb{T}_d$ are responsible for the different behavior?

Amenability

Let $\mathcal{G}(\Lambda, \Delta)$ be a Cayley graph $\mathcal{G}(\Lambda, \Delta)$ and let $d(i, j)$ denote the usual graph distance.

Definition $\mathcal{G}(\Lambda, \Delta)$ is *amenable* if for every $\varepsilon > 0$ there exists a finite nonzero $A \subset \Lambda$ such that

$$\frac{|\partial A|}{|A|} \leq \varepsilon \quad \text{where} \quad \partial A := \big\{ i \notin A : \exists j \in A \text{ s.t. } d(i, j) = 1 \big\}.$$

$\mathcal{G}(\Lambda, \Delta)$ is said to have *exponential growth* (resp. *subexponential growth*) if the limit

$$\lim_{n \to \infty} \frac{1}{n} \log \big| \{ i \in \Lambda : d(0, i) \leq n \} \big|$$

is positive (resp. zero).

$$\text{subexponential growth} \quad \begin{array}{c} \Rightarrow \\ \not\Leftarrow \end{array} \quad \text{amenability.}$$

Branching-coalescing particles

Excursion to another model

- Particles jump from i to j with rate $a(i,j)$.
- Particles split into two with rate $b > 0$.
- Pairs of particles on one site coalesce with rate $c > 0$.

This model survives globally for all $b, c > 0$.

$$b_{\rm c} = b_{\rm c}(a,c) := \inf\{b > 0 : \text{ the process survives locally}\}.$$

For the nearest-neighbor process on Cayley graphs, Frank Schirmeier (Erlangen) has shown:

- If Λ is nonamenable (e.g. a tree), then $b_{\rm c} > 0$.
- If Λ is of subexponential growth (e.g. $\mathbb{Z}^d$), then $b_{\rm c} = 0$.

Open problem Amenable graphs of exponential growth (e.g. the lamplighter group).

The exponential growth rate

It follows from subadditivity that for each (Λ, a, δ)-contact process, the limit

$$r = r(\Lambda, a, \delta) := \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}\big[|\eta_t^{\{0\}}|\big]$$

exists. We call r the *exponential growth rate*.

(i) $r < 0$ iff $\delta > \delta_c$ (Menshikov '86, Aizenman & Barsky '87, Aizenman & Jung '07).

(ii) If $r > 0$ then $\theta > 0$ (i.e., the process survives).

(iii) If Λ is nonamenable and $\theta > 0$, then $r > 0$ (S. 2009).

(iv) For n.n. processes on subexponential groups, $r \leq 0$.

(v) The function $\delta \to r(\Lambda, a, \delta)$ is nonincreasing and continuous.

Parts (ii), (iii) and (v) imply:

- The critical contact process on a nonamenable group dies out.

Infinite starting measures

Let $\mathcal{P}_+ := \{A \subset \Lambda : A \neq \emptyset\}$ and

$$\mu_t := \sum_{i\in\Lambda} \mathbb{P}\big[\eta_t^{\{i\}} \in \cdot\,\big]\big|_{\mathcal{P}_+}$$

be the "law" of the process started with a single, uniformly chosen infected site. This is an infinite measure but conditional probabilities such as

$$\mu_t\big(\,\cdot\,\big|\{A : 0 \in A\}\big)$$

are well-defined. Note that

$$\begin{aligned}\mu_t\big(\{A : 0 \in A\}\big) &= \sum_{i\in\Lambda} \mathbb{P}\big[0 \in \eta_t^{\{i\}}\big] \\ &= \sum_{i\in\Lambda} \mathbb{P}\big[i^{-1} \in \eta_t^{\{0\}}\big] = \mathbb{E}\big[|\eta_t^{\{0\}}|\big] =: E_t.\end{aligned}$$

Palm particles

One has

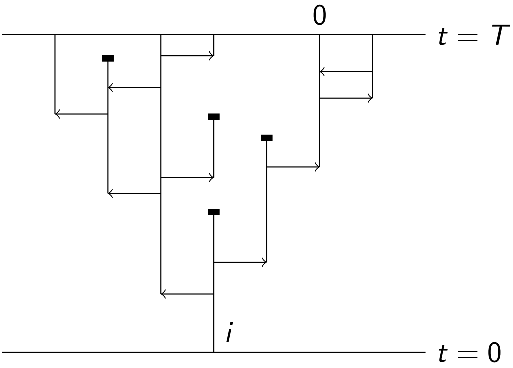
$$\mu_t\big(\,\cdot\,\big|\{A:0\in A\}\big)=\hat{\mathbb{P}}\big[\iota^{-1}\eta_t^{\{0\}}\in\cdot\,\big],$$

where $\hat{\mathbb{P}}$ is a *Palm law*, i.e., $\hat{\mathbb{P}}$ is obtained from the old law $\mathbb{P}$ by size-biasing on $|\eta_t^{\{0\}}|$ and then choosing a 'Palm site' $\iota\in\eta_t^{\{0\}}$ with equal probabilities from $\eta_t^{\{0\}}$.

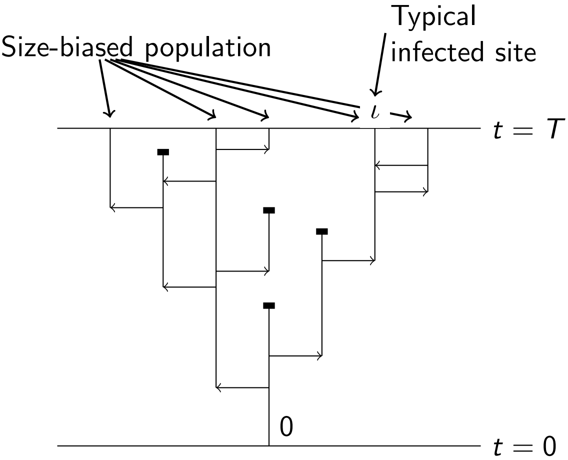
The process seen from a typical infected site.

A typical ancestor

$\mu_t\big(\,\cdot\,\big|\{A : 0 \in A\}\big)$



A typical infected site



Eigenmeasures

Let $\mathcal{P} := \{A : A \subset \Lambda\} \cong \{0,1\}^\Lambda$, equipped with the product topology, and $\mathcal{P}_+ := \{A \in \mathcal{P} : A \neq \emptyset\}$.

Definition An *eigenmeasure* with *eigenvalue* λ is a nonzero, locally finite measure μ on $\mathcal{P}_+$ such that

$$\int \mu(\mathrm{d}A)\mathbb{P}[\eta_t^A \in \cdot\,]\big|_{\mathcal{P}_+} = e^{\lambda t}\mu \qquad (t \geq 0).$$

Conjecture

$$\frac{1}{E_t}\mu_t \underset{t\to\infty}{\Longrightarrow} \mu$$

where $\Rightarrow$ denotes vague convergence on $\mathcal{P}_+$ and μ is an eigenmeasure with eigenvalue r.

Existence

Proposition Let $\hat{\mu}_\alpha$ be the Laplace transform of $(\mu_t)_{t\geq 0}$, i.e.,

$$\hat{\mu}_\alpha := \int_0^\infty \mu_t \, e^{-\alpha t} \mathrm{d}t \qquad (\alpha > r).$$

Then

$$\hat{\mu}_\alpha(\{A : 0 \in A\}) = \int_0^\infty E_t \, e^{-\alpha t} \mathrm{d}t =: \hat{E}_\alpha \qquad (\alpha > r).$$

The measures $(\frac{1}{\hat{E}_\alpha}\hat{\mu}_\alpha)_{\alpha > r}$ are relatively compact in the topology of vague convergence on $\mathcal{P}_+$, and each subsequential limit as $\alpha \downarrow r$ is a homogeneous eigenmeasure with eigenvalue r.

Open Problems Uniqueness. Convergence of $\frac{1}{E_t}\mu_t$.

The case of zero growth

Theorem (S. '09) Assume that the upper invariant law $\overline{\nu}$ is nontrivial. Then $\overline{\nu}$ is up to a multiplicative constant the unique homogeneous eigenmeasure with eigenvalue 0.

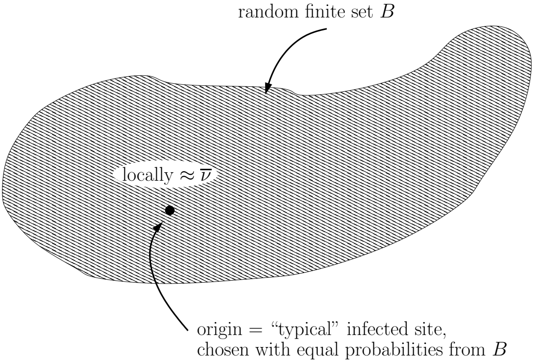
This says that if $r = 0$ and the process survives ($\theta > 0$), then the process seen from a typical infected site at large (Laplace) times looks locally like the upper invariant law.

Note that this also applies to *asymmetric* processes *regardless of local survival*.

Consequence If Λ is nonamenable and $\theta > 0$, then $r > 0$.

Proof of the consequence

Assume that $\overline{\nu}$ is nontrivial and $r = 0$. Consider the law $\hat{\mu}_\alpha$, conditioned on the event $\{A : 0 \in A\}$. This law describes a random finite set B, containing the origin, that looks something like this:



This contradicts nonamenability.

The subcritical case

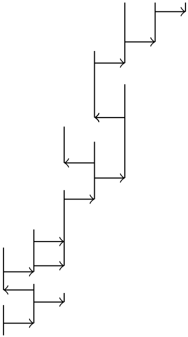
Theorem (Sturm & S. '11) Assume that $r < 0$. Then there exists, up to a multiplicative constant, a unique homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue r. For any nonzero, homogeneous, locally finite measure μ on $\mathcal{P}_+$, one has

$$e^{-rt} \int \mu(\mathrm{d}A)\mathbb{P}[\eta_t^A \in \cdot\,]\big|_{\mathcal{P}_+(\Lambda)} \underset{t\to\infty}{\Longrightarrow} c\,\mathring{\nu},$$

where $\Rightarrow$ denotes vague convergence and $c > 0$.

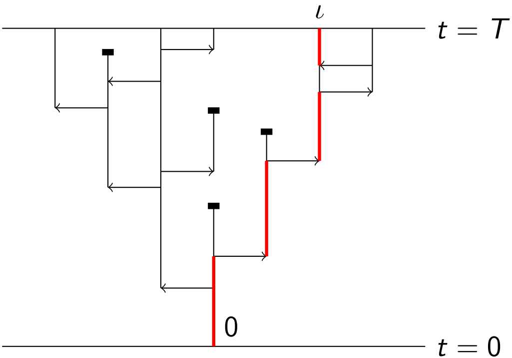
Moreover, $\mathring{\nu}$ is concentrated on finite sets. There are no homogeneous eigenmeasures with eigenvalues other than r.

Backward path



The fact that $\mathring{\nu}$ is concentrated on finite sets means that there is almost a single path leading up to the typical particle.

Russo's formula



Say that a point (i, t) is *pivotal* if all paths $(0,0) \rightsquigarrow (\iota, T)$ pass through it. Russo's formula implies:

$$-\frac{\partial}{\partial \delta} r(\Lambda, a, \delta) = \lim_{T \to \infty} \mathbb{E}\Big[\frac{1}{T}\int_0^T 1_{\{\exists \text{ pivotal at } t\}} \mathrm{d}t\Big].$$

Work in progress

We wish to show that $\delta \mapsto r(\Lambda, a, \delta)$ is continuously differentiable on $(\delta_{\rm c}, \infty)$ and satisfies $\frac{\partial}{\partial\delta} r(\Lambda, a, \delta) < 0$ on $(\delta_{\rm c}, \infty)$. Moreover, we conjecture that

$$-\frac{\partial}{\partial\delta} r(\Lambda, a, \delta) = \frac{\int \mathring{\nu}(\mathrm{d}A) \int \mathring{\nu}^{\dagger}(\mathrm{d}B) 1_{\{A\cap B=\{0\}\}}}{\int \mathring{\nu}(\mathrm{d}A) \int \mathring{\nu}^{\dagger}(\mathrm{d}B) |A\cap B|^{-1} 1_{\{0\in A\cap B\}}},$$

where $\mathring{\nu}$ and $\mathring{\nu}^{\dagger}$ denote the homogeneous eigenmeasures of the (Λ, a, δ)- and $(\Lambda, a^{\dagger}, \delta)$-contact processes, respectively.

Open problem Is $\delta \mapsto r(\Lambda, a, \delta)$ concave?

Related problem Is $\mathring{\nu}$ decreasing in δ, in some sense?