A Fresh Look at R-positivity

Jan M. Swart (ÚTIA AV ČR)

Wednesday, February 28th, 2018

Outline

- The spectral radius of a nonnegative matrix

Outline

- The spectral radius of a nonnegative matrix
- One-dimensional Gibbs measures

Outline

- The spectral radius of a nonnegative matrix
- One-dimensional Gibbs measures
- Equivalence of transfer matrices

Outline

- The spectral radius of a nonnegative matrix
- One-dimensional Gibbs measures
- Equivalence of transfer matrices
- R-recurrence and the Perron-Frobenius theorem

Outline

- The spectral radius of a nonnegative matrix
- One-dimensional Gibbs measures
- Equivalence of transfer matrices
- R-recurrence and the Perron-Frobenius theorem
- Conditions for (strong) R-positivity

Outline

- The spectral radius of a nonnegative matrix
- One-dimensional Gibbs measures
- Equivalence of transfer matrices
- R-recurrence and the Perron-Frobenius theorem
- Conditions for (strong) R-positivity
- Pinning models

The spectral radius

Let $A = (A(x,y))_{x,y \in S}$ be a nonnegative matrix indexed by a countable set S.
We assume that A is *irreducible*, i.e.,
$\forall x, y \in S \ \exists n \geq 0$ s.t. $A^n(x,y) > 0$, and *aperiodic* i.e., the greatest common divisor of $\{n \geq 1 : A^n(x,x) > 0\}$ is one $\forall x$.

[Kingman 1963] The limit

$$\rho(A) := \lim_{n \to \infty} \big(A^n(x,y)\big)^{1/n} \in (0, \infty]$$

exists and does not depend on $x, y \in S$.

We call $\rho(A)$ the *spectral radius* of A.

We assume from now on that $\rho(A) < \infty$.

Gibbs measures

Let Ω^n denote the space of all functions $\omega : \{0, \ldots, n\} \to S$.
Let $\mu_{x,y}^{A,n}$ be the measure on Ω^n defined by

$$\mu_{x,y}^{A,n}(\omega) := 1_{\{\omega_0 = x,\ \omega_n = y\}} \prod_{k=1}^{n} A(\omega_{k-1}, \omega_k).$$

We normalize $\mu_{x,y}^{A,n}$ to a probability measure

$$\overline{\mu}_{x,y}^{A,n} := \frac{1}{A^n(x,y)} \mu_{x,y}^{A,n}.$$

We call $\overline{\mu}_{x,y}^{A,n}$ the *Gibbs measure* on Ω^n with *transfer matrix* A and *boundary conditions* x, y.

Equivalence

[Equivalence of transfer matrices] *Let* A, B *be nonnegative matrices with* $A(x,y) > 0 \Leftrightarrow B(x,y) > 0$ $(x, y \in S)$. *Then the following conditions are equivalent.*

1. $\overline{\mu}^{A,n}_{x,y} = \overline{\mu}^{B,n}_{x,y}$ *for all* x, y, n.
2. *There exists a* $c > 0$ *and* $h : S \to (0, \infty)$ *such that* $B(x,y) := c^{-1} h(x)^{-1} A(x,y) h(y)$.

Moreover, in 2., the matrices A *and* B *determine the constant* c *uniquely and the function* h *uniquely up to a multiplicative constant.*

We write $A \sim_{h,c} B$, $A \sim_c B$, or $A \sim B$ and call A, B *equivalent* if

$$B(x,y) := c^{-1} h(x)^{-1} A(x,y) h(y) \qquad (x, y \in S).$$

One has $A \sim_c B \quad \Rightarrow \quad \rho(A) = c\rho(B)$.

Equivalence

If

$$B(x,y) := c^{-1} h(x)^{-1} A(x,y) h(y) \qquad (x,y \in S),$$

then for $\omega \in \Omega^n$ one has

$$\begin{aligned}
&\prod_{k=1}^{n} B(\omega_{k-1},\omega_k) \\
&\quad = c^{-1} h(\omega_0)^{-1} A(\omega_0,\omega_1) h(\omega_1) c^{-1} h(\omega_1)^{-1} A(\omega_1,\omega_2) h(\omega_2) \\
&\qquad\qquad \cdots c^{-1} h(\omega_{n-1})^{-1} A(\omega_{n-1},\omega_n) h(\omega_n) \\
&\quad = c^{-n} h(\omega_0)^{-1} \prod_{k=1}^{n} A(\omega_{k-1},\omega_k) h(\omega_n).
\end{aligned}$$

Summing over all paths with $\omega_0 = x$, $\omega_n = y$, yields

$$B^n(x,y) := c^{-n} h(x)^{-1} A^n(x,y) h(y) \qquad (x,y \in S).$$

R-recurrence

Observation Let A, P be nonnegative matrices such that $A \sim_{h,c} P$. Then the following conditions are equivalent.

1. $Ah = ch$.
2. P is a probability kernel.

P recurrent $\Rightarrow$ $\rho(P) = 1$ $\Rightarrow$ $\rho(A) = c$.

[David Vere-Jones 1962, 1967] There exists *at most one recurrent* probability kernel P such that $A \sim P$.

Call A *R-recurrent* if such a P exists and *R-transient* otherwise.

[David Vere-Jones 1962, 1967] If A is R-recurrent, then there exists a function $h : S \to (0, \infty)$, unique up to scalar multiples, such that $Ah = \rho(A)h$.

The Perron-Frobenius theorem

It can be shown that every finite nonnegative matrix A is R-recurrent.
Combining this with the result of Vere-Jones, using the fact that finite probability kernels are always recurrent, we obtain:

> **[Perron-Frobenius (1912)]** *Let A be finite. Then there exist a unique constant $c > 0$ and a function $h : S \to (0, \infty)$ that is unique up to scalar multiples, such that $Ah = ch$.*

Where, in fact $c = \rho(A)$.

For infinite matrices, there may exist positive eigenfunctions h corresponding to eigenvalues $c > \rho(A)$. In such cases, the probability kernel defined by $A \sim_{h,c} P$ is transient.

Strong positive recurrence

Let $(X_k)_{k\geq 0}$ be a Markov chain with irreducible transition kernel P. Let $\sigma_x := \inf\{k > 0 : X_k = x\}$ denote the first return time to x.

Def P is *strongly positive recurrent* if for some, and hence for all $x \in S$, there exists an $\varepsilon > 0$ s.t. $\mathbb{E}^x\big[e^{\varepsilon\sigma_x}\big] < \infty$.

[Kendall '59, Vere-Jones '62] Let P be irreducible and aperiodic with invariant law π. Then P is strongly positive recurrent if and only if it is *geometrically ergodic* in the sense that

$$\exists\varepsilon > 0,\ M_{x,y} < \infty \text{ s.t. } \big|P^n(x,y) - \pi(y)\big| \leq M_{x,y}e^{-\varepsilon n}.$$

$(x, y \in S,\ n \geq 0)$.

R-positivity

Let A be R-recurrent and let P be the unique recurrent probability kernel such that $A \sim P$.

We call A *R-null recurrent* if P is null recurrent,
we call A *R-positive* if P is positively recurrent,
we call A *strongly R-positive* if P is strongly positively recurrent.

[David Vere-Jones 1962, 1967] For any reference point x:

$$A \text{ is R-recurrent} \Leftrightarrow \sum_{n=1}^{\infty} \rho(A)^{-n} A^n(x,x) = \infty.$$

$$A \text{ is R-positive} \Leftrightarrow \lim_{n\to\infty} \rho(A)^{-n} A^n(x,x) > 0.$$

Proof $A \sim_c B \;\Rightarrow\; A^n(x,x) = c^n B^n(x,x)$. Now use well-known characterizations of (positive) recurrence. ∎

Finite modifications

Def A and B are *finite modifications* of each other iff
$A(x, y) > 0 \;\Leftrightarrow\; B(x, y) > 0$ $(x, y \in S)$ and
$\big\{(x, y) : A(x, y) < B(x, y)\big\}$ is finite and nonempty.

[Swart 2017] Let $A \leq B$ be finite modifications of each other. Then:

(a) B is strongly R-positive if and only if $\rho(A) < \rho(B)$.

(b) A is R-transient if and only if $\rho(A) = \rho\big(A + \varepsilon(B - A)\big)$ for some $\varepsilon > 0$.

Note: This implies in particular that finite matrices are strongly R-positive.

Def *spectral radius at infinity*

$$\rho_\infty(B) := \inf\{\rho(A) : A \leq B \text{ finite modification}\}.$$

Then B strongly R-positive if and only if $\rho_\infty(B) < \rho(B)$.

Heuristics

Recall spectral radius $\rho(A) := \lim_{n\to\infty} \big(A^n(x,y)\big)^{1/n}$.
Recall unnormalized Gibbs measure

$$\mu^{A,n}_{x,y}(\omega) := 1_{\{\omega_0=x,\ \omega_n=y\}} \prod_{k=1}^{n} A(\omega_{k-1},\omega_k).$$

As $n\to\infty$

$$\sum_{\omega\in\Omega^n} \mu^{A,n}_{x,y}(\omega) = e^{\,n\log\rho(A)\,+\,o(n)}.$$

Likewise, $\rho_\infty(A)$ measures, on an exponential scale, the weight of paths ω that "venture far away from x and y".

If $\rho_\infty(A) < \rho(A)$, paths that stay close to x and y have more weight, on an exponential scale, then paths that venture far away.

The infinite volume limit

[Infinite volume limit] Let A be R-positive and let P be the positive recurrent kernel s.t. $A \sim P$.

Then, for any x, y, one has the one-sided infinite-volume limit

$$\overline{\mu}^{A,n}_{x,y} \underset{n\to\infty}{\Longrightarrow} \nu^P_x ,$$

where ν^P_x is the law of the Markov chain with transition kernel P and initial state x.

Pinning models

Let Q denote the transition kernel of nearest-neighbor random walk on $\mathbb{Z}^d$. Define

$$A_\beta(x,y) := \begin{cases} e^\beta Q(x,y) & \text{if } x = 0, \\ Q(x,y) & \text{otherwise.} \end{cases}$$

[Giacomin, Caravenna, Zambotti 2006] There exists a $-\infty < \beta_{\mathrm{c}} < \infty$ such that:

- A_β is R-transient for $\beta < \beta_{\mathrm{c}}$.
- A_β is R-null recurrent or weakly R-positive for $\beta = \beta_{\mathrm{c}}$.
- A_β is strongly R-positive for $\beta > \beta_{\mathrm{c}}$.

Moreover, $\beta \mapsto \rho(A_\beta)$ is constant on $(-\infty, \beta_{\mathrm{c}}]$ and stricty increasing on $[\beta_{\mathrm{c}}, \infty)$.
One has $\beta_{\mathrm{c}} = 0$ in dimensions $d = 1, 2$ and $\beta_{\mathrm{c}} > 0$ in dimensions $d \geq 3$. In fact, $e^{-\beta_{\mathrm{c}}}$ is the return probability of the random walk.

Bounds on the spectral radius

Sharp upper bounds on $\rho(A)$ can (in principle) be obtained from

$$\rho(A) = \inf\big\{K < \infty : \exists f : S \to (0,\infty) \text{ s.t. } Af \leq Kf\big\}.$$

Let (π, Q) be a pair such that 1. π is a probability measure on some finite $S' \subset S$, 2. Q is a transition kernel on S' with invariant law π. Define a large deviations *rate function*

$$I_A(\pi, Q) := \sum_{x,y} \pi(x) Q(x,y) \log\Big(\frac{Q(x,y)}{A(x,y)}\Big).$$

Sharp lower bounds on $\rho(A)$ can be obtained from

$$\rho(A) = \sup_{(\pi,Q)} e^{-I_A(\pi,Q)}.$$

Open problems

Open problem Necessary and sufficient conditions for strong R-positivity of semigroups $(A_t)_{t\geq 0}$ of nonnegative matrices.

Open problem Prove that $\rho_\infty(A^n) = \rho_\infty(A)^n$.

Open problem A finite modification of $B \not\Rightarrow$ A^n finite modification of B^n. Weaken the concept of "finite modification" so that this holds.

Open problem Show that the contact process modulo translations is strongly R-positive for all $\lambda \neq \lambda_{\rm c}$.

[Sturm & Swart 2014] The contact process modulo translations is R-positive for all $\lambda < \lambda_{\rm c}$.

Proof of the main result

The most interesting part of the main theorem is:

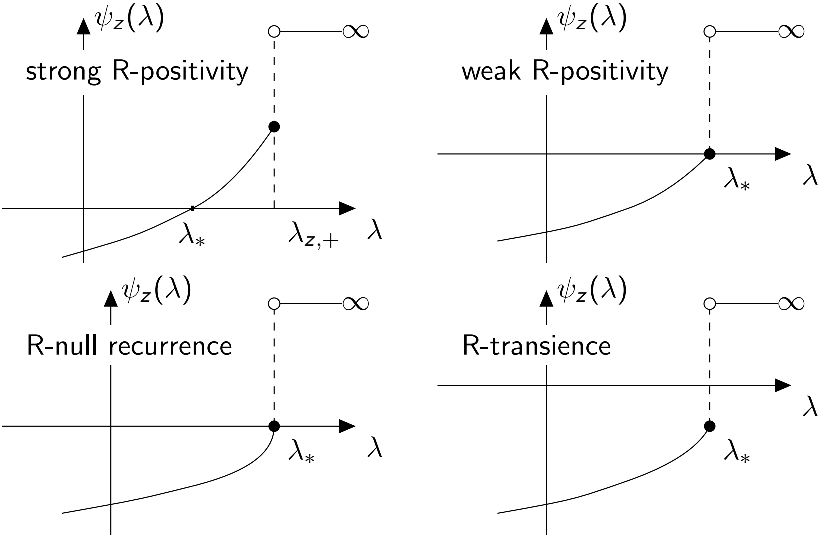
> *Let A be irreducible. If there exists a finite modification $B \leq A$ such that $\rho(B) < \rho(A)$, then $A \sim P$ for some strongly positive recurrent probability kernel P.*

Sketch of the proof Fix a reference point $z \in S$. Let $\widehat{\Omega}_z$ denote the space of all *excursions* away from z, i.e., functions $\omega : \{0, \ldots, n\} \to S$ with $\omega_0 = z = \omega_n$ and $\omega_k \neq z$ for all $0 < k < n$. Let $\ell_\omega := n$ denote the length of ω. Define

$$e^{\psi_z(\lambda)} := \sum_{\omega \in \widehat{\Omega}_z} \nu_\lambda(\omega) \quad \text{with} \quad \nu_\lambda(\omega) := e^{\lambda \ell_\omega} \prod_{k=1}^{\ell_\omega} A(\omega_{k-1}, \omega_k)$$

STEP I: If there exists some λ such that $\psi_z(\lambda) = 0$, then the process that makes i.i.d. excursions away from z with law ν_λ is a recurrent Markov chain with transition kernel $P \sim A$.

The logarithmic moment generating function



Sketch of the proof (continued)

Let G be the directed graph with vertex set S and edge set $\{(x,y) : A(x,y) > 0\}$. For any subgraph $F \subset G$ and vertices $x, y \in F$, define $\psi^F_{x,y}(\lambda) = \log \phi^F_{x,y}(\lambda)$ with

$$\phi^F_{x,y}(\lambda) := \sum_{\omega \in \widehat{\Omega}_{x,y}(F)} e^{\lambda \ell_\omega} \prod_{k=1}^{\ell_\omega} A(\omega_{k-1}, \omega_k),$$

where $\widehat{\Omega}_{x,y}(F)$ denotes the space of excursions away from F starting in x and ending in y.

Removal of an edge Let $F' = F \backslash \{e\}$ be obtained from F by the removal of an edge e. Then

$$\phi^{F'}_{x,y}(\lambda) = \begin{cases} \phi^F_{x,y}(\lambda) + e^{\lambda} A(x,y) & \text{if } e = (x,y), \\ \phi^F_{x,y}(\lambda) & \text{otherwise} \end{cases} \qquad (\lambda \in \mathbb{R}).$$

Sketch of the proof

Removal of an isolated vertex Let $F' = F \backslash \{z\}$ be obtained from F by the removal of an isolated vertex z. Then

$$\phi^{F'}_{x,y}(\lambda) = \phi^{F}_{x,y}(\lambda) + \sum_{k=0}^{\infty} \phi^{F}_{x,z}(\lambda)\phi^{F}_{z,z}(\lambda)^k \phi^{F}_{z,y}(\lambda).$$

Proof: Set $\mathcal{A}(\omega) := \prod_{k=1}^{\ell_\omega} A(\omega_{k-1}, \omega_k)$. Distinguishing excursions away from F' according to how often they visit the vertex z, we have

$$\begin{aligned}\phi^{F'}_{x,y}(\lambda) &= \sum_{\omega_{x,y}} e^{\lambda \ell_{\omega_{x,y}}} \mathcal{A}(\omega_{x,y}) \\ &\quad + \sum_{k=0}^{\infty} \sum_{\omega_{x,z}} \sum_{\omega_{z,y}} \sum_{\omega^1_{z,z}} \cdots \sum_{\omega^k_{z,z}} e^{\lambda(\ell_{\omega_{x,z}} + \ell_{\omega_{z,y}} + \ell_{\omega^1_{z,z}} + \cdots + \ell_{\omega^k_{z,z}})} \\ &\qquad\qquad \times \mathcal{A}(\omega_{x,z})\mathcal{A}(\omega_{z,y})\mathcal{A}(\omega^1_{z,z}) \cdots \mathcal{A}(\omega^k_{z,z}),\end{aligned}$$

where we sum over $\omega_{x,y} \in \widehat{\Omega}_{x,y}(F)$ etc.

Sketch of the proof

Rewriting gives

$$\phi^{F'}_{x,y}(\lambda) = \sum_{\omega_{x,y}} e^{\lambda \ell_{\omega_{x,y}}} \mathcal{A}(\omega_{x,y}) +$$

$$\Big(\sum_{\omega_{x,z}} e^{\lambda \ell_{\omega_{x,z}}} \mathcal{A}(\omega_{x,z})\Big)\Big(\sum_{\omega_{z,y}} e^{\lambda \ell_{\omega_{z,y}}} \mathcal{A}(\omega_{z,y})\Big) \sum_{k=0}^{\infty} \Big(\sum_{\omega_{z,z}} e^{\lambda \ell_{\omega_{z,z}}} \mathcal{A}(\omega_{z,z})\Big)^{k}.$$

Sketch of the proof

Lemma (Exponential moments of excursions) Let P be an irreducible subprobability kernel. Set

$$\lambda^F_{x,y} := \sup\{\lambda : \psi^F_{x,y}(\lambda) < \infty\}.$$

Then, if

$$\lambda^F_{x,y,+} > 0 \text{ for all } x, y \in F \cap S$$

holds for some finite nonempty subgraph F of G, it holds for all such subgraphs.

Proof: By induction, removing edges and isolated vertices.

Sketch of the proof

Lemma Set $\lambda_* := \sup\{\lambda : \psi_z(\lambda) = 0\}$. Then $\lambda_* = -\log \rho(A)$.

Proof of the theorem: Assume that A is not strongly positive recurrent. Let $A' \leq A$ be a finite modification. We must show that $\rho(A') = \rho(A)$. By a similarity transformation, we may assume w.l.o.g. that A is a subprobability kernel and $\lambda_* = 0$. We need to show $\lambda'_* = 0$. It suffices to show that for the subgraph $F = \{z\}$, we have $\lambda'_{z,+} = 0$. Since A is not strongly positive, we have $\lambda_{z,+} = 0$. Since B is a finite modification, we can choose a finite subgraph F such that $\lambda^F_{x,y,+}$ is the same for A and A'. Now

$$\lambda_{z,+} \leq 0 \quad \Leftrightarrow \quad \lambda^F_{x,y,+} \leq 0 \text{ for some } x, y \in F \quad \Leftrightarrow \quad \lambda'_{z,+} \leq 0.$$

∎