Sharpness of the phase transition for the contact process

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The contact process

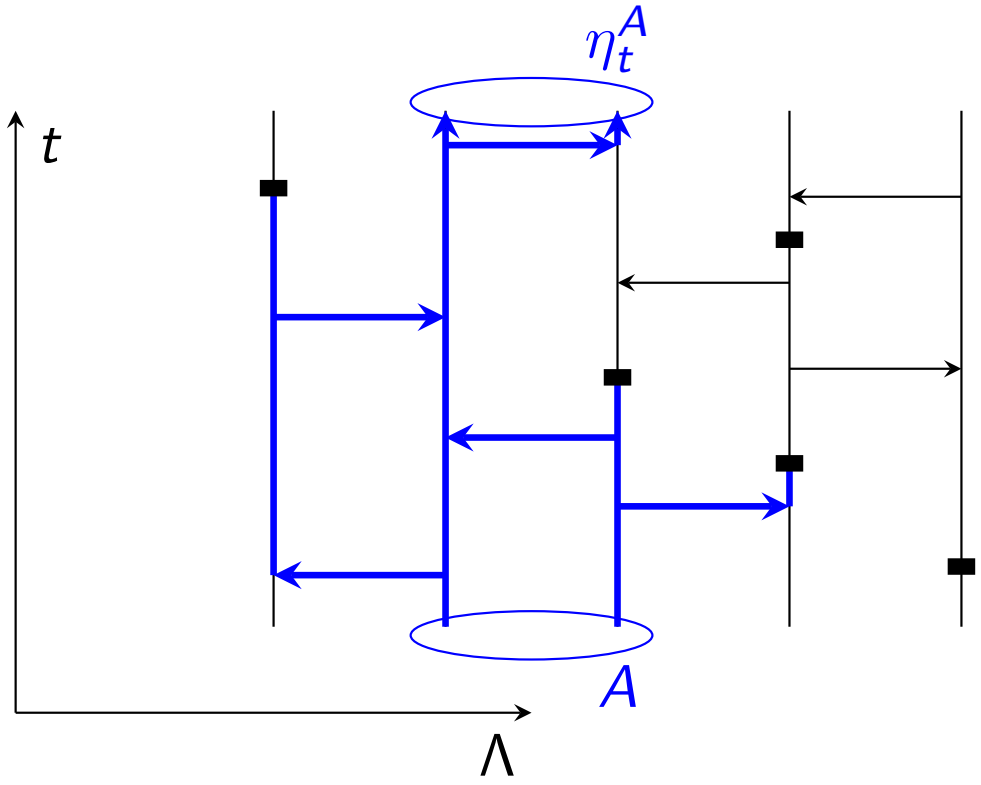
Λ *Lattice* e.g. $\Lambda = \mathbb{Z}^d$, more generally any infinite graph.
Usually, with a translation-invariant structure, e.g. Cayley graph.

Definition The contact process $(\eta_t)_{t\geq 0}$ with infection rate λ is a Markov process taking values in the subets of Λ. Sites $i \in \eta_t$ are called *infected*.

- An infected site at i infects each neighboring healthy site j with rate λ.
- Infected sites recover with rate one.

Graphical representation

Draw recovery symbols ▬ with Poisson rate 1.
Draw an arrow from i to neighbor j with rate λ.

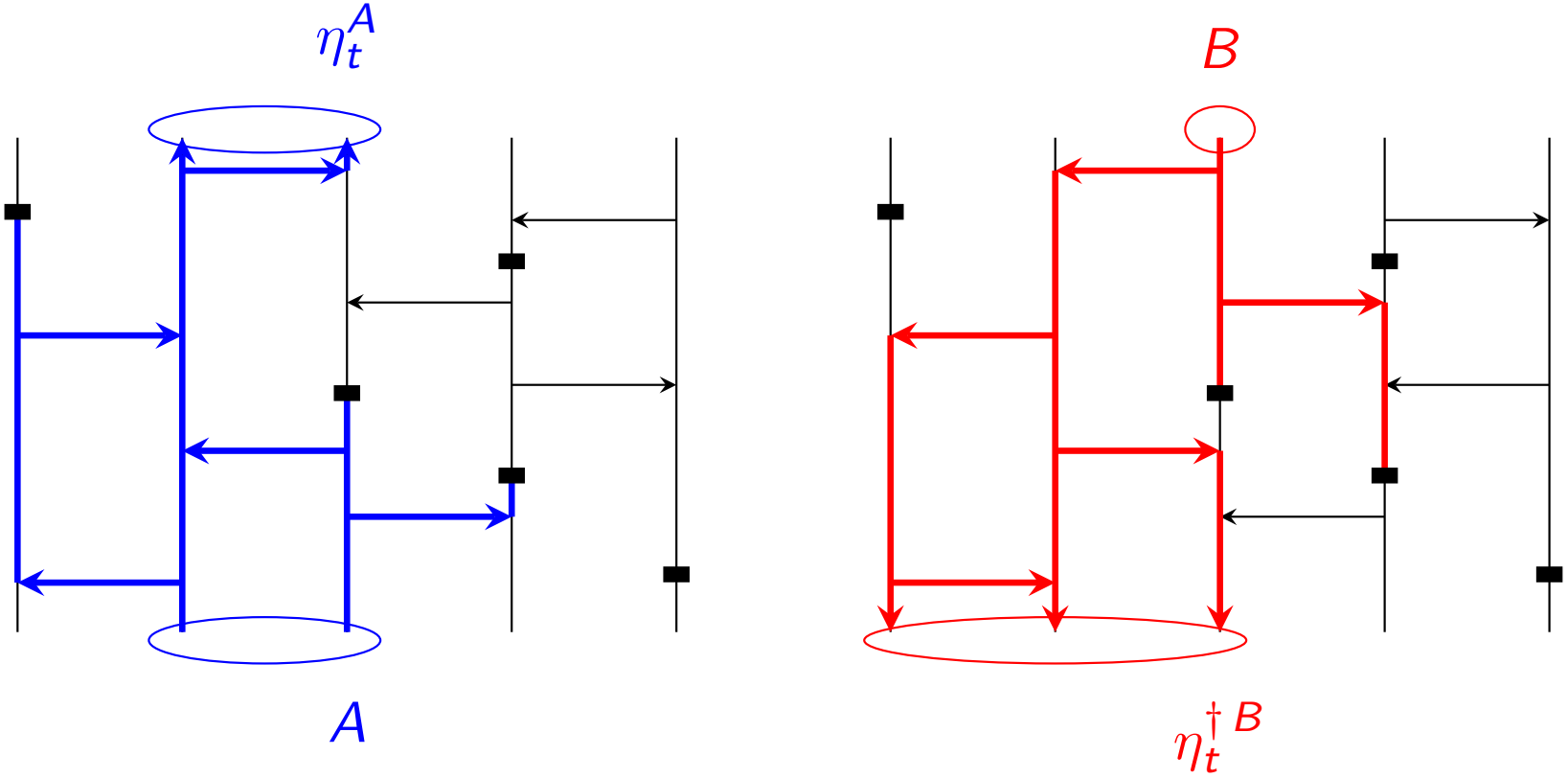


$$\eta_t^A = \{j \in \Lambda : (i,0) \rightsquigarrow (j,t) \text{ for some } i \in A\}.$$

Open paths may follow arrows but must avoid recovery symbols.

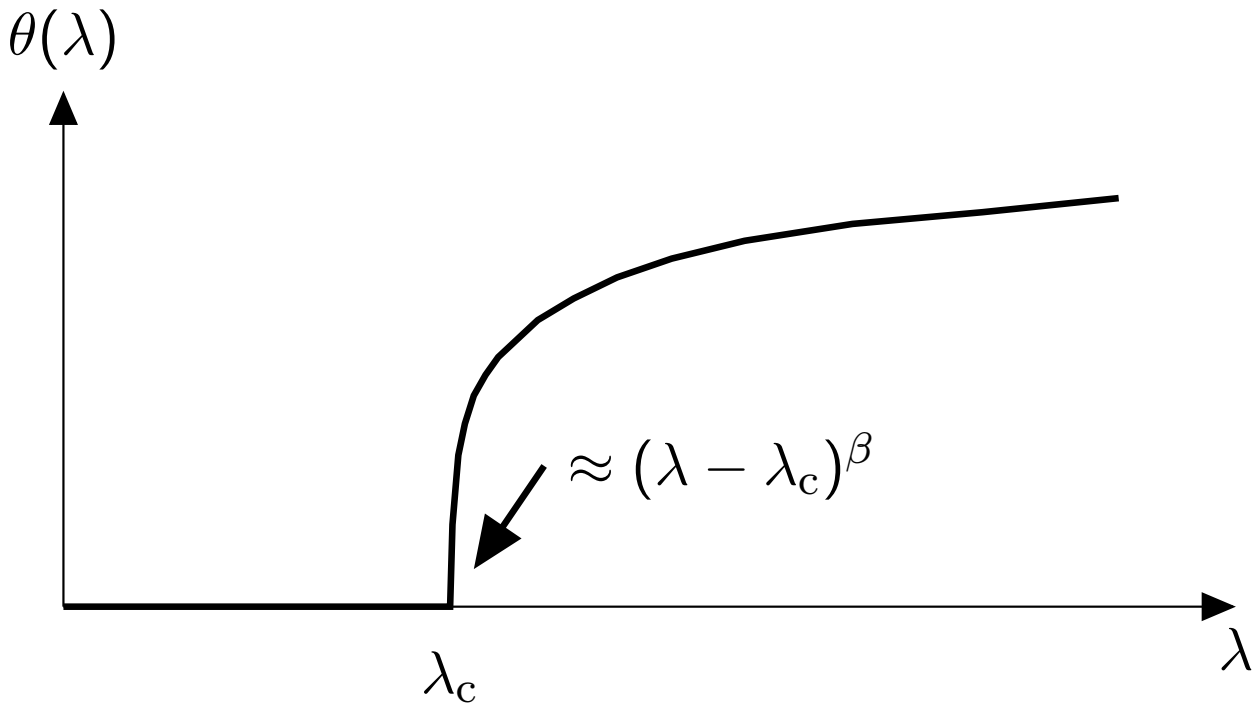
Duality

For the dual process $\eta_t^{\dagger\, B}$ time runs backwards and all arrows are reversed.



$$\{\eta_t^A \cap B \neq \emptyset\} = \{\exists \text{ open path from } A \text{ to } B\} = \{A \cap \eta_t^{\dagger\, B} \neq \emptyset\}.$$

The percolation probability



$$\theta(\lambda) := \mathbb{P}[(0,0) \rightsquigarrow \infty] = \mathbb{P}\big[\eta_t^{\{0\}} \neq \emptyset \ \forall t \geq 0\big] = \lim_{t\to\infty} \mathbb{P}\big[\eta_t^{\dagger\,\Lambda} \ni 0\big].$$

Exponential growth or decay rate

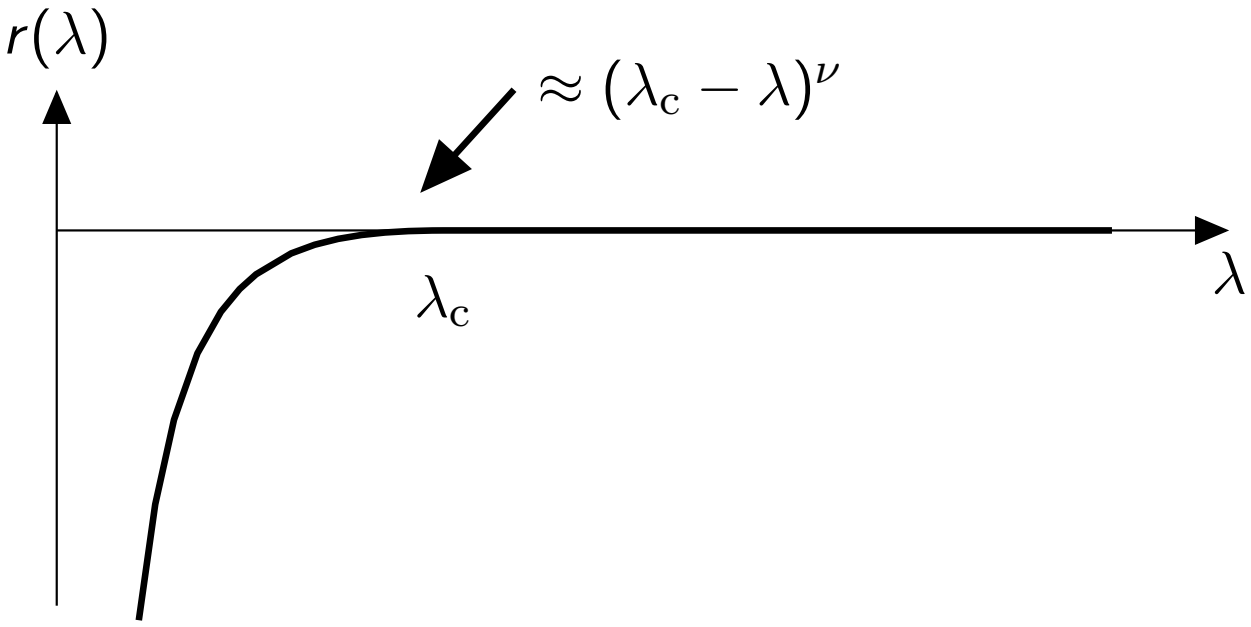
A simple subadditivity argument proves the existence of the limit

$$r(\lambda) = r := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[|\eta_t^{\{0\}}|\big].$$

For processes on $\Lambda = \mathbb{Z}^d$ and more generally on *Cayley graphs of subexponential growth*, one has $r \leq 0$.

On the other hand, on *nonamenable graphs*, it is known that $\theta(\lambda) > 0$ implies $r(\lambda) > 0$ [Swa09].

Sharpness of the phase transition



On general graphs, it is known that $r(\lambda) < 0$ iff $\lambda < \lambda_c$.

Sharpness of the phase transition.

Proofs of sharpness of the phase transition

Proof strategies:

I Assume $\theta(\lambda_*) = 0$, conclude $r(\lambda) < 0$ for $\lambda < \lambda_*$.

II Assume $r(\lambda_*) = 0$, conclude $\theta(\lambda) > 0$ for $\lambda > \lambda_*$.

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For unoriented percolation:

- Menshikov (1986) $\approx$ Strategy **I**.
- Aizenman & Barsky (1987) Strategy **II**.
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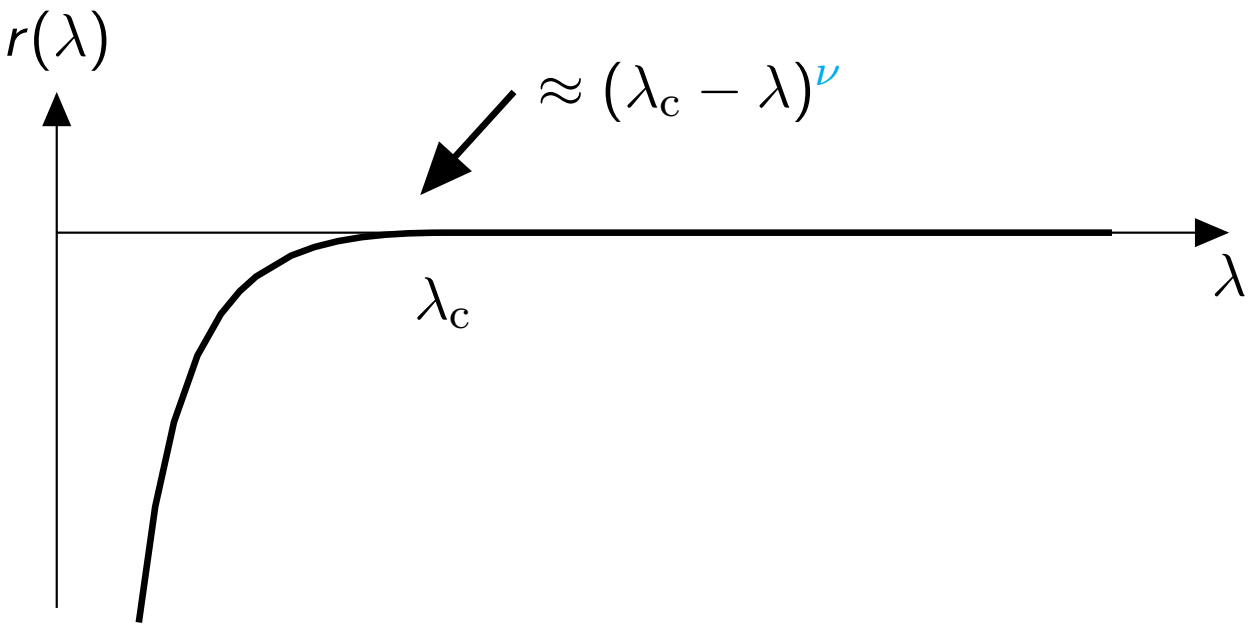
For oriented percolation & the contact process

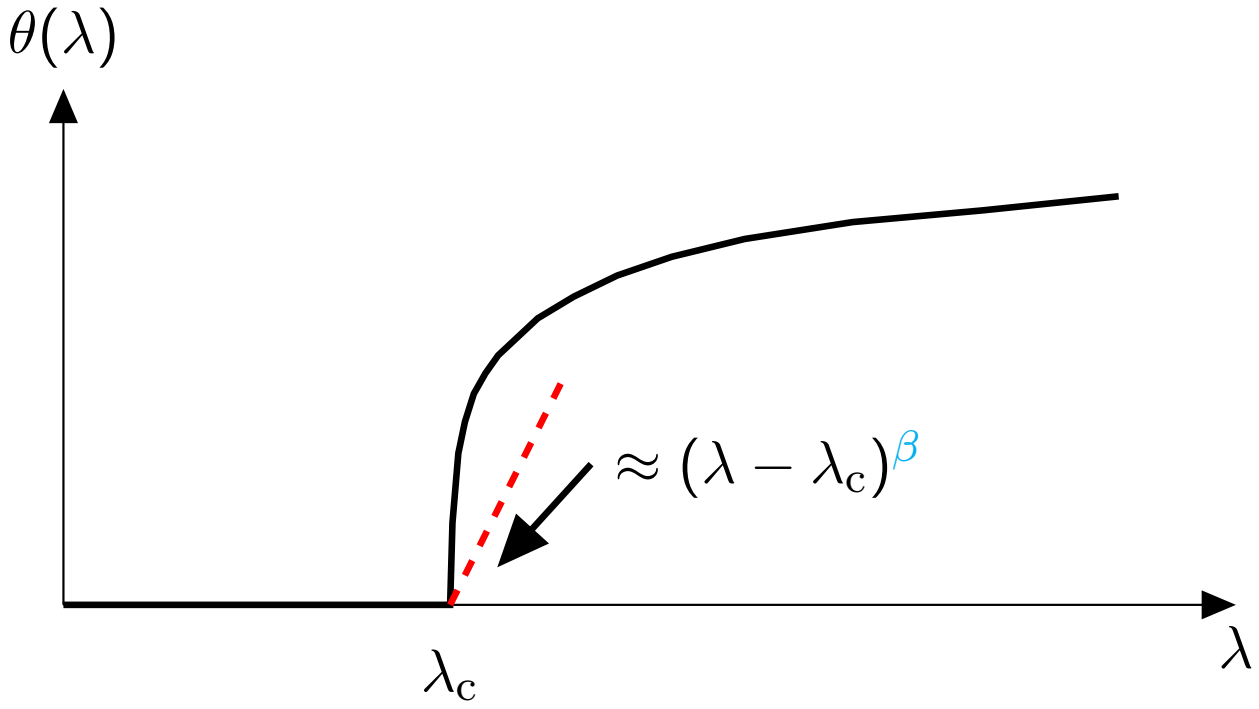
- Bezuidenhout & Grimmett (1991) adapted the method of Aizenman & Barsky (1987).
- Method of Duminil-Copin & Tassion (2016) carries over without a change to oriented percolation; with some work also to the contact process.
- S. (2016) method based on harmonic functions & eigenmeasures.

Proofs of sharpness of the phase transition



The problem with Strategy I seems to be that it is hard to get universal upper bounds on $r(\lambda)$...

Proofs of sharpness of the phase transition



...whereas there seems to be hope to prove universal lower bounds on $\theta(\lambda)$. Indeed, all known proofs yield as a side result $\beta \leq 1$.

Proofs of sharpness of the phase transition

The method of Aizenman & Barsky (1987) requires the introduction of an *external field / spontaneous disease* and depends on *differential inequalities* involving the two parameters (infection and spontaneous disease) of the process.

The method of Duminil-Copin & Tassion (2016) does away with the external field and depends on a single differential inequality involving only the infection rate.

The process modulo translations

Define an equivalence relation on the set of all finite subsets of Λ by $A \sim B \iff A$ is a translation of B.

Let $\tilde{A}$ denote the equivalence class containing A. Let $\tilde{\mathcal{P}}_{\mathrm{fin},+}$ denote the space of finite, nonzero subsets of Λ modulo translation.

[Sturm & S. '14] If $r < 0$, then the *contact process modulo translations* $(\tilde{\eta}_t)_{t \geq 0}$ has a unique quasi-invariant law μ. Moreover,

$$e^{-rt}\mathbb{P}\big[\tilde{\eta}_t^{\{0\}} \in \cdot\,\big]\Big|_{\tilde{\mathcal{P}}_{\mathrm{fin},+}} \underset{t\to\infty}{\Longrightarrow} \mu.$$

Eigenmeasures

A different way to view the previous result is as follows. Let $\mathcal{P}_+$ denote the space of all nonempty subsets of the lattice. Then

$$e^{-rt} \sum_{i \in \Lambda} \mathbb{P}[\eta_t^{\{i\}} \in \cdot\,]\Big|_{\mathcal{P}_+} \underset{t\to\infty}{\Longrightarrow} \nu,$$

where $\Rightarrow$ denotes vague convergence and ν is a locally finite measure on $\mathcal{P}_+$ that evolves under the semigroup $(P_t)_{t\geq 0}$ of the contact process as

$$\nu P_t = e^{-rt}\nu \quad (t \geq 0),$$

i.e., ν is an *eigenmeasure* with *eigenvalue* r.

This different point of view is valid even if $r \geq 0$:

[S. '09] Each translation-invariant contact process defined on a countable group Λ has a translation-invariant eigenmeasure with eigenvalue r.

An eigenfunction

If two Markov processes X and Y are dual, then invariant laws of X give rise to harmonic functions of Y.

Similarly, an eigenmeasure $\nu^\dagger$ for the dual contact process $\eta^\dagger$ gives rise to an eigenfunction for the generator G of the contact process η through the formula

$$h(A) := \int \nu^\dagger(\mathrm{d}B) 1_{\{A \cap B \neq \emptyset\}} \qquad (A \in \mathcal{P}_{\mathrm{fin}}).$$

This satisfies $Gh = rh$ and moreover:

$h(\emptyset) = 0$	
$h(A) \leq h(A)\ \forall A \subset B$	monotone
$h(A \cup B) \leq h(A) + h(B)$	subadditive
$h(\{0\}) = 1$	normalization
$h(i + A) = h(A)$	translation invariance.

Harmonic function

In particular, if $r = 0$, then $Gh = 0$, i.e., h is a harmonic function. This is good news for:

Strategy II Assume $r(\lambda_*) = 0$, conclude $\theta(\lambda) > 0$ for $\lambda > \lambda_*$.

The harmonic function h for λ_* turns into a subharmonic function for $\lambda > \lambda_*$.
Let G_λ denote the generator of the contact process with infection rate λ.

Lemma For each $\varepsilon > 0$, there exists a $\delta > 0$ such that $G_{\lambda_*} h = 0$ implies $G_{\lambda_*+\varepsilon} f_\delta \geq 0$, where

$$f_\delta := \delta^{-1}(1 - e^{-\delta h}).$$

Consequence:

$$\mathbb{P}[\eta_t^A \neq \emptyset \ \forall t \geq 0] \geq \delta f_\delta(A).$$

Additive particle systems

Both the method of Duminil-Copin & Tassion and the method with harmonic functions work more generally.

Additive particle systems can be constructed with a graphical representation involving infection arrows and recovery symbols. One can expect sharpness of the phase transition if, fixing all other parameters, the system goes through a phase transition at some critical recovery rate $\delta_c > 0$.

The method of Duminil-Copin & Tassion confirms this if connection probabilities inside and outside space-time boxes are positively correlated.

The method with harmonic functions confirms this provided there is only a single parameter describing the proportion of infection/recovery.

Additive particle systems

Example of a result that can be proved using Duminil-Copin & Tassion but not using harmonic functions:
For a range-two contact process, sharpness as we increase the nearest-neighbor infection rate while keeping the infection rate at distance two constant.

Example of a result that can be proved using harmonic functions but not using Duminil-Copin & Tassion:
Sharpness for a contact process where two neighboring sites always recover together.

The method using harmonic functions is technically easier in a continuous-time setting.

Open problem Monotone systems that are not additive.