Self-organized criticality on the stock market

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Prague, January 5th, 2014.

Some classical ecomomic theory

In classical economic theory, the *price* of a commodity is determined by *demand* and *supply*.

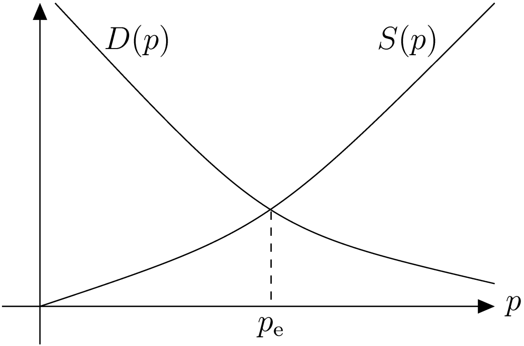
Let $D(p)$ (resp. $S(p)$) be the total *demand* (resp. *supply*) for a commodity at price level p, i.e., the total amount that could be sold (resp. bought), per unit of time, for a price of at most (resp. at least) p per unit.

Assumption $S(p)$ is strictly increasing in p, $D(p)$ is strictly decreasing in p, $S(0) = 0$, $\lim_{p\to\infty} D(p) = 0$.

Consequence There is a unique $0 < p_{\rm e} < \infty$ such that $D(p_{\rm e}) = S(p_{\rm e})$.

Postulate In an equilibrium market, the commodity is traded at the *equilibrium prize* $p_{\rm e}$.

Some classical ecomomic theory



Questions

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- How is equilibrium maintained?
- How does the system evolve from one equilibrium state to another, if demand or supply change?

Stock & Commodity Exchanges & the Order Book

Stocks, as well as other derivates such as options, are usually traded at *stock exchanges*. In addition, (futures on) commodities are commonly traded at *commodity exchanges*.

On a stock exchange or commodity exchange, buyers and sellers commonly interact by means of an *order book*.

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- Place a **market order**, i.e., either *buy* (*buy market order*) or *sell* (*sell market order*) n units of the asset at the best price available in the order book.

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Market orders are matched to existing limit orders according to a mechanism that depends on the trading system.

Bid, ask, spread, midprice

- The **bid price** at time t, denoted $b(t)$, is equal to the highest price among all buy limit orders in the limit order book.
- The **ask price** at time t, denoted $a(t)$, is equal to the lowest price among all sell limit orders in the limit order book.
- The bid-ask **spread** at time t, denoted $s(t)$, is the difference between the ask and bid price: $s(t) = a(t) - b(t)$.
- The **mid price** at time t, denoted $m(t)$, is the arithemtic mean of the ask and bid price: $m(t) = (a(t) + b(t))/2$.

In real markets, the spread is most of the time small and all prices are roughly the same.

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- If the order book contains a suitable offer, then the trader places a *market order*, i.e., sells to the highest bidder or buys from the cheapest seller.
- If the order book contains no suitable offer, then the trader places a *limit order* at his/her minimal sell or maximal buy price.

Plačková's model

Unrealistic elements of the model:

- One item per trader.
- Start with an empty order book.
- Limit orders are never cancelled.
- Uniform distribution.
- Independence, and more...

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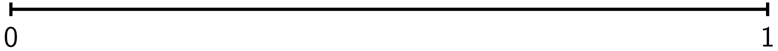
Or not??

Numerical simulation



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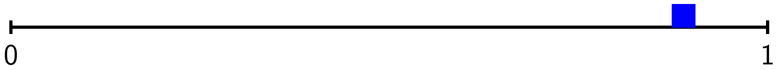

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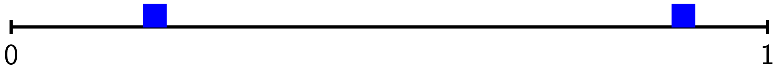

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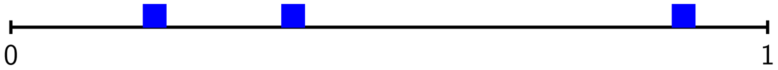

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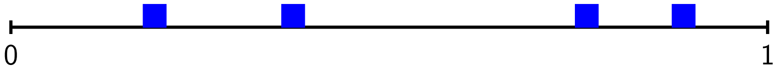

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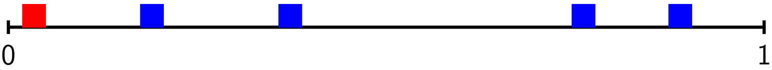

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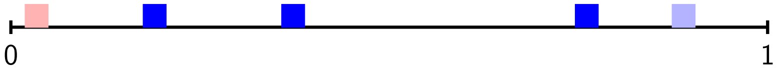

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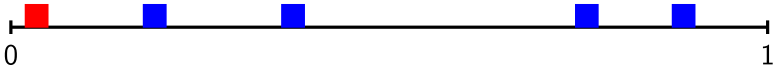

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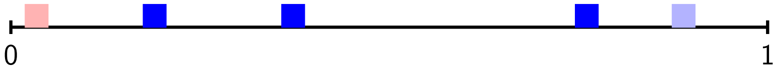

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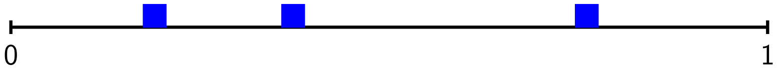

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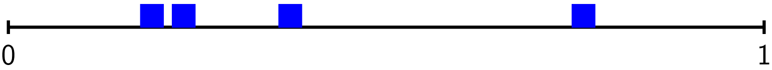

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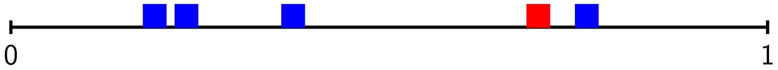

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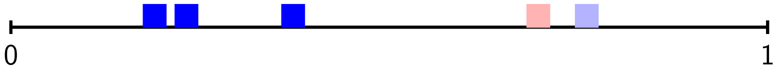

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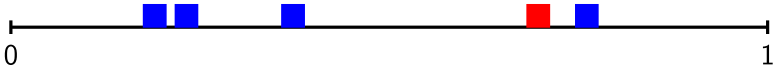

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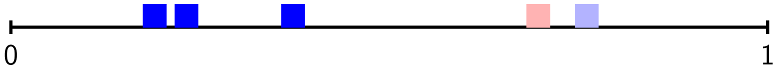

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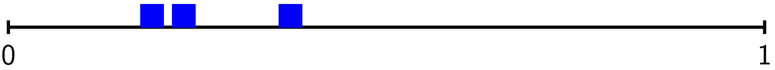

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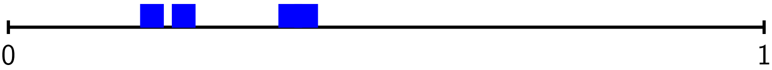

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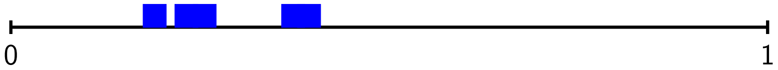

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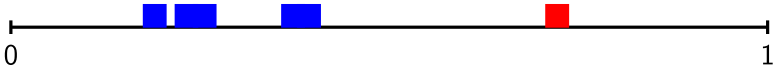

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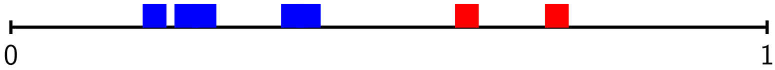

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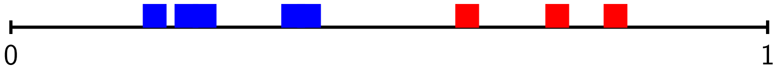

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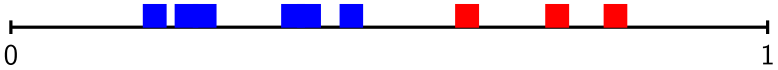

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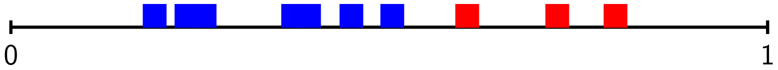

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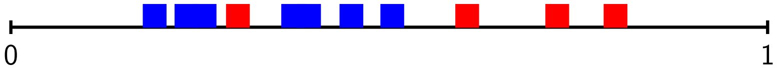

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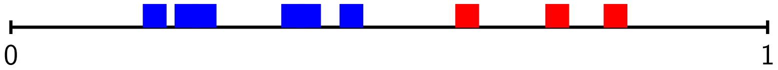

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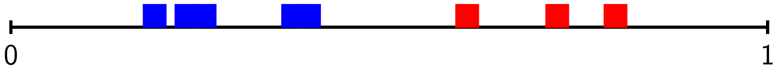

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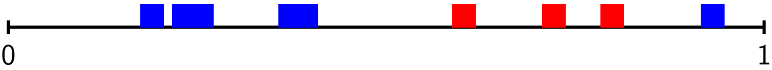

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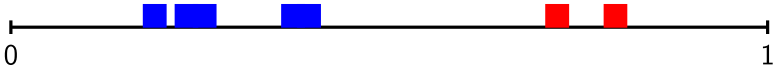

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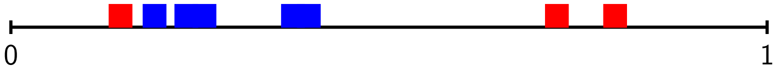

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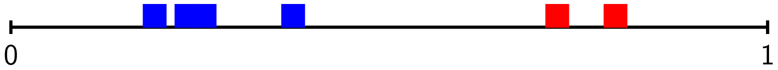

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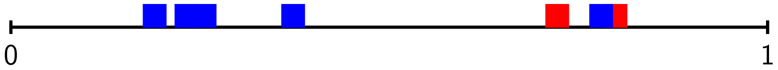

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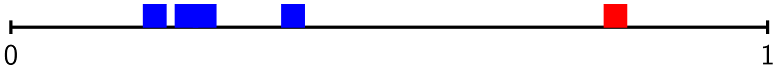

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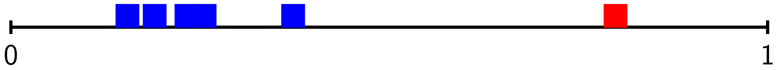

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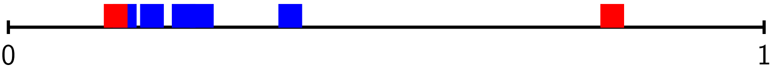

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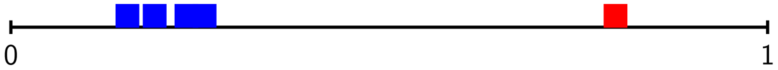

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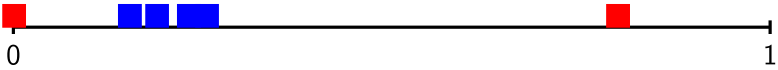

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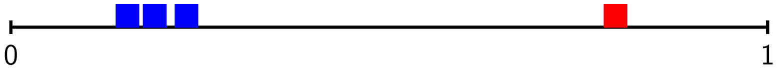

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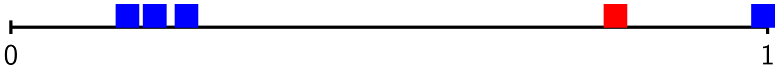

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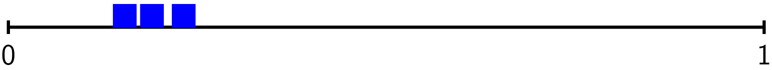

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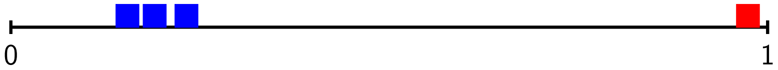

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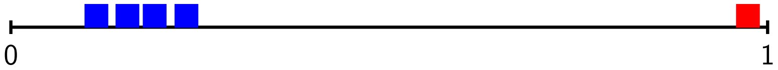

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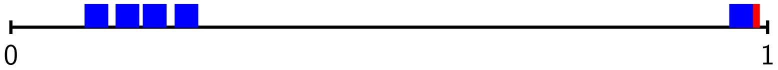

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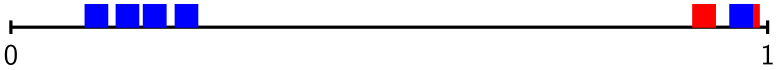

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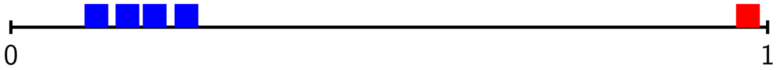

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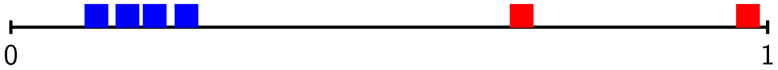

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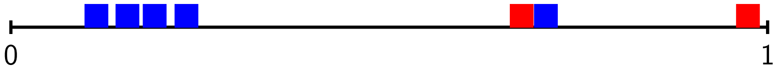

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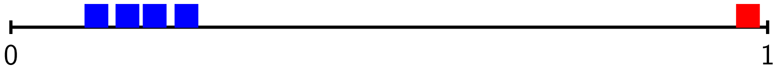

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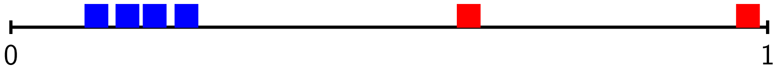

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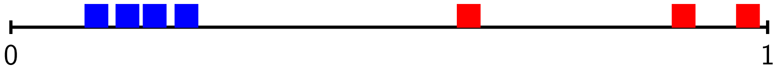

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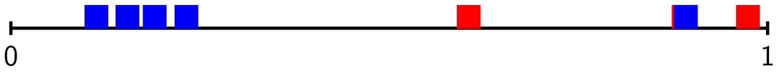

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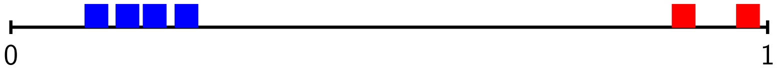

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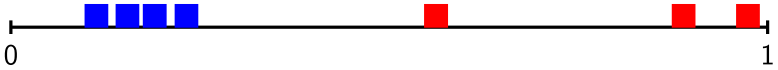

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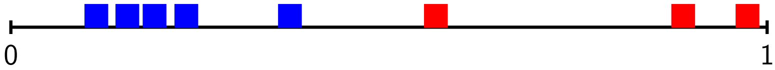

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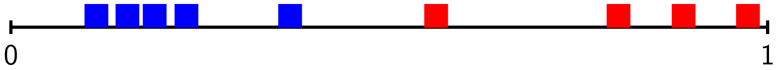

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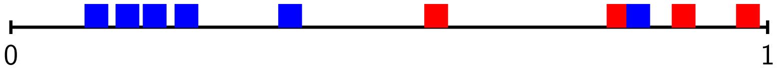

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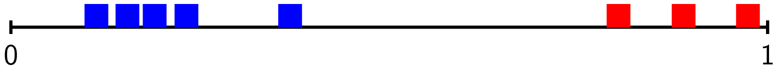

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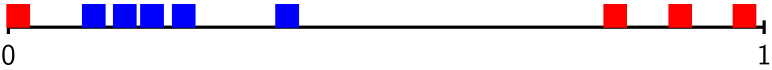

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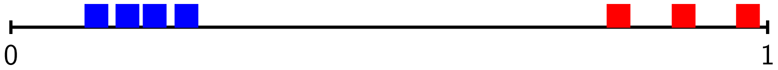

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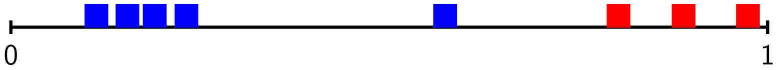
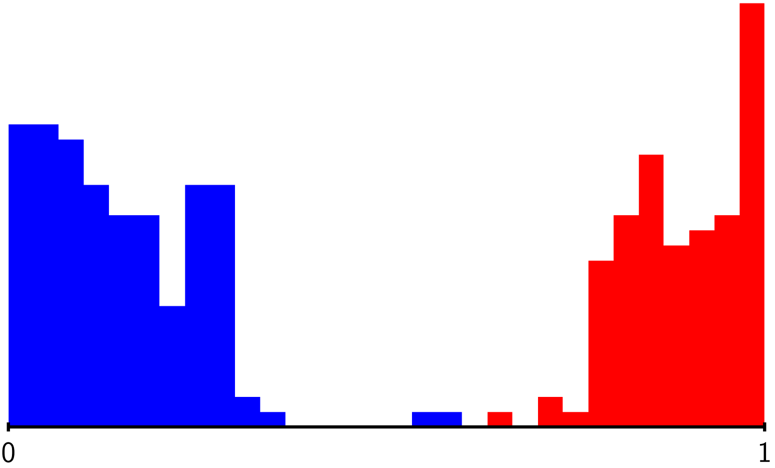


Numerical simulation

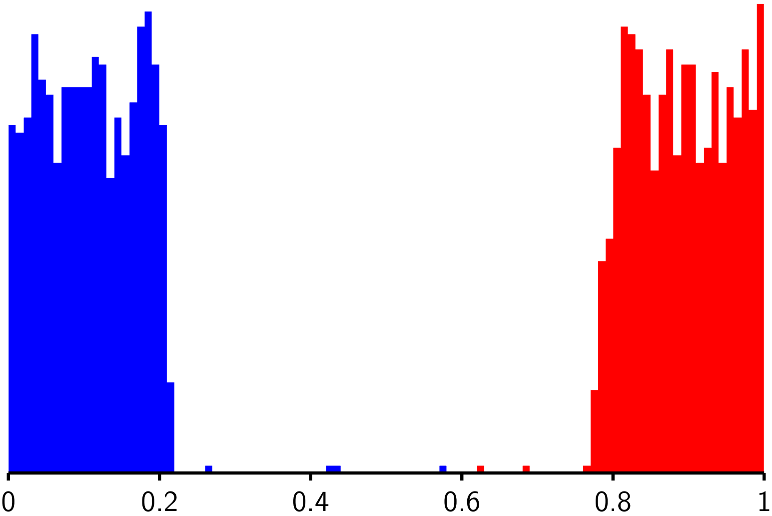
The order book after the arrival of 100 traders.

Numerical simulation



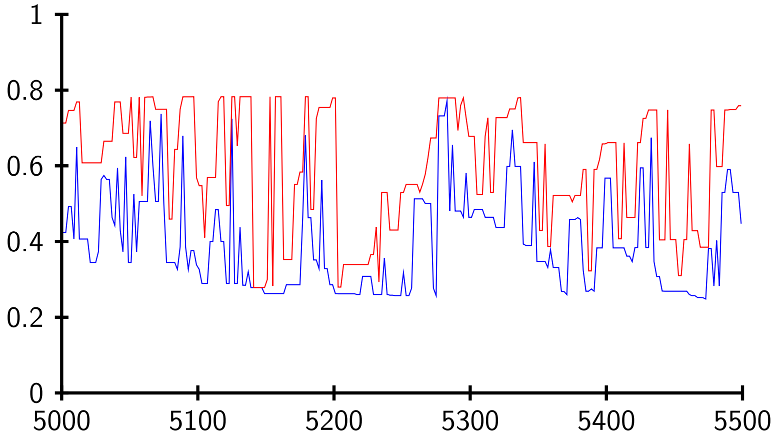
The order book after the arrival of 1000 traders.

Numerical simulation



The order book after the arrival of 10,000 traders.

Numerical simulation



Evolution of the bid and ask prices between the arrivals of the 5000th and 5500th trader.

Our naive guess was wrong.

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The theoretical equilibrium price $p_{\mathrm{e}} = 0.5$ is never attained.

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- Sell limit orders at a price above $1 - q_{\mathrm{c}}$ are never matched.
- The bid and ask prices keep fluctuating between q_{c} and $1 - q_{\mathrm{c}}$.
- The spread is huge, most of the time.

A conjecture

Conjecture $q_{\rm c} := 1 + 1/z$ with z the unique solution of the equation $1 + z + e^z = 0$.

Numerically, $q_{\rm c} \approx 0.21781170571980$.

Let M_k^L and M_k^R denote the bid and ask price after the arrival of the k-th trader. Then, almost surely

$$\liminf_{k\to\infty} M_k^L = \liminf_{k\to\infty} M_k^R = q_{\rm c}, \quad \limsup_{k\to\infty} M_k^L = \limsup_{k\to\infty} M_k^R = 1 - q_{\rm c},$$

while for each $q_{\rm c} < q < 1 - q_{\rm c}$, both M_k^L and M_k^R spend a positive fraction of time on either side of q.

Outline

The plan for the rest of this talk is as follows:

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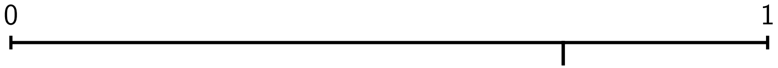
- Self-organized criticality.
- Solution of the one-sided canyon model.
- Partial solution of Plačková's model.

A model for canyon formation

We start with a flat rock profile.

A model for canyon formation



The river cuts into the rock at a uniformly chosen point.

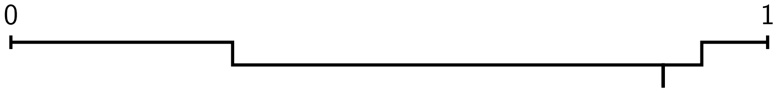
A model for canyon formation

Rock between a next point and the river is eroded one step down.

A model for canyon formation

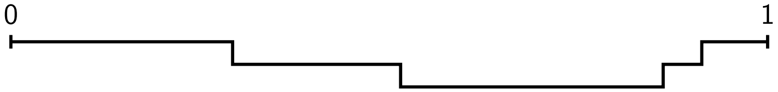
We continue in this way.

A model for canyon formation



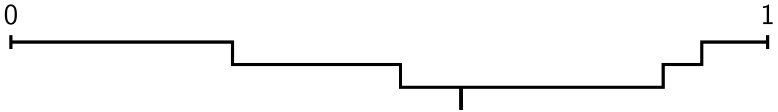
Either the river cuts deeper in the rock.

A model for canyon formation



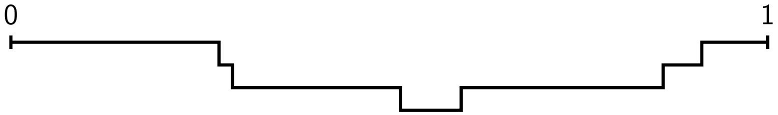
Or one side of the river is eroded down.

A model for canyon formation



We are interested in the limit profile.

A model for canyon formation



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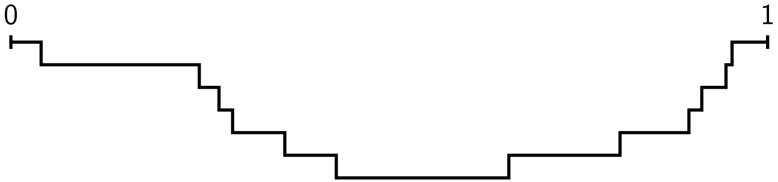

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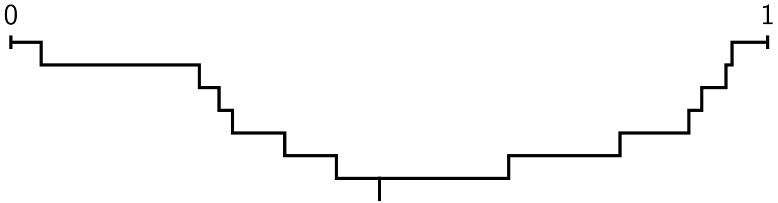

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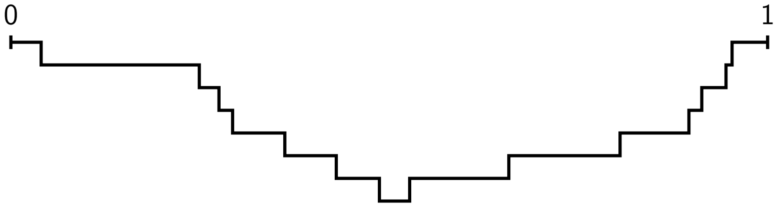

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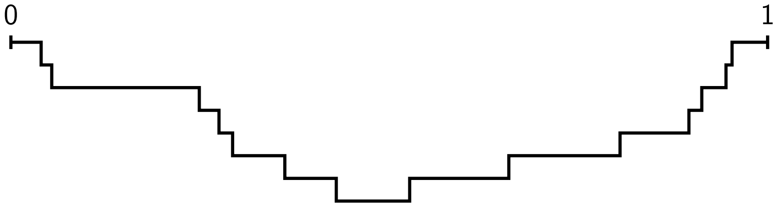

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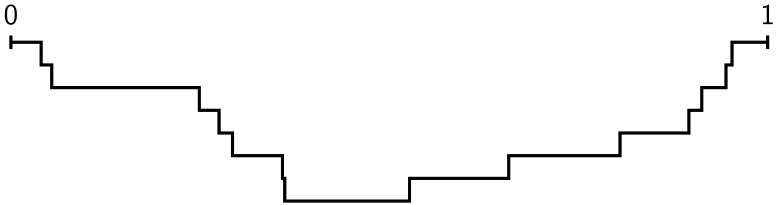

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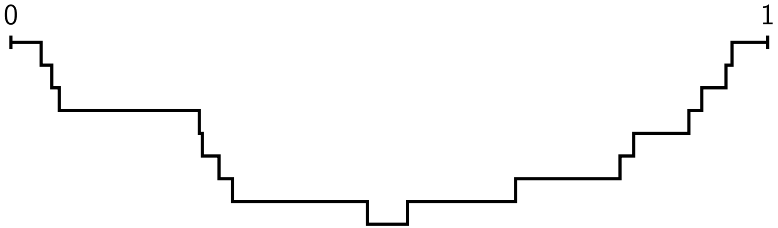

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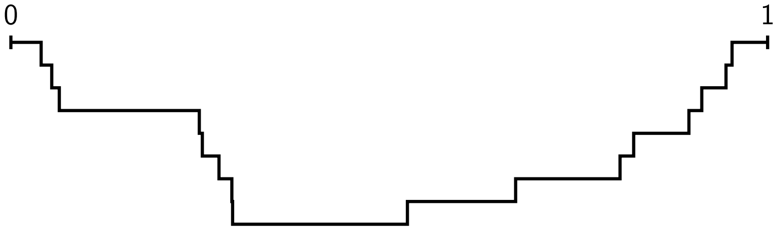

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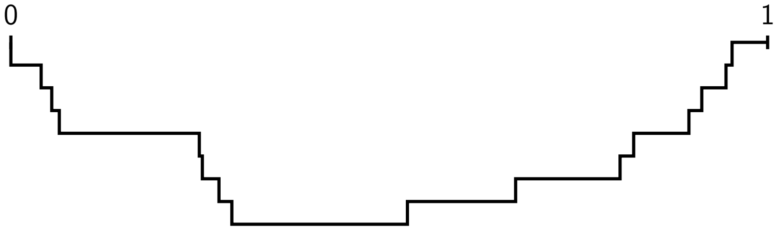

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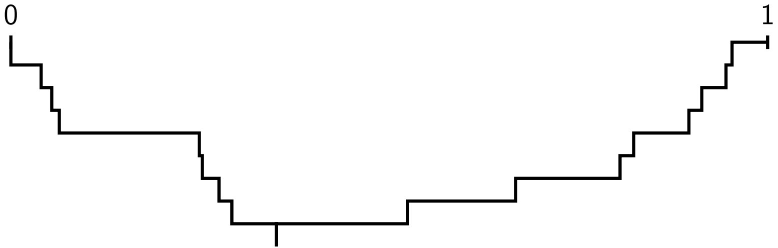

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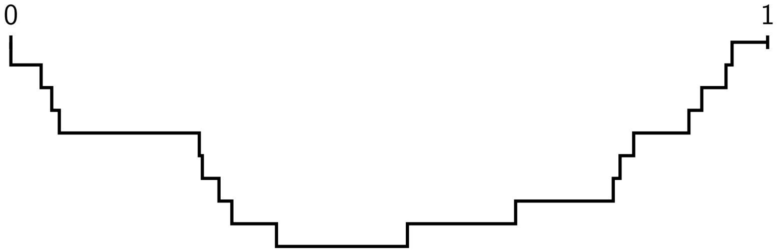

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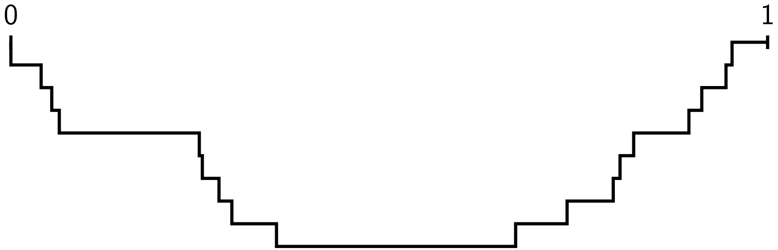

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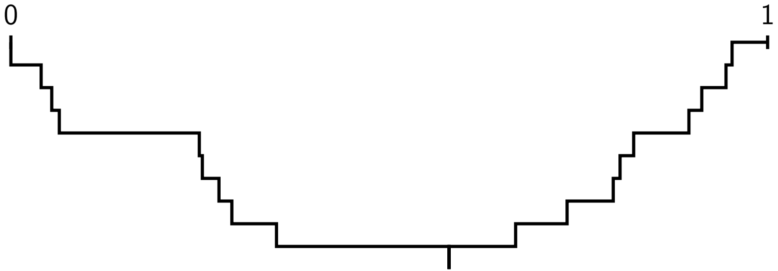

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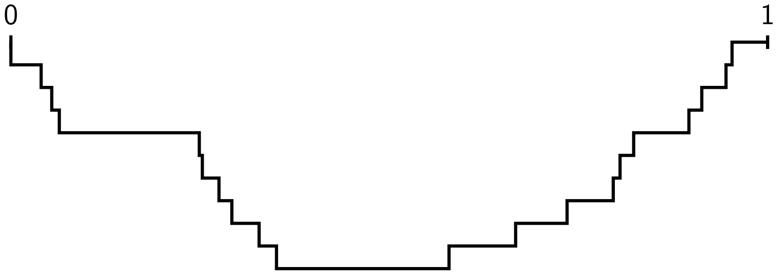

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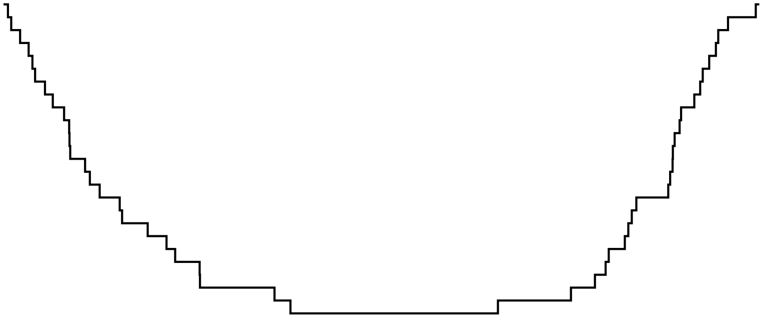
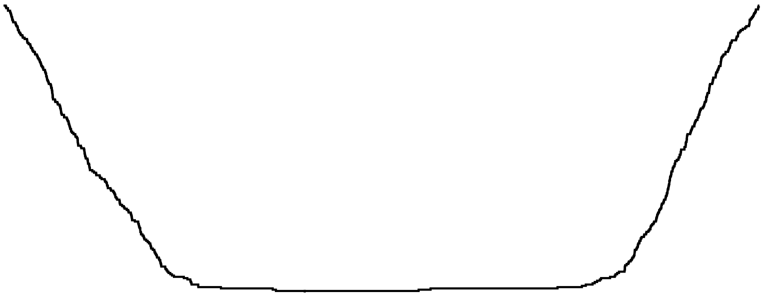


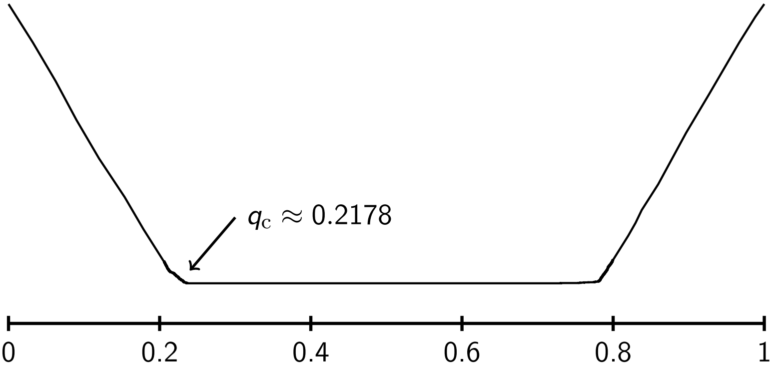
The profile after 100 steps.

A model for canyon formation



The profile after 1000 steps.

A model for canyon formation



The profile after 10,000 steps.

A model for canyon formation

We find the same critical point $q_{\rm c}$ as for Plačková's model.
In fact, the models are very similar:

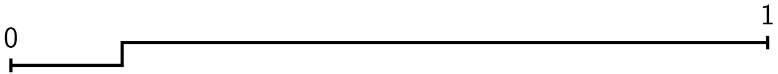
- In Plačková's model, interpret a buy limit order as an increment -1 and interpret a sell limit order as an increment $+1$.
- Assume that each trader places *both* a buy and sell limit order, at the (almost) same price, but with the sell order infinitesimally on the right of the buy order.

Then we obtain the canyon model.

A one-sided canyon model

We can also model a single shore.

A one-sided canyon model



We can also model a single shore.

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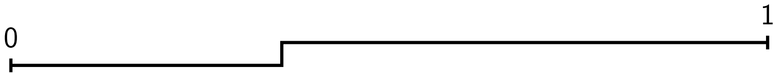

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A one-sided canyon model

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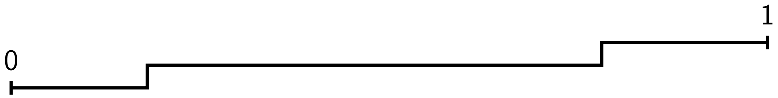
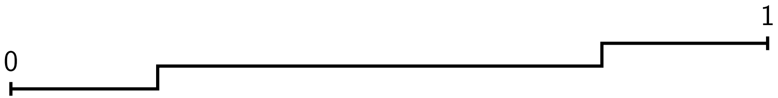


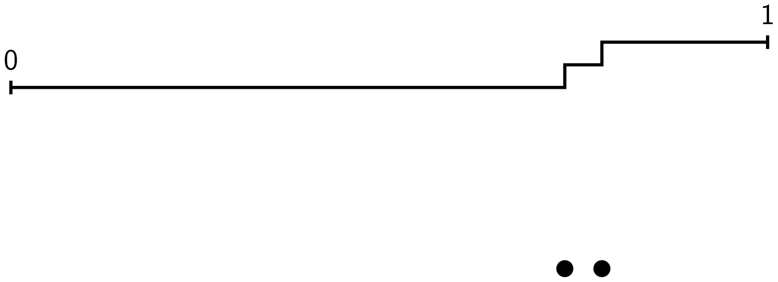
We either make the river deeper. . .

A one-sided canyon model



. . . or we erode the shore,

A one-sided canyon model



...depending on where the new point falls.

A one-sided canyon model

Points on the left of all others are simply added.

A one-sided canyon model

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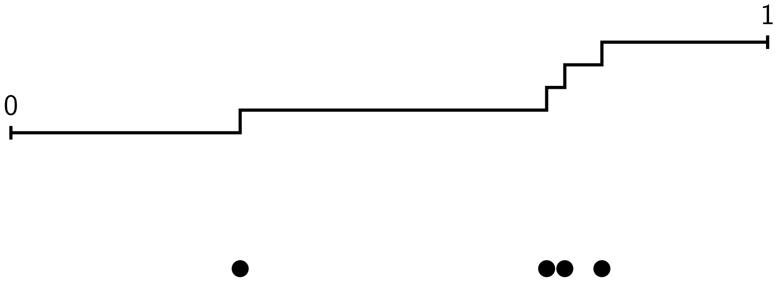
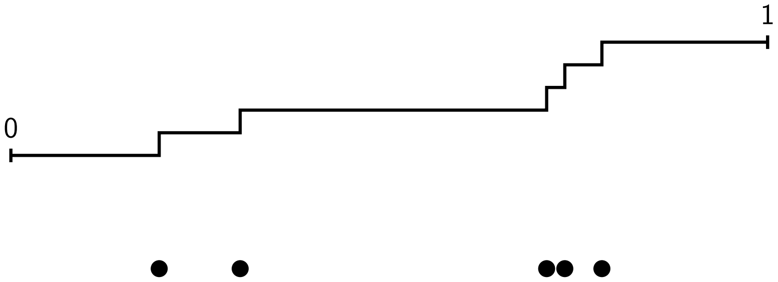
A one-sided canyon model



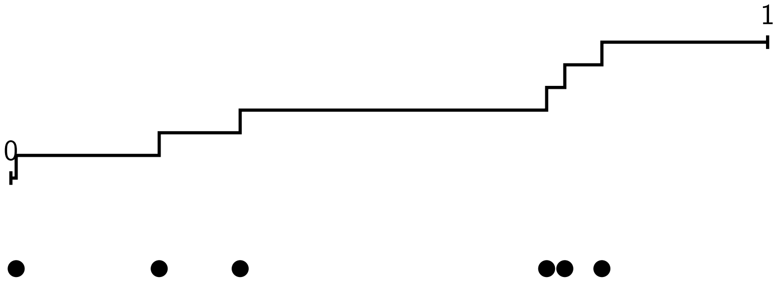
Otherwise, we remove the left-most point.

A one-sided canyon model



In other words, we always add the new point.

A one-sided canyon model



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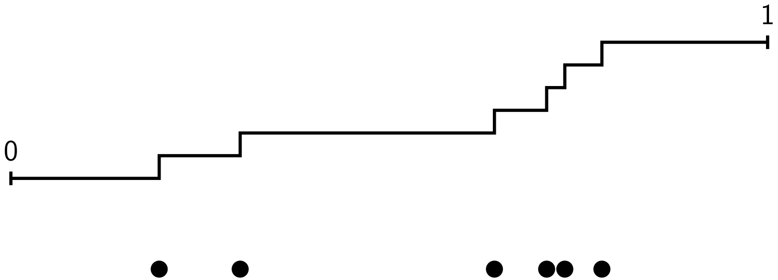
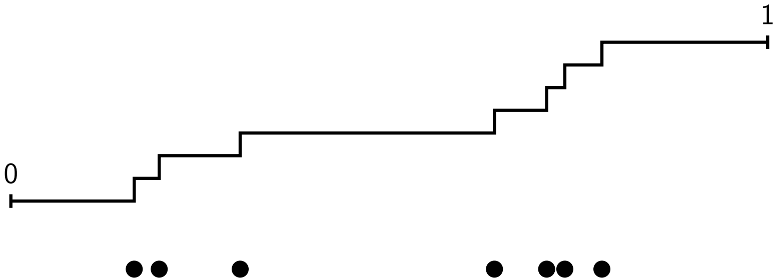


If the new point is not the left-most, then we remove the left-most.

A one-sided canyon model



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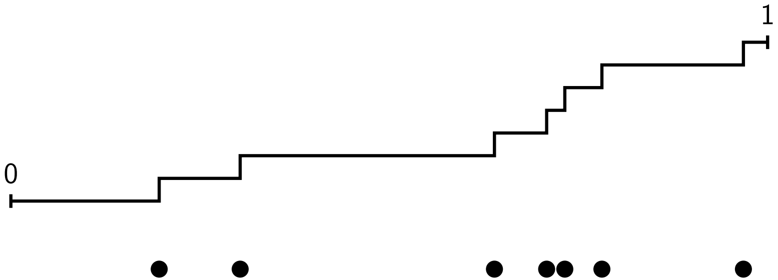

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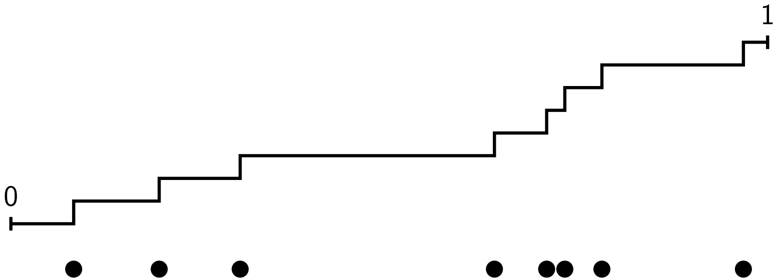

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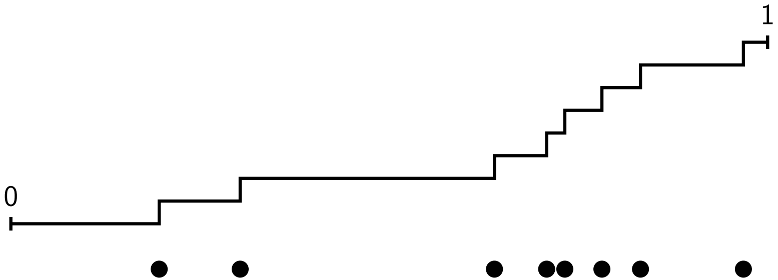

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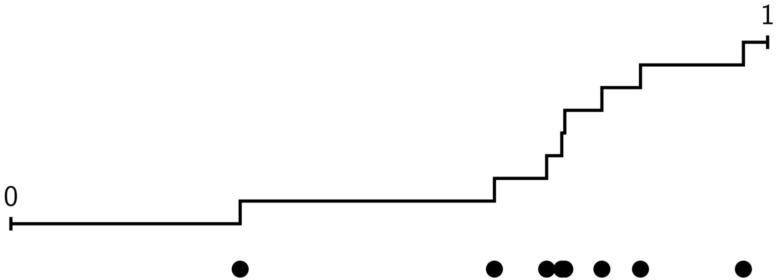

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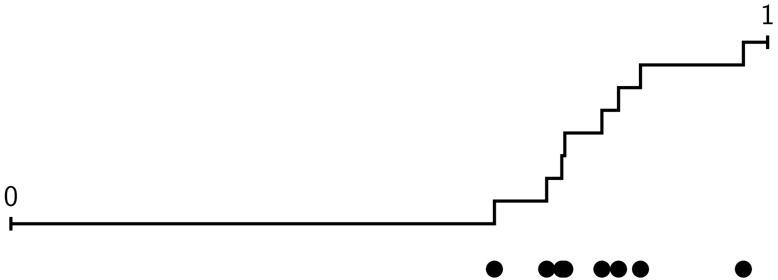

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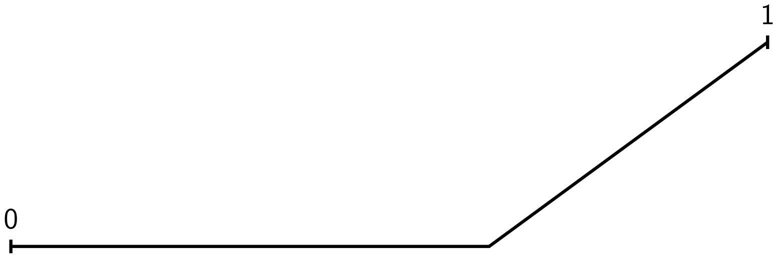


In this model, the critical point is $p_{\rm c} = 1 - e^{-1} \approx 0.63212$.

Barabási's queueing model

Consider a person who receives emails according to a Poisson process with intensity $\lambda_{\rm in}$ and who answers emails according to a Poisson process with intensity $\lambda_{\rm out} < \lambda_{\rm in}$.
Obviously, only a $\lambda_{\rm out}/\lambda_{\rm in}$ fraction of all emails will be answered in the long run.

The person assigns to each email a *priority* and always answers the email in the queue with the highest priority.
We assume that priorities are independent and uniformly distributed on $[0, 1]$.

A discrete time model similar to this has been studied by Gabrielli and Caldarelli (2009), inspired by earlier work of Barabási (2005).

The Bak Sneppen model

Introduced by Bak & Sneppen (1993).

Consider an ecosystem with N species. Each species has a fitness in $[0,1]$.

In each step, the species $i \in \{1,\ldots,N\}$ with the lowest fitness dies out, together with its neighbors $i-1$ and $i+1$ (with periodic b.c.), and all three are replaced by species with new, i.i.d. uniformly distributed fitnesses.

There is a critical fitness $f_{\rm c} \approx 0.6672(2)$ such that when N is large, after sufficiently many steps, the fitnesses are approximately uniformly distributed on $(f_{\rm c},1]$ with only a few smaller fitnesses. Moreover, for each $\varepsilon > 0$, the lowest fitness spends a positive fraction of time above $f_{\rm c}-\varepsilon$, uniformly as $N \to \infty$.

Self-organized criticality

All these models share some common features:

- Only the relative order of the limit orders, points of increase, priorities, or fitnesses matter. As a result, replacing the uniform distribution with any other atomless law basically yields the same model (up to a transformation of space).
- All models use some version of the rule "kill the minimal element".
- All models exhibit *self-organized criticality*.

Self-organized criticality

Physical systems with second order phase transitions exhibit *critical behavior* at the point of the phase transition, which is characterized by:

- Scale invariance.
- Power law decay of quantities.
- Critical exponents.

Usually, critical behavior is only observed when the parameter(s) of the system, such as the temperature, have just the right value so that we are at the point of the phase transition, also called (in this context) the *critical point*.

Self-organized criticality

Some physical systems show critical behavior even without the necessity to tune a parameter to exactly the right value.

In particular, this happens for systems whose dynamics find the critical point themselves. Such systems are said to exhibit *self-organized criticality*.

A classical example are sandpiles, which automatically find the maximal slope that is still stable. Adding a single grain to such a sandpile causes an avalanche whose size has a power-law distribution.

The Bak Sneppen model is another classical example of self-organized criticality and a cornerstone of Bak's (1996) book.

In Barabási's queueing model, the distribution of serving times (of answered emails) has a power-law tail. (As opposed to the more usual exponential tails in queueing theory.)

The one-sided canyon model

Given a finite subset $X_0 \subset [0,1]$ and an i.i.d. sequence $(U_k)_{k\geq 1}$ of uniformly distributed $[0,1]$-valued random variables, we define $(X_k)_{k\geq 0}$ inductively by $M_{k-1} := \min(X_{k-1} \cup \{1\})$ and

$$X_k := \begin{cases} X_{k-1} \cup \{U_k\} & \text{if } U_k < M_{k-1}, \\ (X_{k-1} \cup \{U_k\}) \backslash \{M_{k-1}\} & \text{if } U_k > M_{k-1}. \end{cases} \qquad (k \geq 1).$$

In words, X_k is constructed from X_{k-1} by adding U_k, and in case that the previous minimal element M_{k-1} is less than U_k, removing M_{k-1} from X_{k-1}.

The one-sided canyon model

For each $0 < q < 1$, we observe that the *restricted process*

$$\big(X_k \cap [0,q]\big)_{k\geq 0}$$

is a Markov chain.

Theorem The restricted process is positively recurrent for $q < 1 - e^{-1}$ and transient for $q > 1 - e^{-1}$.

Conjecture The process with $q = p_{\rm c} := 1 - e^{-1}$ is null recurrent. Started from $X_0 = \emptyset$, one has

$$\mathbb{P}\big[X_k \cap [0,p_{\rm c}] = \emptyset\big] \sim k^{-3/2} \quad \text{as } k \to \infty.$$

Self-organized criticality.

Proof of the theorem Since only the relative order of the points matters, we may without loss of generality assume that the $(U_k)_{k\geq 1}$ are i.i.d. exponentially distributed with mean one and $X_k \subset [0, \infty]$.

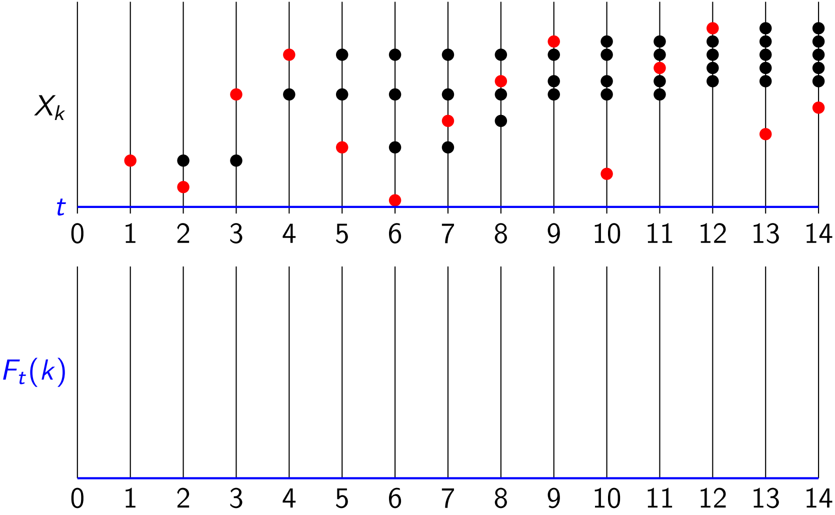
For the modified model, we must prove that $p_{\rm c} = 1$.

Start with $X_0 = \emptyset$ and define

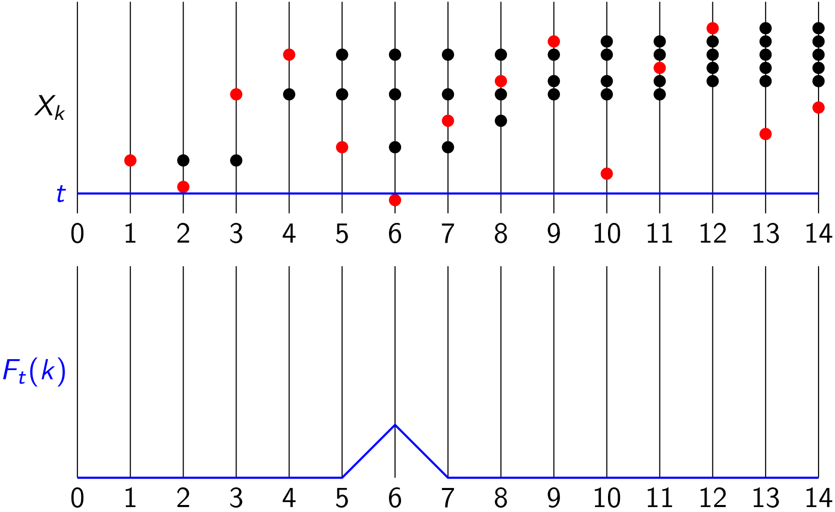
$$F_t(k) := \big|X_k \cap [0, t]\big| \qquad (k \geq 0,\ t \geq 0).$$

Claim $(F_t)_{t\geq 0}$ is a continuous-time Markov process taking values in the functions $F : \mathbb{N} \to \mathbb{N}$.

The point-counting function



The point-counting function



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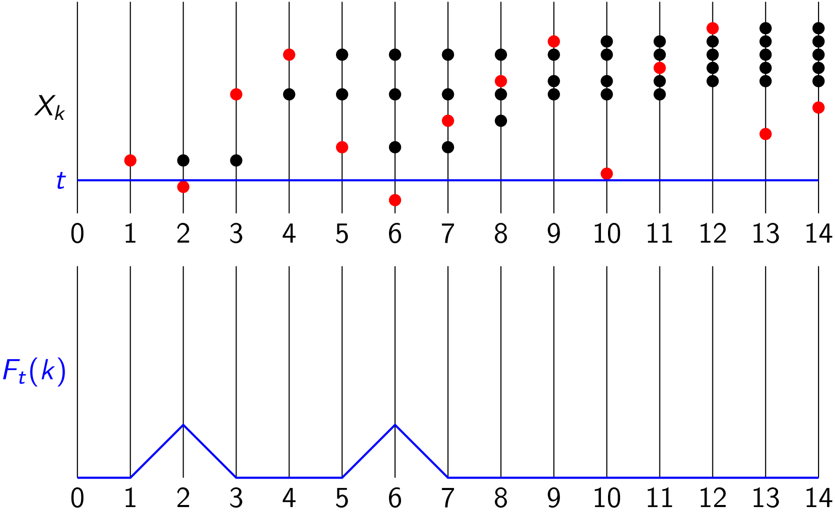

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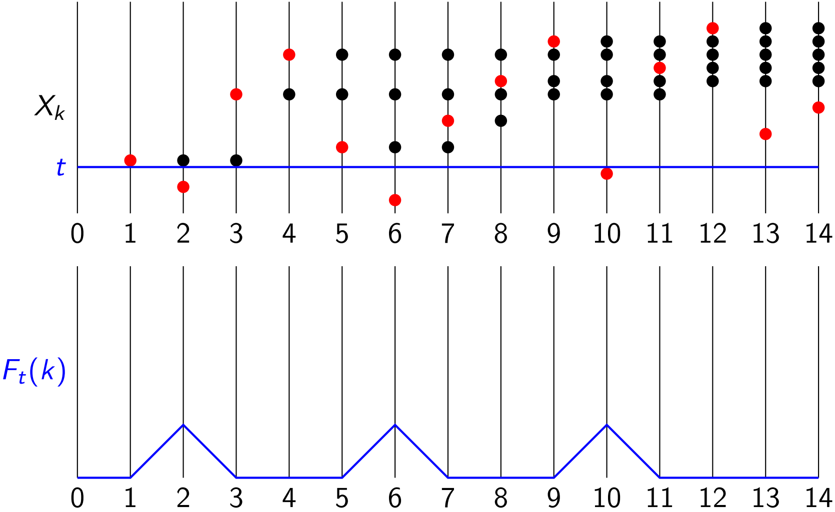

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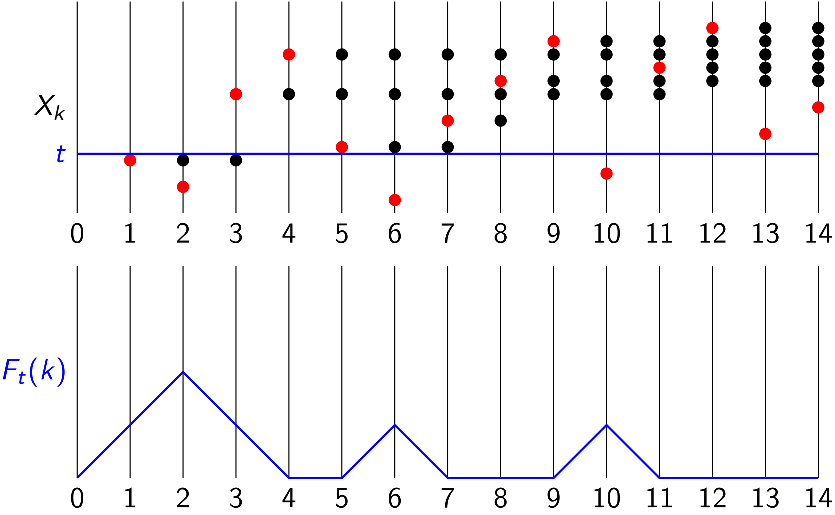

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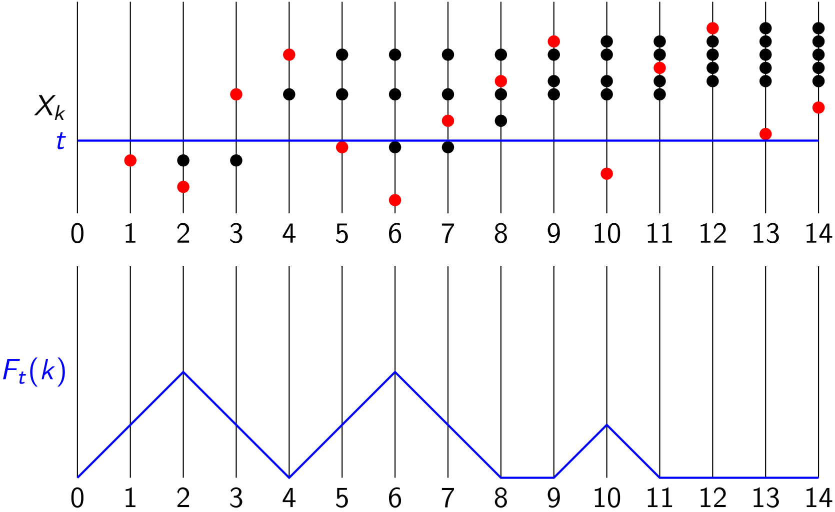

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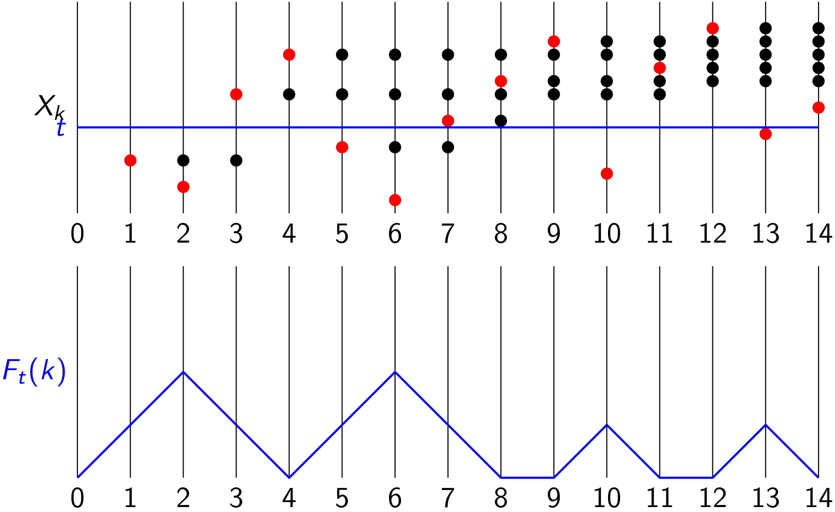

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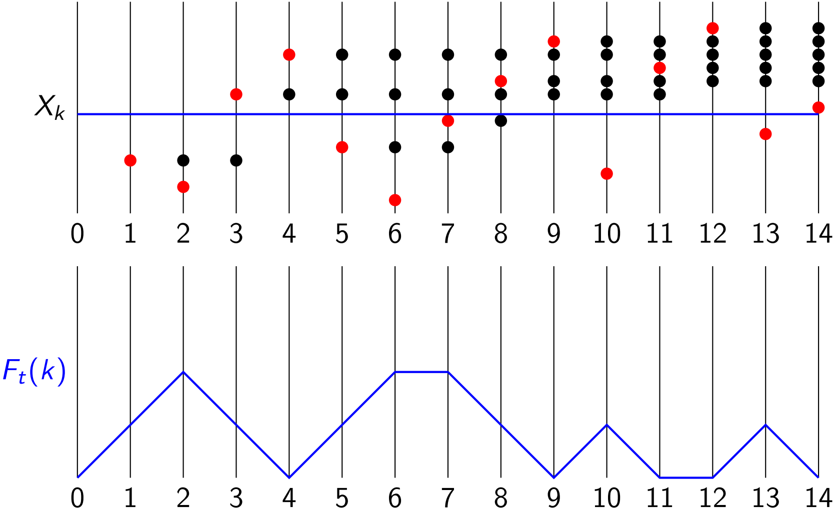

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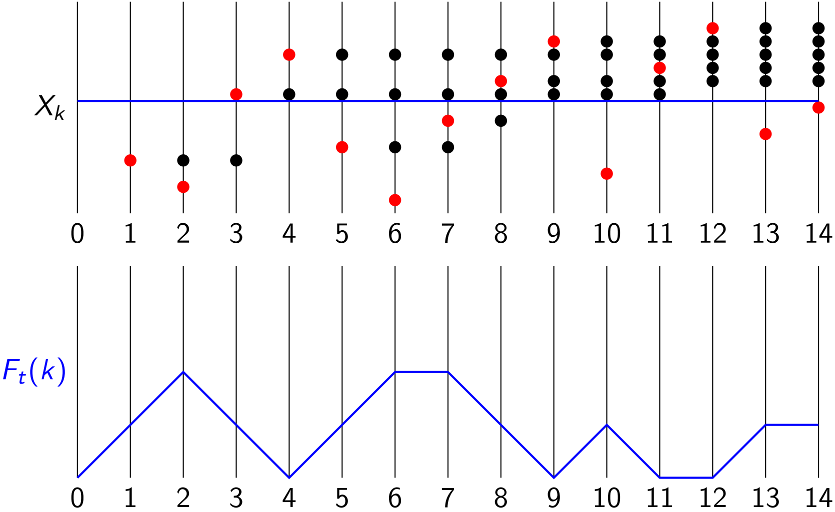

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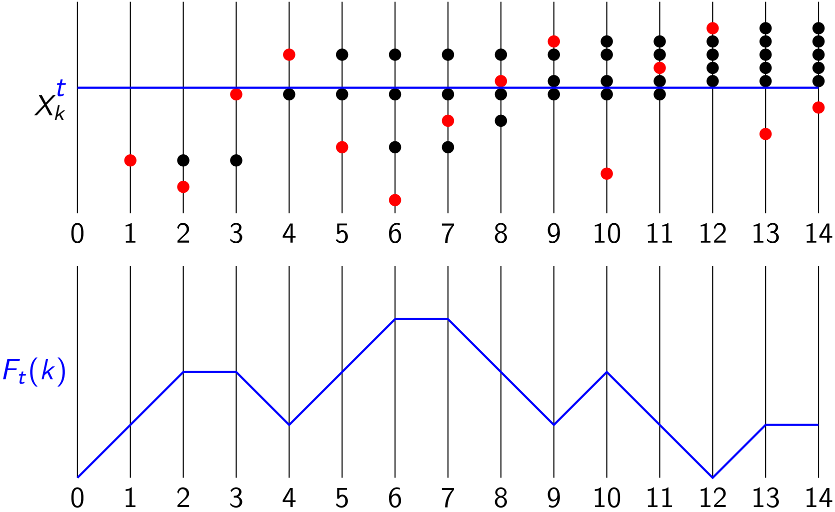

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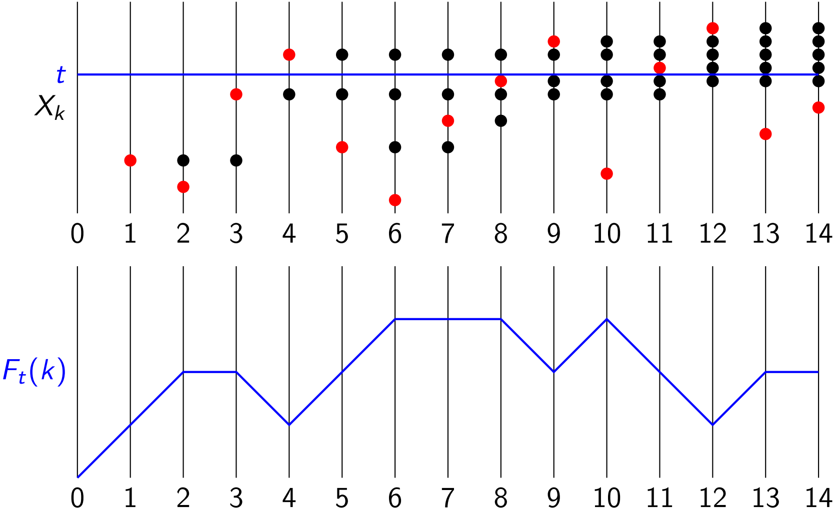

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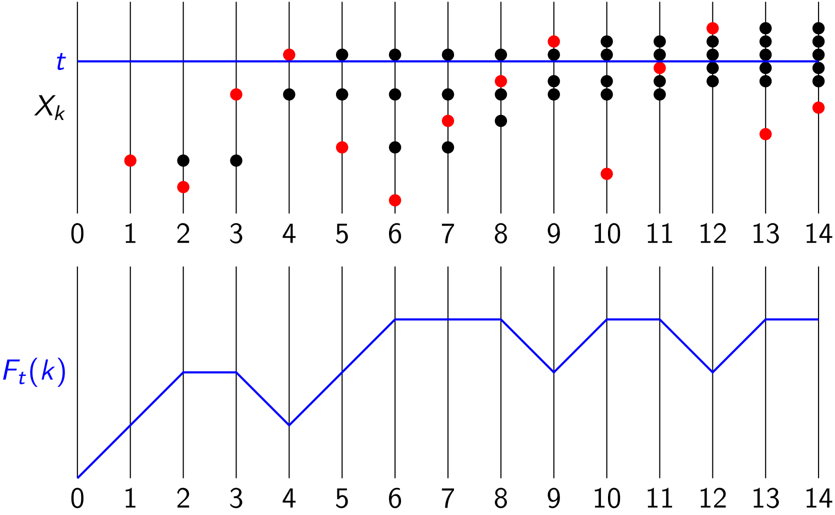

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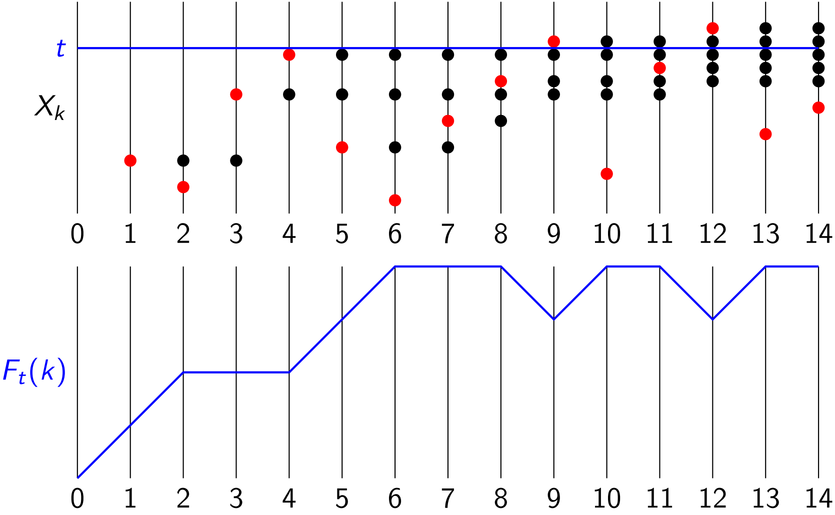

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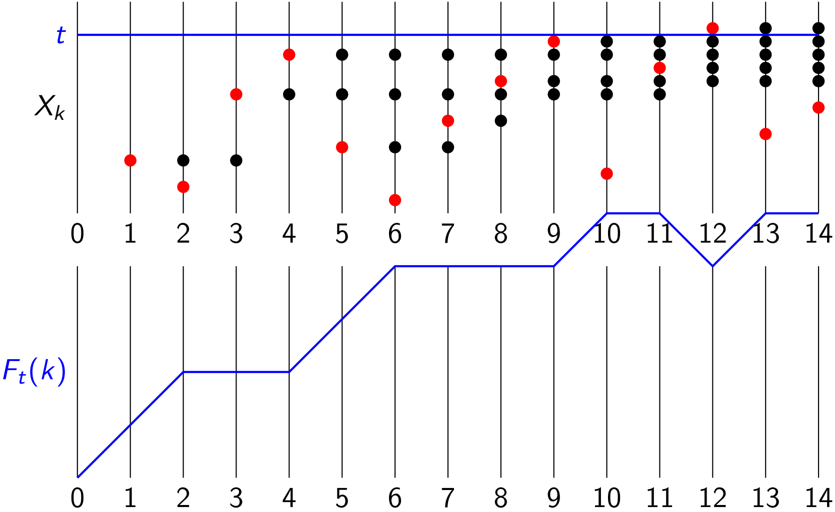

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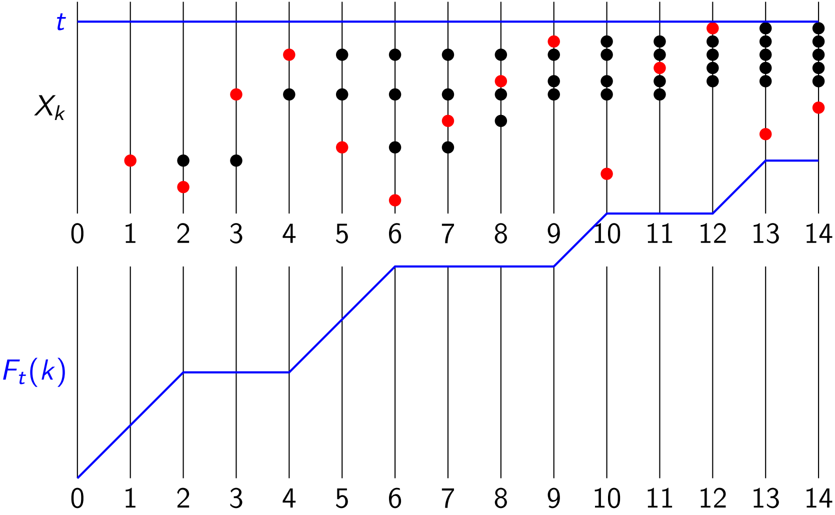


Increments

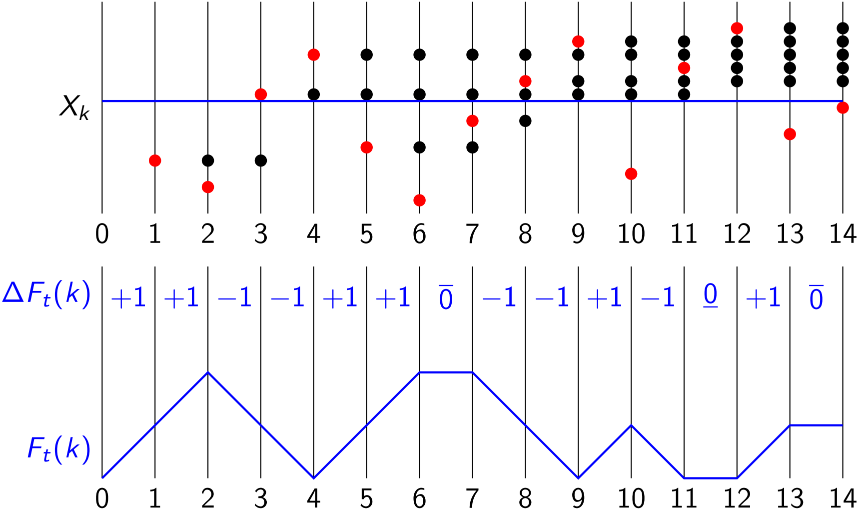
Define

$$\Delta F_t(k) := \begin{cases} \underline{0} & \text{if } F_t(k) = F_t(k-1) = 0, \\ \overline{0} & \text{if } F_t(k) = F_t(k-1) > 0, \\ -1 & \text{if } F_t(k) = F_t(k-1) - 1, \\ +1 & \text{if } F_t(k) = F_t(k-1) + 1. \end{cases}$$

At the exponentially distributed time $t = U_k$, the increment $\Delta F_t(k)$ changes from $\underline{0}$ to $+1$ or from -1 to $\overline{0}$.

At the same time, the next $\underline{0}$ to the right of k, if there is one, is changed into a -1.

The point-counting function



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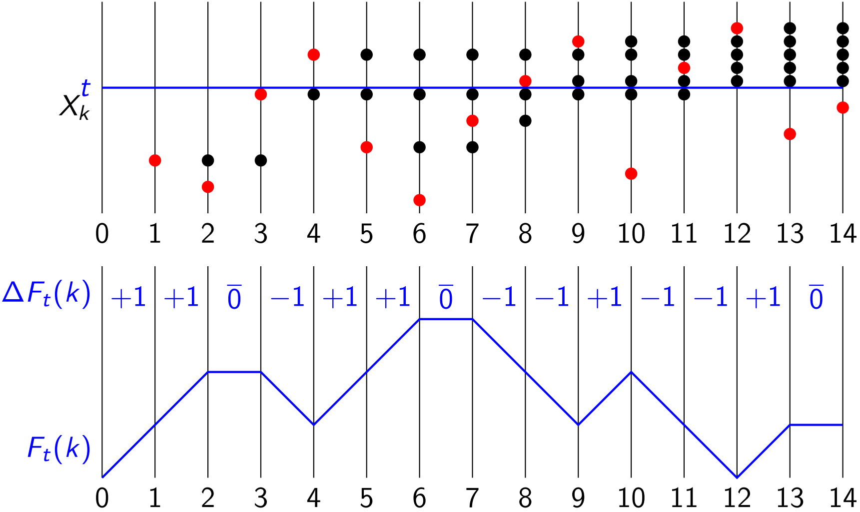


A stationary increment process

We can define the Markov process $(\Delta F_t)_{t\geq 0}$ also on $\mathbb{Z}$ instead of on $\mathbb{N}_+$.

As long as the density of $\underline{0}$'s is nonzero, the process started in $\Delta F_0(k) = \underline{0}$ ($k \in \mathbb{Z}$) satisfies

$$\begin{aligned} \tfrac{\partial}{\partial t}\mathbb{P}[\Delta F_t(k) = \underline{0}] &= -2\mathbb{P}[\Delta F_t(k) = \underline{0}] - \mathbb{P}[\Delta F_t(k) = -1], \\ \tfrac{\partial}{\partial t}\mathbb{P}[\Delta F_t(k) = -1] &= \mathbb{P}[\Delta F_t(k) = \underline{0}], \end{aligned}$$

from which we derive that the $\underline{0}$'s run out at $t_{\rm c} = 1$ and

$$\mathbb{P}[\Delta F_t(1) = \underline{0}] = (1-t)e^{-t} \quad \text{and} \quad \mathbb{P}[\Delta F_t(1) = -1] = te^{-t}$$

$(0 \leq t \leq 1)$.

The increment process

The process $(F_t)_{t\geq 0}$, both on $\mathbb{N}_+$ and $\mathbb{Z}$, makes i.i.d. excursions away from 0.

For the process started in $X_0 = \emptyset$, define the return time

$$\tau_t^{\emptyset} := \inf\big\{k \geq 1 : X_k \cap [0, t] = \emptyset\big\}.$$

From the density of $\underline{0}$'s for the process $(F_t)_{t\geq 0}$ on $\mathbb{Z}$ we deduce that

$$\mathbb{E}[\tau_t^{\emptyset}] = (1 - t)^{-1} \qquad (0 \leq t < 1).$$

This proves positive recurrence $\Leftrightarrow$ $t < 1$.
It is not hard to derive from this that the restricted process $\big(X_k \cap [0, t]\big)_{k\geq 0}$ is transient for $t > 1$.

Null recurrence at $t = 1$ is so far an open problem.

A stationary process

Let $t_c := 1$.

Theorem The restricted process $\big(X_k \cap [0, t_c]\big)_{k \geq 0}$ is ergodic. There exists a random, infinite, but locally finite subset $X_\infty \subset [0, t_c)$ such that

$$\mathbb{P}\big[X_k \cap [0, t] \in \cdot\,\big] \underset{k \to \infty}{\longrightarrow} \mathbb{P}\big[X_\infty \cap [0, t] \in \cdot\,\big] \qquad (t < t_c),$$

where the convergence is in total variation norm distance.

Plačková's model

Plačková's model is a Markov chain $(L_k, R_k)_{k\geq 0}$ where L_k, R_k are finite subsets of $[0,1]$ representing buy and sell limit orders.

Let $(U_k)_{k\geq 1}$ be i.i.d. uniformly distributed $[0,1]$-valued random variables, representing the prices of each trader, and let $(B_k)_{k\geq 1}$ be i.i.d. uniformly distributed $\{-1,+1\}$-valued random variables that determine whether a trader wants to buy or sell. Then

$$(L_k, R_k) := \begin{cases} (L_{k-1}\cup\{U_k\}, R_{k-1}) & \text{if } B_k = -1,\ U_k < M^R_k, \\ (L_{k-1}, R_{k-1}\backslash\{M^R_k\}) & \text{if } B_k = -1,\ M^R_k < U_k, \\ (L_{k-1}\backslash\{M^L_k\}, R_{k-1}) & \text{if } B_k = +1,\ U_k < M^L_k, \\ (L_{k-1}, R_{k-1}\cup\{U_k\}) & \text{if } B_k = +1,\ M^L_k < U_k, \end{cases}$$

where $M^L_k := \sup(L_k\cup\{0\})$ and $M^R_k := \inf(R_k\cup\{1\})$ $(k\geq 0)$.

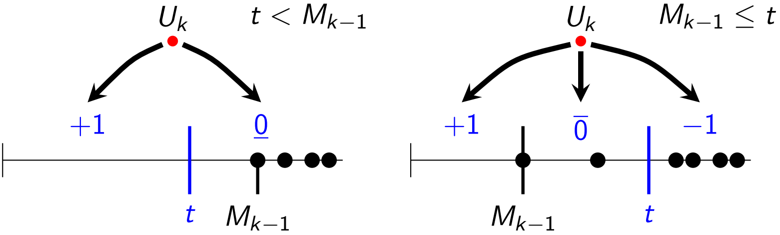
A cut-off version of Plačková's model

The restriction of Plačková's model to a subinterval of $[0,1]$ is in general *not* a Markov chain.

Nevertheless, we can define a "cut-off" version of the process on an interval $[q_-, q_+] \subset [0,1]$ if we change the dynamics in such a way that at q_- and q_+ there are infinite stacks of sell and buy limit orders, that are never depleted.

Theorem Assume that for some q_-, q_+, the cut-off process has an invariant law with a.s. locally finitely many limit orders in (q_-, q_+). Assume moreover that the bid and ask prices never reach q_- and q_+, respectively, so that the limit orders at q_- and q_+ are never matched. Then we must have $q_- = q_{\rm c}$ and $q_+ = 1 - q_{\rm c}$ with $q_{\rm c}$ as before.

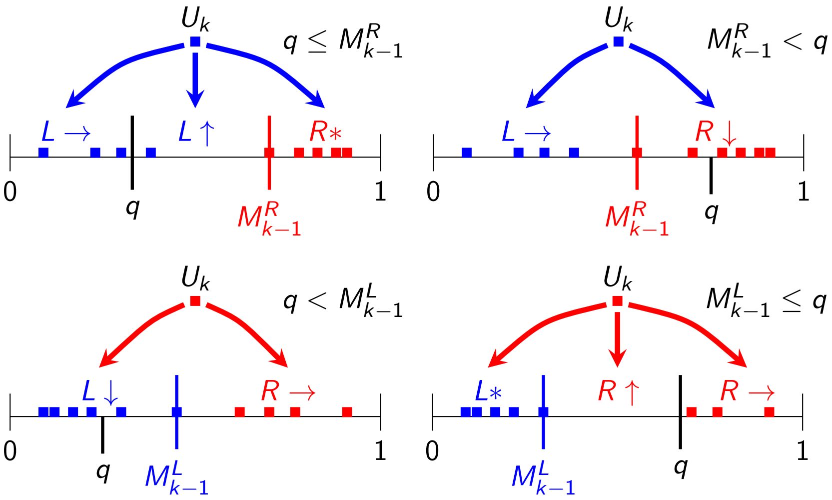
The one-sided model revisited



For the one-sided canyon model, we can read off the value of $\Delta F_t(k)$ from t, U_k, and X_{k-1}.

For the stationary process, we can derive a differential equation that tells us how the frequencies of $\underline{0}, \overline{0}, -1, +1$ change if we raise the level t.

Quantities for Plačková's model



Quantities for Plačková's model

If a cut-off version of Plačková's model has an invariant law, then

$$
\begin{aligned}
\mathbb{P}[L \rightarrow] &= \tfrac{1}{2}p_R - g_L(q) & \qquad \mathbb{P}[R \rightarrow] &= \tfrac{1}{2}(1-p_L) - g_R(q),\\
\mathbb{P}[L \uparrow] &= g_L(q) & \mathbb{P}[R \uparrow] &= g_R(q),\\
\mathbb{P}[L*] &= \tfrac{1}{2}p_L - g_L(q) & \mathbb{P}[R*] &= \tfrac{1}{2}(1-p_R) - g_R(q),\\
\mathbb{P}[L \downarrow] &= g_L(q) & \mathbb{P}[R \downarrow] &= g_R(q).
\end{aligned}
$$

Here

$$p_L := \mathbb{E}[M_k^L] \quad \text{and} \quad p_R := \mathbb{E}[M_k^R]$$

and g_L, g_R are functions that are continuously differentiable on (q_-, q_+) and...

Quantities for Plačková's model

. . . solve the differential equations

$$\begin{aligned}\frac{\partial}{\partial q} g_L(q) &= -(1-q)^{-1}\Big[\tfrac{1}{2}(1-p_R) + g_L(q) - g_R(q)\Big]\\ \frac{\partial}{\partial q} g_R(q) &= q^{-1}\Big[\tfrac{1}{2}p_L + g_R(q) - g_L(q)\Big]\end{aligned}$$

with the boundary conditions

$$\begin{aligned} g_L(q_-) &= \tfrac{1}{2}(p_R - q_-) \qquad & g_L(q_+) &= 0,\\ g_R(q_-) &= 0 & g_R(q_+) &= \tfrac{1}{2}(q_+ - p_L).\end{aligned}$$

Moreover, given q_- and q_+, there exists at most one quadruple (p_L, p_R, g_L, g_R) satisfying this differential equation and boundary conditions.

The critical value for Plačková's model

Set

$$p_\Delta := p_R - p_L \quad \text{and} \quad g_\Delta(q) := g_R(q) - g_L(q).$$

In the symmetric case $q_+ = 1 - q_-$, our differential equation simplifies to

$$\frac{\partial}{\partial q} g_\Delta(q) = \tfrac{1}{4}(1-p_\Delta)\big\{q^{-1}+(1-q)^{-1}\big\}+\big\{q^{-1}-(1-q)^{-1}\big\}g_\Delta(q).$$

with the boundary conditions

$$g_\Delta(\tfrac{1}{2}) = 0 \quad \text{and} \quad g_\Delta(q_+) = \tfrac{1}{2}q_+ - \tfrac{1}{4}(1-p_\Delta),$$

which can be solved explicitly as

$$g_\Delta(q) = \tfrac{1}{4}(1-p_\Delta)q(1-q)\big\{1/(1-q)-1/q-2\log(1-q)+2\log(q)\big\}.$$

The critical value for Plačková's model

The number of sell limit orders in (q_-, q_+) decreases with probability $\frac{1}{2}\mathbb{P}[M_k^R < q_+]$ if a limit order arrives on the right of q_+ and increases with probability $\frac{1}{2}$ if a limit order arrives between M_k^L and M_k^R. Limit orders arriving elsewhere have on average no effect. It follows that

$$\tfrac{1}{2}(1-q_+)\mathbb{P}[M_k^R < q_+] = \tfrac{1}{2}p_\Delta = \tfrac{1}{2}q_-\mathbb{P}[M_k^L < q_-].$$

Using also the conditions that

$$\mathbb{P}[M_k^R < q_+] = 1 = \mathbb{P}[q_- < M_k^L],$$

it follows that $q_- = 1 - q_+$. Using the explicit formula for g_Δ, one can now derive that $z := -1/q_+$ solves

$$1 + z + e^z = 0.$$