Interface tightness

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One-dimensional voter models

$\{0,1\}^{\mathbb{Z}}$ = the space of all functions $x : \mathbb{Z} \to \{0,1\}$. Interpretation:

$$x = \cdots 0000110101000110101110011111 \cdots$$

models the distribution of two genetic types of a plant, living in a one-dimensional environment (coastline, river).

$(X_t)_{t\geq 0}$ with $X_t = \big(X_t(i)\big)_{i\in\mathbb{Z}}$ continuous-time Markov process with state space $\{0,1\}^{\mathbb{Z}}$.
Dynamics: each plant lives an exponential time with mean 1, and upon death is immediately replaced by a clone of a near-by plant, at a distance chosen according to a probability distribution p.

In other words, if the present state is x, then $x(i)$ jumps:

$$\begin{array}{lll} 0 \mapsto 1 & \text{with rate} & \displaystyle\sum_{j\in\mathbb{Z}} p(j-i) 1_{\{x(j)=1\}}, \\ 1 \mapsto 0 & \text{with rate} & \displaystyle\sum_{j\in\mathbb{Z}} p(j-i) 1_{\{x(j)=0\}}. \end{array}$$

Interfaces

$$
\begin{aligned}
S^{01}_{\rm int} := \Big\{x \in \{0,1\}^{\mathbb{Z}} : \exists i < j \text{ s.t. } & x(i') = 0 \ \forall i' \leq i, \\
& x(j') = 1 \ \forall j' \geq j\Big\}.
\end{aligned}
$$

Interpretation: $x \in S^{01}_{\rm int}$ describes the *interface* between two infinite populations of 0's and 1's:

$$\cdots 0000000000000\,\underbrace{1011000110100}_{\text{interface}}\,11111111111111 \cdots$$

Lemma
If $\displaystyle\sum_k p(k)|k| < \infty$, then $X_0 \in S^{01}_{\rm int}$ implies $X_t \in S^{01}_{\rm int}$ $\forall t \geq 0$ a.s.

Question Starting from the *Heaviside configuration*

$$x_0 := \cdots 00000000000000000000001111111111111111111111 \cdots$$

does the size of the interface keep growing, or does it reach some finite equilibrium size?

Numerics

A voter model on $\{1, \ldots, 500\}$ with periodic boundary conditions, and p the uniform distribution on $\{-2, -1, 1, 2\}$.
Total time elapsed 600.

Interface tightness

Def $x \sim y$ if $\exists j$ s.t. $x(i) = y(i+j)$ $(i \in \mathbb{Z})$.

Def $\overline{x} := \{y : y \sim x\}$ and $\overline{S}^{01}_{\rm int} := \{\overline{x} : x \in S^{01}_{\rm int}\}$.

Observation The *voter model modulo translations* $(\overline{X}_t)_{t \geq 0}$ is a Markov process.

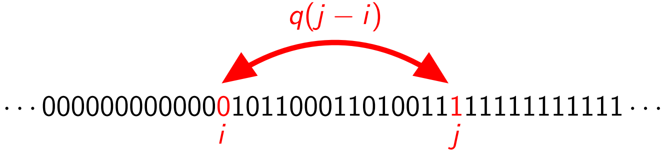
Def A voter model exhibits *interface tightness on* $S^{01}_{\rm int}$ if $\overline{x}_0$ is a positive recurrent state for the Markov process $(\overline{X}_t)_{t \geq 0}$.

Theorem If $\displaystyle\sum_k p(k)|k|^2 < \infty$, then interface tightness holds on $S^{01}_{\rm int}$ and $S^{10}_{\rm int}$.

Proved when $\sum_k p(k)|k|^3 < \infty$ by Cox and Durrett (1995) and in general by Belhaouari, Mountford and Valle (2007), who moreover showed that the second moment condition is optimal.

Voter-exclusion models

In an *exclusion process*, the states of two sites i and j are interchanged with rate $q(j-i)$, where $q:\mathbb{Z}\to[0,\infty)$ is a symmetric function.



Voter-exclusion models

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$$q(j-i)$$

$$\cdots 000000000000\underset{i}{1}101100011010011\underset{j}{0}111111111111\cdots$$

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$q(j-i)$

$$\cdots 000000000000\underset{i}{1}10110001101001\underset{j}{0}11111111111\cdots$$

Theorem [Sturm & S. '08] If $\sum_k p(k)|k|^2<\infty$ and $\sum_k q(k)|k|^2<\infty$, then interface tightness on $S^{01}_{\rm int}$ and $S^{10}_{\rm int}$ holds for voter-exclusion processes.

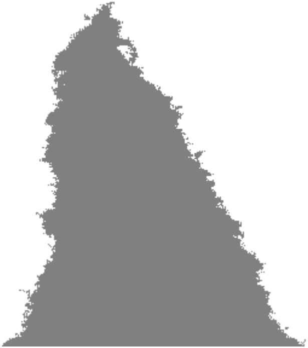
Biased voter models

In the *biased voter model* with *bias* $\varepsilon \in [0,1]$, $x(i)$ jumps:

$$\begin{aligned} 0 \mapsto 1 \quad & \text{with rate} \quad & \sum_{j\in\mathbb{Z}} p(j-i) 1_{\{x(j)=1\}}, \\ 1 \mapsto 0 \quad & \text{with rate} \quad & (1-\varepsilon)\sum_{j\in\mathbb{Z}} p(j-i) 1_{\{x(j)=0\}}. \end{aligned}$$

Theorem [Sun, S. & Yu '18] If $\sum_{k<0} p(k)|k| < \infty$ and $\sum_{k>0} p(k)|k|^2 < \infty$, then interface tightness holds on $S^{01}_{\rm int}$.

Numerics



A biased voter model with bias $\varepsilon = 0.3$.

Rebellious voter models

Let $f_\sigma(i) := \sum_{j\in\mathbb{Z}} p(j-i)1_{\{x(j)=\sigma\}} \quad (\sigma = 0,1).$

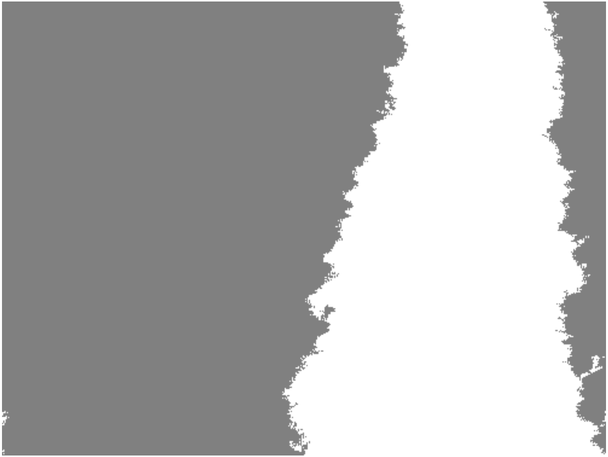
In the neutral *Neuhauser-Pacala model* (1999) with *competition parameter* $\alpha \geq 0$,

$$x(i) \text{ flips } \begin{cases} 0 \mapsto 1 \text{ with rate } f_1(i)\big(f_0(i) + \alpha f_1(i)\big), \\ 1 \mapsto 0 \text{ with rate } f_0(i)\big(f_1(i) + \alpha f_0(i)\big). \end{cases}$$

Note: $f_0(i) + f_1(i) = 1$. For $\alpha = 1$ this is the voter model.
For $\alpha < 1$, you die faster if your neighbors are of *your own* type.
For $\alpha > 1$, you die faster if your neighbors are of *the other* type.

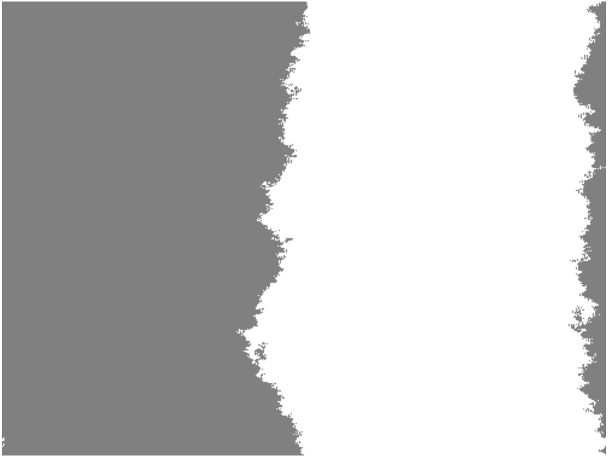
Conjecture There exists an $\alpha_c < 1$ such that interface tightness holds for $\alpha > \alpha_c$, but not for $\alpha \leq \alpha_c$.

Numerics



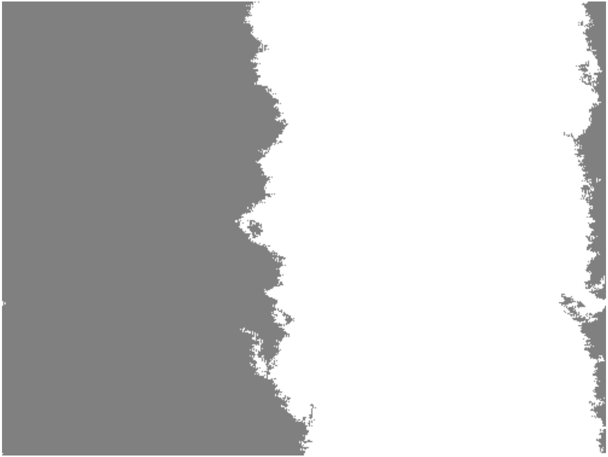
A neutral Neuhauser-Pacala model with $\alpha = 0.9$.

Numerics



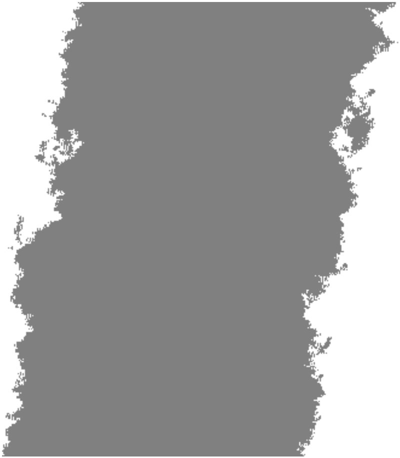
A neutral Neuhauser-Pacala model with $\alpha = 0.8$.

Numerics



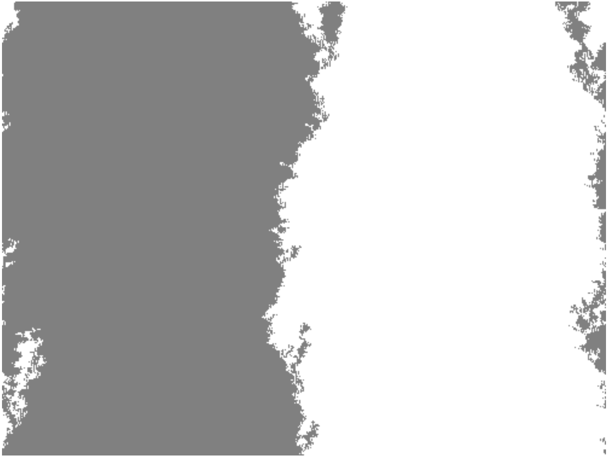
A neutral Neuhauser-Pacala model with $\alpha = 0.7$.

Numerics



A neutral Neuhauser-Pacala model with $\alpha = 0.6$.

Numerics



A neutral Neuhauser-Pacala model with $\alpha = 0.5$.

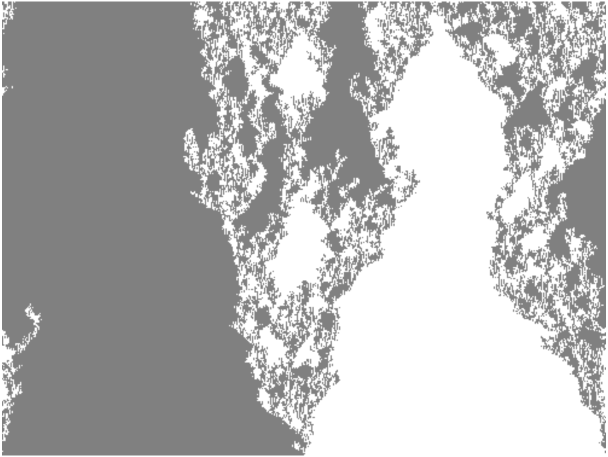
Numerics

A neutral Neuhauser-Pacala model with $\alpha = 0.4$.

Numerics

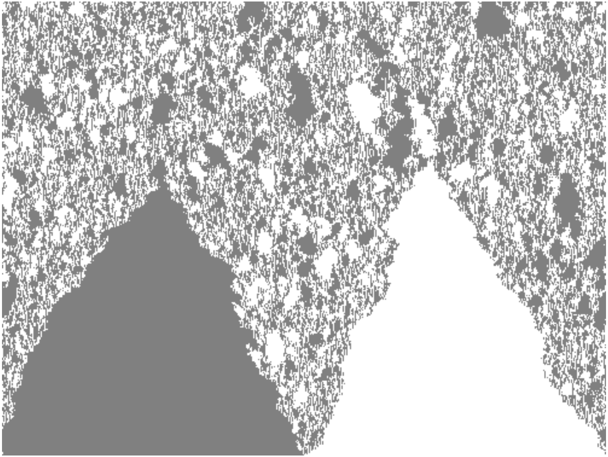
A neutral Neuhauser-Pacala model with $\alpha = 0.3$.

Numerics



A neutral Neuhauser-Pacala model with $\alpha = 0.2$.

Numerics

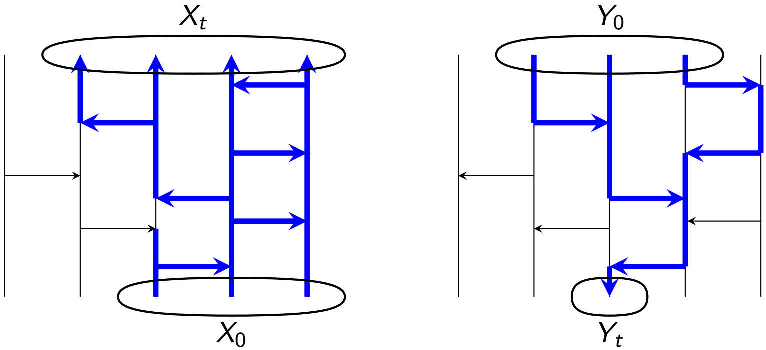


A neutral Neuhauser-Pacala model with $\alpha = 0.1$.

Other models

- Asymmetric exclusion processes. [Bramson, Liggett & Mountford '02].
- Neutral two-type contact processes. [Mountford & Valesin '10, '16].
- Biased two-type contact processes. [Andjel, Mountford, Pimentel & Valesin '10]

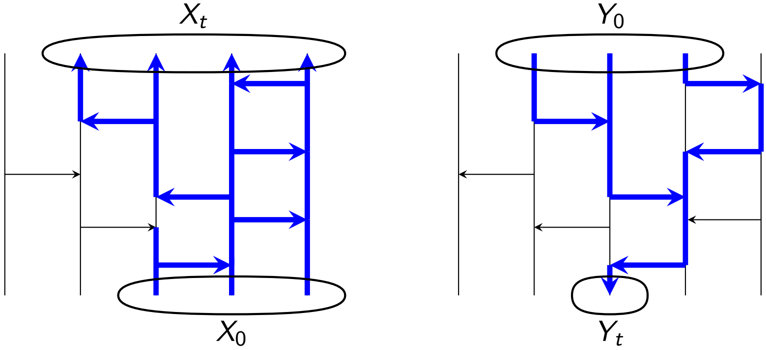
Duality



A voter model X is dual to a system of coalescing random walks Y:

$$\mathbb{P}\big[X_t \wedge Y_0 \neq 0\big] = \mathbb{P}\big[X_0 \wedge Y_1 \neq 0\big] \quad (t \geq 0).$$

Duality



Interpretation: Set $\mathcal{Y}_t := \{i : Y_t(i) = 1\}$.

Then $\mathcal{Y}_t$ are the ancestors of $\mathcal{Y}_0$.

$X_t(i) = 1$ for some $i \in \mathcal{Y}_0$ iff $X_0(i) = 1$ for some $i \in \mathcal{Y}_t$.

A useful function

Cox and Durrett (1995) look at the function

$$h(x) := \sum_{i<j} 1_{\{x(i) > x(j)\}} \qquad \big(x \in S^{01}_{\rm int}\big),$$

which counts the *number of inversions.* For the process started in the Heaviside state x_0, they used duality to prove

$$\sup_{t \geq 0} \mathbb{P}\big[h(X_t) \geq N\big] \underset{N\to\infty}{\longrightarrow} 0.$$

The function h also plays a key role in the proofs of Belhaouari, Mountford and Valle (2007).

The width of the interface

Let $\ell(x) := \sup\{i : x(i) = 0\} - \inf\{i : x(i) = 1\} + 1$ denote the *width of the interface*.

$$\cdots 0000000000000\underbrace{10110}_{\ell(x)=5}11111111111111\cdots$$

If interface tightness holds, then $\overline{X}_t$, started in $\overline{x}_0$, converges in law as $t \to \infty$ to some $\overline{X}_\infty$. Cox and Durrett (Theorem 6) prove that

$$\mathbb{E}\big[\ell(\overline{X}_\infty)\big] = \infty.$$

Belhaouari, Mountford, Sun and Valle (2006, Theorem 1.4) have shown that

$$\mathbb{E}\big[\ell(\overline{X}_\infty) \geq L\big] \asymp L^{-1}.$$

A pseudo-Lyapunov function

The process modulo translations $\overline{X}_t$ is a countinuous-time Markov chain with countable state space $\overline{S}^{01}_{\rm int}$.

By Foster's theorem, positive recurrence is equivalent to the existence of a Lyapunov function $V : \overline{S}^{01}_{\rm int} \to [0, \infty)$ such that

$$\begin{aligned} GV(x) &< \infty && \text{for all } x \in \overline{S}^{01}_{\rm int}, \\ GV(x) &\leq -1 && \text{for all but finitely many } x \in \overline{S}^{01}_{\rm int}, \end{aligned}$$

where G is the *generator* of $\overline{X}_t$.

For the voter model modulo translations, no such Lyapunov function has been found explicitly.

A pseudo-Lyapunov function

Sturm & S. (2008) have shown that the *number of inversions* $h(x)$ is "almost" a Lyapunov function.

More precisely,

$$Gh(x) = \frac{1}{2}\sum_{k\in\mathbb{Z}} p(k)|k|^2 - \frac{1}{2}\sum_{k\in\mathbb{Z}} p(k) I_k(x),$$

where

$$I_k(x) := \sum_{i\in\mathbb{Z}} 1_{\{x(i)\neq x(i+k)\}}$$

denotes the number of *k-boundaries.*

Since $\{\overline{x} : x \in S^{01}_{\rm int},\ Gh(x) \not\leq -1\}$ is in general not finite (except when p is almost nearest neighbor), this is not a Lyapunov function.

A pseudo-Lyapunov function

Nevertheless, it is almost as good as a Lyapunov function. One can show that if interface tightness does not hold, then

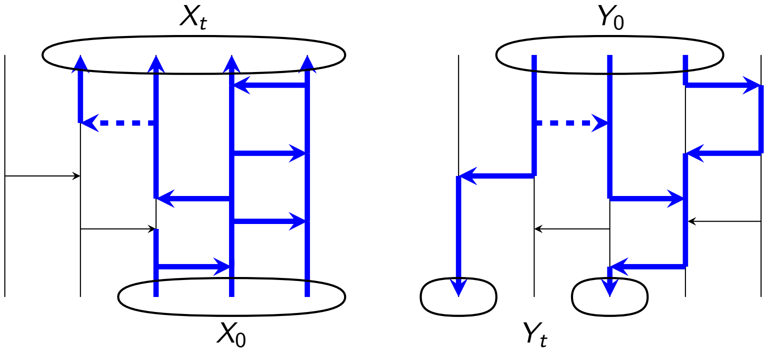
$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\ \mathbb{P}\big[I_k(X_t)<N\big]=0 \qquad (N,k\geq 1),$$

i.e., *most of the time, there are lots of k-boundaries.*

As a result, most of the time $Gh(X_t)\leq -1$, while the rest of the time $Gh(X_t)\leq\frac{1}{2}\sum_{k\in\mathbb{Z}}p(k)|k|^2<\infty$.

This means that if interface tightness does not hold, then over long time intervals, $h(X_t)$ decreases more than it increases. Since $h\geq 0$, we arrive at a contradiction. ∎

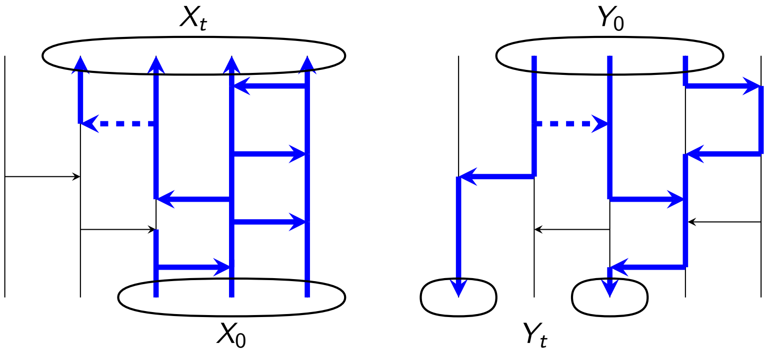
Duality for biased voter models



A biased voter model X has a branching-coalescing dual Y:

$$\mathbb{P}\big[X_t \wedge Y_0 \neq 0\big] = \mathbb{P}\big[X_0 \wedge Y_1 \neq 0\big] \quad (t \geq 0).$$

Duality for biased voter models



Interpretation: Set $\mathcal{Y}_t := \{i : Y_t(i) = 1\}$.
Then $\mathcal{Y}_t$ are the *potential ancestors* of $\mathcal{Y}_0$.
$X_t(i) = 1$ for some $i \in \mathcal{Y}_0$ iff $X_0(i) = 1$ for some $i \in \mathcal{Y}_t$.

Interface tightness for biased voter models

Sun, S. & Yu (2018) prove interface tightness for biased voter models using the pseudo-Lyapunov function technique of Sturm & S.

Set $i_0(x) := \inf\{i \in \mathbb{Z} : x(i) = 1\}$, $i_{n+1} := \inf\{i > i_n : x(i) = 1\}$.

A suitable pseudo-Lyapunov function turns out to be the *weighted number of inversions*

$$h_\varepsilon(x) := \sum_{n=0}^{\infty} (1-\varepsilon)^n \sum_{j > i_n} 1_{\{x(j) = 0\}}.$$

Scaling limit

Let L_t and R_t denote the left and right boundaries

$$L_t := \inf\{i \in \mathbb{Z} : X_t(i) = 1\} \quad \text{and} \quad R_t := \sup\{i \in \mathbb{Z} : X_t(i) = 0\}.$$

For the unbiased voter model, it has been proved by Belhaouari, Mountford, Sun and Valle (2006) that

$$\big(\varepsilon L_{\varepsilon^{-2}t}, \varepsilon R_{\varepsilon^{-2}t}\big) \underset{\varepsilon\to 0}{\Longrightarrow} \big(B_t, B_t\big)_{t\geq 0},$$

where B_t is a Brownian motion with variance $\sum_k p(k)|k|^2$.

For the biased voter model with bias ε, a similar result has been conjectured where B_t is a drifted Brownian motion.

Heuristics

Recall that the equilibrium interface width is heavy-tailed:

$$\mathbb{E}\big[\ell(\overline{X}_\infty) \geq L\big] \asymp L^{-1}.$$

On the other hand, numerics suggest that the equilibrium number of 1-boundaries has an exponential tail:

$$\mathbb{E}\big[I_1(\overline{X}_\infty) \geq N\big] \asymp e^{-cN} \quad \text{for some } c > 0.$$

Conjecture:

$$\lim_{L\to\infty} \mathbb{P}\big[I_1(\overline{X}_\infty) \in \cdot \,\big|\, \ell(\overline{X}_\infty) \geq L\big]$$

exists, and equals the law of $I^1 + I^2 + I^3$, where I^1, I^2, I^3 are i.i.d. and equally distributed with $I_1(\overline{X}_\infty)$.

Proof?

Three Brownian motions

Let $(B^1_t, B^2_t, B^3_t)_{t\geq 0}$ be three independent Brownian motions started from $B^i_0 = x_i$, with $x_1 < x_2 < x_3$. Set

$$\tau := \tau_{12} \wedge \tau_{23} \quad \text{with} \quad \tau_{ij} := \inf\{t \geq 0 : B^i_t = B^j_t\}.$$

Then it is known (Grabiner 1999) that

$$\mathbb{P}[\tau > t] \sim \frac{3}{4\sqrt{\pi}} \prod_{1\leq i<j\leq 3} (x_j - x_i)\, t^{-3/2} \qquad \text{as } t \to \infty.$$

In particular, τ has finite mean.

Heuristics

We want to heuristically "deduce" that

$$\mathbb{E}\big[\ell(\overline{X}_\infty) \geq L\big] \asymp L^{-1}.$$

By definition, a macroscopic "excursion" of duration τ is an event where a microscopic interface splits into three interfaces. After time τ, two of these interfaces annihilate each other, leaving only one microscopic interface.

For a given excursion, $\mathbb{P}[\tau > T] \sim T^{-3/2}$.

In equilibrium, the duration $\hat{\tau}$ of the excursion that takes place at time zero has a size-biased law, so $\mathbb{P}[\hat{\tau} > T] \sim T^{-1/2}$.

The width ℓ of an excursion of duration $\hat{\tau}$ should be of order $\ell = \sqrt{\hat{\tau}}$, so

$$\mathbb{P}[\ell > L] \approx \mathbb{P}[\sqrt{\hat{\tau}} > L] = \mathbb{P}[\hat{\tau} > L^2] \sim L^{-1}.$$