The Brownian net

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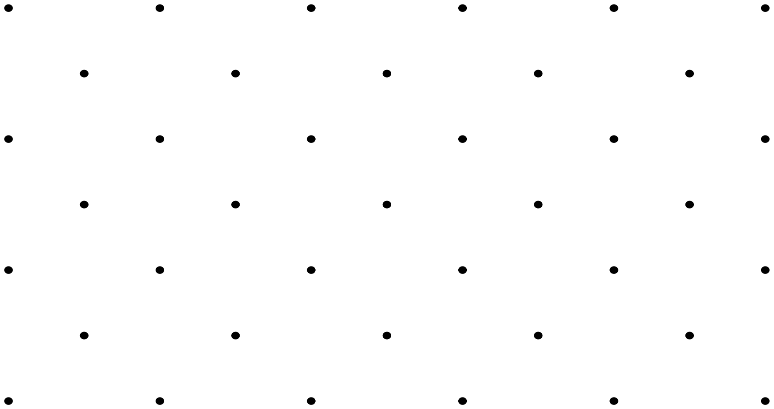
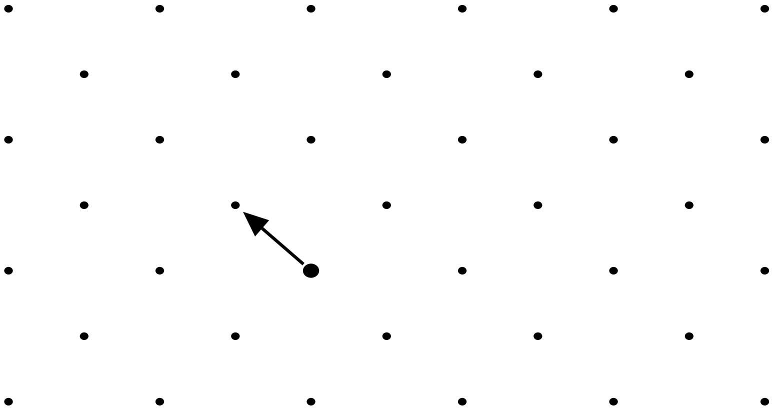
Arrow configurations



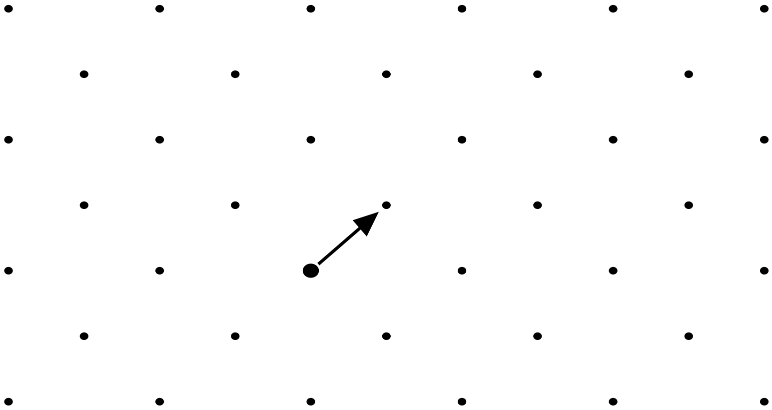
$$\mathbb{Z}^2_{\mathrm{even}} := \{(x, t) \in \mathbb{Z}^2 : x + t \text{ is even}\}.$$

Arrow configurations



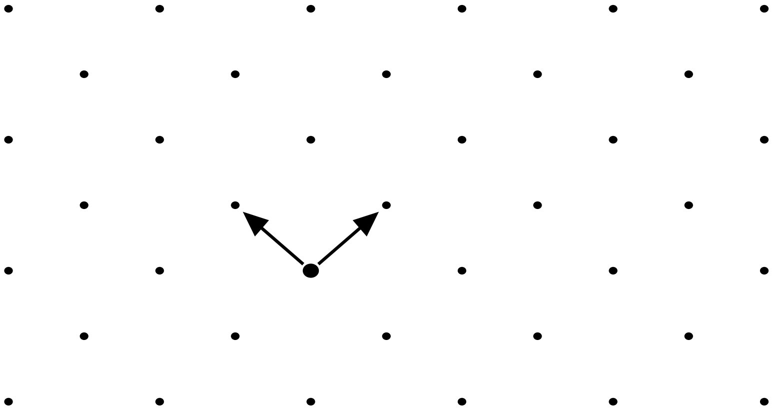
With probability p_{l} we draw an arrow to the left.

Arrow configurations



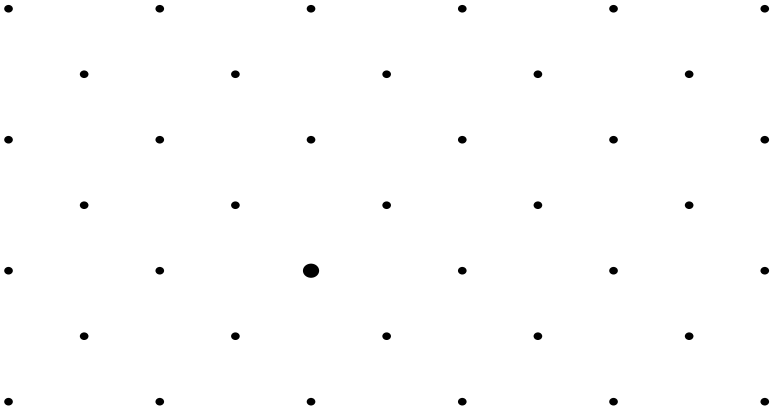
With probability p_{r} we draw an arrow to the right.

Arrow configurations



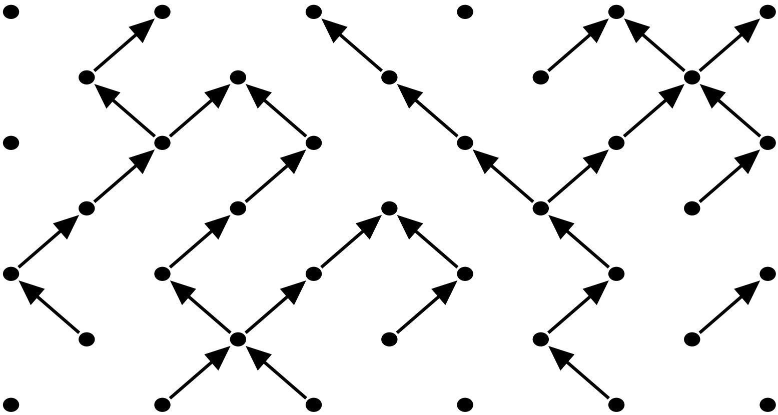
With probability p_{b} we draw two arrows.

Arrow configurations



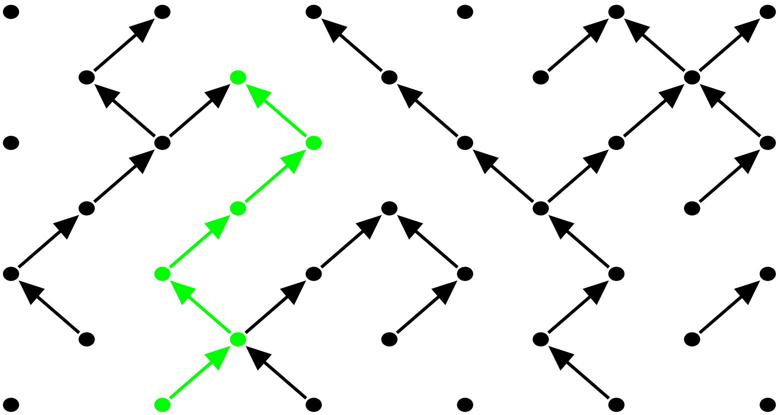
And with probability p_{k} we draw no arrows at all.

Arrow configurations



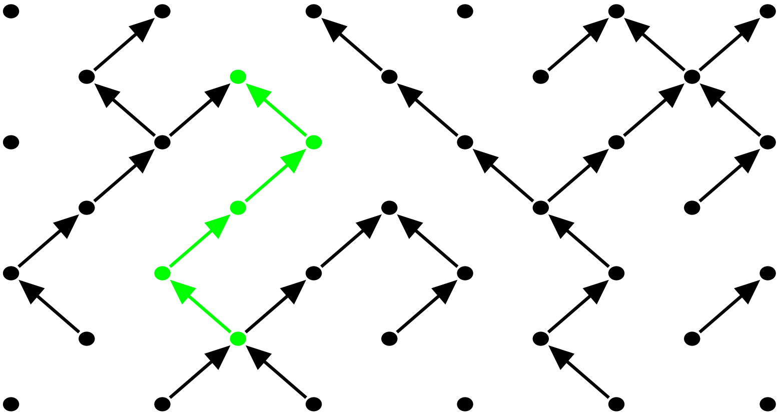
We do this independently for each point.

Arrow configurations



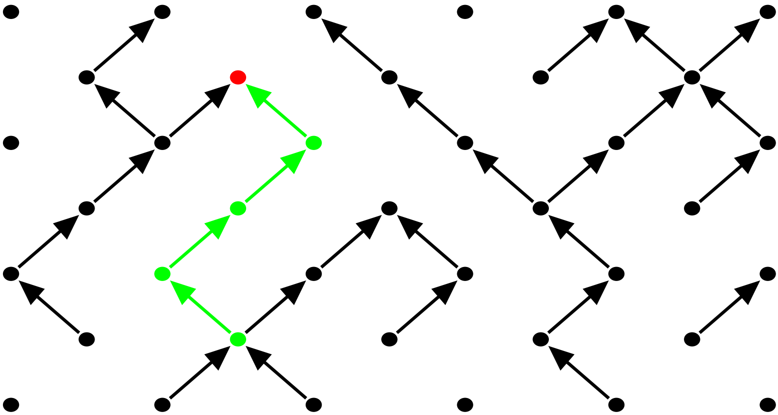
We are interested in *open paths*.

Arrow configurations



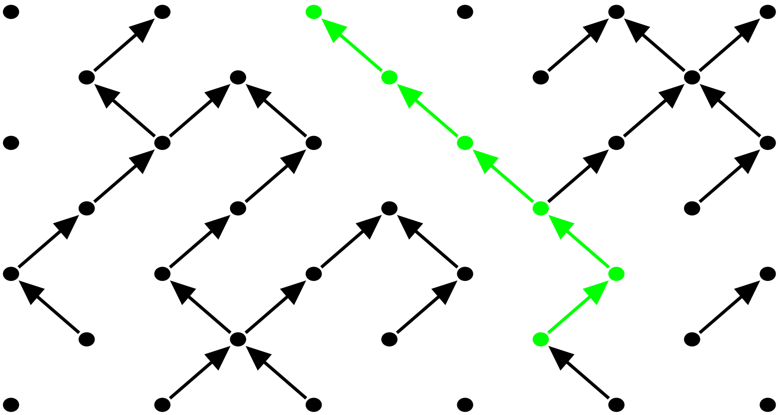
Open paths can start at any point in $\mathbb{Z}^2_{\text{even}}$.

Arrow configurations

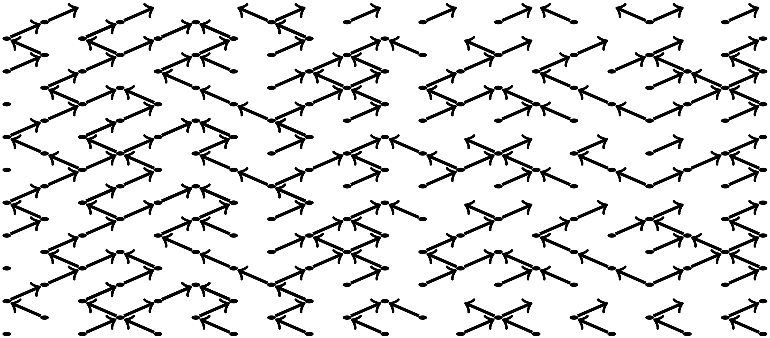


Open paths either end at killing points. . .

Arrow configurations



. . . or carry on forever.



We rescale diffusively, multiplying all spatial distances with ε and all temporal distances with ε^2.

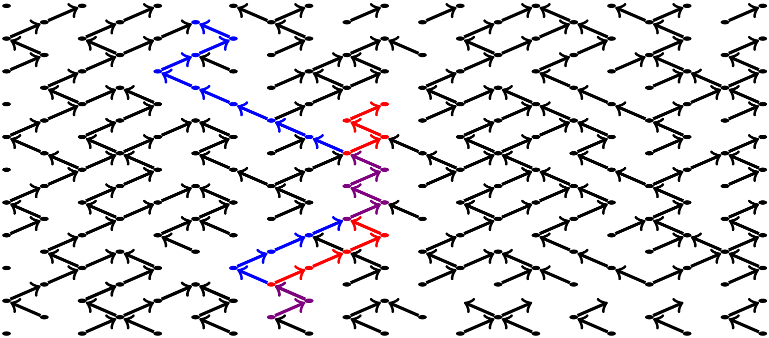
Scaling limit

Claim Assume that

$$\begin{aligned} \varepsilon^{-1}(p_{\mathrm{r}} - p_{\mathrm{l}} - p_{\mathrm{b}}) &\to \beta_{-}, \\ \varepsilon^{-1}(p_{\mathrm{r}} - p_{\mathrm{l}} + p_{\mathrm{b}}) &\to \beta_{+}, \\ \varepsilon^{-2} p_{\mathrm{k}} &\to \delta. \end{aligned}$$

Then the collection $\mathcal{U}$ of all open paths converges to a diffusive scaling limit $\mathcal{N}^{\delta}_{\beta_{-},\beta_{+}}$.

Scaling limit



At each point $z \in \mathbb{Z}^2_{\mathrm{even}}$ there starts an a.s. unique left-most open path l_z and right-most open path r_z.

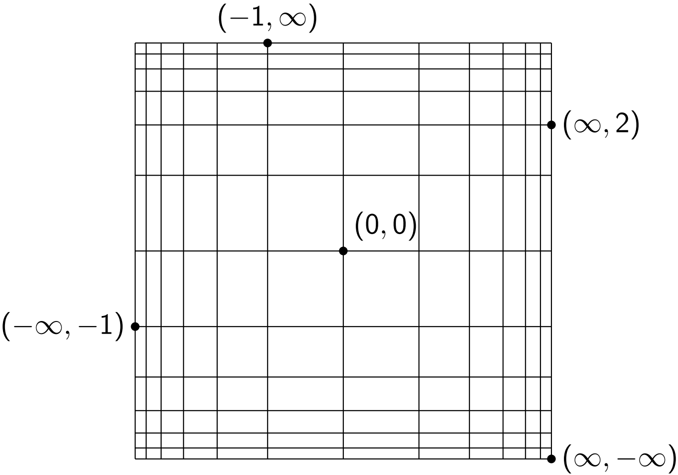
Scaling limit

Under the assumptions

$$\begin{aligned} \varepsilon^{-1}(p_{\mathrm{r}} - p_{\mathrm{l}} - p_{\mathrm{b}}) &\to \beta_-, \\ \varepsilon^{-1}(p_{\mathrm{r}} - p_{\mathrm{l}} + p_{\mathrm{b}}) &\to \beta_+, \\ \varepsilon^{-2} p_{\mathrm{k}} &\to \delta, \end{aligned}$$

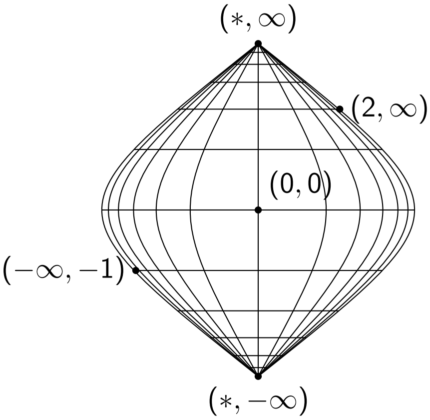
left- and right-most open paths converge to Brownian motions with drift β_- and β_+, respectively, and exponential lifetimes with mean $1/\delta$.

Topological matters



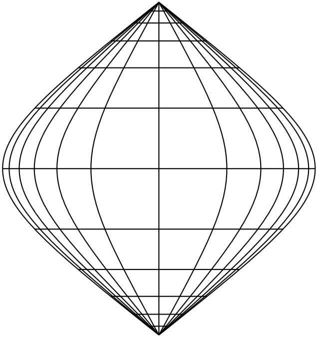
We first compactify $\mathbb{R}^2$ to $[-\infty,\infty]^2$...

Topological matters



...and then contract $[-\infty, \infty] \times \{-\infty\}$ and $[-\infty, \infty] \times \{\infty\}$ to single points.

Topological matters

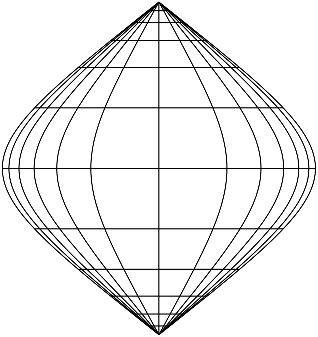


Alternatively, map $\mathbb{R}^2$ into itself with the map

$$\Theta(x,t) := \Big(\frac{\tanh(x)}{1+|t|}, \tanh(t)\Big),$$

and take the closure.

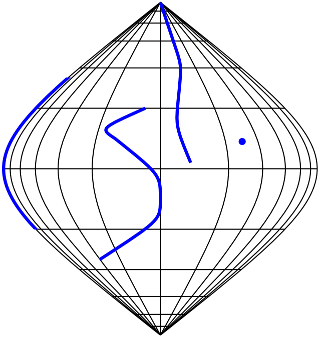
Topological matters



Another equivalent formulation is: take the completion of $\mathbb{R}^2$ w.r.t. the metric

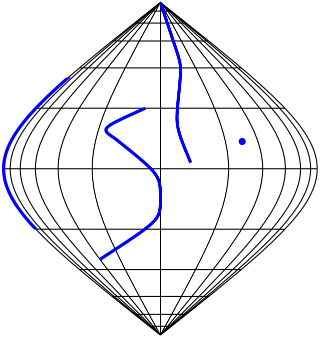
$$d(z,z') := \big|\Theta(z) - \Theta(z')\big|.$$

Topological matters



A *path* is a continuous function $\pi : [\sigma_\pi, \tau_\pi] \to [-\infty, \infty]$,
with $-\infty \le \sigma_\pi \le \tau_\pi \le \infty$.

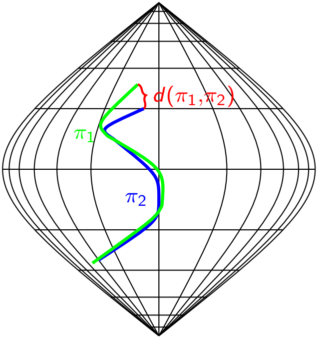
Topological matters



We identify a path with its graph

$$\{(\pi(t), t) : t \in [\sigma_\pi, \tau_\pi]\}.$$

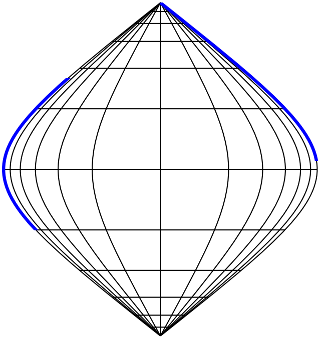
Topological matters



We equip the space Π of all paths with the Hausdorff metric

$$d(\pi_1, \pi_2) = \sup_{z_1 \in \pi_1} \inf_{z_2 \in \pi_2} d(z_1, z_2) \vee \sup_{z_2 \in \pi_2} \inf_{z_1 \in \pi_1} d(z_1, z_2).$$

Topological matters



By adding trivial paths that are constantly $-\infty$ or $+\infty$, we can make the set $\mathcal{U}$ of open paths into a compact subset of Π.

Topological matters

We equip the space $\mathcal{K}(\Pi)$ of all compact subsets of the space of paths Π with the Hausdorff metric

$$d(\mathcal{U}_1, \mathcal{U}_2) = \sup_{\pi_1 \in \mathcal{U}_1} \inf_{\pi_2 \in \mathcal{U}_2} d(\pi_1, \pi_2) \vee \sup_{\pi_2 \in \mathcal{U}_2} \inf_{\pi_1 \in \mathcal{U}_1} d(\pi_1, \pi_2).$$

We define a diffusive scaling map S_ε by

$$S_\varepsilon(x, t) := (\varepsilon x, \varepsilon^2 t).$$

Topological matters

Theorem Let $\varepsilon_n \downarrow 0$ and let $\mathcal{U}_n$ be the sets of open paths in arrow configurations with parameters satisfying

$$\begin{aligned} \varepsilon_n^{-1}\big(p_{\rm r}(n) - p_{\rm l}(n) - p_{\rm b}(n)\big) &\to \beta_-, \\ \varepsilon_n^{-1}\big(p_{\rm r}(n) - p_{\rm l}(n) + p_{\rm b}(n)\big) &\to \beta_+, \\ \varepsilon_n^{-2} p_{\rm k}(n) &\to \delta. \end{aligned}$$

Then

$$\mathbb{P}[S_{\varepsilon_n}(\mathcal{U}_n) \in \cdot\,] \underset{n\to\infty}{\Longrightarrow} \mathbb{P}[\mathcal{N}^{\delta}_{\beta_-,\beta_+} \in \cdot\,],$$

where $\Rightarrow$ denotes weak convergence of probability laws on $\mathcal{K}(\Pi)$. The limiting object is a *Brownian net with killing.*

The Brownian web

If $\beta = \beta_- = \beta_+$ and $\delta = 0$, then the limiting object $\mathcal{W}_\beta := \mathcal{N}^0_{\beta,\beta}$ is a *Brownian web* with drift β. In particular, $\mathcal{W} := \mathcal{W}_0$ is the *standard Brownian web.*

- For each deterministic $z \in \mathbb{R}^2$, almost surely there is a unique open path $p_z \in \mathcal{W}$.

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- For each deterministic $z \in \mathbb{R}^2$, almost surely there is a unique open path $p_z \in \mathcal{W}$.
- For any deterministic finite set of points $z_1, \ldots, z_k \in \mathbb{R}^2$, the collection $(p_{z_1}, \ldots, p_{z_k})$ is distributed as coalescing Brownian motions.

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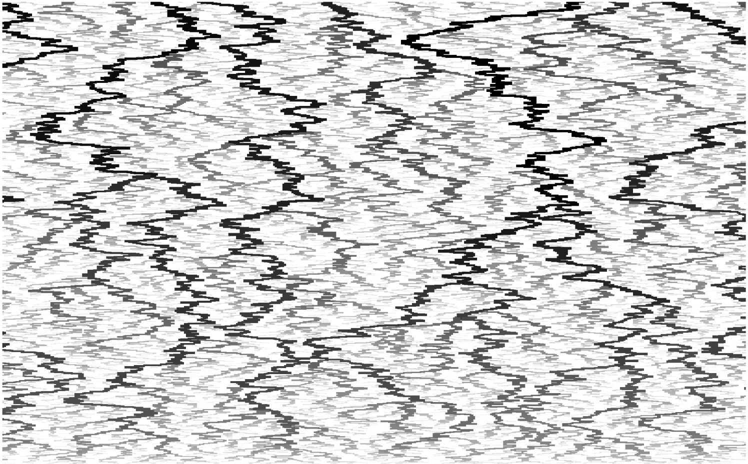
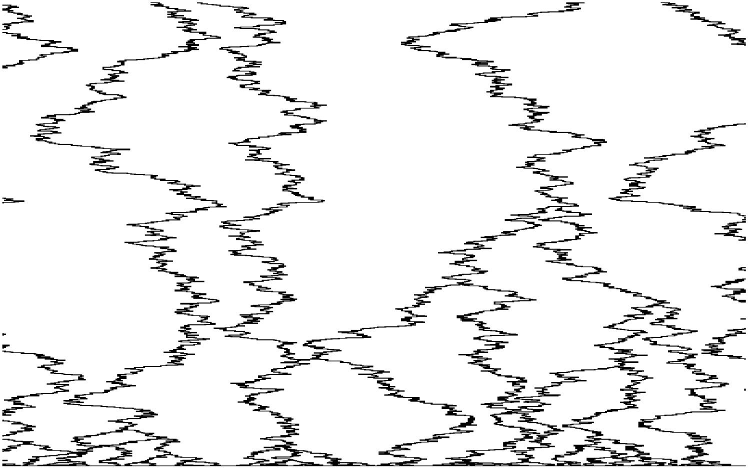
- For each deterministic $z \in \mathbb{R}^2$, almost surely there is a unique open path $p_z \in \mathcal{W}$.
- For any deterministic finite set of points $z_1, \ldots, z_k \in \mathbb{R}^2$, the collection $(p_{z_1}, \ldots, p_{z_k})$ is distributed as coalescing Brownian motions.
- For any deterministic countable dense subset $\mathcal{D} \subset \mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure of $\{p_z : z \in \mathcal{D}\}$.

The Brownian web

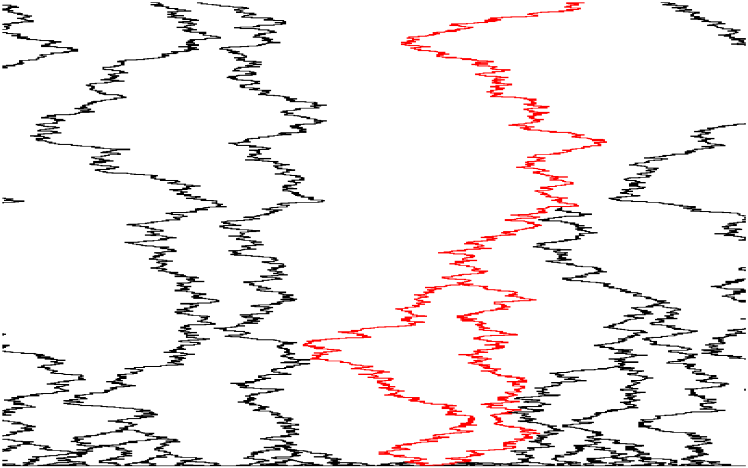


Artist's impression of the Brownian web.

The Brownian web

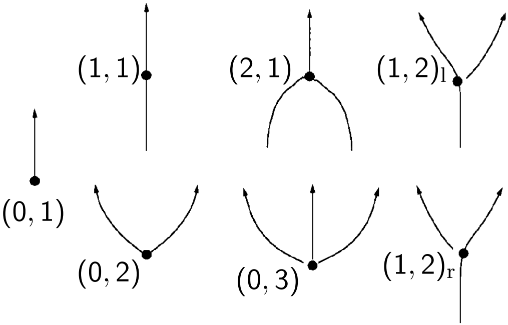


Open paths started at time zero.



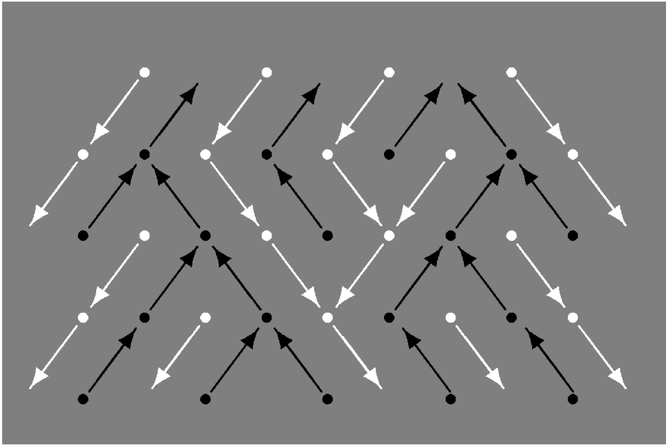
There exists random points where two open paths start.

Special points



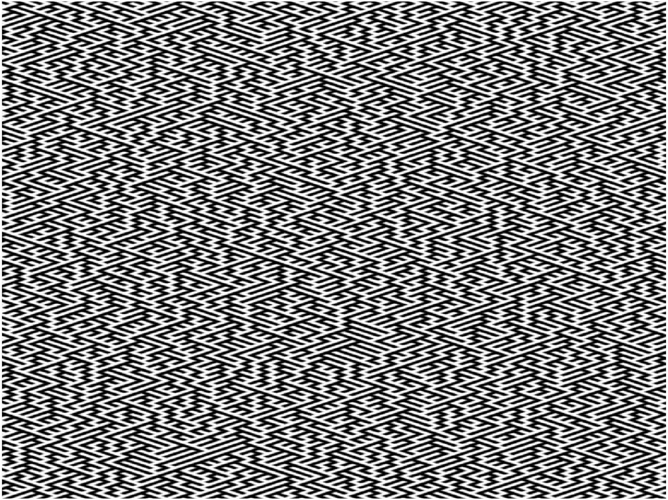
Special points are classified according to the number of incoming and outgoing open paths. There exists 7 types of special points.

Dual arrows



Forward and dual arrows.

Dual Brownian web



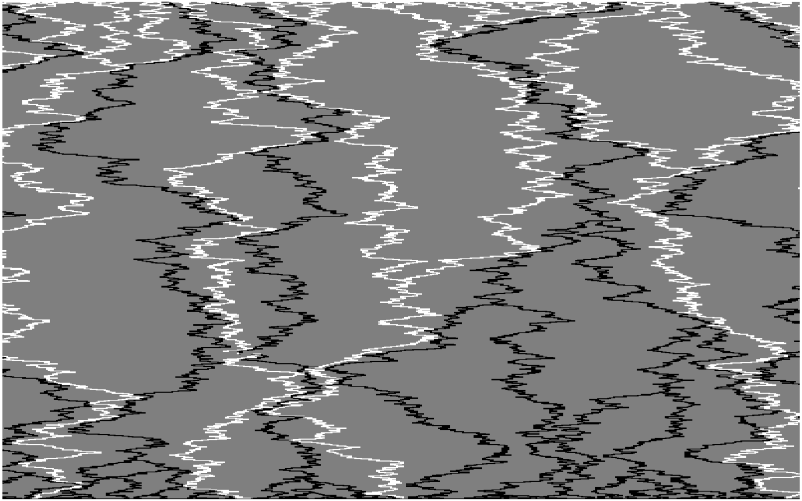
Approximation of the forward and dual Brownian web.

Dual Brownian web

To each Brownian web $\mathcal{W}$, we can associate an a.s. unique *dual web* $\hat{\mathcal{W}}$ that is equally distributed with $\mathcal{W}$ except for a rotation over 180°.

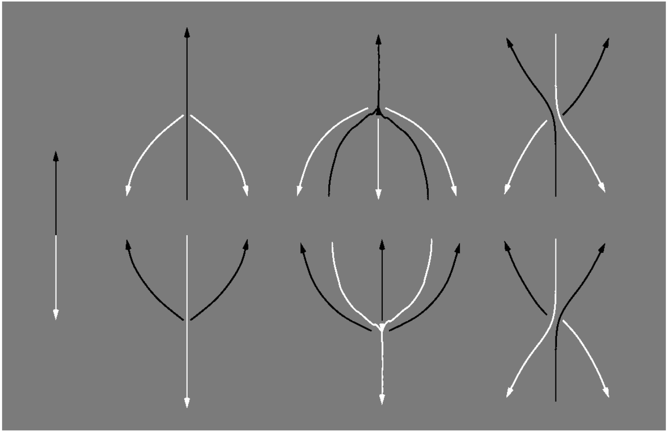
Fix a deterministic finite set of starting points and condition on the forward open paths starting at these points.
Then open paths of the dual web are Brownian motions with immediate reflection off the fixed forward open paths.

Dual Brownian web



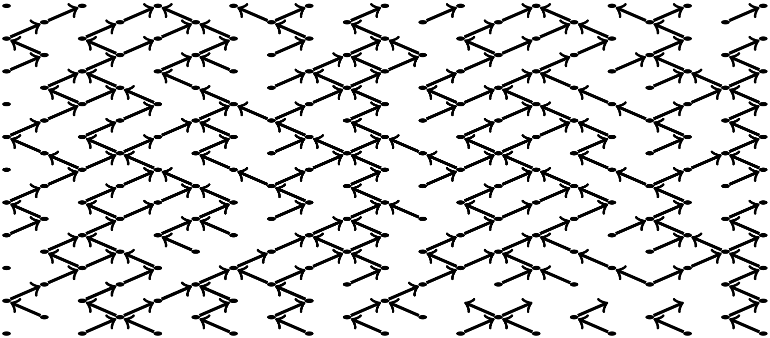
Forward and dual open paths started from fixed times.

Special points revisited



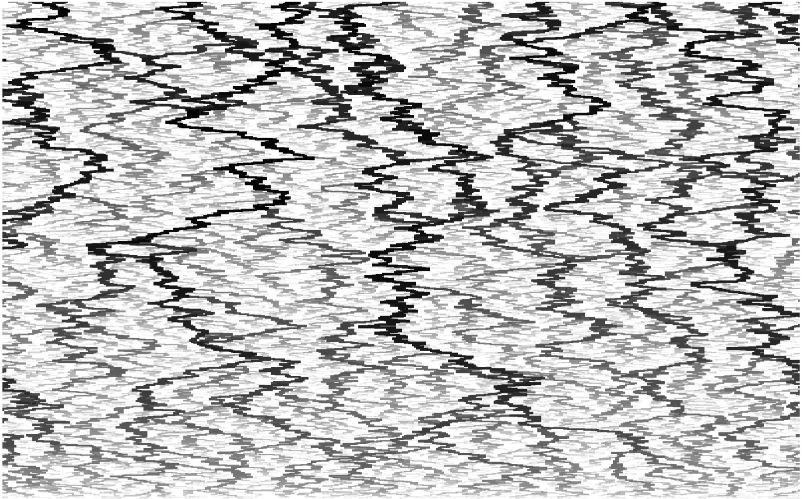
Structure of dual open paths at special points.

Left- and right-most open paths



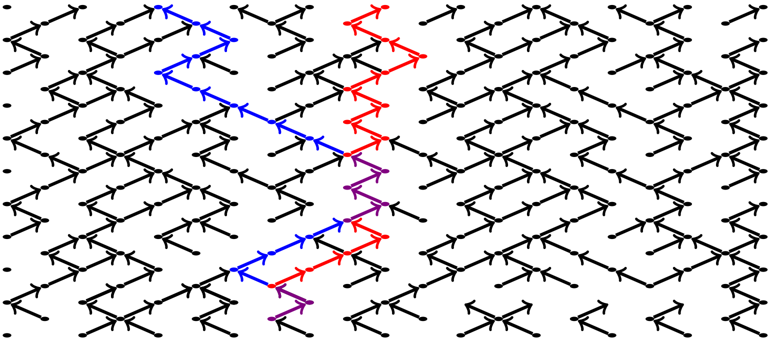
Consider an arrow configuration with
branching probability $p_\mathrm{b} > 0$ but killing probability $p_\mathrm{k} = 0$.

Left- and right-most open paths



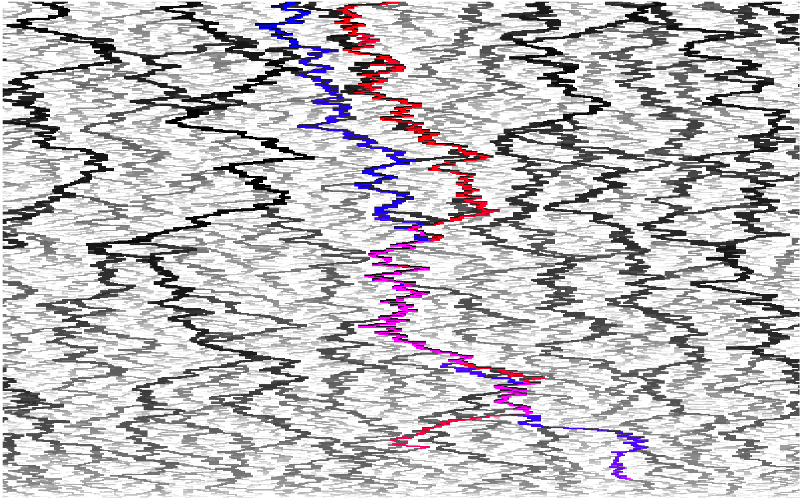
Artist's impression of the Brownian net.

Left- and right-most open paths



Left- and right-most open paths interact with a form of sticky interaction.

Left- and right-most open paths



In the limit, left- and right-most open paths are Brownian motions with drift $\beta_- < \beta_+$.

Left- and right-most open paths

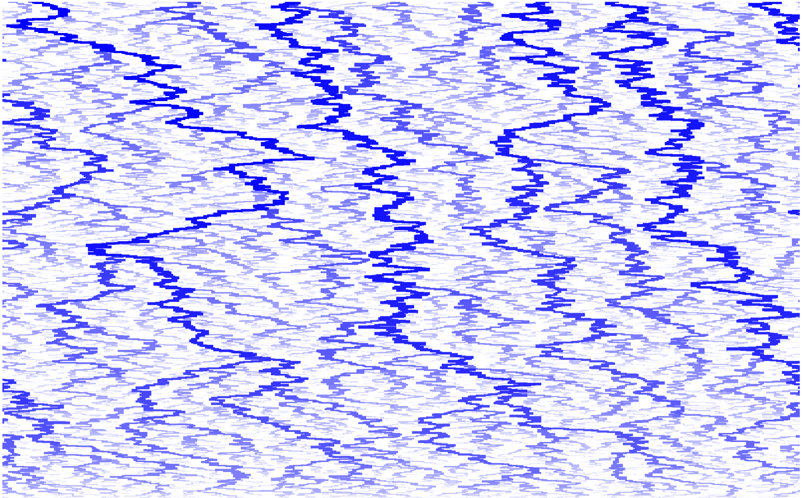
The interaction between left-most and right-most open paths is described by the stochastic differential equation (SDE):

$$\begin{aligned} \mathrm{d}L_t &= 1_{\{L_t \neq R_t\}} \mathrm{d}B^{\mathrm{l}}_t + 1_{\{L_t = R_t\}} \mathrm{d}B^{\mathrm{s}}_t + \beta_- \mathrm{d}t, \\ \mathrm{d}R_t &= 1_{\{L_t \neq R_t\}} \mathrm{d}B^{\mathrm{r}}_t + 1_{\{L_t = R_t\}} \mathrm{d}B^{\mathrm{s}}_t + \beta_+ \mathrm{d}t, \end{aligned}$$

where $B^{\mathrm{l}}_t, B^{\mathrm{r}}_t, B^{\mathrm{s}}_t$ are independent Brownian motions, and L_t and R_t are subject to the constraint that $L_t \leq R_t$ for all $t \geq \tau := \inf\{u \geq 0 : L_u = R_u\}$.

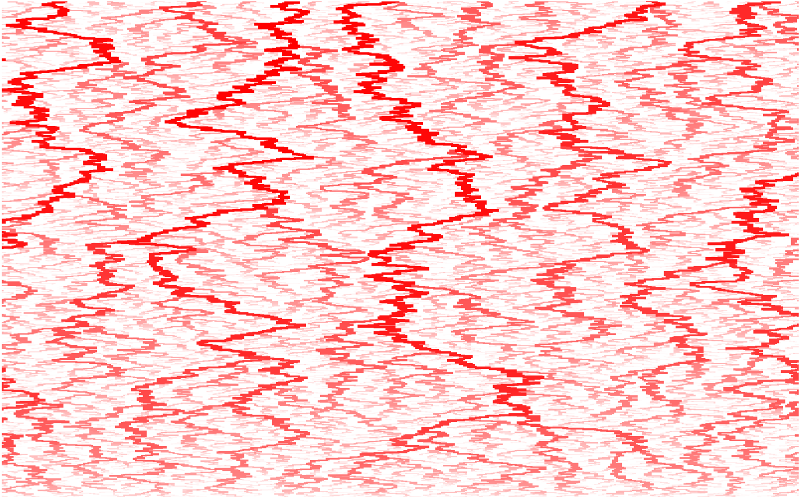
The set $\{t : L_t = R_u\}$ is nowhere dense and has positive Lebesgue measure whenever it is nonempty.

The left Brownian web



The left-most open paths converge to a left Brownian web. . .

The right Brownian web



. . . and the right-most open paths to a right Brownian web.

By definition, an *intersection time* of two paths π_1, π_2 is a time $t > \sigma_{\pi_1} \vee \sigma_{\pi_2}$ such that $\pi_1(t) = \pi_2(t)$.
We may concatenate two paths at an intersection time by putting

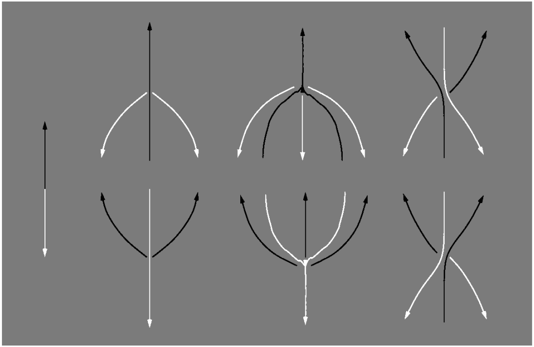
$$\pi(s) := \begin{cases} \pi_1(s) & \big(s \in [\sigma_{\pi_1}, t]\big), \\ \pi_2(s) & \big(s \in [t, \infty]\big). \end{cases}$$

Let $(\mathcal{W}^{\mathrm{l}}, \mathcal{W}^{\mathrm{r}})$, be a *left-right Brownian web*.
Let $\mathcal{D} \subset \mathbb{R}^2$ be deterministic, countable, and dense and let $\mathcal{W}^{\mathrm{l}}(\mathcal{D})$ and $\mathcal{W}^{\mathrm{r}}(\mathcal{D})$ denote the left- and right-most open paths started from $\mathcal{D}$.

Let $\mathtt{Hop}\big(\mathcal{W}^{\mathrm{l}}(\mathcal{D}) \cup \mathcal{W}^{\mathrm{r}}(\mathcal{D})\big)$ denote the smallest set containing $\mathcal{W}^{\mathrm{l}}(\mathcal{D}) \cup \mathcal{W}^{\mathrm{r}}(\mathcal{D})$ that is closed under concatenation of open paths at intersection times.

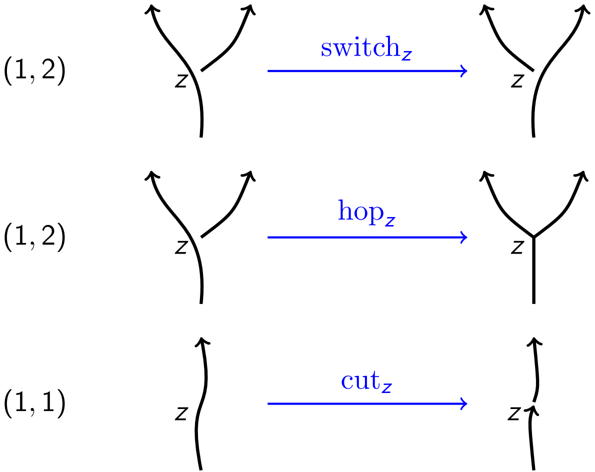
Hopping construction $\mathcal{N}^0_{\beta_-,\beta_+} = \overline{\mathtt{Hop}\big(\mathcal{W}^{\mathrm{l}}(\mathcal{D}) \cup \mathcal{W}^{\mathrm{r}}(\mathcal{D})\big)}$.

Marking constructions



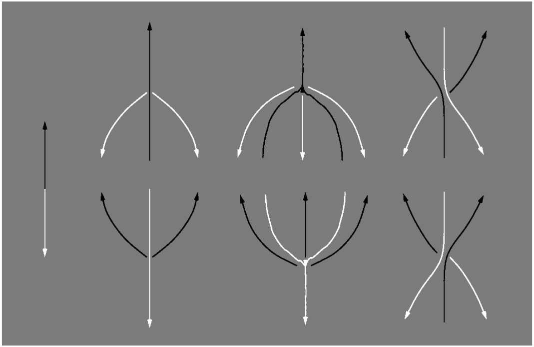
Recall that points of the Brownian web are classified according to the number of incoming and outgoing open paths $(m_{\mathrm{in}}, m_{\mathrm{out}})$.

Modifying a Brownian web



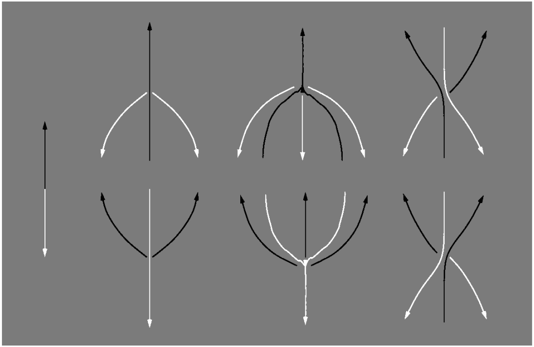
We can modify a Brownian web by changing the structure at some (finitely many) special points.

Marking constructions



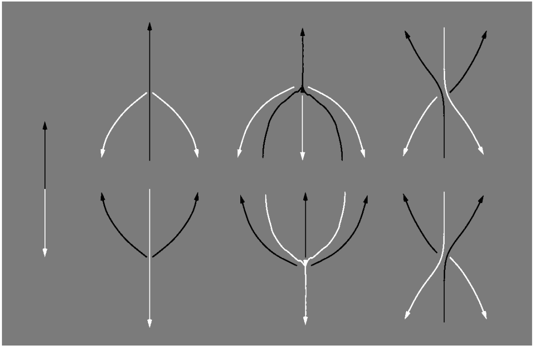
With respect to Lebesgue measure, a.e. point is of type $(0, 1)$.

Marking constructions



With respect to the *length measure* μ_{length}
of the forward web, a.e. point is of type $(1, 1)$.

Marking constructions



With respect to the *intersection local measure* μ_{int} of the forward and dual webs, a.e. point is of type $(1, 2)$.

Marking constructions

The *length measure* μ_{length} is a measure on $\mathbb{R}^2$ that is concentrated on points of type $(1,1)$ such that for every path $\pi \in \mathcal{W}$ and $\sigma_\pi \leq s \leq u < \infty$,

$$\mu_{\mathrm{length}}(\{(\pi(t),t) : t \in [s,u]\}) = u - s.$$

The *intersection local measure* μ_{int} is a measure on $\mathbb{R}^2$ that is concentrated on points of type $(1,2)$ such that for every two paths $\pi \in \mathcal{W}$ and $\hat{\pi} \in \hat{\mathcal{W}}$,

$$\begin{aligned}\mu_{\mathrm{int}}\big(\big\{(x,t) \in \mathbb{R}^2 : \sigma_\pi < t < \hat{\sigma}_{\hat{\pi}},\ \pi(t) = x = \hat{\pi}(t)\big\}\big) \\ = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}\big|\big\{t \in \mathbb{R} : \sigma_\pi < t < \hat{\sigma}_{\hat{\pi}},\ |\pi(t) - \hat{\pi}(t)| \leq \varepsilon\big\}\big|.\end{aligned}$$

These measures are σ-finite, but not locally finite; they give infinite measure to any nonempty open subset of $\mathbb{R}^2$.

Marking constructions

Let $\mu^{\rm l}_{\rm int}$ and $\mu^{\rm r}_{\rm int}$ be the restrictions of $\mu_{\rm int}$ to the set of points of type $(1,2)_{\rm l}$ and $(1,2)_{\rm r}$, respectively.

Modified web Let $\mathcal{W}$ be a Brownian web with drift β and let S be a Poisson set with intensity $c_{\rm l}\mu^{\rm l}_{\rm int}+c_{\rm r}\mu^{\rm r}_{\rm int}$. Then, for any finite $\Delta_n\uparrow S$, the limit

$$\mathcal{W}' := \lim_{\Delta_n\uparrow S} \mathtt{switch}_{\Delta_n}(\mathcal{W})$$

exists and is a Brownian web with drift $\beta'=\beta+c_{\rm l}-c_{\rm r}$.

In particular, if $c_{\rm r}=0$, then $(\mathcal{W},\mathcal{W}')$ is a left-right Brownian web.

Marking constructions

Let $\mathcal{W}$ be a "reference" Brownian web with drift β.
Let S_{12} be a Poisson set with intensity $c_{\rm l}\mu^{\rm l}_{\rm int}+c_{\rm r}\mu^{\rm r}_{\rm int}$.
Let S_{11} be a Poisson set with intensity $\delta\mu_{\rm length}$.

Marking construction For any finite $\Delta_n\uparrow S_{12}$, the limit

$$\mathcal{N}:=\lim_{\Delta_n\uparrow S_{12}}\mathtt{hop}_{\Delta_n}(\mathcal{W})$$

exists and is a Brownian net (without killing) with left and right drifts

$$\beta_-=\beta-c_{\rm r}\quad\text{and}\quad\beta_+=\beta+c_{\rm l}.$$

Moreover, $\mathrm{cut}_{S_{11}}(\mathcal{N})$ is a Brownian net with left and right drifts β_-,β_+ and killing rate δ.

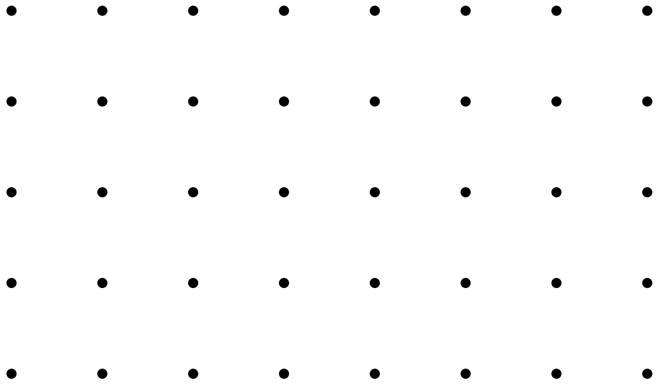
Historical notes

- R. Arratia ('79,'81), motivated by scaling limits of the 1D voter model, studies coalescing Brownian motions started from each point in space and time.
- B. Tóth and W. Werner ('98) arrive at the same object by studying the true self-repellent motion. They classify special points and use right-continuity to choose a unique open path at points of multiplicity.
- F. Soucaliuc, B. Tóth, and W. Werner ('00) prove that open paths in the dual web are reflected off forward open paths.
- L. Fontes, M. Isopi, C. Newman, and K. Ravishankar ('04) invent the name "Brownian web", viewed this as a compact set of paths, and prove weak convergence w.r.t. to the Hausdorff topology.
- C. Newman, K. Ravishankar, and R. Sun ('05) prove convergence of coalescing non-nearest neighbor random walks to the Brownian web.

Historical notes

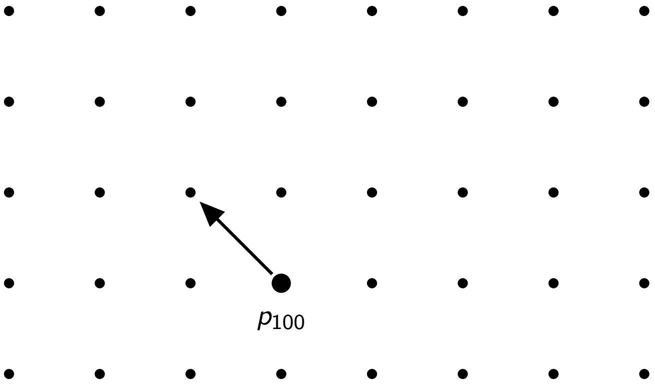
- R. Sun and J.S. ('08) invent the name Brownian net and the hopping, wedge, and mesh constructions, which are all based on the left-right SDE.
- E. Schertzer, R. Sun and J.S. ('09) classify special points of the Brownian net.
- C. Howitt and J. Warren ('09) construct sticky pairs of Brownian webs by means of a martingale problem.
- C. Newman, K. Ravishankar, and E. Schertzer ('10) publish the marking construction of the Brownian net, conceived around '05.
- C. Newman, K. Ravishankar, and E. Schertzer ('13) construct the Brownian net with killing.
- E. Schertzer, R. Sun and J.S. ('14) study stochastic flows using marked webs.
- R. Sun, J. Yu and J.S. ('17?) study convergence of non-nearest neighbor arrow configurations to the Brownian net.

Arrow configurations revisited



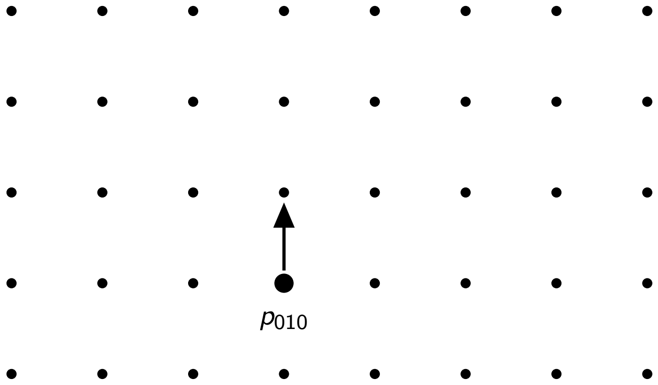
Consider the lattice $\mathbb{Z}^2$.

Arrow configurations revisited



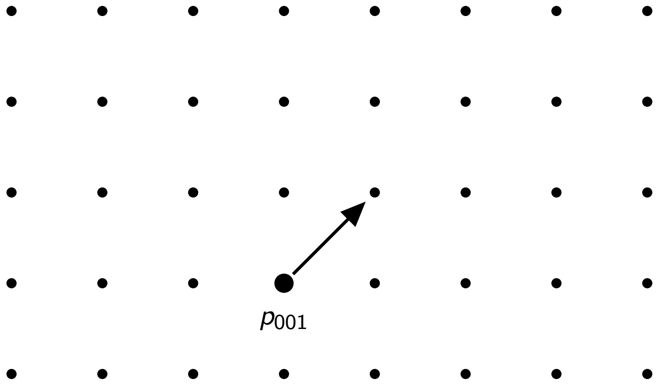
Draw an arrow to the left with probability p_{100}...

Arrow configurations revisited



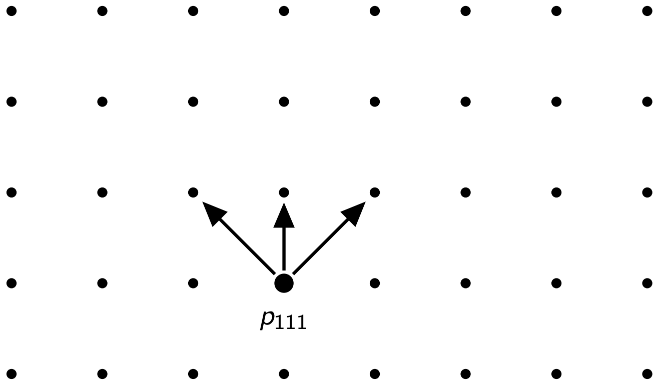
...draw an arrow straight up with probability p_{010}...

Arrow configurations revisited



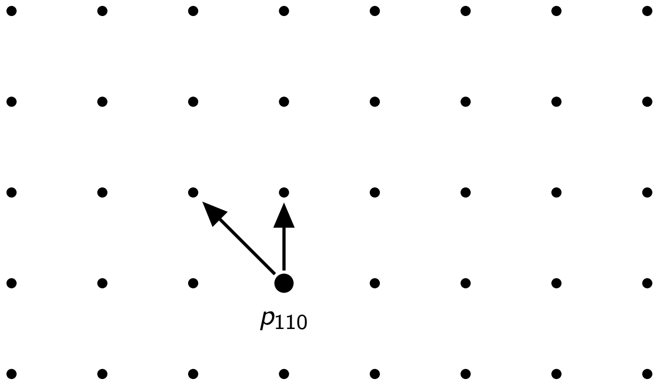
. . . and draw an arrow to the right with probability p_{001}.

Arrow configurations revisited



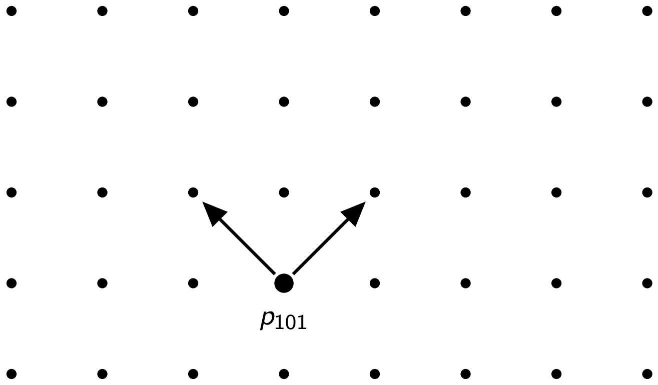
Also draw 3, 2, or zero arrows with certain probabilities.

Arrow configurations revisited



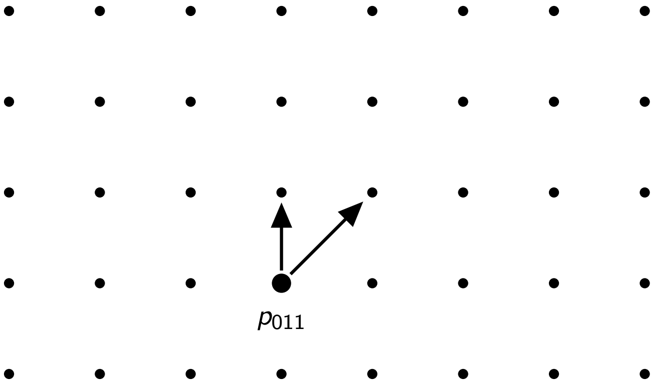
Also draw 3, 2, or zero arrows with certain probabilities.

Arrow configurations revisited



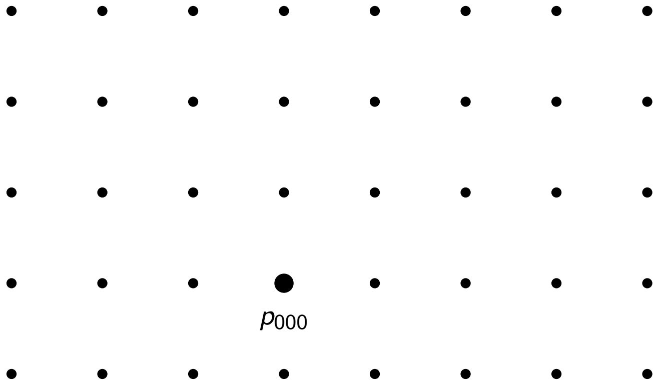
Also draw 3, 2, or zero arrows with certain probabilities.

Arrow configurations revisited



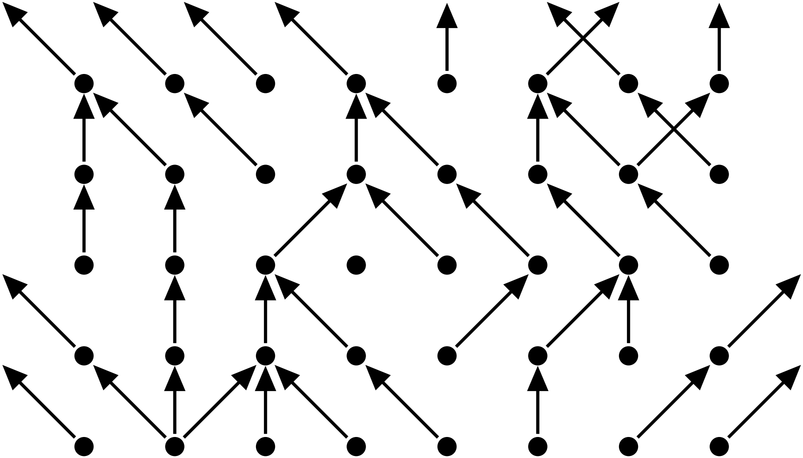
Also draw 3, 2, or zero arrows with certain probabilities.

Arrow configurations revisited



Also draw 3, 2, or zero arrows with certain probabilities.

Arrow configurations revisited



Do this independently for each point.

A conjecture

Rescale diffusively with ε and assume that

$$\begin{aligned} p_{001} - p_{100} &= O(\varepsilon), \\ p_{111}, p_{110}, p_{101}, p_{011} &= O(\varepsilon), \\ p_{000} &= O(\varepsilon^2). \end{aligned}$$

Conjecture This should converge to a Brownian net.

So far, only an incomplete proof for a special class of distributions $p_{000}, \ldots, p_{111}$.

Difficulty: Arrows can cross. No dual arrow configuration.

Branching-coalescing point set

For any closed subset $A \subset \mathbb{R}$,

$$\xi_t := \{\pi(t) : \exists \pi \in \mathcal{N}^{\delta}_{\beta_-,\beta_+} \text{ s.t. } \sigma_\pi = 0,\ \pi(0) \in A\}$$

defines a Feller process taking values in the closed subsets of $\mathbb{R}$.
For $\delta = 0$ (no killing):

(i) Reversible invariant law: the law of a Poisson point set with intensity $\beta_+ - \beta_-$.

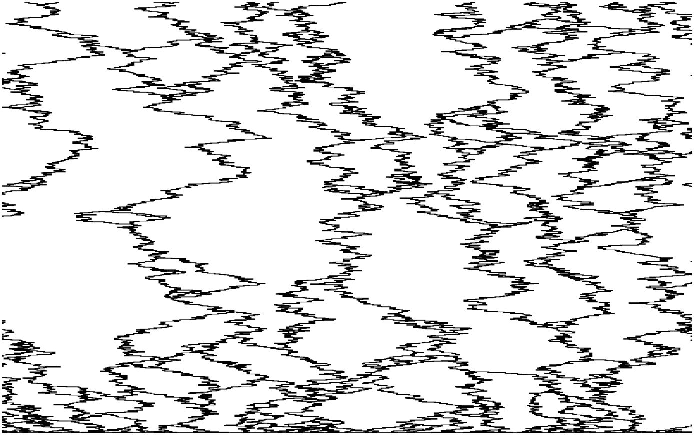
(ii) For deterministic $t > 0$, a.s. ξ_t is a locally finite subset of $\mathbb{R}$.

(iii) There exists a dense set of random times $\tau > 0$ such that ξ_τ has no isolated points.

Open problem: generator characterization!

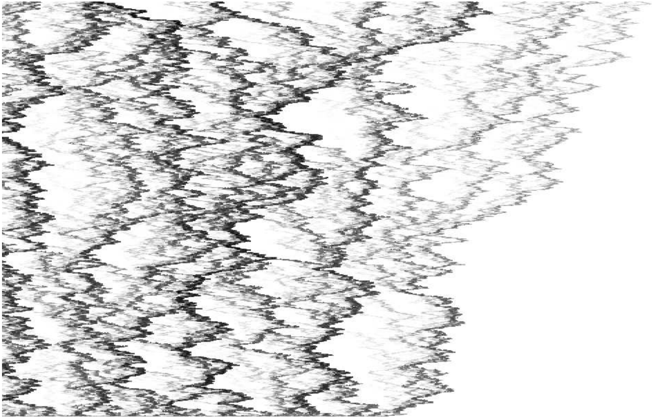
Thm Phase transition between survival and extinction at some $\delta_{\rm c}$.

The branching-coalescing point set



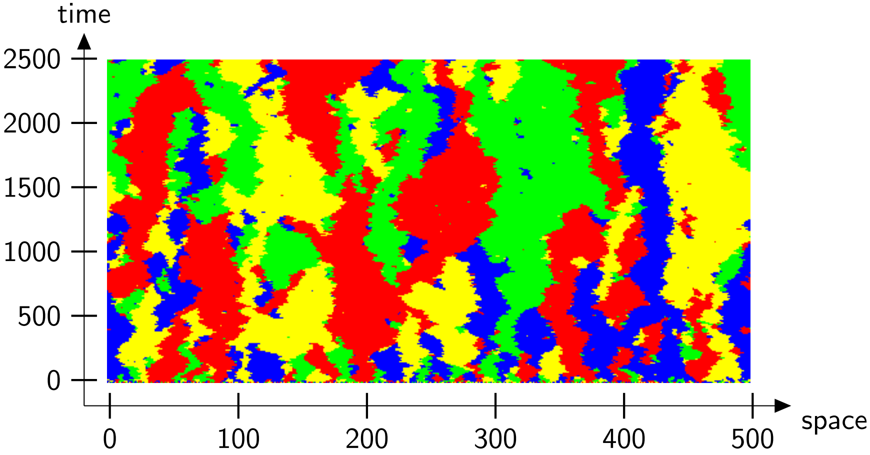
The branching-coalescing point set with
$\beta_- = -1$, $\beta_+ = 1$, $\delta = 0$ started in $\xi_0 = \mathbb{R}$.

Howitt-Warren flows



A one-sided erosion flow.

A one-dimensional Potts model



A low-temperature one-dimensional Potts model.
[C. Newman, K. Ravishankar, and E. Schertzer ('16)]