The Brownian net

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Kohútka, February 8, 2017
The Brownian net

Arrow configurations

\[ \mathbb{Z}_\text{even}^2 := \{(x, t) \in \mathbb{Z}^2 : x + t \text{ is even}\}. \]
Arrow configurations

With probability $p_1$ we draw an arrow to the left.
Arrow configurations

With probability $p_r$ we draw an arrow to the right.
With probability $p_b$ we draw two arrows.
And with probability $p_k$ we draw no arrows at all.
Arrow configurations

We do this independently for each point.
We are interested in open paths.
Open paths can start at any point in $\mathbb{Z}_2^{\text{even}}$. 
Open paths either end at killing points...
Arrow configurations

... or carry on forever.
Scaling limit

We rescale diffusively, multiplying all spatial distances with $\varepsilon$ and all temporal distances with $\varepsilon^2$. 
**Claim** Assume that

\[ \varepsilon^{-1}(p_r - p_l - p_b) \to \beta_-, \]
\[ \varepsilon^{-1}(p_r - p_l + p_b) \to \beta_+, \]
\[ \varepsilon^{-2}p_k \to \delta. \]

Then the collection \( \mathcal{U} \) of all open paths converges to a diffusive scaling limit \( \mathcal{N}^{\delta}_{\beta_-, \beta_+} \).
At each point $z \in \mathbb{Z}^2_{\text{even}}$ there starts an a.s. unique left-most open path $l_z$ and right-most open path $r_z$. 
Under the assumptions

\[ \varepsilon^{-1}(p_r - p_l - p_b) \to \beta_- , \]

\[ \varepsilon^{-1}(p_r - p_l + p_b) \to \beta_+ , \]

\[ \varepsilon^{-2}p_k \to \delta , \]

left- and right-most open paths converge to Brownian motions with drift \( \beta_- \) and \( \beta_+ \), respectively, and exponential lifetimes with mean \( 1/\delta \).
We first compactify $\mathbb{R}^2$ to $[-\infty, \infty]^2$.

Topological matters
Topological matters

... and then contract $[-\infty, \infty] \times \{-\infty\}$ and $[-\infty, \infty] \times \{\infty\}$ to single points.
Alternatively, map $\mathbb{R}^2$ into itself with the map

$$\Theta(x, t) := \left( \frac{\tanh(x)}{1 + |t|}, \tanh(t) \right),$$

and take the closure.
Another equivalent formulation is: take the completion of $\mathbb{R}^2$ w.r.t. the metric
\[ d(z, z') := |\Theta(z) - \Theta(z')|. \]
A *path* is a continuous function \( \pi : [\sigma_\pi, \tau_\pi] \to [-\infty, \infty] \), with \( -\infty \leq \sigma_\pi \leq \tau_\pi \leq \infty \).
We identify a path with its graph

\[ \{(\pi(t), t) : t \in [\sigma_\pi, \tau_\pi]\} . \]
We equip the space $\Pi$ of all paths with the Hausdorff metric

$$d(\pi_1, \pi_2) = \sup_{z_1 \in \pi_1} \inf_{z_2 \in \pi_2} d(z_1, z_2) \lor \sup_{z_2 \in \pi_2} \inf_{z_1 \in \pi_1} d(z_1, z_2).$$
By adding trivial paths that are constantly $-\infty$ or $+\infty$, we can make the set $\mathcal{U}$ of open paths into a compact subset of $\Pi$. 
We equip the space $\mathcal{K}(\Pi)$ of all compact subsets of the space of paths $\Pi$ with the Hausdorff metric

$$d(U_1, U_2) = \sup_{\pi_1 \in U_1} \inf_{\pi_2 \in U_2} d(\pi_1, \pi_2) \lor \sup_{\pi_2 \in U_2} \inf_{\pi_1 \in U_1} d(\pi_1, \pi_2).$$

We define a diffusive scaling map $S_\epsilon$ by

$$S_\epsilon(x, t) := (\epsilon x, \epsilon^2 t).$$
**Theorem** Let $\varepsilon_n \downarrow 0$ and let $\mathcal{U}_n$ be the sets of open paths in arrow configurations with parameters satisfying

\[
\varepsilon_n^{-1}(p_{\text{r}}(n) - p_{\text{l}}(n) - p_{\text{b}}(n)) \to \beta_-, \\
\varepsilon_n^{-1}(p_{\text{r}}(n) - p_{\text{l}}(n) + p_{\text{b}}(n)) \to \beta_+, \\
\varepsilon_n^{-2}p_k(n) \to \delta.
\]

Then

\[
\mathbb{P}[S_{\varepsilon_n}(\mathcal{U}_n) \in \cdot] \xrightarrow{n \to \infty} \mathbb{P}[\mathcal{N}^{\delta}_{\beta_-, \beta_+} \in \cdot],
\]

where $\Rightarrow$ denotes weak convergence of probability laws on $\mathcal{K}(\Pi)$. The limiting object is a *Brownian net with killing*. 

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**The Brownian net**
The Brownian web

If $\beta = \beta_- = \beta_+$ and $\delta = 0$, then the limiting object $\mathcal{W}_\beta := \mathcal{N}_{\beta,\beta}^0$ is a Brownian web with drift $\beta$. In particular, $\mathcal{W} := \mathcal{W}_0$ is the standard Brownian web.

- For each deterministic $z \in \mathbb{R}^2$, almost surely there is a unique open path $p_z \in \mathcal{W}$. 

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The Brownian net
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- For each deterministic $z \in \mathbb{R}^2$, almost surely there is a unique open path $p_z \in \mathcal{W}$.
- For any deterministic finite set of points $z_1, \ldots, z_k \in \mathbb{R}^2$, the collection $(p_{z_1}, \ldots, p_{z_k})$ is distributed as coalescing Brownian motions.
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- For any deterministic finite set of points $z_1, \ldots, z_k \in \mathbb{R}^2$, the collection $(p_{z_1}, \ldots, p_{z_k})$ is distributed as coalescing Brownian motions.
- For any deterministic countable dense subset $\mathcal{D} \subset \mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure of $\{p_z : z \in \mathcal{D}\}$.
Artist’s impression of the Brownian web.
Open paths started at time zero.
There exists random points where two open paths start.
Special points are classified according to the number of incoming and outgoing open paths. There exists 7 types of special points.
Forward and dual arrows.
Dual Brownian web

Approximation of the forward and dual Brownian web.
To each Brownian web $\mathcal{W}$, we can associate an a.s. unique dual web $\hat{\mathcal{W}}$ that is equally distributed with $\mathcal{W}$ except for a rotation over $180^\circ$.

Fix a deterministic finite set of starting points and condition on the forward open paths starting at these points. Then open paths of the dual web are Brownian motions with immediate reflection off the fixed forward open paths.
Forward and dual open paths started from fixed times.
Structure of dual open paths at special points.
Consider an arrow configuration with branching probability $p_b > 0$ but killing probability $p_k = 0$. 
Left- and right-most open paths

Artist's impression of the Brownian net.
Left- and right-most open paths interact with a form of sticky interaction.
In the limit, left- and right-most open paths are Brownian motions with drift $\beta_- < \beta_+$. 
The interaction between left-most and right-most open paths is described by the stochastic differential equation (SDE):

\[ \mathrm{d}L_t = 1_{\{L_t \neq R_t\}} \mathrm{d}B^1_t + 1_{\{L_t = R_t\}} \mathrm{d}B^s_t + \beta_- \mathrm{d}t, \]
\[ \mathrm{d}R_t = 1_{\{L_t \neq R_t\}} \mathrm{d}B^r_t + 1_{\{L_t = R_t\}} \mathrm{d}B^s_t + \beta_+ \mathrm{d}t, \]

where \( B^1_t, B^r_t, B^s_t \) are independent Brownian motions, and \( L_t \) and \( R_t \) are subject to the constraint that \( L_t \leq R_t \) for all \( t \geq \tau := \inf\{u \geq 0 : L_u = R_u\} \).

The set \( \{ t : L_t = R_u \} \) is nowhere dense and has positive Lebesgue measure whenever it is nonempty.
The left Brownian web

The left-most open paths converge to a left Brownian web...
The right Brownian web

... and the right-most open paths to a right Brownian web.
By definition, an *intersection time* of two paths \( \pi_1, \pi_2 \) is a time \( t > \sigma_{\pi_1} \lor \sigma_{\pi_2} \) such that \( \pi_1(t) = \pi_2(t) \).

We may concatenate two paths at an intersection time by putting

\[
\pi(s) := \begin{cases} 
\pi_1(s) & (s \in [\sigma_{\pi_1}, t]), \\
\pi_2(s) & (s \in [t, \infty]).
\end{cases}
\]

Let \( (\mathcal{W}_l, \mathcal{W}_r) \) be a *left-right Brownian web*.

Let \( \mathcal{D} \subset \mathbb{R}^2 \) be deterministic, countable, and dense and let \( \mathcal{W}_l(\mathcal{D}) \) and \( \mathcal{W}_r(\mathcal{D}) \) denote the left- and right-most open paths started from \( \mathcal{D} \).

Let \( \text{Hop}(\mathcal{W}_l(\mathcal{D}) \cup \mathcal{W}_r(\mathcal{D})) \) denote the smallest set containing \( \mathcal{W}_l(\mathcal{D}) \cup \mathcal{W}_r(\mathcal{D}) \) that is closed under concatenation of open paths at intersection times.

**Hopping construction** \( \mathcal{N}^{0}_{\beta_-, \beta_+} = \overline{\text{Hop}(\mathcal{W}_l(\mathcal{D}) \cup \mathcal{W}_r(\mathcal{D}))} \).
Recall that points of the Brownian web are classified according to the number of incoming and outgoing open paths \((m_{\text{in}}, m_{\text{out}})\).
We can modify a Brownian web by changing the structure at some (finitely many) special points.
Marking constructions

With respect to Lebesgue measure, a.e. point is of type \((0,1)\).
With respect to the *length measure* $\mu_{\text{length}}$ of the forward web, a.e. point is of type $(1, 1)$. 
With respect to the intersection local measure $\mu_{\text{int}}$ of the forward and dual webs, a.e. point is of type $(1, 2)$. 
The **length measure** $\mu_{\text{length}}$ is a measure on $\mathbb{R}^2$ that is concentrated on points of type $(1, 1)$ such that for every path $\pi \in \mathcal{W}$ and $\sigma_\pi \leq s \leq u < \infty$,

$$\mu_{\text{length}}(\{(\pi(t), t) : t \in [s, u]\}) = u - s.$$  

The **intersection local measure** $\mu_{\text{int}}$ is a measure on $\mathbb{R}^2$ that is concentrated on points of type $(1, 2)$ such that for every two paths $\pi \in \mathcal{W}$ and $\hat{\pi} \in \hat{\mathcal{W}}$,

$$\mu_{\text{int}}\left(\{(x, t) \in \mathbb{R}^2 : \sigma_\pi < t < \hat{\sigma}_{\hat{\pi}}, \pi(t) = x = \hat{\pi}(t)\}\right) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \left|\left\{ t \in \mathbb{R} : \sigma_\pi < t < \hat{\sigma}_{\hat{\pi}}, |\pi(t) - \hat{\pi}(t)| \leq \varepsilon \right\}\right|.$$  

These measures are $\sigma$-finite, but not locally finite; they give infinite measure to any nonempty open subset of $\mathbb{R}^2$. 

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**The Brownian net**
Let $\mu^1_{\text{int}}$ and $\mu^r_{\text{int}}$ be the restrictions of $\mu_{\text{int}}$ to the set of points of type $(1, 2)_l$ and $(1, 2)_r$, respectively.

**Modified web** Let $\mathcal{W}$ be a Brownian web with drift $\beta$ and let $S$ be a Poisson set with intensity $c_l\mu^1_{\text{int}} + c_r\mu^r_{\text{int}}$. Then, for any finite $\Delta_n \uparrow S$, the limit

$$\mathcal{W}' := \lim_{\Delta_n \uparrow S} \text{switch}_{\Delta_n}(\mathcal{W})$$

exists and is a Brownian web with drift $\beta' = \beta + c_l - c_r$.

In particular, if $c_r = 0$, then $(\mathcal{W}, \mathcal{W}')$ is a left-right Brownian web.
Marking constructions

Let $\mathcal{W}$ be a “reference” Brownian web with drift $\beta$.
Let $S_{12}$ be a Poisson set with intensity $c_l \mu_{\text{int}}^l + c_r \mu_{\text{int}}^r$.
Let $S_{11}$ be a Poisson set with intensity $\delta \mu_{\text{length}}$.

**Marking construction** For any finite $\Delta_n \uparrow S_{12}$, the limit

$$\mathcal{N} := \lim_{\Delta_n \uparrow S_{12}} \text{hop}_{\Delta_n}(\mathcal{W})$$

exists and is a Brownian net (without killing) with left and right drifts

$$\beta_- = \beta - c_r \quad \text{and} \quad \beta_+ = \beta + c_l.$$ 

Moreover, $\text{cut}_{S_{11}}(\mathcal{N})$ is a Brownian net with left and right drifts $\beta_-, \beta_+$ and killing rate $\delta$. 

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The Brownian net
R. Arratia ('79,'81), motivated by scaling limits of the 1D voter model, studies coalescing Brownian motions started from each point in space and time.

B. Tóth and W. Werner ('98) arrive at the same object by studying the true self-repellent motion. They classify special points and use right-continuity to choose a unique open path at points of multiplicity.

F. Soucaliuc, B. Tóth, and W. Werner ('00) prove that open paths in the dual web are reflected off forward open paths.

L. Fontes, M. Isopi, C. Newman, and K. Ravishankar ('04) invent the name “Brownian web”, viewed this as a compact set of paths, and prove weak convergence w.r.t. to the Hausdorff topology.

C. Newman, K. Ravishankar, and R. Sun ('05) prove convergence of coalescing non-nearest neighbor random walks to the Brownian web.
Historical notes

- R. Sun and J.S. ('08) invent the name Brownian net and the hopping, wedge, and mesh constructions, which are all based on the left-right SDE.
- E. Schertzer, R. Sun and J.S. ('09) classify special points of the Brownian net.
- C. Howitt and J. Warren ('09) construct sticky pairs of Brownian webs by means of a martingale problem.
- C. Newman, K. Ravishankar, and E. Schertzer ('10) publish the marking construction of the Brownian net, conceived around '05.
- C. Newman, K. Ravishankar, and E. Schertzer ('13) construct the Brownian net with killing.
- E. Schertzer, R. Sun and J.S. ('14) study stochastic flows using marked webs.
- R. Sun, J. Yu and J.S. ('17?) study convergence of non-nearest neighbor arrow configurations to the Brownian net.
Consider the lattice $\mathbb{Z}^2$. 
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The Brownian net

Arrow configurations revisited

$p_{100}$

Draw an arrow to the left with probability $p_{100}$...
...draw an arrow straight up with probability $p_{010}$...
Arrow configurations revisited

...and draw an arrow to the right with probability $p_{001}$. 

$p_{001}$
Arrow configurations revisited

Also draw 3, 2, or zero arrows with certain probabilities.
Arrow configurations revisited

Also draw 3, 2, or zero arrows with certain probabilities.
Arrow configurations revisited

Also draw 3, 2, or zero arrows with certain probabilities.

$p_{101}$
Arrow configurations revisited

Also draw 3, 2, or zero arrows with certain probabilities.
Arrow configurations revisited

Also draw 3, 2, or zero arrows with certain probabilities.
Do this independently for each point.
Rescale diffusively with $\varepsilon$ and assume that

$$p_{001} - p_{100} = O(\varepsilon),$$

$$p_{111}, p_{110}, p_{101}, p_{011} = O(\varepsilon),$$

$$p_{000} = O(\varepsilon^2).$$

**Conjecture** This should converge to a Brownian net.

So far, only an incomplete proof for a special class of distributions $p_{000}, \ldots, p_{111}$.

*Difficulty*: Arrows can cross. No dual arrow configuration.
Branching-coalescing point set

For any closed subset $A \subset \mathbb{R}$,

$$
\xi_t := \{ \pi(t) : \exists \pi \in \mathcal{N}_{\beta_-}^{\delta}, \beta_+ \text{ s.t. } \sigma_\pi = 0, \pi(0) \in A \}
$$

defines a Feller process taking values in the closed subsets of $\mathbb{R}$. For $\delta = 0$ (no killing): 

(i) Reversible invariant law: the law of a Poisson point set with intensity $\beta_+ - \beta_-$. 

(ii) For deterministic $t > 0$, a.s. $\xi_t$ is a locally finite subset of $\mathbb{R}$. 

(iii) There exists a dense set of random times $\tau > 0$ such that $\xi_\tau$ has no isolated points. 

Open problem: generator characterization! 

**Thm** Phase transition between survival and extinction at some $\delta_c$. 

The branching-coalescing point set with \( \beta_- = -1, \beta_+ = 1, \delta = 0 \) started in \( \xi_0 = \mathbb{R} \).

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The Brownian net
A one-sided erosion flow.
A low-temperature one-dimensional Potts model.
[C. Newman, K. Ravishankar, and E. Schertzer ('16)