Recursive tree processes and the mean-field limit of stochastic flows

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Kohútka, February 7, 2019
Mean-field equations

Basic ingredients

(i) Polish space $S$ local state space.

(ii) $(\Omega, \mathcal{B}, r)$ Polish space with Borel $\sigma$-field and finite measure: source of external randomness.

(iii) $\kappa : \Omega \to \mathbb{N}$ measurable function.

(iv) For each $\omega \in \Omega$, a measurable function $\gamma[\omega] : S^{\kappa(\omega)} \to S$.

Def $\mathcal{P}(S) :=$ the space of probability measures on $S$.

Def $T : \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$T(\mu) := \text{the law of } \gamma[\omega](X_1, \ldots, X_{\kappa(\omega)}),$$

where $\omega$ is an $\Omega$-valued random variable with law $|r|^{-1}r$ and $(X_i)_{i \geq 1}$ are i.i.d. with law $\mu$. We are interested in mean-field equations of the form

$$\frac{\partial}{\partial t} \mu_t = |r| \left\{ T(\mu_t) - \mu_t \right\} \quad (t \geq 0). \quad (1)$$
Define a *cooperative branching* map and *death* map by:

\[ \text{cob} : S^3 \to S \quad \text{with} \quad \text{cob}(x_1, x_2, x_3) := x_1 \lor (x_2 \land x_3), \]

\[ \text{dth} : S^0 \to S \quad \text{with} \quad \text{dth}(\emptyset) := 0, \]

and set \( S = \{0, 1\}, \ \Omega = \{1, 2\}, \)

\[ \gamma[1] = \text{cob} : S^3 \to S, \quad \kappa(1) = 3, \quad r(\{1\}) = \alpha, \]

\[ \gamma[2] = \text{dth} : S^0 \to S, \quad \kappa(2) = 0, \quad r(\{2\}) = 1. \]
We can rewrite the mean-field equation as

$$\frac{\partial}{\partial t} \mu_t = \alpha \left\{ T_{\text{cob}}(\mu_t) - \mu_t \right\} + \left\{ T_{\text{dth}}(\mu_t) - \mu_t \right\}, \quad (2)$$

with

$$T_g(\mu) := \text{the law of } g(X_1, \ldots, X_\kappa(\omega)),$$

where \((X_i)_{i \geq 1}\) are i.i.d. with law \(\mu\).

Define a (nonlinear) semigroup \((T_t)_{t \geq 0}\) of operators acting on probability measures by

$$T_t(\mu) := \mu_t \quad \text{where } (\mu_t)_{t \geq 0} \text{ solves } (2) \text{ with } \mu_0 = \mu.$$

**Claim** \((T_t)_{t \geq 0}\) is similar to the semigroup of a Markov chain, except that *time has a tree-like structure*. 

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Fix $d$ such that $\kappa(\omega) \leq d$ for all $\omega \in \Omega$.

Let $\mathbb{T}^d$ denote the space of all words $\mathbf{i} = i_1 \cdots i_n$ made from the alphabet $\{1, \ldots, d\}$ (if $d < \infty$) resp. $\mathbb{N}_+$ (if $d = \infty$).
We attach i.i.d. \((\omega_i)_{i \in T}\) with law \(|r|^{-1}r\) to each node, which translate into maps \((\gamma[\omega_i])_{i \in T}\).
Let $\mathcal{S}$ be the random subtree of $T$ defined as

$$\mathcal{S} := \{i_1 \cdots i_n \in T : i_m \leq \kappa(\omega_{i_1 \cdots i_{m-1}}) \forall 1 \leq m \leq n\}.$$
For any rooted subtree $U \subset S$, let

$$\nabla U := \{i_1 \cdots i_n \in S : i_1 \cdots i_{n-1} \in U, \ i_1 \cdots i_n \not\in U\}$$

denote the boundary of $U$ relative to $S$. 
Given \((X_i)_{i \in \mathcal{U}}\), we inductively define \((X_i)_{i \in \mathcal{U}}\) by

\[
X_i = \gamma[\omega_i](X_{i_1}, \ldots, X_{i_{\kappa(\omega)}}) \quad (i \in \mathcal{U}).
\]
Given \((X_i)_{i \in \nabla U}\), we inductively define \((X_i)_{i \in U}\) by

\[
X_i = \gamma[\omega_i](X_{i1}, \ldots, X_{i\kappa(\omega)}) \quad (i \in U).
\]
Given \((X_i)_{i \in \nabla \U}\), we inductively define \((X_i)_{i \in \U}\) by

\[
X_i = \gamma[\omega_i](X_{i1}, \ldots, X_{i\kappa(\omega)}) \quad (i \in \U).
\]
Given \((X_i)_{i \in \nabla U}\), we inductively define \((X_i)_{i \in U}\) by

\[ X_i = \gamma[\omega_i](X_{i1}, \ldots, X_{i\kappa(\omega)}) \quad (i \in U). \]
A recursive tree representation

Setting

\[ G_U \left( (X_i)_{i \in \nabla U} \right) := X_{\emptyset} \]

defines a random map

\[ G_U : \mathcal{S}^\nabla U \to \mathcal{S} \]

that is the concatenation of the maps \((\gamma[\omega_i])_{i \in U}\) according to the tree structure of \(U\).

Let \(|i_1 \cdots i_n| := n\) denote the length of a word \(i\) and set

\[ \mathcal{S}(n) := \{ i \in \mathcal{S} : |i| < n \} \quad \text{and} \quad \nabla \mathcal{S}(n) = \{ i \in \mathcal{S} : |i| = n \}. \]

Aldous and Bandyopadyay (2005) observed that

\[ T^n(\mu) := \text{the law of } G_{\mathcal{S}(n)} \left( (X_i)_{i \in \nabla \mathcal{S}(n)} \right), \]

where \((X_i)_{i \in \nabla \mathcal{S}(n)}\) are i.i.d. with law \(\mu\) and independent of \((\omega_i)_{i \in \mathcal{S}(n)}\).
We add independent exponentially distributed lifetimes to the nodes.
A recursive tree representation

Let \((\sigma_i)_{i \in \mathbb{T}}\) be i.i.d. exponentially distributed with mean \(|r|^{-1}\), independent of \((\omega_i)_{i \in \mathbb{T}}\), and set

\[
\tau_i^* := \sum_{m=1}^{n-1} \sigma_{i_1 \ldots i_m} \quad \text{and} \quad \tau_i^\dagger := \tau_i^* + \sigma_i \quad (i = i_1 \cdot \cdot \cdot i_n),
\]

\[
S_t := \{i \in S : \tau_i^\dagger \leq t\} \quad \text{and} \quad \nabla S_t = \{i \in S : \tau_i^* \leq t < \tau_i^\dagger\}.
\]

Let \(F_t\) be the filtration

\[
F_t := \sigma(\nabla S_t, (\omega_i, \sigma_i)_{i \in S_t}) \quad (t \geq 0).
\]

**Theorem [Mach, Sturm, S. ’18]**

\[
T_t(\mu) := \text{the law of } G_{S_t}((X_i)_{i \in \nabla S_t}),
\]

where \((X_i)_{i \in \nabla S_t}\) are i.i.d. with law \(\mu\) and independent of \(F_t\).
A recursive tree representation

\[(X_i)_{i \in \nabla S_t} \]
i.i.d. law \(\mu_0\)

\[\text{Law } \mu_t = T_t(\mu)\]
The mean-field equation

Theorem [Mach, Sturm, S. ’18] Assume that

$$
\int_{\Omega} r(d\omega) \kappa(\omega) < \infty
$$

(3)

Then for each initial state, the mean-field equation (1) has a unique solution.

In our example, the mean-field equation is

$$
\frac{\partial}{\partial t} \mu_t = \alpha \{ T_{\text{cob}}(\mu_t) - \mu_t \} + \{ T_{\text{dth}}(\mu_t) - \mu_t \}.
$$

Rewriting this in terms of $p_t := \mu_t(\{1\})$ yields

$$
\frac{\partial}{\partial t} p_t = \alpha p_t^2 (1 - p_t) - p_t =: F_\alpha(p_t) \quad (t \geq 0).
$$
Cooperative branching

For $\alpha < 4$, the equation $\frac{\partial}{\partial t} p_t = F_\alpha(p_t)$ has a single, stable fixed point $p = 0$. 

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Cooperative branching

For $\alpha = 4$, a second fixed point appears at $p = 0.5$. 
For $\alpha > 4$, there are two stable fixed points and one unstable fixed point, which separates the domains of attraction of the other two.
Cooperative branching

Fixed points of $\frac{\partial}{\partial t} \mathbf{p}_t = F_\alpha(\mathbf{p}_t)$ for different values of $\alpha$. 
A Recursive Distributional Equation is an equation of the form

\[ X \overset{d}{=} \gamma[\omega](X_1, \ldots, X_{\kappa(\omega)}) \] (RDE),

where \( X_1, X_2, \ldots \) are i.i.d. copies of \( X \), independent of \( \omega \). A law \( \nu \) solves (RDE) iff

(i) \( T_t(\nu) = \nu \quad (t \geq 0) \) or (ii) \( T(\nu) = \nu \).

We can view \( \nu \) as the “invariant law” of a “Markov chain” where time has a tree-like structure.

In our example, solutions to the RDE are the Bernoulli distributions \( \nu_{\text{low}}, \nu_{\text{mid}}, \nu_{\text{upp}} \) with density \( z_{\text{low}}, z_{\text{mid}}, z_{\text{upp}} \).
Recursive Tree Processes

For each solution \( \nu \) of (RDE), there exists a Recursive Tree Process (RTP) \((\omega_i, X_i)_{i \in \mathbb{T}}\), unique in law, such that:

(i) \((\omega_i)_{i \in \mathbb{T}}\) are i.i.d. with law \(|r|^{-1}r\).

(ii) For finite \( U \subset \mathbb{T} \), the r.v.’s \((X_i)_{i \in \partial U}\) are i.i.d. with \( \nu \) and independent of \((\omega_i)_{i \in U}\).

(iii) \[ X_i = \gamma[\omega_i](X_{i1}, \ldots, X_{i\kappa(\omega_i)}) \quad (i \in \mathbb{T}). \]

If we add independent exponentially distributed lifetimes, then:

- Conditional on \( \mathcal{F}_t \), the r.v.’s \((X_i)_{i \in \Delta S_t}\) are i.i.d. with law \( \nu \).

Aldous and Bandyopadyay (RDE) say that an RTP is endogenous if \( X_\emptyset \) is measurable w.r.t. the \( \sigma \)-field generated by \((\omega_i)_{i \in \mathbb{T}}\).

They showed that endogeny is equivalent to bivariate uniqueness.
For each $n \geq 1$, a measurable map $g : S^k \to S$ gives rise to an $n$-variate map $g^{(n)} : (S^n)^k \to S^n$ defined as

$$g^{(n)}(x_1, \ldots, x_k) = g^{(n)}(x^1, \ldots, x^n) := (g(x^1), \ldots, g(x^n)),$$

with $x = (x^m_i)_{i=1}^{n}, x_i = (x^1_i, \ldots, x^n_i), x^m = (x^m_1, \ldots, x^m_k)$.

We define an $n$-variate map

$$T^{(n)}(\mu^{(n)}) := \text{the law of } \gamma^{(n)}[\omega](X_1, \ldots, X_{\kappa}(\omega)),$$

which acts on probability measures $\mu^{(n)}$ on $S^n$.

The $n$-variate mean-field equation

$$\frac{\partial}{\partial t} \mu^{(n)}_t = |r| \left\{ T^{(n)}(\mu^{(n)}_t) - \mu^{(n)}_t \right\} \quad (t \geq 0).$$

describes the mean-field limit of $n$ coupled processes that are constructed using the same random maps.
n-Variate processes

\[ \mathcal{P}(S) \] space of probability measures on \( S \).

\[ \mathcal{P}_{\text{sym}}(S^n) \] space of probability measures on \( S^n \) that are symmetric under a permutation of the coordinates.

\[ S^n_{\text{diag}} \] \( \{x \in S^n : x_1 = \cdots = x_n\} \)

\[ \mathcal{P}(S^n)_\mu \] space of probability measures on \( S^n \) whose one-dimensional marginals are all equal to \( \mu \).

- If \( (\mu_t^{(n)})_{t \geq 0} \) solves the \( n \)-variate equation, then its \( m \)-dimensional marginals solve the \( m \)-variate equation.

- \( \mu_0^{(n)} \in \mathcal{P}_{\text{sym}}(S^n) \) implies \( \mu_t^{(n)} \in \mathcal{P}_{\text{sym}}(S^n) \) (\( t \geq 0 \)).

- \( \mu_0^{(n)} \in \mathcal{P}(S^n_{\text{diag}}) \) implies \( \mu_t^{(n)} \in \mathcal{P}(S^n_{\text{diag}}) \) (\( t \geq 0 \)).

- If \( \mathbf{T}(\nu) = \nu \), then \( \mu_0^{(n)} \in \mathcal{P}(S^n)_\nu \) implies \( \mu_t^{(n)} \in \mathcal{P}(S^n)_\nu \).
If $\nu = \mathbb{P}[X \in \cdot]$ solves the RDE $T(\nu) = \nu$, then

$$\bar{\nu}^{(n)} := \mathbb{P}\left[(X, \ldots, X) \in \cdot \right] \quad n \text{ times}$$

solves the $n$-variate RDE $T^{(n)}(\nu^{(n)}) = \nu^{(n)}$.

Questions:
- Is $\bar{\nu}^{(n)}$ a stable fixed point of the $n$-variate equation?
- Is $\bar{\nu}^{(n)}$ the only solution in $\mathcal{P}_{\text{sym}}(S^n)_\nu$ of the $n$-variate RDE?
n-Variate processes

Let \((\omega_i, X_i)_{i \in \mathbb{T}}\) be the RTP corresponding to a solution \(\nu\) of the RDE. Recall that the RTP is *endogenous* if

\[ X_{\emptyset} \text{ is measurable w.r.t. the } \sigma\text{-field generated by } (\omega_i)_{i \in \mathbb{T}}. \]

**Theorem [AB ’05 & MSS ’18]** The following statements are equivalent:

(i) The RTP corresponding to \(\nu\) is endogenous.

(ii) \(T_t^{(n)}(\mu) \xrightarrow{t \to \infty} \overline{\nu}^{(n)}\) for all \(\mu \in \mathcal{P}(S^n)_\nu\) and \(n \geq 1\).

(iii) \(\overline{\nu}^{(2)}\) is the only solution in \(\mathcal{P}_{\text{sym}}(S^2)_\nu\) of the bivariate RDE.
n-Variate processes

Fixed points of \( \frac{\partial}{\partial t} p_t = F_\alpha(p_t) \) for different values of \( \alpha \).
n-Variate processes

The RDE $T(\nu) = \nu$ has three solutions $\nu_{\text{low}}, \nu_{\text{mid}},$ and $\nu_{\text{upp}},$ where $\nu_{\ldots}$ is the probability measure on $\{0, 1\}$ with mean $\nu_{\ldots}(\{1\}) = z_{\ldots} \ (\ldots = \text{low, mid, upp}),$ which give rise to solutions $\nu_{\text{low}}^{(2)}, \nu_{\text{mid}}^{(2)},$ and $\nu_{\text{upp}}^{(2)}$ of the bivariate RDE.

**Proposition [Mach, Sturm, S. '18]** Apart from $\nu_{\text{low}}^{(2)}, \nu_{\text{mid}}^{(2)}, \nu_{\text{upp}}^{(2)},$ the bivariate RDE has one more solution $\nu_{\text{mid}}^{(2)}$ in $\mathcal{P}_{\text{sym}}(S^2)$. The domains of attraction are:

- $\nu_{\text{low}}^{(2)}$: \{ $\mu_0^{(2)} : \mu_0^{(1)}(\{1\}) < z_{\text{mid}}$ \},
- $\nu_{\text{mid}}^{(2)}$: \{ $\mu_0^{(2)} : \mu_0^{(1)}(\{1\}) = z_{\text{mid}}, \mu_0^{(2)} \neq \nu_{\text{mid}}^{(2)}$ \},
- $\nu_{\text{mid}}^{(2)}$: \{ $\nu_{\text{mid}}^{(2)}$ \},
- $\nu_{\text{upp}}^{(2)}$: \{ $\mu_0^{(2)} : \mu_0^{(1)}(\{1\}) > z_{\text{mid}}$ \}.

The RTPs for $\nu_{\text{low}}, \nu_{\text{upp}}$ are endogenous, but the RTP corresponding to $\nu_{\text{mid}}$ is not.
The $n$-variate map $\mathbf{T}^{(n)}$ is defined even for $n = \infty$, and $\mathbf{T}^{(\infty)}$ maps $\mathcal{P}_{\text{sym}}(S_{N+}^\infty)$ into itself.

By De Finetti’s theorem, $(X_i)_{i \in N^+}$ have a law in $\mathcal{P}_{\text{sym}}(S_{N+}^\infty)$ if and only if there exists a random probability measure $\xi$ on $S$ such that conditional on $\xi$, the $(X_i)_{i \in N^+}$ are i.i.d. with law $\xi$.

Let $\rho := \mathbb{P}[\xi \in \cdot]$ the law of $\xi$. Then $\rho \in \mathcal{P}(\mathcal{P}(S))$.

In view of this, $\mathcal{P}_{\text{sym}}(S_{N+}^\infty) \cong \mathcal{P}(\mathcal{P}(S))$.

The map $\mathbf{T}^{(\infty)} : \mathcal{P}_{\text{sym}}(S_{N+}^\infty) \to \mathcal{P}_{\text{sym}}(S_{N+}^\infty)$ corresponds to a higher-level map $\tilde{\mathbf{T}} : \mathcal{P}(\mathcal{P}(S)) \to \mathcal{P}(\mathcal{P}(S))$. 
The higher-level equation

For any measurable map \( g : S^k \to S \), define \( \check{g} : \mathcal{P}(S)^k \to \mathcal{P}(S) \) by
\[
\check{g} := \text{the law of } g(X_1, \ldots, X_k),
\]
where \((X_1, \ldots, X_k)\) are independent with laws \( \mu_1, \ldots, \mu_k \).

Then
\[
\check{T}(\rho) := \text{the law of } \check{\gamma}[\omega](\xi_1, \ldots, \xi_{\kappa(\omega)}),
\]
with \( \omega \) as before and \( \xi_1, \xi_2, \ldots \) i.i.d. with law \( \rho \).

Define \( n \)-th moment measures
\[
\rho^{(n)} := \mathbb{E}[\xi \otimes \cdots \otimes \xi] \quad \text{where } \xi \text{ has law } \rho.
\]

**Proposition [MSS ’18]** If \((\rho_t)_{t \geq 0}\) solves the higher-level mean-field equation, then its \( n \)-th moment measures \((\rho_t^{(n)})_{t \geq 0}\) solve the \( n \)-variate equation.
The higher-level equation

Equip $\mathcal{P}(\mathcal{P}(S))_\nu = \{\rho : \rho^{(1)} = \nu\}$ with the **convex order**

$$\rho_1 \leq_{cv} \rho_2 \text{ iff } \int \phi \, d\rho_1 \leq \int \phi \, d\rho_2 \quad \forall \text{ convex } \phi.$$ 

**[Strassen '65]** $\rho_1 \leq_{cv} \rho_2$ iff there exist a r.v. $X$ and $\sigma$-fields $\mathcal{H}_1 \subset \mathcal{H}_2$ s.t. $\rho_i = \mathbb{P}[\mathbb{P}[X \in \cdot | \mathcal{H}_i] \in \cdot]$ ($i = 1, 2$).

Define $\nu := \mathbb{P}[\delta_X \in \cdot]$ with $\mathbb{P}[X \in \cdot] = \nu$. Maximal and minimal elements:

$$\delta_\nu \leq_{cv} \rho \leq_{cv} \overline{\nu} \quad \forall \rho \in \mathcal{P}(\mathcal{P}(S))_\nu.$$ 

**Proposition [MSS '18]** $\mathcal{T}$ is monotone w.r.t. the convex order. There exists a solution $\nu$ to the higher-level RDE s.t.

$$\mathcal{T}^n(\delta_\nu) \xrightarrow{n \to \infty} \nu \quad \text{and} \quad \mathcal{T}_t(\delta_\nu) \xrightarrow{t \to \infty} \nu$$

and any solution $\rho \in \mathcal{P}(\mathcal{P}(S))_\nu$ to the higher-level RDE satisfies

$$\nu \leq_{cv} \rho \leq_{cv} \overline{\nu} \quad \forall \rho \in \mathcal{P}(\mathcal{P}(S))_\nu.$$
Proposition [MSS '18]

Let \((\omega_i, X_i)_{i \in T}\) be the RTP corresponding to \(\gamma\) and \(\nu\). Set

\[
\xi_i := \mathbb{P}[X_i \in \cdot \mid (\omega_{ij})_{j \in T}].
\]

Then \((\omega_i, \xi_i)_{i \in T}\) is an RTP corresponding to \(\tilde{\gamma}\) and \(\nu\).

Also, \((\omega_i, \delta X_i)_{i \in T}\) is an RTP corresponding to \(\tilde{\gamma}\) and \(\overline{\nu}\).

**Corollary** The RTP is endogenous iff \(\nu = \overline{\nu}\).
Theorem [Mach, Sturm, S. ’18] One has

\[ \nu_{\text{low}} = \bar{\nu}_{\text{low}}, \quad \nu_{\text{upp}} = \bar{\nu}_{\text{upp}}, \quad \text{but} \quad \nu_{\text{mid}} \neq \bar{\nu}_{\text{mid}}. \]

These are all solutions to the higher-level RDE. Any solution \((\rho_t)_{t \geq 0}\) to the higher-level mean-field equation converges to one of these fixed points. The domains of attraction are:

\[ \bar{\nu}_{\text{low}} : \quad \{ \rho_0 : \rho_0^{(1)}(\{1\}) < z_{\text{mid}} \}, \]
\[ \nu_{\text{mid}} : \quad \{ \rho_0 : \rho_0^{(1)}(\{1\}) = z_{\text{mid}}, \rho_0 \neq \bar{\nu}_{\text{mid}} \}, \]
\[ \bar{\nu}_{\text{mid}} : \quad \{ \bar{\nu}_{\text{mid}} \}, \]
\[ \bar{\nu}_{\text{upp}} : \quad \{ \rho_0 : \rho_0^{(1)}(\{1\}) > z_{\text{mid}} \}. \]
The map \( \mu \mapsto \mu(\{1\}) \) defines a bijection \( \mathcal{P}(\{0,1\}) \cong [0,1] \), and hence \( \mathcal{P}(\mathcal{P}(\{0,1\})) \cong \mathcal{P}[0,1] \).

Then the higher-level RDE takes the form

\[
\eta \overset{d}{=} \chi \cdot (\eta_1 + (1 - \eta_1)\eta_2\eta_3),
\]

where \( \eta \) takes values in \([0,1]\), \( \eta_1, \eta_2, \eta_3 \) are independent copies of \( \eta \) and \( \chi \) is an independent Bernoulli r.v. with \( \mathbb{P}[\chi = 1] = \alpha/(\alpha + 1) \).

This RDE has three “trivial” solutions

\[
\bar{\nu}_{\ldots} = (1 - z_{\ldots})\delta_0 + z_{\ldots}\delta_1 \quad (\ldots = \text{low, mid, upp}),
\]

and a nontrivial solution

\[
\nu_{\text{mid}} = \lim_{n \to \infty} \mathcal{T}^n(\delta z_{\text{mid}}).
\]
Numerical results

\[ \delta_{z_{\text{mid}}} \]

\[ 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1 \]

\[ \alpha = 4.1 \]

\[ \alpha = 4.2 \]

\[ \alpha = 4.3 \]

\[ \alpha = 4.4 \]

\[ \alpha = 4.5 \]

\[ \alpha = 5.5 \]

\[ \alpha = 6 \]

\[ \alpha = 7 \]

\[ \alpha = 8 \]

\[ \alpha = 10 \]

\[ \alpha = 12 \]

\[ \alpha = 15 \]
Numerical results

\[ \tilde{T}(\delta_{\text{z,mid}}) \]
Numerical results

\[ T^3(\delta_{z_{\text{mid}}}) \]
Numerical results

\[ \tilde{T}_n^4(\delta_{z_{\text{mid}}}) \]

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Numerical results

\[ \hat{T}^5(\delta_{z_{mid}}) \]

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Numerical results

\[
\tilde{T}^7(\delta_{z_{\text{mid}}})
\]
Numerical results

\[
\tilde{T}^{10}(\delta_{\text{mid}})
\]
Numerical results

\[ \hat{T}_{15}(\delta_{z_{\text{mid}}}) \]
Numerical results

\[ \mathcal{T}^{25}\left(\delta_{z_{\text{mid}}}\right) \]
Numerical results

\[ \bar{T}_{50}(\delta_{z_{\text{mid}}}) \]
Numerical results

\[ \tilde{T}^{100}(\delta_{z_{\text{mid}}}) \]
Numerical results

\[ \alpha = 4.1 \]
Numerical results

\[ \alpha = 4.2 \]
Numerical results

\[ \delta z_{\text{mid}} \]

\[ \tilde{T}(\delta z_{\text{mid}}) \]

\[ \tilde{T}_2(\delta z_{\text{mid}}) \]

\[ \tilde{T}_3(\delta z_{\text{mid}}) \]

\[ \tilde{T}_4(\delta z_{\text{mid}}) \]

\[ \tilde{T}_5(\delta z_{\text{mid}}) \]

\[ \tilde{T}_{10}(\delta z_{\text{mid}}) \]

\[ \tilde{T}_{15}(\delta z_{\text{mid}}) \]

\[ \tilde{T}_{25}(\delta z_{\text{mid}}) \]

\[ \tilde{T}_{50}(\delta z_{\text{mid}}) \]

\[ \tilde{T}_{100}(\delta z_{\text{mid}}) \]

\[ \alpha = 4.3 \]

\[ \alpha = 4.1 \]

\[ \alpha = 4.2 \]

\[ \alpha = 4.3 \]

\[ \alpha = 4.4 \]

\[ \alpha = 4.5 \]

\[ \alpha = 5.5 \]

\[ \alpha = 6 \]

\[ \alpha = 7 \]

\[ \alpha = 10 \]

\[ \alpha = 12 \]

\[ \alpha = 15 \]
Numerical results

\[ \delta z_{mid} \] (\[ \bar{T} \])

\[ \delta z_{mid} \] (\[ \bar{T}^2 \])

\[ \delta z_{mid} \] (\[ \bar{T}^3 \])

\[ \delta z_{mid} \] (\[ \bar{T}^4 \])

\[ \delta z_{mid} \] (\[ \bar{T}^5 \])

\[ \delta z_{mid} \] (\[ \bar{T}^7 \])

\[ \delta z_{mid} \] (\[ \bar{T}^{10} \])

\[ \delta z_{mid} \] (\[ \bar{T}^{15} \])

\[ \delta z_{mid} \] (\[ \bar{T}^{25} \])

\[ \delta z_{mid} \] (\[ \bar{T}^{50} \])

\[ \delta z_{mid} \] (\[ \bar{T}^{100} \])

\[ \alpha = 4.4 \]

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Numerical results

\[ \alpha = 4.5 \]
Numerical results

\[ \alpha = 4.7 \]
Numerical results

\[\delta z_{\text{mid}} \hat{T}(\delta z_{\text{mid}}) \hat{T}_2(\delta z_{\text{mid}}) \hat{T}_3(\delta z_{\text{mid}}) \hat{T}_4(\delta z_{\text{mid}}) \hat{T}_5(\delta z_{\text{mid}}) \hat{T}_7(\delta z_{\text{mid}}) \hat{T}_{10}(\delta z_{\text{mid}}) \hat{T}_{15}(\delta z_{\text{mid}}) \hat{T}_{25}(\delta z_{\text{mid}}) \hat{T}_{50}(\delta z_{\text{mid}}) \hat{T}_{100}(\delta z_{\text{mid}}) \]

\[\alpha = 4.1, 4.2, 4.3, 4.4, 4.5, 5, 6, 7, 10, 12, 15\]

\[\alpha = 5\]
Numerical results

\[ \delta_{\text{mid}} \bar{T}(\delta_{\text{mid}}) \bar{T}_2(\delta_{\text{mid}}) \bar{T}_3(\delta_{\text{mid}}) \bar{T}_4(\delta_{\text{mid}}) \bar{T}_5(\delta_{\text{mid}}) \bar{T}_7(\delta_{\text{mid}}) \bar{T}_{10}(\delta_{\text{mid}}) \bar{T}_{15}(\delta_{\text{mid}}) \bar{T}_{25}(\delta_{\text{mid}}) \bar{T}_{50}(\delta_{\text{mid}}) \bar{T}_{100}(\delta_{\text{mid}}) \]

\[ \alpha = 4 \]

\[ \alpha = 4 \]

\[ \alpha = 4 \]

\[ \alpha = 4 \]

\[ \alpha = 4 \]

\[ \alpha = 4 \]

\[ \alpha = 4 \]

\[ \alpha = 4 \]

\[ \alpha = 5 \]

\[ \alpha = 5 \]

\[ \alpha = 6 \]

\[ \alpha = 6 \]

\[ \alpha = 6 \]

\[ \alpha = 7 \]

\[ \alpha = 8 \]

\[ \alpha = 10 \]

\[ \alpha = 12 \]

\[ \alpha = 15 \]

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\[ \alpha = 7 \]
Numerical results

\[ \delta z^{\text{mid}} \]  
\[ \tilde{T}(\delta z^{\text{mid}}) \]  
\[ \tilde{T}_2(\delta z^{\text{mid}}) \]  
\[ \tilde{T}_3(\delta z^{\text{mid}}) \]  
\[ \tilde{T}_4(\delta z^{\text{mid}}) \]  
\[ \tilde{T}_5(\delta z^{\text{mid}}) \]  
\[ \tilde{T}_7(\delta z^{\text{mid}}) \]  
\[ \tilde{T}_{10}(\delta z^{\text{mid}}) \]  
\[ \tilde{T}_{15}(\delta z^{\text{mid}}) \]  
\[ \tilde{T}_{25}(\delta z^{\text{mid}}) \]  
\[ \tilde{T}_{50}(\delta z^{\text{mid}}) \]  
\[ \tilde{T}_{100}(\delta z^{\text{mid}}) \]  

\[ \alpha = 4 \]  
\[ \alpha = 8 \]  
\[ \alpha = 10 \]  
\[ \alpha = 12 \]  
\[ \alpha = 15 \]  

\[ \alpha = 5 \]  
\[ \alpha = 8 \]  

\[ \alpha = 5 \]  
\[ \alpha = 10 \]  

\[ \alpha = 8 \]  
\[ \alpha = 12 \]  
\[ \alpha = 15 \]  

\[ \alpha = 8 \]  

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Numerical results

Recursive tree processes and the mean-field limit

\[ \delta_z \mid \hat{T}(\delta_z), \hat{T}_2(\delta_z), \hat{T}_3(\delta_z), \hat{T}_4(\delta_z), \hat{T}_5(\delta_z), \hat{T}_7(\delta_z), \hat{T}_{10}(\delta_z), \hat{T}_{15}(\delta_z), \hat{T}_{25}(\delta_z), \hat{T}_{50}(\delta_z), \hat{T}_{100}(\delta_z) \]

\[ \alpha = 4.1, 4.2, 4.3, 4.4, 4.5, 5, 6, 7, 10, 12, 15 \]

\[ \alpha = 10 \]
Numerical results

\[ \delta z_{mid} \]

\[ \hat{T}(\delta z_{mid}) \]

\[ \hat{T}_2(\delta z_{mid}) \]

\[ \hat{T}_3(\delta z_{mid}) \]

\[ \hat{T}_4(\delta z_{mid}) \]

\[ \hat{T}_5(\delta z_{mid}) \]

\[ \hat{T}_7(\delta z_{mid}) \]

\[ \hat{T}_{10}(\delta z_{mid}) \]

\[ \hat{T}_{15}(\delta z_{mid}) \]

\[ \hat{T}_{25}(\delta z_{mid}) \]

\[ \hat{T}_{50}(\delta z_{mid}) \]

\[ \hat{T}_{100}(\delta z_{mid}) \]

\[ \alpha = 4.1 \]

\[ \alpha = 4.2 \]

\[ \alpha = 4.3 \]

\[ \alpha = 4.4 \]

\[ \alpha = 4.5 \]

\[ \alpha = 5.5 \]

\[ \alpha = 6 \]

\[ \alpha = 7 \]

\[ \alpha = 10 \]

\[ \alpha = 12 \]

\[ \alpha = 15 \]
Numerical results

\[ \delta z_{\text{mid}} \]

\[ \hat{T}(\delta z_{\text{mid}}) \]

\[ \hat{T}^2(\delta z_{\text{mid}}) \]

\[ \hat{T}^3(\delta z_{\text{mid}}) \]

\[ \hat{T}^4(\delta z_{\text{mid}}) \]

\[ \hat{T}^7(\delta z_{\text{mid}}) \]

\[ \hat{T}^{10}(\delta z_{\text{mid}}) \]

\[ \hat{T}^{15}(\delta z_{\text{mid}}) \]

\[ \hat{T}^{25}(\delta z_{\text{mid}}) \]

\[ \hat{T}^{50}(\delta z_{\text{mid}}) \]

\[ \hat{T}^{100}(\delta z_{\text{mid}}) \]

\[ \alpha = 4 \]

\[ \alpha = 4.1 \]

\[ \alpha = 4.2 \]

\[ \alpha = 4.3 \]

\[ \alpha = 4.4 \]

\[ \alpha = 4.5 \]

\[ \alpha = 4.7 \]

\[ \alpha = 5 \]

\[ \alpha = 5.5 \]

\[ \alpha = 6 \]

\[ \alpha = 7 \]

\[ \alpha = 10 \]

\[ \alpha = 12 \]

\[ \alpha = 15 \]

\[ \alpha = 17 \]

\[ \alpha = 20 \]

\[ \alpha = 25 \]

\[ \alpha = 50 \]

\[ \alpha = 100 \]

\[ \alpha = 150 \]

\[ \alpha = 200 \]

\[ \alpha = 250 \]

\[ \alpha = 500 \]

\[ \alpha = 1000 \]