Antiferromagnetic Potts models and random colorings of planar graphs.

(For subcritical contact processes see the talk by Anja Sturm.)

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Antiferromagnetic Potts models

Let $G = (V, E)$ be a finite graph. For each *spin configuration* $\sigma : V \to \{1, \ldots, q\}$, define a *Hamiltonian*

$$H(\sigma) := \sum_{\{x,y\} \in E} 1_{\{\sigma(x)=\sigma(y)\}},$$

and for each *inverse temperature* $\beta \geq 0$, define a *Gibbs measure*

$$\mu_\beta(\sigma) := \frac{1}{Z_\beta} e^{-\beta H(\sigma)},$$

where the *partition sum* $Z_\beta := \sum_\sigma e^{-\beta H(\sigma)}$ is just a normalization constant. Then the probability measure μ_β is the law of an *antiferromagnetic q-state Potts model*.

Zero temperature

In the zero temperature limit $\beta \to \infty$, we obtain the uniform distribution on all *q-colorings* of the graph G, if any exist. I.e., μ_∞ is uniformly distributed on configurations σ such that $\sigma(x) \neq \sigma(y)$ for each edge $\{x, y\} \in E$.

By contrast, for the *ferromagnetic model* (the model with H replaced by $-H$), the *ground states* are the constant configurations $\sigma(x) = \sigma(y)$ for each $\{x, y\} \in E$.

Boundary conditions

Fix $\Lambda \subset V$ and a configuration τ. Then the conditional law

$$\mu_\beta\big(\sigma \,|\, \sigma = \tau \text{ on } V\backslash\Lambda\big)$$

is a Gibbs measure corresponding to the Hamiltonian

$$H_\Lambda(\sigma|\tau) := \sum_{\substack{\{x,y\}\in E\\ x,y\in\Lambda}} 1_{\{\sigma(x)=\sigma(y)\}} + \sum_{\substack{\{x,y\}\in E\\ x\in\Lambda,\ y\in V\backslash\Lambda}} 1_{\{\sigma(x)=\tau(y)\}}.$$

This can be used to define *infinite volume Gibbs measures* through the *DLR conditions*.

Uniqueness of the infinite volume Gibbs measure is equivalent to the effect of the boundary conditions going to zero as $\Lambda \uparrow V$.

Ferromagnetic model

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- Phase transition of second order for small q and first order for large q.
- For $\mathbb{Z}^2$: second order for $q < 4$ and first order for $q > 4$ (proved for $q = 2$ and $q > 25$).

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- For $\mathbb{Z}^2$, it is believed that $q_{\rm c} = 3$ and the 3-state model is critical at zero temperature.

Height mapping

Let $h : \mathbb{Z}^d \to \mathbb{Z}$ satisfy

$$|h(x) - h(y)| = 1 \quad \text{if} \quad |x - y| = 1.$$

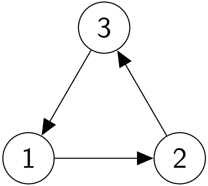
Then

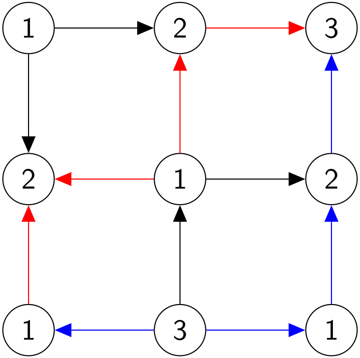
$$\sigma(x) := h(x) \mod(3)$$

is a 3-coloring.

Fact: The mapping $h \mapsto \sigma$ is a *bijection*, i.e., we can recover h from σ.

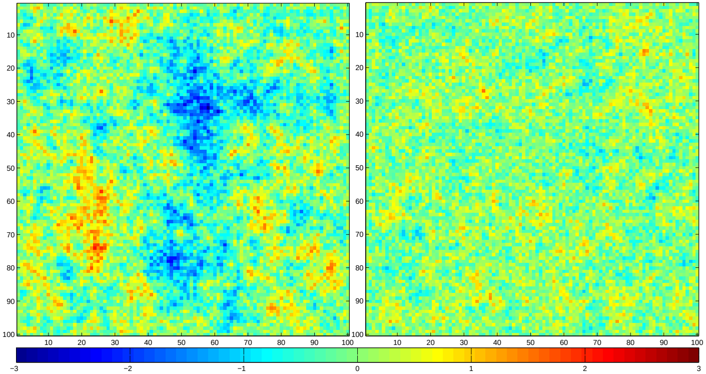
Height mapping





The red path can be deformed into the blue path so that the height difference between the endpoints stays the same.

Height mapping



Simulations by Ron Peled of a random height mapping on a 100×100 square and the middle layer of a $100 \times 100 \times 100$ cube. Simulated using Propp-Wilson's coupling from the past.

High dimension versus dimension two

Ron Peled (preprint 2010) has proved that for sufficiently high d, a typical height-configuration is flat.
This implies (some form of) long-range order for the zero-temperature, 3-state antiferromagnetic Potts model on $\mathbb{Z}^d$.

On the other hand, on $\mathbb{Z}^2$, the fluctuations of the height model are believed to be of order log(system size). This is similar to wat is known for dimer models (R. Kenyon).

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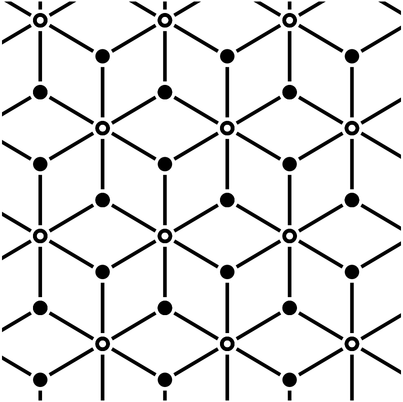
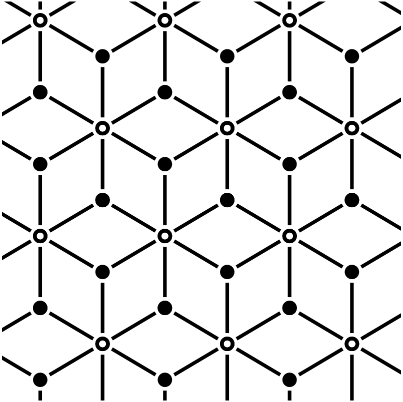
NO.

The diced lattice



Theorem (R. Kotecký, J. Salas & A.D. Sokal, 2008): The 3-state antiferromagnetic Potts model on the diced lattice has long-range order for β sufficiently large.

The diced lattice



The diced lattice:

- Is bipartite.
- Is a quadrangulation.
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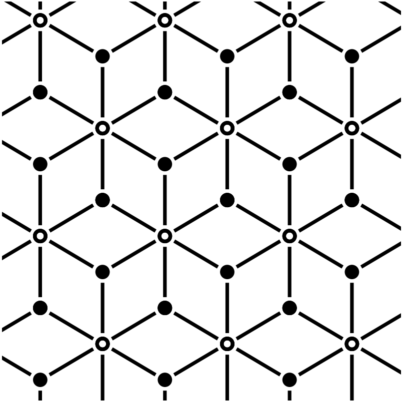

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- Is a quadrangulation.
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So why is it different from $\mathbb{Z}^2$*?*

Asymmetry

Explanation 1: different densities of sublattices

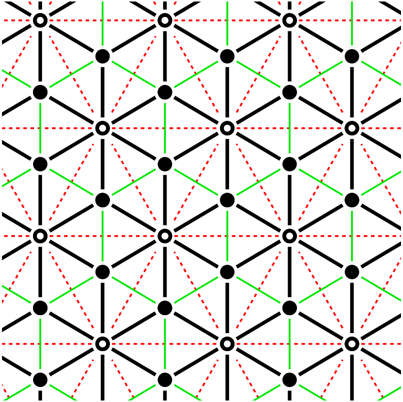
For any bipartite graph, we can construct special 3-colorings by using one color for one sublattice and reserving the other two colors for the other sublattice.

This happens *locally* on $\mathbb{Z}^2$, but on larger scales, we see infinitely many switchings between regions where one or the other sublattice is monotonely colored.

For the diced lattice, the spatial density of points of one sublattice is *twice as high* as for the other sublattice. Therefore, we can make many more configurations if we reserve two colors for this sublattice.

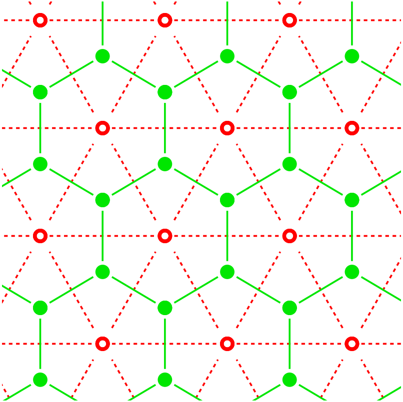
Effectively, this is like applying an external field that favors one sublattice.

Explanation 2: contour model



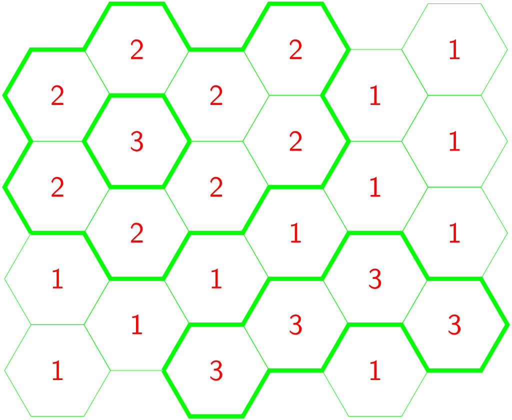
We may view the sublattices as graphs on their own, connecting vertices along the diagonals of quadrilaterals.

Contour model



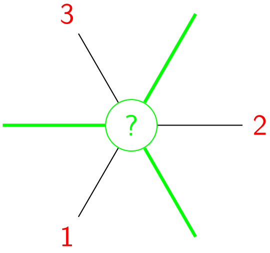
The two sublattices are dual in the sense of planar graph duality.

Contour model



We separate vertices of with different spins in the red sublattice by contours in the green sublattice.

Contour model



Contours are collections of simple cycles, since vertices in the green sublattice cannot be surrounded by three different types in the red sublattice.

Peierls argument

For vertices on a contour, only one type is available, while for vertices that are not on a contour, 2 types are available. As a result, the probability of a given cycle γ being present is less or equal than $2^{-|\gamma|}$, where $|\gamma|$ is the length of γ.

The expected number of cycles surrounding a given vertex can be estimated by

$$\sum_{L=6}^{\infty} N(L) 2^{-L},$$

where $N(L)$ denotes the number of cycles of length L surrounding a given vertex. Duminil-Copin and Smirnov (2010) have proved that the connective constant of the honeycomb lattice is $\sqrt{2+\sqrt{2}}$. It follows that

$$N(L) \leq \text{constant} \times \big(\sqrt{2+\sqrt{2}}\big)^{L}.$$

Peierls argument

Using moreover explicit counting of cycles up to length 140 due to Jensen (2006), Kotecký, Salas & Sokal (2008) were able to prove that for any vertex x in the red sublattice

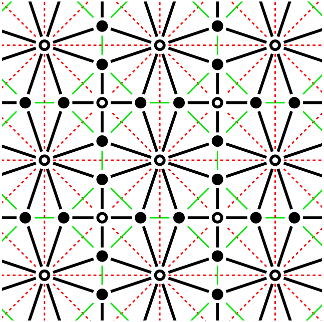
$$\mathbb{P}[x \text{ is surrounded by a cycle}] < \frac{2}{3}.$$

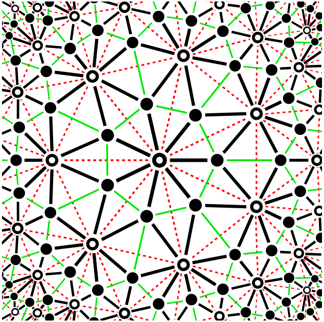
Using 1-boundary conditions on the red sublattice and letting the box size to infinity, it follows that there exists a zero-temperature infinite-volume Gibbs measure μ_∞ such that

$$\mu_\infty\big(\sigma(x) = 1\big) > \frac{1}{3}.$$

In particular, this 'positive magnetization' proves Gibbs state multiplicity and long range order.

More general lattices





We can prove Gibbs state multiplicity for more general lattices, as long as the red sublattice is a triangulation.

More general lattices

If the red sublattice is a triangulation, then each vertex in the green sublattice has degree three.
Green cycles have at each vertex 2 choices where to go.
With a bit of work, this can be used to show that the connective constant α of the green sublattice must be strictly less than 2.
As a result, the Peierls sum is finite:

$$\sum_{L=3}^{\infty} N(L)2^{-L} \leq \text{constant} \times \sum_{L=3}^{\infty} \alpha^L 2^{-L} < \infty.$$

More general lattices

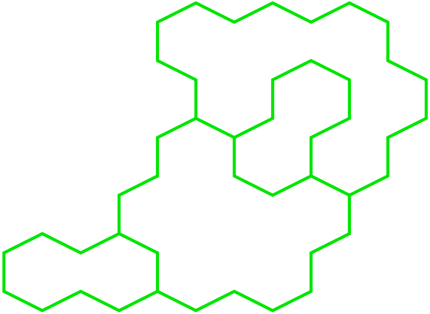
This does not prove positive magnetization, but it does show that very large cycles are unlikely.

As a result, we can show that for a sufficiently large, finite block Δ, there exists a zero-temperature infinite-volume Gibbs measure μ_∞ such that

$$\mu_\infty\big(\sigma(x) = 1 \ \forall x \in \Delta\big) \gg \mu_\infty\big(\sigma(x) = i \ \forall x \in \Delta\big) \qquad (i = 2, 3),$$

which is enough to prove Gibbs state multiplicity and long-range order.

Positive temperature



$N(L,T)$
$\le (L^T/T!)^2 C^T(\alpha+\varepsilon)^L,$

where T is the number of triple points.

The argument can be extended to small positive temperature by a careful counting of non-simple contours.

Open problems

- Prove positive magnetization.
- How different do the sublattices have to be to obtain a phase transition?

and one more question for the experts...

Our proofs apply whenever the red sublattice is a quasi-transitive, 3-connected, planar graph with one end, such that each face is bounded by exactly three edges. This includes certain hyperbolic lattices.

Does every quasi-transitive, 3-connected, planar graph with one end have a periodic embedding in $\mathbb{R}^2$ or the hyperbolic plane?