Markov Process Duality

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Markov Chains

S finite set. $\mathbb{R}^S$ space of functions $f : S \to \mathbb{R}$.
For probability kernel $P = (P(x,y))_{x,y\in S}$ and $f \in \mathbb{R}^S$ define left and right multiplication as

$$Pf(x) := \sum_y P(x,y)f(y) \quad \text{and} \quad fP(x) := \sum_y f(y)P(y,x).$$

(I do not distinguish row and column vectors.)
Def Chain $X = (X_k)_{k\geq 0}$ of S-valued r.v.'s is *Markov chain* with *transition kernel* P and *state space* S if

$$\begin{aligned} & \mathbb{E}\big[f(X_{k+1})\,\big|\,(X_0,\ldots,X_k)\big] = Pf(X_k) \quad \text{a.s.} \quad (f \in \mathbb{R}^S) \\ \Leftrightarrow \quad & \mathbb{P}\big[(X_0,\ldots,X_k) = (x_0,\ldots,x_k)\big] \\ & = \mathbb{P}[X_0 = x_0]P(x_0,x_1)\cdots P(x_{k-1},x_k). \end{aligned}$$

Write $\mathbb{P}^\mu, \mathbb{E}^\mu$ for process with initial law $\mu = \mathbb{P}^\mu[X_0 \in \cdot\,]$.
$\mathbb{P}^x := \mathbb{P}^{\delta_x}$ with $\delta_x(y) := 1_{\{x=y\}}$. $\mathbb{E}^x$ similar.

Markov Chains

Set

$$\mu_k := \mu P^k(x) = \mathbb{P}^\mu[X_k = x] \quad \text{and} \quad f_k := P^k f(x) = \mathbb{E}^x[f(X_k)].$$

Then the *forward* and *backward equations* read

$$\mu_{k+1} = \mu_k P \quad \text{and} \quad f_{k+1} = P f_k.$$

In particular π *invariant law* iff $\pi P = \pi$.
Function h *harmonic* iff $Ph = h \Leftrightarrow h(X_k)$ martingale.

Random mapping representation

$(Z_k)_{k \geq 1}$ i.i.d. with common law ν, take values in $(E, \mathcal{E})$.
$\phi : S \times E \to S$ measurable

$$P(x, y) = \mathbb{P}[\phi(x, Z_1) = y].$$

Random mapping representation $(E, \mathcal{E}, \nu, \phi)$ always exists, highly non-unique.
X_0 independent of $(Z_k)_{k \geq 1}$, then

$$X_k := \phi(X_{k-1}, Z_k) \qquad (k \geq 1)$$

defines Markov chain with transition kernel P.
Example

```
if rand < 0.3
 X = X + 1
else
 X = X - 1
end
```

Continuous time

Markov semigroup $(P_t)_{t\geq 0}$ satisfies $P_s P_t = P_{s+t}$, $\lim_{t\downarrow 0} P_t = P_0 = 1$. Given by

$$P_t = e^{tG} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n G^n,$$

where *generator* G satisfies $G(x,y) \geq 0$ for $x \neq y$ and $\sum_y G(x,y) = 0$.
Def Process $X = (X_t)_{t\geq 0}$ is *Markov* with *semigroup* $(P_t)_{t\geq 0}$ and *generator* G if

$$\mathbb{E}\big[f(X_u)\,\big|\,(X_s)_{0\leq s\leq t}\big] = P_{u-t} f(X_t) \quad \text{a.s.} \quad (f \in \mathbb{R}^S).$$

$P_\varepsilon(x,y) = 1_{\{x=y\}} + \varepsilon G(x,y) + O(\varepsilon^2)$ with $G(x,y)$ *jump rate*.

$$\mu_t := \mu P_t(x) = \mathbb{P}^\mu[X_t = x] \quad \text{and} \quad f_t := P_t f(x) = \mathbb{E}^x[f(X_t)]$$

satisfy the *forward* and *backward equations*

$$\tfrac{\partial}{\partial t}\mu_t = \mu_t G \quad \text{and} \quad \tfrac{\partial}{\partial t} f_t = G f_t.$$

Also $\frac{\partial}{\partial t} P_t = G P_t = P_t G$.

Random mapping representation

Write

$$Gf(x) = \sum_{m \in \mathcal{M}} r_m \big(f(m(x)) - f(x) \big)$$

with $\mathcal{M}$ collection of maps $m : S \to S$ and $(r_m)_{m \in \mathcal{M}}$ nonnegative rates. Let Δ be a Poisson point subset of $\mathcal{M} \times \mathbb{R}$ with local intensity $r_m \mathrm{d}t$, and set

$$\begin{aligned} \Delta_{s,u} &:= \{(m,t) : s < t \leq u\} \\ &=: \{(m_1, t_t), \ldots, (m_n, t_n)\}, \quad t_1 < \cdots < t_n. \end{aligned}$$

Then

$$\Phi_{s,u} := m_n \circ \cdots \circ m_1 \quad \text{satisfy} \quad \Phi_{t,u} \circ \Phi_{s,t} = \Phi_{s,u}.$$

If X_0 independent of Δ, then

$$X_t := \Phi_{0,t}(X_0) \qquad (t \geq 0)$$

Markov process with generator G.

Duality

$X = (X_t)_{t\geq 0}$ Markov with state space S, generator G, semigroup $(P_t)_{t\geq 0}$.
$Y = (Y_t)_{t\geq 0}$ Markov with state space R, generator H, semigroup $(Q_t)_{t\geq 0}$.

Def X and Y *dual* with *duality function* $\psi : S \times R \to \mathbb{R}$ iff

$$\mathbb{E}^x[\psi(X_t, y)] = \mathbb{E}^y[\psi(x, Y_t)] \qquad (t \geq 0).$$

Implies more generally, if X and Y independent, then

$$\mathbb{E}[\psi(X_s, Y_{t-s})] \text{ does not depend on } s \in [0, t].$$

Equivalent formulations (with $A^\dagger(x, y) := A(y, x)$):

- $\displaystyle\sum_{x'} P_t(x, x')\psi(x', y) = \sum_{y'} \psi(x, y')Q_t(y, y'),$

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- $P_t\psi = \psi Q_t^\dagger$,
- $G\psi = \psi H^\dagger$.

Duality

If the matrix ψ is invertible, then

$$P_t = \psi Q_t^\dagger \psi^{-1},$$

which relates the *backward evolution* of X to the *forward evolution* of Y. In general

$$\pi \text{ invariant for } Y \quad \Rightarrow \quad \psi\pi \text{ harmonic for } X.$$

Proof $P_t\psi\pi = \psi Q_t^\dagger \pi = \psi(\pi Q_t) = \psi\pi$. ∎

Similar: h harmonic for $Y \Rightarrow \psi h$ invariant under right-multiplication with P_t (in particular, if ψh is a probabiity distribution, then it is an invariant law).

Pathwise Duality

Def Maps $m : S \to S$ and $\hat{m} : R \to R$ are *dual* w.r.t. ψ if

$$\psi\big(m(x), y\big) = \psi\big(x, \hat{m}(y)\big) \quad \forall x, y.$$

Let

$$Gf(x) = \sum_{m \in \mathcal{M}} r_m \big(f(m(x)) - f(x)\big),$$
$$Hf(y) = \sum_{m \in \mathcal{M}} r_m \big(f(\hat{m}(y)) - f(y)\big).$$

Lemma For each $t > 0$, X and Y can be coupled such that $(X_u)_{0 \leq u \leq s}$ and $(Y_u)_{0 \leq u \leq t-s}$ independent and

$$\psi\big(X_{s-}, Y_{t-s}\big) \quad \text{a.s.} \quad \text{does not depend on } s \in [0, t].$$

Proof Set

$$\begin{aligned}\Delta_{s-,u-} &:= \{(m,t) : s \leq t < u\} \\ &=: \{(m_1,t_t),\ldots,(m_n,t_n)\}, \quad t_1 < \cdots < t_n.\end{aligned}$$

Then

$$\hat{\Phi}_{s-,u-} := \hat{m}_1 \circ \cdots \circ \hat{m}_n \quad \text{dual to} \quad \Phi_{s-,u-} := m_n \circ \cdots \circ m_1.$$

For fixed $t > 0$, observe that $Y_s := \hat{\Phi}_{(t-s)-,t-}(Y_0)$ $(s \geq 0)$ Markov with generator H. Then

$$\psi(X_{s-}, Y_{t-s}) = \psi\big(\Phi_{0-,s-}(X_0), \hat{\Phi}_{s-,t-}(Y_0)\big) = \psi\big(\Phi_{0-,t-}(X_0), Y_0\big)$$

does not depend on $s \in [0,t]$. ∎

A formal dual

$\mathcal{P}(S) :=$ set of all subsets of S. For $m : S \to S$ define $m^{-1} : \mathcal{P}(S) \to \mathcal{P}(S)$ by $m^{-1}(A) := \{x : m(x) \in A\}$ *inverse image*.

Observe m^{-1} dual to m w.r.t. $\psi(x, A) := 1_{\{x \in A\}}$:

$$\psi(m(x), A) = 1_{\{m(x) \in A\}} = 1_{\{x \in m^{-1}(A)\}} = \psi(x, m^{-1}(A)).$$

Consequence X dual to set-valued process $\mathcal{X}$ with generator

$$\mathcal{G}f(A) = \sum_{m \in \mathcal{M}} r_m \big(f(m^{-1}(A)) - f(A) \big).$$

Question Does the large ($|\mathcal{P}(S)| = 2^{|S|}$) space $\mathcal{P}(\Lambda)$ contain any useful subspaces that are invariant under the dynamics of $\mathcal{X}$?

Partial Order

Recall that a *partial order* over S is a relation $\leq$ s.t.

- $x \leq x$,

A partial order is a *total order* if

Let S, S' be partially ordered. Then $m : S \to S'$ is *monotone* if

$$x \leq y \quad \Rightarrow \quad m(x) \leq m(y).$$

A set $A \subset S$ is *increasing* (*decreasing*) if $1_A : S \to \{0, 1\}$ monotone (resp. $1 - 1_A$ monotone).

Observe $m : S \to S$ monotone iff

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A set $A \subset S$ is *increasing* (*decreasing*) if $1_A : S \to \{0,1\}$ monotone (resp. $1 - 1_A$ monotone).

Observe $m : S \to S$ monotone iff

- A increasing $\Rightarrow$ $m^{-1}(A)$ increasing,
- A decreasing $\Rightarrow$ $m^{-1}(A)$ decreasing.

Partial Order

Def $x \in A$ *minimal element* if

$$x' \neq x,\ x' \leq x \ \Rightarrow\ x' \notin A.$$

Def The *episet* of a set B is the increasing set

$$B^{\uparrow} := \{y : y \geq x \text{ for some } x \in B\}.$$

Lemma A *finite* increasing set, $A_{\rm min}$ set of minimal elements, then

$$A = (A_{\rm min})^{\uparrow}.$$

Def $\mathcal{P}_{\rm inc}(\Lambda) :=$ set of increasing subsets of Λ,
$\mathcal{P}_{\rm !inc}(\Lambda) :=$ set of increasing subsets of Λ
that have a *unique* minimal element.

Similarly *maximal element*, *hyposet* $B^{\downarrow}$, $\mathcal{P}_{\rm dec}(\Lambda)$, $\mathcal{P}_{\rm !dec}(\Lambda)$.

Monotone Systems Duality

Observe The condition

$$(*) \quad A \in \mathcal{P}_{!\mathrm{inc}} \ \Rightarrow \ m^{-1}(A) \in \mathcal{P}_{!\mathrm{inc}}$$

is *stronger* than saying that m is monotone. But if S totally ordered almost the same since $\mathcal{P}_{!\mathrm{inc}}(\Lambda) = \mathcal{P}_{\mathrm{inc}}(\Lambda) \backslash \{\emptyset\}$.

Proposition For each $m : S \to S$ satisfying $(*)$, there exists a unique $\hat{m} : S \to S$ such that

$$\psi\big(m(x), y\big) = \psi\big(x, \hat{m}(y)\big) \quad \text{with} \quad \psi(x, y) := 1_{\{x \geq y\}}.$$

Moreover, $\hat{m}$ satisfies

$$(\dagger) \quad A \in \mathcal{P}_{!\mathrm{dec}} \ \Rightarrow \ \hat{m}^{-1}(A) \in \mathcal{P}_{!\mathrm{dec}}.$$

Proof We need

$$1_{\{m(x)\geq y\}} = 1_{\{x\geq \hat{m}(y)\}} \quad \forall x, y,$$

which says that

$$m^{-1}\big(\{y\}^{\uparrow}\big) = \{x : m(x) \geq y\} = \{x : x \geq \hat{m}(y)\} = \{\hat{m}(y)\}^{\uparrow}.$$

A map $\hat{m}$ with this property exists iff m satisfies $(*)$, and $\hat{m}$ is clearly unique. Moreover

$$\hat{m}^{-1}(\{x\}^{\downarrow}) = \{y : \hat{m}(y) \leq x\} = \{y : y \leq m(x)\} = \{m(x)\}^{\downarrow},$$

which proves that $\hat{m}$ maps the space $\mathcal{P}_{!\mathrm{dec}}(S)$ into itself. ∎

Monotone Systems Duality

Def A Markov process X *monotone* if generator of the form

$$Gf(x) = \sum_{m \in \mathcal{M}} r_m \big(f(m(x)) - f(x) \big)$$

with $\mathcal{M}$ a collection of *monotone* maps.

Observe If moreover each $m \in \mathcal{M}$ maps $\mathcal{P}_{!\mathrm{inc}}(S)$ into itself, then X is pathwise dual to the process Y with generator

$$Hf(y) = \sum_{m \in \mathcal{M}} r_m \big(f(\hat{m}(y)) - f(y) \big)$$

in the sense that for each $t > 0$, X, Y can be coupled s.t.

$$\{X_{s-} \geq Y_{t-s}\}$$

a.s. does not depend on $s \in [0, t]$.

Birth-and-death processes

Let $S := \{0, \ldots, n\}$ and define

$$\mathrm{birth}_z(x) := \begin{cases} x+1 & \text{if } x+1 = z, \\ x & \text{otherwise,} \end{cases}$$

$$\mathrm{death}_z(x) := \begin{cases} x-1 & \text{if } x = z, \\ x & \text{otherwise.} \end{cases}$$

Then

$$\widehat{\mathrm{birth}}_z = \mathrm{death}_z \quad \text{and} \quad \widehat{\mathrm{death}}_z = \mathrm{birth}_{z+1}.$$

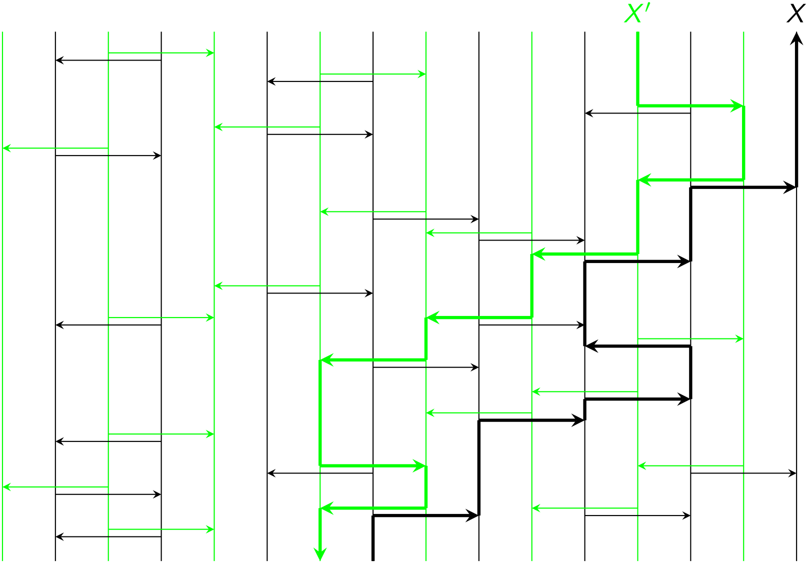
Birth-and-death process X with generator

$$Gf(x) = \sum_{z=1}^{n} b_z \big(f(\mathrm{birth}_z(x)) - f(x) \big) + \sum_{z=1}^{n-1} d_z \big(f(\mathrm{death}_z(x)) - f(x) \big)$$

dual to process X' with

$$d'_z = b_z \quad \text{and} \quad b'_{z+1} = d_z.$$

Birth-and-death processes



Additive Systems Duality

Let $S = \mathcal{P}(\Lambda)$ where Λ is a finite set.
Assume Z and Y have generators

$$Gf(z) = \sum_{n \in \mathcal{M}} r_n \big(f(n(z)) - f(z) \big),$$
$$Hf(y) = \sum_{n \in \mathcal{M}} r_n \big(f(\hat{n}(y)) - f(y) \big)$$

where each n^{-1} maps $\mathcal{P}_{!\mathrm{inc}}(\Lambda)$ into itself and each $\hat{n}^{-1}$ maps $\mathcal{P}_{!\mathrm{dec}}(\Lambda)$ into itself, so Z and Y dual w.r.t.

$$\psi(z, y) := 1_{\{z \geq y\}}.$$

Replace Z_t by $X_t := Z_t^{\mathrm{c}} = \Lambda \backslash Z_t$, replace map n by

$$m(x) := n(x^{\mathrm{c}})^{\mathrm{c}},$$

and set $m^{\dagger} := \hat{n}$. Then $m, m^{\dagger}$ both map $\mathcal{P}_{!\mathrm{dec}}(\Lambda)$ into itself and X, Y are dual w.r.t.

$$\psi(x, y) := 1_{\{x \cap y \neq \emptyset\}}.$$

Additive Systems Duality

Def $m : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda)$ is *additive* if

$$m(\emptyset) = \emptyset \quad \text{and} \quad m(x \cup y) = m(x) \cup m(y) \qquad \big(x, y \in \mathcal{P}(\Lambda)\big).$$

Proposition Then the following statements are equivalent.

(i) $m^{-1}(A) \in \mathcal{P}_{!\mathrm{dec}}(\mathcal{P}(\Lambda))$ for all $A \in \mathcal{P}_{!\mathrm{dec}}(\mathcal{P}(\Lambda))$.

Def A Markov process X *additive* if generator of the form

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(ii) There exists a unique $m^\dagger$ such that $1_{\{m(x) \cap y \neq \emptyset\}} = 1_{\{x \cap m^\dagger(y) \neq \emptyset\}}$.

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(iii) m is additive.

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Additive Systems Duality

Proof (i)⇒(ii): monotone systems duality applied to X^{c} and Y.

(ii)⇒(iii):

$$m(\emptyset) = \{i \in \Lambda : \{i\} \cap m(\emptyset) \neq \emptyset\} = \{i \in \Lambda : m^\dagger(\{i\}) \cap \emptyset \neq \emptyset\} = \emptyset,$$

and

$$\begin{aligned} m(x \cup x') &= \{i \in \Lambda : \{i\} \cap m(x \cup x') \neq \emptyset\} = \{i \in \Lambda : m^\dagger(\{i\}) \cap (x \cup x') \neq \emptyset\} \\ &= \{i \in \Lambda : m^\dagger(\{i\}) \cap x \neq \emptyset\} \cup \{i \in \Lambda : m^\dagger(\{i\}) \cap x' \neq \emptyset\} = m(x) \cup m(x'). \end{aligned}$$

(iii)⇒(i): Setting $\hat{m}(y) := \{i \in \Lambda : m(\{i\}) \subset y\}$, one has

$$m^{-1}(\{y\}^\downarrow) = \{x : m(x) \subset y\} = \{x : \bigcup_{i \in x} m(\{i\}) \subset y\} = \{\hat{m}(y)\}^\downarrow,$$

proving that m^{-1} maps $\mathcal{P}_{!\mathrm{dec}}(\mathcal{P}(\Lambda))$ into itself. ∎

Additive Systems Duality

Def $m(i,j) := 1_{\{j \in m(\{i\})\}}$. Then: **Lemma** $m^\dagger(i,j) = m(j,i)$.

In the *graphical representation*, we draw Λ horizontaly, time vertically, and for each $(m,t) \in \Delta$, we draw:

an arrow from (i,t) to (j,t) for each $i,j \in \Lambda$, $i \neq j$ such that $m(i,j) = 1$,

a blocking symbol ▬ at (i,t) for each $i \in \Lambda$ such that $m(i,i) = 0$.

Write $(i,s) \rightsquigarrow (j,t)$ if there is an open path γ from $\gamma_s = i$ to $\gamma_t = j$ that may use arrows and avoids blocking symbols. Then

$$X_s := \big\{ j \in \Lambda : \exists i \in X_0 \text{ s.t. } (i,0) \rightsquigarrow (j,s) \big\},$$
$$Y_{s-} := \big\{ i \in \Lambda : \exists j \in Y_0 \text{ s.t. } (i,t-s) \rightsquigarrow (j,t) \big\}.$$

The dual process runs downward in time and uses arrows in the reverse order.

The voter model

Define

$$\mathrm{vot}_{i,j}(x) := \begin{cases} x \cup \{j\} & \text{if } i \in x, \\ x \backslash \{j\} & \text{if } i \notin x. \end{cases}$$

Fix $p(i,j) \geq 0$. In the *voter model* with generator

$$G_{\mathrm{vot}} f(x) := \sum_{i \neq j} p(i,j) \big(f(\mathrm{vot}_{i,j}(x)) - f(x) \big),$$

site j adopts the type of site j with rate $p(i,j)$.
Dual map

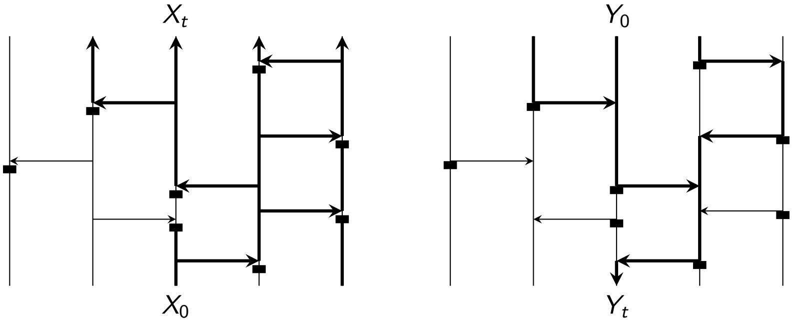
$$\mathrm{rw}_{j,i}(x) := \begin{cases} (x \backslash \{j\}) \cup \{i\} & \text{if } j \in x, \\ x & \text{if } j \notin x. \end{cases}$$

Dual process Y with generator

$$G_{\mathrm{rw}} f(y) := \sum_{i \neq j} p(i,j) \big(f(\mathrm{rw}_{j,i}(y)) - f(y) \big)$$

is *system of coalescing random walks.*

The voter model



$$\{X_t \cap Y_0 \neq \emptyset\} = \{\exists \text{ open path from } X_0 \text{ to } Y_0\} = \{X_0 \cap Y_t \neq \emptyset\}.$$

The contact process

Interpret $X_t =$ set of infected sites.

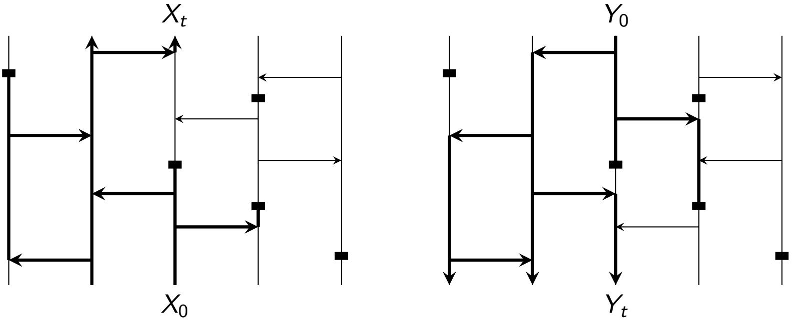
$$\begin{aligned}
\mathrm{rec}_i(x) &:= x\backslash\{i\} && (i \in \Lambda),\\
\mathrm{inf}_{i,j}(x) &:= \begin{cases} x \cup \{j\} & \text{if } i \in \Lambda \\ x & \text{otherwise,} \end{cases} && (i,j \in \Lambda,\ i \neq j).
\end{aligned}$$

The *contact process* with *recovery rate* δ and *infection rates* $\lambda(i,j)$ has generator

$$G_{\mathrm{cont}} f(x) := \delta \sum_i \big(f(\mathrm{rec}_i(x)) - f(x)\big) + \sum_{i \neq j} \lambda(i,j)\big(f(\mathrm{inf}_{i,j}(x)) - f(x)\big).$$

(Self-) dual to process with reversed infection rates $\lambda^\dagger(i,j) := \lambda(j,i)$.

The contact process



$$\{X_t \cap Y_0 \neq \emptyset\} = \{\exists \text{ open path from } X_0 \text{ to } Y_0\} = \{X_0 \cap Y_t \neq \emptyset\}.$$

Linear systems duality

Let S be (a subspace of) $\mathbb{R}^\Lambda$, with Λ a finite set.
Def A Markov process X is *linear* if its generator has a representation

$$Gf(x) = \sum_{m \in \mathcal{M}} r_m \big(f(m(x)) - f(x) \big)$$

with each $m \in \mathcal{M}$ a linear map $m : \mathbb{R}^\Lambda \to \mathbb{R}^\Lambda$. The *adjoint* $m^\dagger(i,j) := m(j,i)$ is dual w.r.t. the duality function

$$\psi(x,y) := \langle x, y \rangle := \sum_{i \in \Lambda} x(i) y(i).$$

Graphical representation

an arrow with *weight* $m(i,j)$ from (i,t) to (j,t)

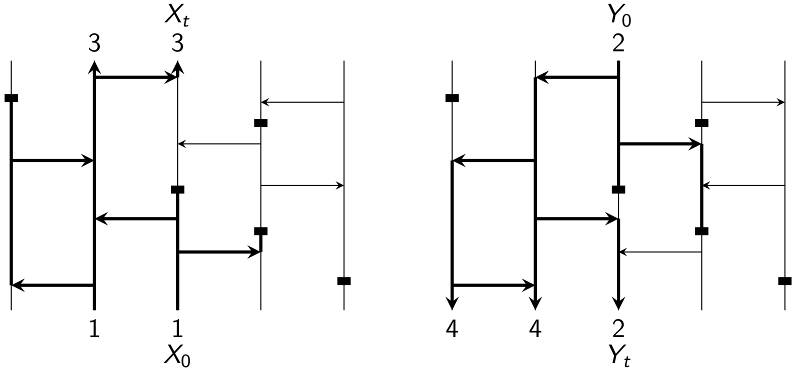
for each $i,j \in \Lambda$ with $i \neq j$ such that $m(i,j) \neq 0$,

a symbol ▬ with *weight* $m(i,i)$ at (i,t)

for each $i \in \Lambda$ such that $m(i,i) \neq 1$.

Each path has weight $=$ product of arrows and ▬ on the path.

The contact path process



$$\langle X_t, Y_0 \rangle = \langle X_0, Y_t \rangle$$

$$= \sum_{i,j} X_0(i) \cdot \#\{\text{open paths } (i,0) \rightsquigarrow (j,t)\} \cdot Y_0(j).$$

Cancellative Systems Duality

The set $\{0,1\}$ with the usual product and with addition modulo 2, denoted by $\oplus$, is a *finite field*.

We may view $\{0,1\}^\Lambda \cong \mathcal{P}(\Lambda)$ as a *linear space* over $\{0,1\}$.

A map $m : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$ is linear iff

$$mx(i) = \bigoplus_j m(i,j)x(j),$$

where $m(i,j) \in \{0,1\}$ form the matrix of m. Adjoint matrix $m^\dagger$ dual w.r.t.

$$\psi(x,y) = \langle x,y \rangle := \bigoplus_i x(i)y(i).$$

In the graphical representation, each arrow has weight 1 and each ▬ has weight 0.

The voter model revisited

The voter model map $\mathrm{vot}_{i,j}$ is linear mod 2 and dual to

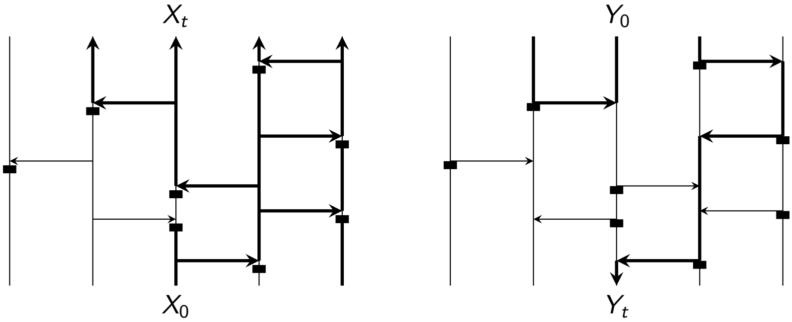
$$\mathrm{ann}_{i.j}(y)(k) = \begin{cases} 0 & \text{if } k = i, \\ y(i) \oplus y(j) & \text{if } k = j, \\ y(k) & \text{otherwise,} \end{cases}$$

Dual process Y with generator

$$G_{\mathrm{ann}} f(y) := \sum_{i \neq j} p(i,j)\big(f(\mathrm{ann}_{j,i}(y)) - f(y)\big)$$

is *system of annihilating random walks*.

The voter model revisited



$$\langle X_t, Y_0 \rangle = \langle X_0, Y_t \rangle$$

$$= 1_{\{\#\text{paths from } X_0 \text{ to } Y_0 \text{ is odd}\}}.$$

Applications of duality

- Characterization of a 'difficult' invariant law (e.g. the upper invariant law of the contact process) in terms of a 'simple' harmonic function of the dual process (e.g. the survival probability).

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- Finding 'difficult' harmonic functions in terms of 'simple' invariant laws of the dual process. (E.g. Vandermonde determinant based on noncrossing duality, strong interface tightness implies noncoexistence.)
- Subinvariant laws (Holley-Liggett upper bound on critical point for the contact process).

Lloyd-Sudbury duals

Let Λ be an undirected graph. Let X be a Markov process with state space $\mathcal{P}(\Lambda) \cong \{0,1\}^\Lambda$ such that for each edge $\{i,j\}$, the local state $(x(i),x(j))$ performs

$$
\begin{array}{rll}
\text{annihilation} & 11 \mapsto 00 & \text{with rate } a, \\
\text{branching} & 01 \mapsto 11 & \text{with rate } b, \\
\text{coalescence} & 11 \mapsto 01 & \text{with rate } c, \\
\text{death} & 01 \mapsto 00 & \text{with rate } d, \\
\text{exclusion} & 01 \mapsto 10 & \text{with rate } e,
\end{array}
$$

with similar rates for transitions that are mirror images of these.
This is the most general interacting particle system with only two-point interactions, for which $\emptyset$ is a trap.

Lloyd-Sudbury duals

[Lloyd and Sudbury ('95, '97, '00)] Let X and X' be given by rates $a, b, c, d, e \geq 0$ resp. $a', b', c', d', e' \geq 0$ satisfying

$$a' = a + 2q\gamma, \quad b' = b + \gamma, \quad c' = c - (1+q)\gamma, \quad d' = d + \gamma, \quad e' = e - \gamma,$$

where $\gamma := (a + c - d + qb)/(1 - q)$. Then

$$\mathbb{E}\big[q^{\,|X_t \cap X'_0|}\big] = \mathbb{E}\big[q^{\,|X_0 \cap X'_t|}\big]$$

Example 1 $q = 0$ gives

$$0^{\,|x \cap y|} = 1_{\{x \cap y = \emptyset\}} \qquad \text{additive duality.}$$

Example 2 $q = -1$ gives

$$(-1)^{\,|x \cap y|} = 1 - 2 \bigoplus_i x(i)y(i) \qquad \text{cancellative duality.}$$

Lloyd-Sudbury duals

Proof (sketch) Write the space of all functions $f : \{0,1\}^\Lambda \to \mathbb{R}$ as a tensor product

$$\mathbb{R}^S = \mathbb{R}^{\{0,1\}^\Lambda} \cong \bigotimes_{i\in\Lambda} \mathbb{R}^{\{0,1\}}.$$

Write the generator G as $G = \sum_{\{i,j\}} G_{ij}$ where we sum over all edges of the graph and G_{ij} acts only on the coordinates i and j, and similarly $H = \sum_{\{i,j\}} H_{ij}$.
Write ψ as the commutative product $\psi = \prod_i \psi_i$ where ψ_i is an operator that acts only on coordinate i.
For $k \neq i, j$, ψ_k commutes with G_{ij}, so suffices to check for each edge $\{i,j\}$

$$G_{ij}\psi_i\psi_j = \psi_i\psi_j H_{ij}^\dagger.$$

Lloyd-Sudbury duals

$$G_{ij} = \begin{pmatrix} \cdot & 0 & 0 & 0 \\ d & \cdot & e & b \\ d & e & \cdot & b \\ a & c & c & \cdot \end{pmatrix} \quad \text{and} \quad H^{\dagger}_{ij} = \begin{pmatrix} \cdot & d' & d' & a' \\ 0 & \cdot & e' & c' \\ 0 & e' & \cdot & c' \\ 0 & b' & b' & \cdot \end{pmatrix}$$

$$\psi_i = \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix} \quad \text{and} \quad \psi_i \psi_j = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & q & 1 & q \\ 1 & 1 & q & q \\ 1 & q & q & q^2 \end{pmatrix}.$$

Now brutal calculation. Can simplify a bit by using

$$G_{ij} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} H^{\dagger}_{ij}.$$

Duals of the voter model

Voter model X has

$$a = 0, \quad b = 1, \quad c = 0, \quad d = 1, \quad e = 0.$$

For each $0 \leq \alpha \leq 1$ q-dual with $q := -\alpha$ to the process Y with generator

$$Hf(y) = \sum_{\{i,j\}} \big\{\alpha\big(f(\mathrm{ann}_{i,j}(y)) - f(y)\big) + (1-\alpha)\big(f(\mathrm{rw}_{i,j}(y)) - f(y)\big)\big\}.$$

$\alpha = 0$ gives coalescing random walks, $\alpha = 1$ gives annihilating random walks.
Extension to biased voter model and branching-coalescing-annihilating random walk (exercise).

Applications of intertwining

- Interlacing of non-crossing random walks (Patrik Ferrari).

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- Processes with multiple time scales.

Autonomous Markov chain

Let $X = (X_k)_{k\geq 0}$ a Markov chain with state space S and transition kernel P, an let $f : S \to R$ be surjective.

Def $(Y_k)_{k\geq 0} = (f(X_k))_{k\geq 0}$ is *autonomous* (also called *lumpable*) if

$$f(x) = f(x') \quad \text{implies} \quad \mathbb{P}^x[f(X_1) = y] = \mathbb{P}^{x'}[f(X_1) = y].$$

Lemma Y autonomous $\Rightarrow$ Y on its own Markov with transition kernel

$$Q(y, y') := \mathbb{P}^x[f(X_1) = y'] = \sum_{x' \in S} 1_{\{f(x')=y\}} P(x, x').$$

(Y is sometimes called a *lumped* Markov chain.)

Markov functionals

X Markov chain with state space S and transition kernel P.

[Rogers & Pitman '81] Let $f : S \to R$ be surjective and let $K(y, x)$ be a probability kernel from R to S s.t.

$$K(y, x) = 0 \quad \text{whenever} \quad f(x) \neq y.$$

Assume

$$QK = KP.$$

Then

$$\mathbb{P}[X_0 = x \,|\, Y_0] = K(Y_0, x) \quad \text{a.s.} \qquad (x \in S),$$

implies

$$\mathbb{P}[X_k = x \,|\, (Y_0, \ldots, Y_k)] = K(Y_k, x) \quad \text{a.s.} \qquad (x \in S),$$

and Y, on its own, is a Markov chain with transition kernel Q.

Markov functionals

Proof Set

$$\pi(x \mid y_0, \ldots, y_k) := \mathbb{P}\big[X_k = x \,\big|\, (Y_0, \ldots, Y_k) = (y_0, \ldots, y_k)\big].$$

We wish to prove that

$$\pi(x \mid y_0, \ldots, y_k) = K(x, y_k) \qquad (k \geq 1),$$

given that this holds at $k = 0$. The *filtering equations* tell us that

$$\pi(x \mid y_0, \ldots, y_{k+1}) = \frac{\sum_{x' \in S} P(x', x; y_{k+1}) \pi(x' \mid y_0, \ldots, y_k)}{\sum_{x', x'' \in S} P(x', x''; y_{k+1}) \pi(x' \mid y_0, \ldots, y_k)},$$

where

$$P(x, x'; y) := 1_{\{f(x') = y\}} P(x, x') \qquad (x, x' \in S, \ y \in R).$$

Markov functionals

Our assumptions on K imply that

$$\sum_{x\in S} K(y,x)P(x,x';y') = 1_{\{f(x')=y'\}}(KP)(y,x') = 1_{\{f(x')=y'\}}(QK)(y,x')$$
$$= \sum_{y''\in R} Q(y,y'')K(y'',x')1_{\{f(x')=y'\}} = Q(y,y')K(y',x')$$

Using this, by induction,

$$\pi(x\,|\,y_0,\ldots,y_{k+1}) = \frac{\sum_{x'\in S} P(x',x;y_{k+1})K(y_k,x')}{\sum_{x',x''\in S} P(x',x'';y_{k+1})K(y_k,x')}$$
$$= \frac{Q(y_k,y_{k+1})K(y_{k+1},x)}{\sum_{x''\in S} Q(y_k,y_{k+1})K(y_{k+1},x'')} = K(y_{k+1},x).$$

Markov functionals

Now, by the Markov property of X and what we have already proved

$$
\begin{aligned}
&\mathbb{P}\big[Y_{k+1} = y \,\big|\, (Y_0, \ldots, Y_k) = (y_0, \ldots, y_k)\big] \\
&\quad = \sum_{x \in S} \mathbb{P}\big[Y_{k+1} = y \,\big|\, X_k = x,\ (Y_0, \ldots, Y_k) = (y_0, \ldots, y_k)\big] \\
&\qquad\qquad \cdot \mathbb{P}\big[X_k = x \,\big|\, (Y_0, \ldots, Y_k) = (y_0, \ldots, y_k)\big] \\
&\quad = \sum_{x \in S} \mathbb{P}\big[Y_{k+1} = y \,\big|\, X_k = x\big] \pi(x \,|\, y_0, \ldots, y_k) \\
&\quad = \sum_{x, x' \in S} P(x, x'; y) K(y_k, x) = \sum_{x' \in S} Q(y_k, y) K(y_k, x') = Q(y_k, y),
\end{aligned}
$$

proving that Y is a Markov chain with transition kernel Q. ∎

Intertwining

Let P, Q be transition kernels on S, R, and let K be a kernel from R to S.

[Diaconis & Fill '90] Assume that

$$QK = KP.$$

Then there exists a Markov chain $(X, Y) = (X_k, Y_k)_{k \geq 0}$ with state space $\hat{S} := \{(x, y) \in S \times R : K(y, x) > 0\}$ such that

1. X is autonomous with transition kernel P,

and moreover, the condition

$$\mathbb{P}[X_0 = x \,|\, Y_0] = K(Y_0, x) \quad \text{a.s.} \qquad (x \in S) \tag{1}$$

implies that

2. Y, on its own, is a Markov chain with transition kernel Q,
3. $\mathbb{P}[X_k = x \,|\, (Y_0, \ldots, Y_k)] = K(Y_k, x) \quad \text{a.s.} \qquad (k \geq 0,\ x \in S)$.

Proof (sketch) Set

$$Q_{x'}(y,y') := \frac{Q(y,y')K(y',x')}{QK(y,x')} \qquad (QK(y,x') > 0),$$

and make an arbitrary choice for $Q_{x'}(y,\,\cdot\,)$ if $QK(y,x') = 0$.
Check that

$$\hat{P}(x,y;x',y') := P(x,x')Q_{x'}(y,y')$$

unambiguously defines a transition kernel on $\hat{S}$ which satisfies

$$Q\hat{K} = \hat{K}\hat{P}$$

with

$$\hat{K}(y;x',y') := K(y,x')1_{\{y=y'\}}.$$

Apply Rogers & Pitman's result to Q, $\hat{P}$, $\hat{K}$, and the function $f:\hat{S}\to R$ be defined by $f(x,y) := y$. ∎

Intertwining

Remark 1 Compared to duality, there are two differences: 1. The intertwiner is necessarily a probability kernel. 2. We link the forward equation of one process to the forward equation of another.

Remark 2 It seems the first use of the term 'intertwining' in the context of Markov chains was by Marc Yor ('88, unpublished).

Remark 3 Diaconis and Fill's result contains Rogers & Pitman's as a special case. Indeed, $\hat{S} \cong S$ if there exists a function $f : S \to R$ such that $K(y,x) = 0$ unless $f(x) = y$.

Remark 4 The condition $\mathbb{P}[X_0 = x \,|\, Y_0] = K(Y_0, x)$ a.s. puts restrictions on the law of X_0 but not on Y_0. We can read the proposition as saying that Y, started in any initial law, can be coupled to a process X such that $\mathbb{P}[X_k = x \,|\, (Y_0, \ldots, Y_k)] = K(Y_k, x)$ a.s. $(k \geq 0)$.

Remark 5 Since the inverse of a probability kernel K is not a probability kernel, intertwining of Markov chains is not a symmetric relation. We will say that X sits *on top* of Y. (Because we view X as extra structure added 'on top' of Y.)

Intertwining

Remark 6 Athreya & S. '10 proved a generalization of Diaconis and Fill's result where X need not be autonomous. They applied this in a case where X is 'almost' autonomous.

Continuous time

Let G, H be generators of Markov processes with state spaces S, R, and let K be a probability kernel from R to S.

[Fill '92] Assume that

$$HK = KG.$$

Then there exists a Markov process $(X, Y) = (X_t, Y_t)_{t \geq 0}$ with state space $\hat{S} := \{(x, y) \in S \times R : K(y, x) > 0\}$ such that

1. X is autonomous with generator G,

and moreover, the condition

$$\mathbb{P}[X_0 = x \,|\, Y_0] = K(Y_0, x) \quad \text{a.s.} \qquad (x \in S)$$

implies that

2. Y, on its own, is a Markov process with generator H,
3. $\mathbb{P}[X_t = x \,|\, (Y_s)_{0 \leq s \leq t}] = K(Y_t, x) \quad \text{a.s.} \qquad (t \geq 0, \; x \in S).$

Thinning

Let Λ be a finite set and let $x \in \{0,1\}^\Lambda \cong \mathcal{P}(\Lambda)$. Let $\chi \in \mathcal{P}(\Lambda)$ be independent of x and assume that $(\chi(i))_{i\in\Lambda}$ are i.i.d. with $\mathbb{P}[\chi(i) = 1] = p$. Then

$$\mathrm{Thin}_p(x) := x \cap \chi$$

is called a *p-thinning* of x. We define a thinning kernel T_p on $\mathcal{P}(\Lambda)$ by

$$T_p(x,y) := \mathbb{P}[\mathrm{Thin}_p(x) = y] \qquad \big(x, y \in \mathcal{P}(\Lambda)\big),$$

Call processes X and Y *q-dual* if they are dual w.r.t. the duality function

$$\psi_q(x,y) := q^{|x\cap y|}.$$

Thinning

[Lloyd & Sudbury '97] Let X, X', Y be $\mathcal{P}(\Lambda)$-valued Markov processes with generators G, G', H. Assume that X is a q-dual of Y and that X' is a q'-dual of Y, for constants $q, q' \neq 1$ satisfying

$$p := \frac{1-q}{1-q'} \in [0,1].$$

Then the generators of X and X' satisfy the intertwining relation

$$GT_p = T_p G'.$$

In particular, the process X, started in an arbitrary initial law, can be coupled to a process X' such that

1. X' is an autonomous Markov process with generator G,
2. $\mathbb{P}[X'_t \in \cdot \,|\, (X_s)_{0 \leq s \leq t}] = T_p(X_t, \cdot\,) \quad \text{a.s.} \qquad (t \geq 0).$

We say that X' is a *p-thinning* of X.

Thinning

Proof We claim that

$$\psi_q \psi_{q'}^{-1} = T_p \quad \text{provided that} \quad p = \frac{1-q}{1-q'} \in [0,1].$$

Since both ψ_q and T_p are products of commuting operators acting on a single sites, it suffices to prove the claim for single sites. Then

$$\psi_q = \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix} \quad \text{and} \quad \psi_{q'}^{-1} = (1-q')^{-1} \begin{pmatrix} -q' & 1 \\ 1 & -1 \end{pmatrix},$$

which implies that

$$\psi_q \psi_{q'}^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{q-q'}{1-q'} & \frac{1-q}{1-q'} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1-p & p \end{pmatrix} = T_p.$$

Thinning

Now duality says that

$$G\psi_q = \psi_q H^\dagger \quad \text{and} \quad G'\psi_{q'} = \psi_{q'} H^\dagger,$$

which implies that $\psi_{q'}^{-1} G' = H^\dagger \psi_{q'}^{-1}$ and

$$GT_p = G\psi_q\psi_{q'}^{-1} = \psi_q H^\dagger \psi_{q'}^{-1} = \psi_q\psi_{q'}^{-1} G' = T_p G'.$$

$\blacksquare$

Remark We have never used that H is a Markov generator. It is therefore sufficient if Y is only a 'formal dual'.

Annihilating and coalescing random walks

Let X^α be the process with generator

$$G_\alpha f(x) = \sum_{\{i,j\}} \big\{\alpha\big(f(\mathrm{ann}_{i,j}(x)) - f(x)\big) + (1-\alpha)\big(f(\mathrm{rw}_{i,j}(x)) - f(x)\big)\big\},$$

i.e., these are random walks that when on the same site annihilate with probability $0 \leq \alpha \leq 1$ and coalesce with probab. $1-\alpha$.

Since X^α is q-dual to the voter model with $q = -\alpha$, we obtain that for any $0 \leq \alpha \leq \alpha' \leq 1$, the process X^α can be coupled to $X^{\alpha'}$ s.t.

$$\mathbb{P}[X^{\alpha'}_t \in \cdot \,|\, (X^\alpha_s)_{0\leq s\leq t}] = T_{(1+\alpha)/(1+\alpha')}(X^\alpha_t, \,\cdot\,) \quad \text{a.s.} \qquad (t \geq 0).$$

In particular, annihilating random walks are a $1/2$-thinning of coalescing random walks.

This can be extended to systems with branching (exercise).

Intertwining of birth and death processes

[Karlin & McGregor '59] Let Z be a Markov process with state space $\{0, 1, 2, \ldots\}$, started in $Z_0 = 0$, that jumps $k - 1 \mapsto k$ with rate $b_k > 0$ and $k \mapsto k - 1$ with rate $d_k > 0$ $(k \geq 1)$. Then

$$\tau_N := \inf\{t \geq 0 : Z_t = N\}$$

is distributed as a sum of independent exponentially distributed random variables whose parameters $\lambda_1 < \cdots < \lambda_N$ are the negatives of the eigenvalues of the generator of the process stopped in N.

[Diaconis & Miclos '09] Let $X_t := Z_{t \wedge \tau_N}$ be the stopped process and let $0 > -\lambda_1 > \cdots > -\lambda_N$ be its eigenvalues. Let X^+ be a pure birth process with birth rates $b_1, \ldots, b_N$ given by $\lambda_N, \ldots, \lambda_1$. Then it is possible to couple the processes X and X^+, both started in zero, in such a way that $X_t \leq X_t^+$ for all $t \geq 0$ and both processes arrive in N at the same time.

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Idea of the proof Let G, G^+ be the generators of X, X^+. Then one can show that there exists a kernel K^+ such that

$$\begin{aligned} &K^+(x, \{0, \ldots, x\}) = 1 \qquad (0 \leq x \leq N), \\ &K^+(N, N) = 1, \end{aligned}$$

and moreover

$$K^+ G = G^+ K^+.$$

This can be proved by induction, using the Perron-Frobenius theorem in each step. ∎

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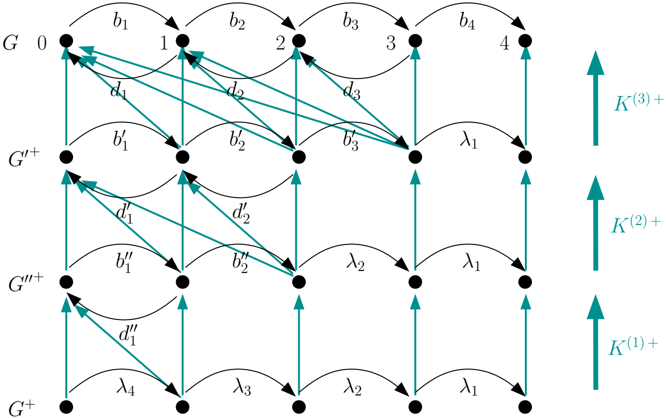

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[S. '10] Let X_t and $\lambda_1, \ldots, \lambda_N$ be as before. Let X^- be a pure birth process with birth rates $b_1, \ldots, b_N$ given by $\lambda_1, \ldots, \lambda_N$. Then it is possible to couple the processes X and X^-, both started in zero, in such a way that $X^-_t \leq X_t$ for all $t \geq 0$ and both processes arrive in N at the same time.

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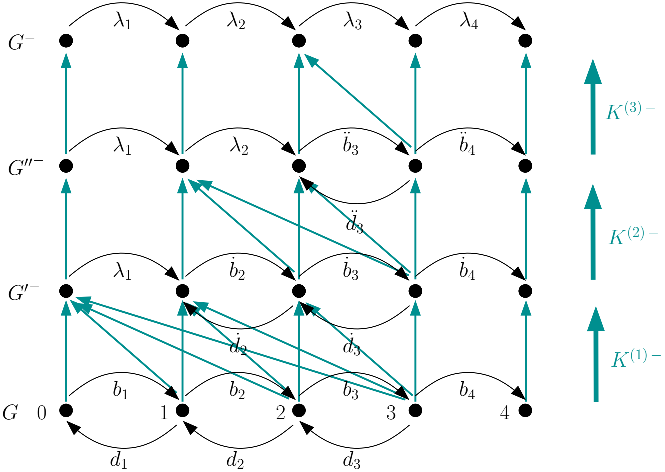

The complete figure