Frozen percolation on the binary tree

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(Budapest)

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Percolation on 3-regular graphs

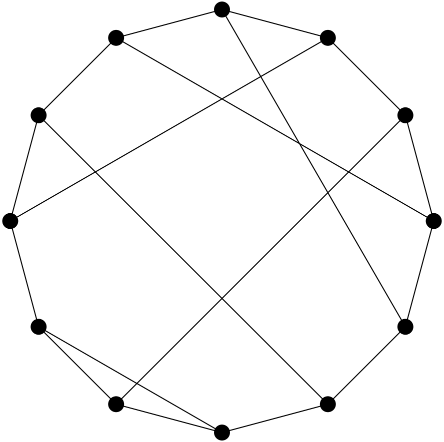
Let $G_n = (V_n, E_n)$ be random, uniformly chosen, 3-regular graphs with n vertices (n is even).

Let $(\tau_e)_{e \in E_n}$ be i.i.d. uniformly distributed $[0,1]$-valued random variables attached to the edges.

Initially, all edges are closed. At time τ_e, the edge e opens.

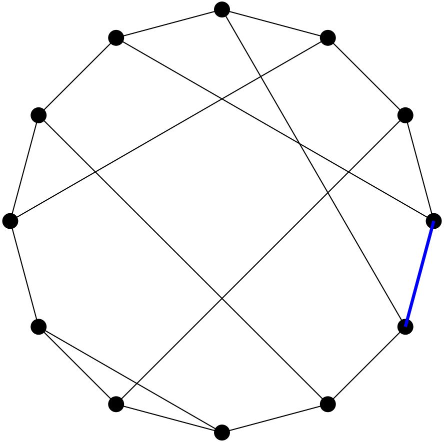
Known fact For large n, a giant component forms at time $t = \frac{1}{2}$.

Percolation on 3-regular graphs



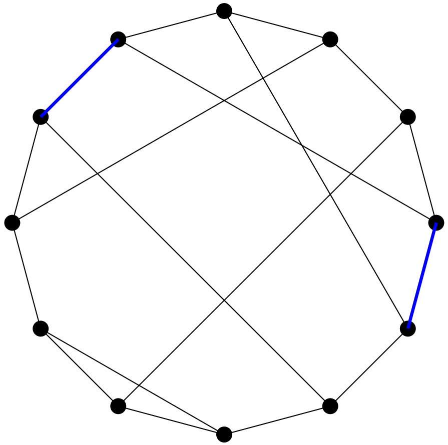
$t = 0$

Percolation on 3-regular graphs



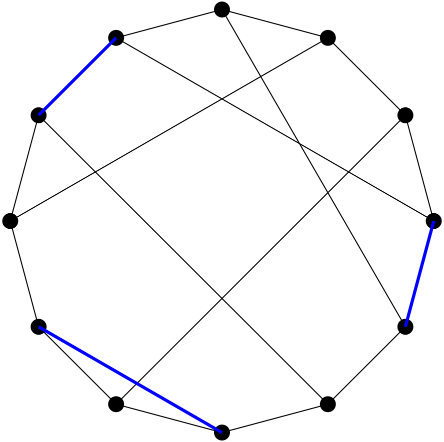
$t = 0.06$

Percolation on 3-regular graphs



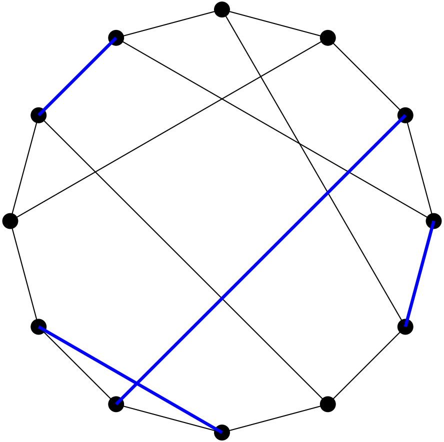
$$t = 0.11$$

Percolation on 3-regular graphs



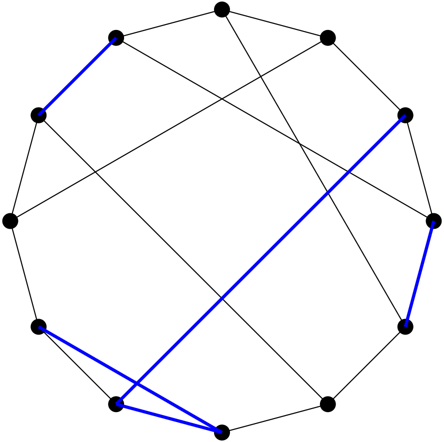
$t = 0.17$

Percolation on 3-regular graphs



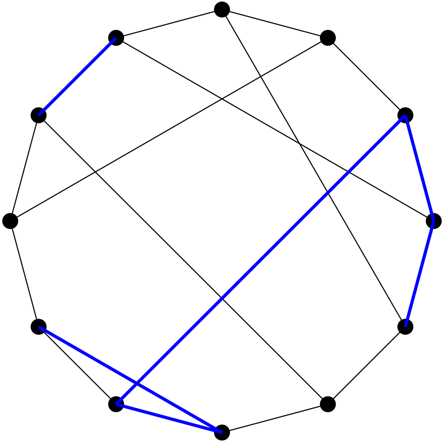
$t = 0.22$

Percolation on 3-regular graphs



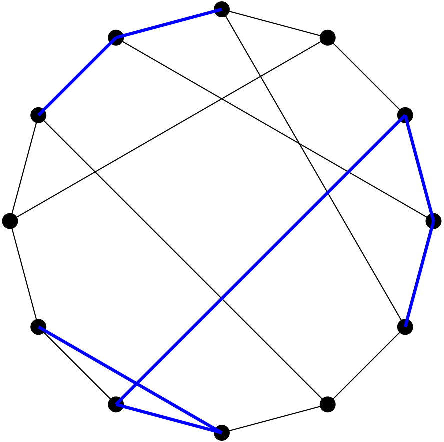
$t = 0.28$

Percolation on 3-regular graphs



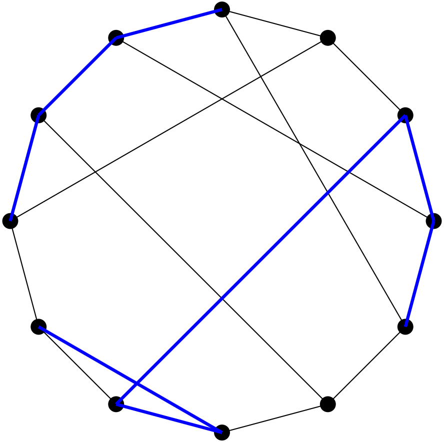
$t = 0.33$

Percolation on 3-regular graphs



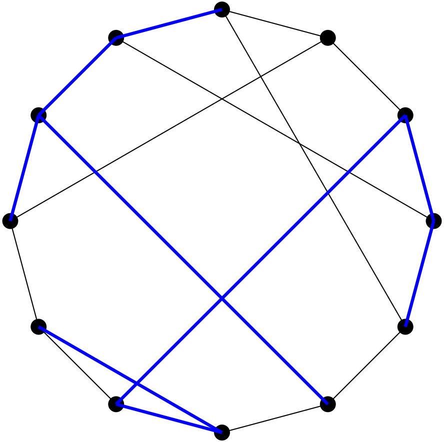
$t = 0.39$

Percolation on 3-regular graphs



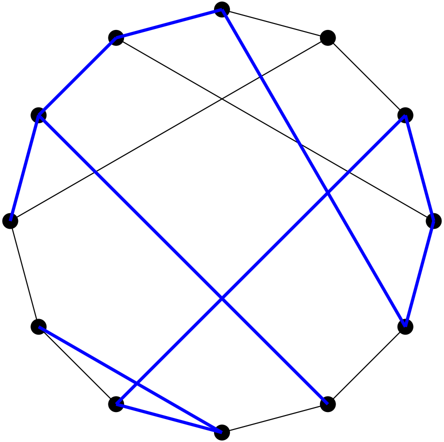
$t = 0.45$

Percolation on 3-regular graphs



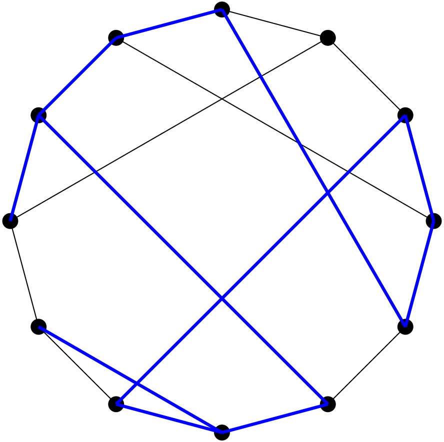
$t = 0.50$

Percolation on 3-regular graphs



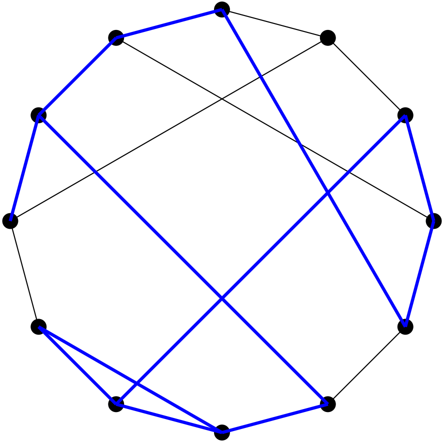
$t = 0.56$

Percolation on 3-regular graphs



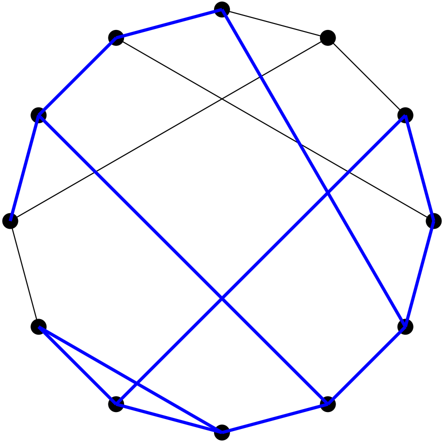
$t = 0.61$

Percolation on 3-regular graphs



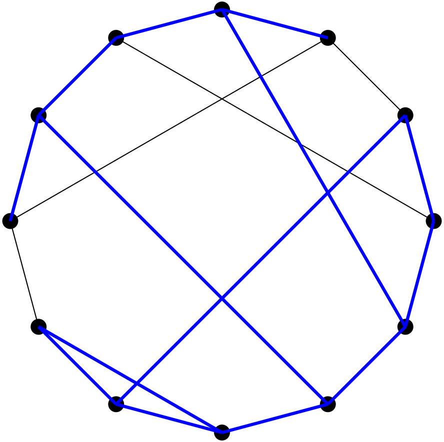
$t = 0.67$

Percolation on 3-regular graphs



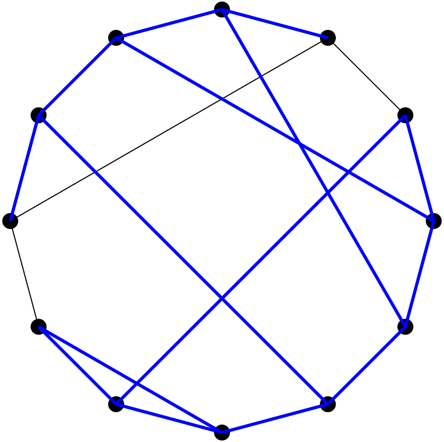
$t = 0.72$

Percolation on 3-regular graphs



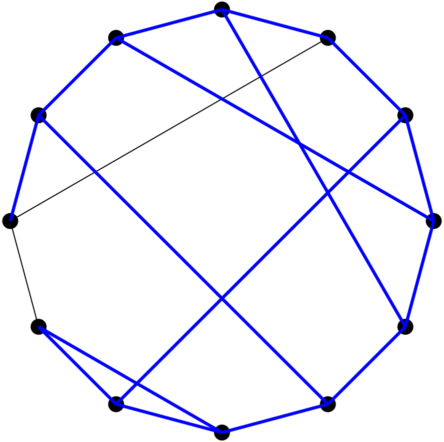
$t = 0.78$

Percolation on 3-regular graphs



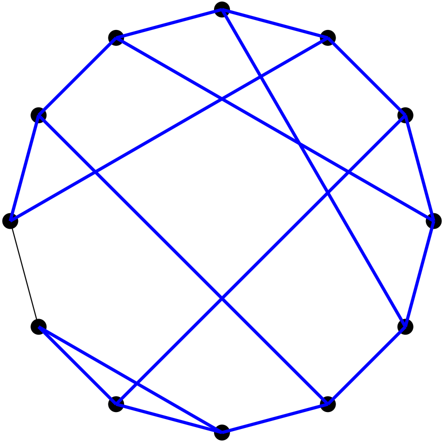
$t = 0.83$

Percolation on 3-regular graphs



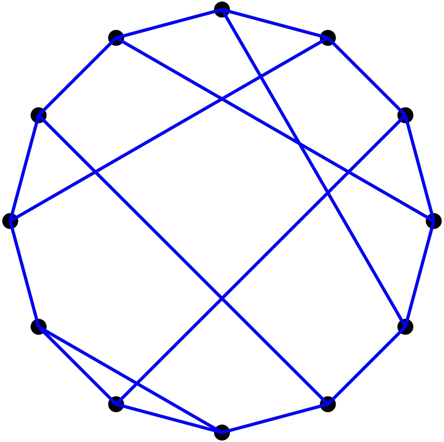
$t = 0.89$

Percolation on 3-regular graphs



$t = 0.93$

Percolation on 3-regular graphs



$t = 1$

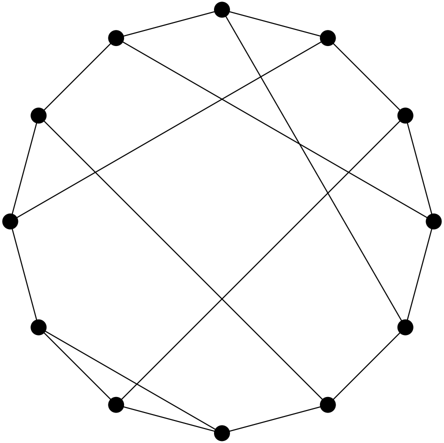
Frozen percolation on 3-regular graphs

Let $(\sigma_v)_{v \in V_n}$ be i.i.d. exponentially distributed times with mean $1/\lambda_n$, attached to the vertices.

- At time σ_v, all vertices in the open component containing v *freeze*.
- At time τ_e, the edge e opens *only* if neither of its endvertices is frozen.

We are interested in $n^{-1} \ll \lambda_n \ll 1$, which means that w.h.p., small components do not freeze, but giant components freeze immediately.

Frozen percolation on 3-regular graphs



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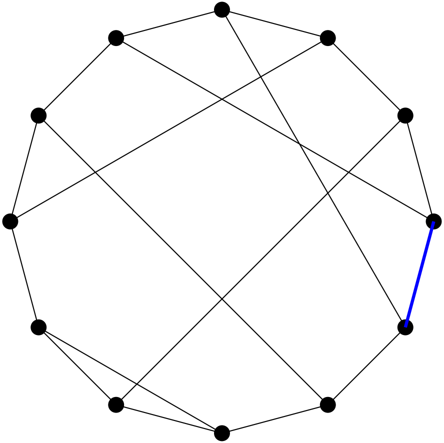

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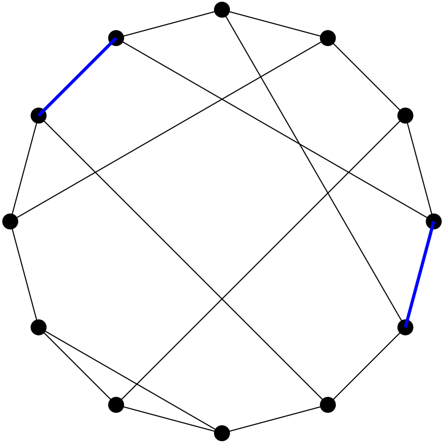

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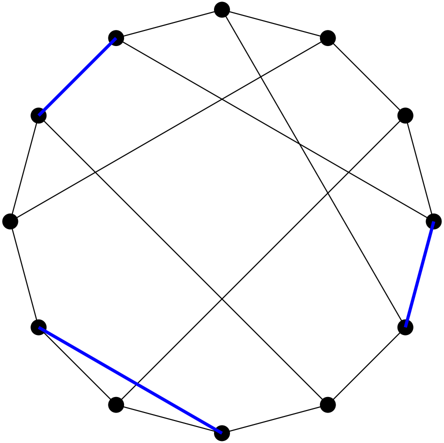

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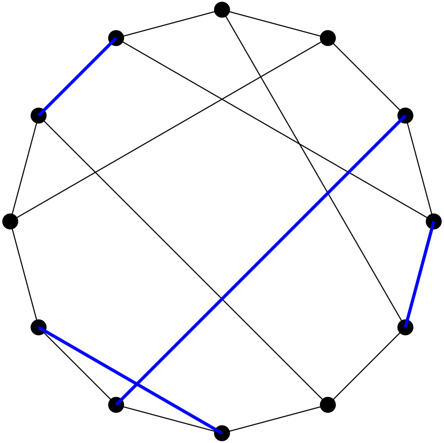

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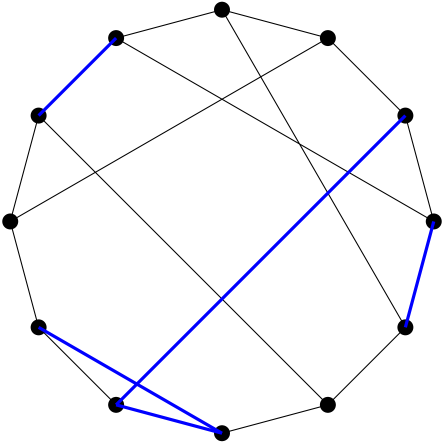

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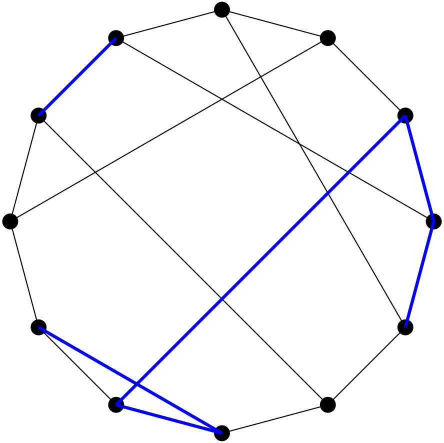

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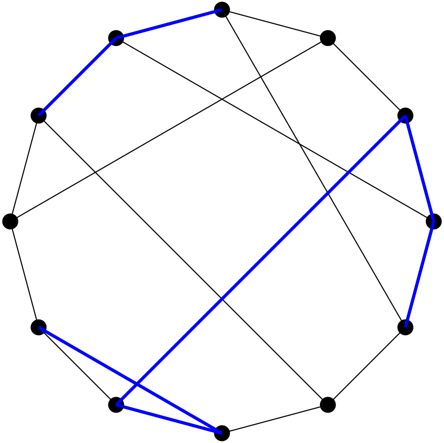

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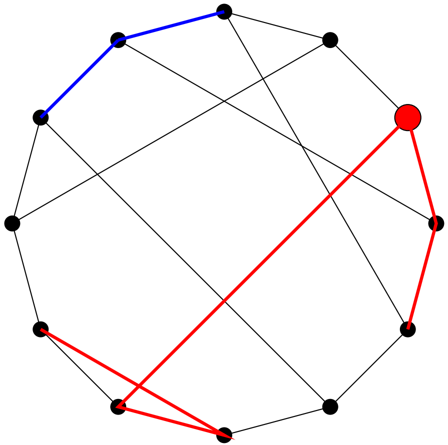

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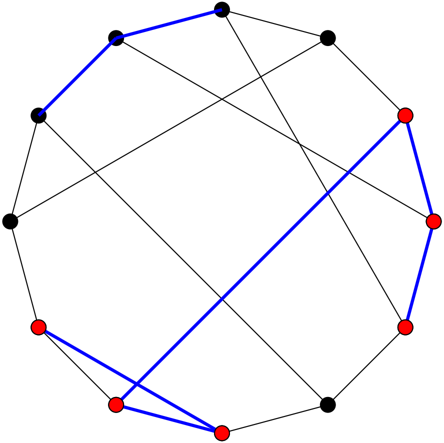

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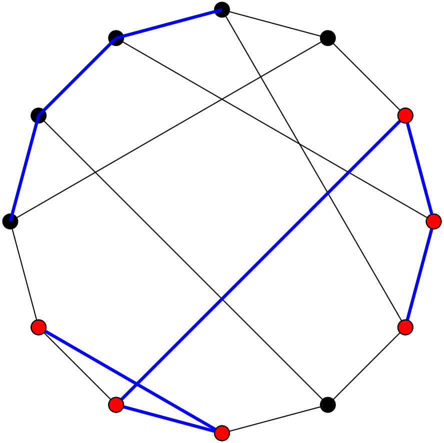

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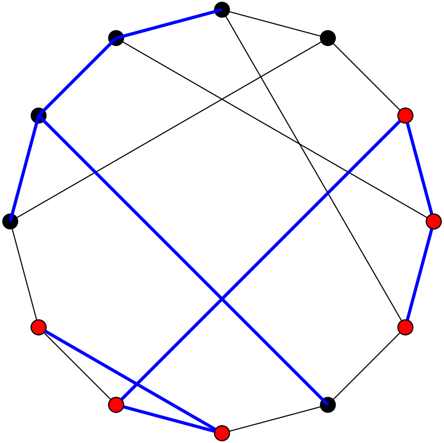

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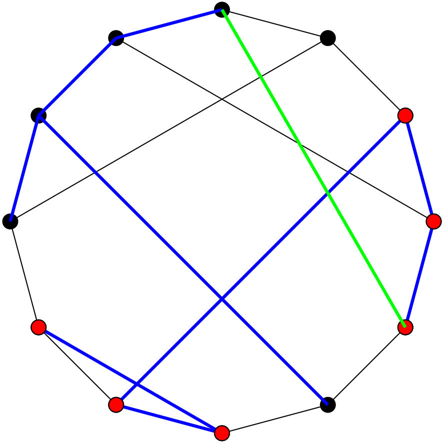

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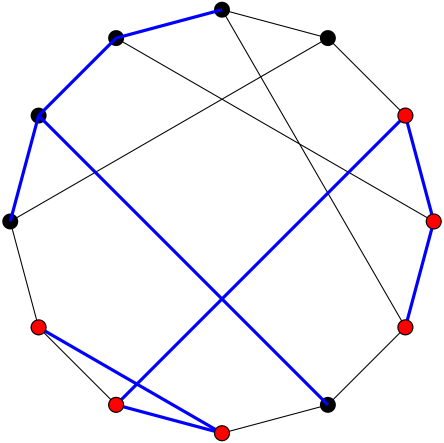

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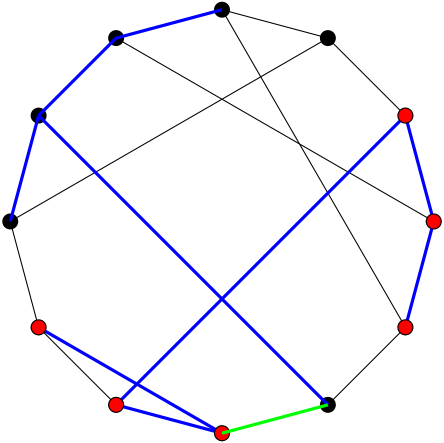

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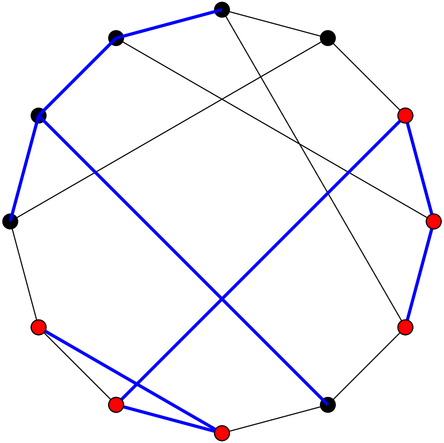

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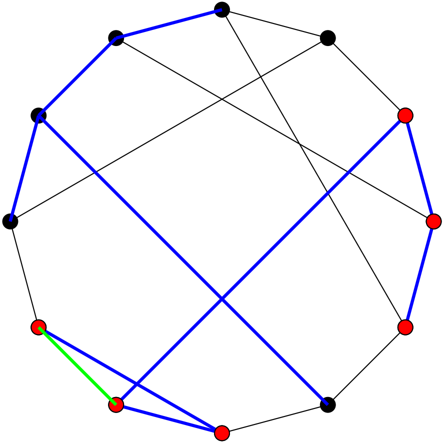

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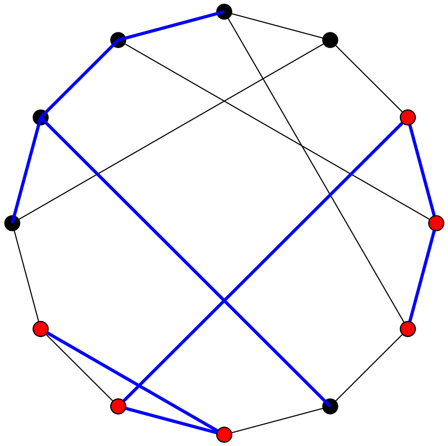

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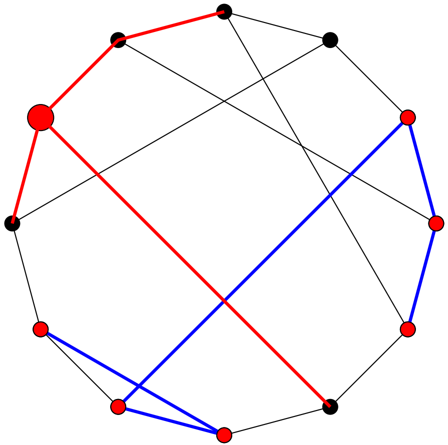

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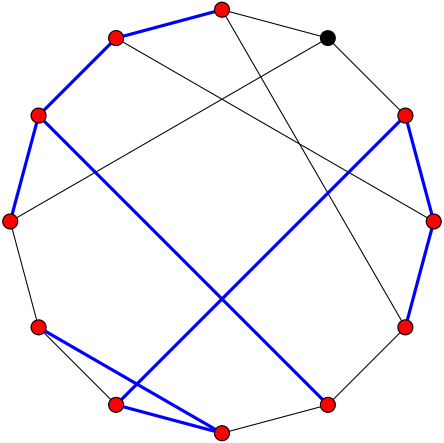

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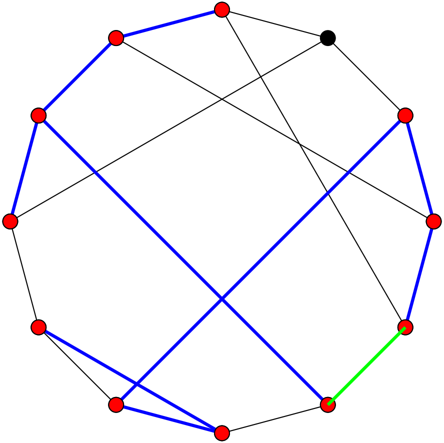

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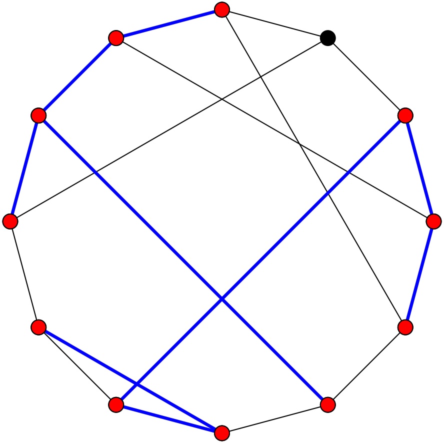


The *local weak limit* of G_n is the infinite 3-regular tree $G = (V, E)$.

Let $(\tau_e)_{e \in E}$ be i.i.d. uniformly distributed $[0, 1]$-valued times attached to the edges.

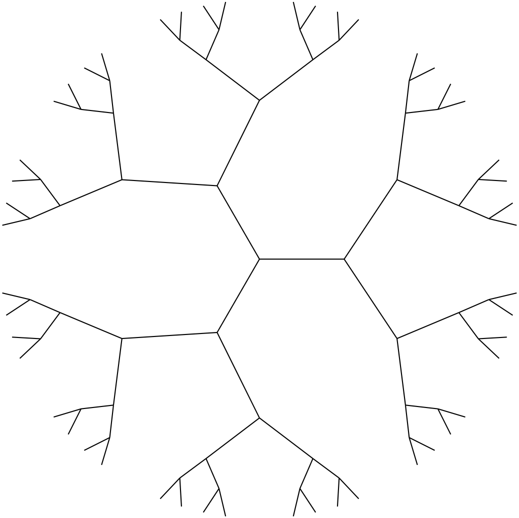
Aldous (2000) has constructed a process $(F_t)_{t \in [0,1]}$ of frozen vertices $F_t \subset V$ such that:

- As soon as an open component reaches infinite size, all its vertices are frozen.
- At time τ_e, the edge e opens if and only if neither of its endvertices is frozen.

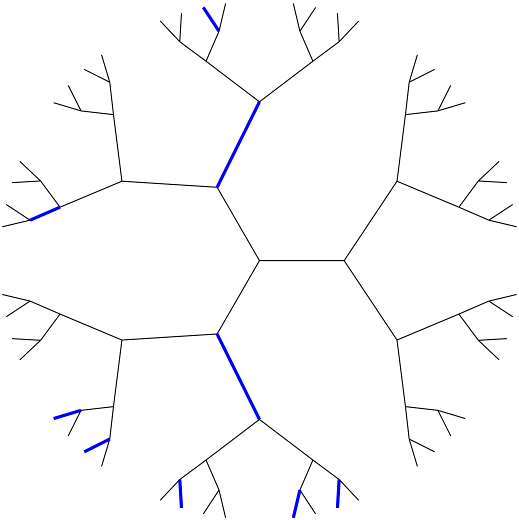
Question Given, $(\tau_e)_{e \in E}$, is $(F_t)_{t \in [0,1]}$ a.s. unique?

Short answer No.

Frozen percolation on the 3-regular tree



Frozen percolation on the 3-regular tree



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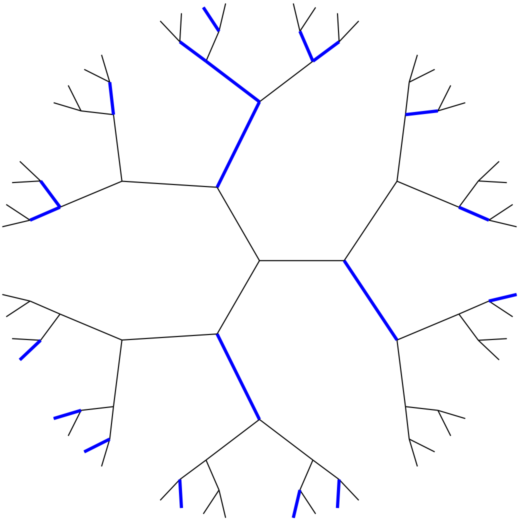

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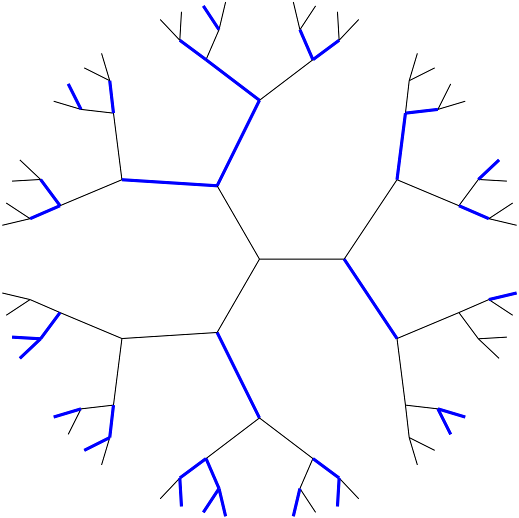

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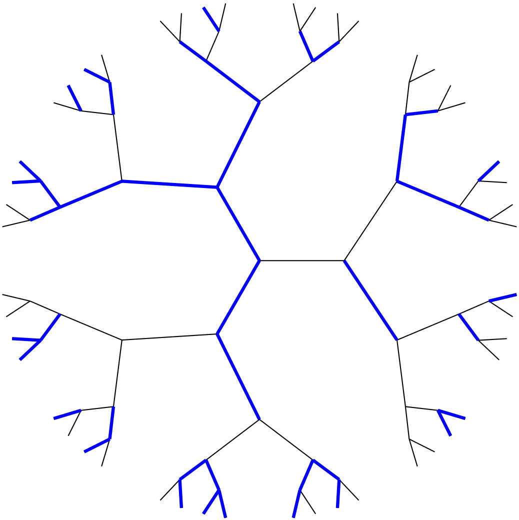

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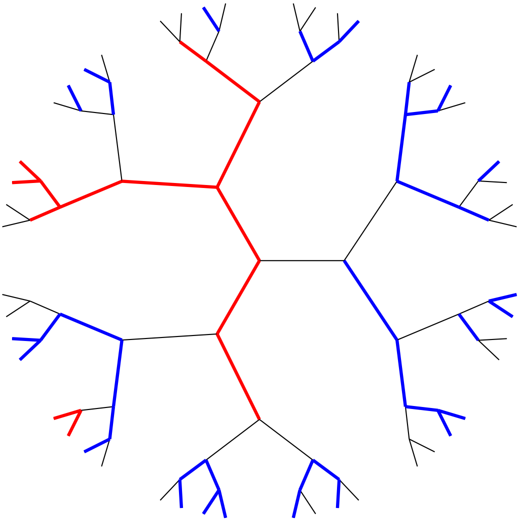

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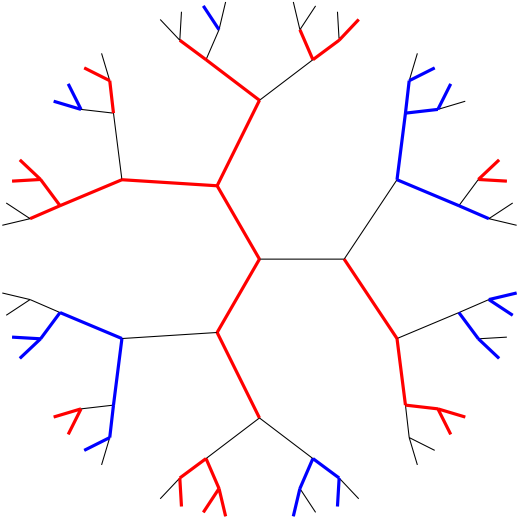

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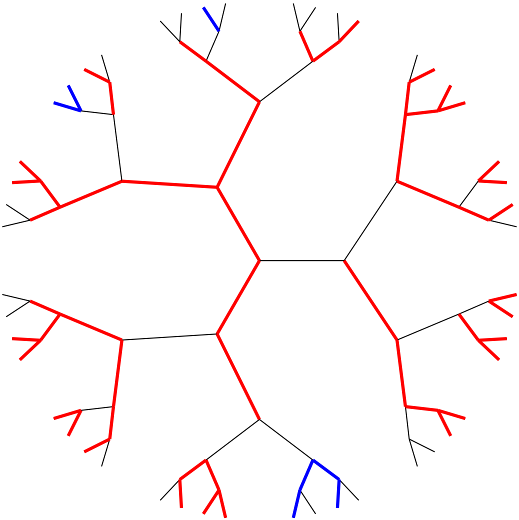

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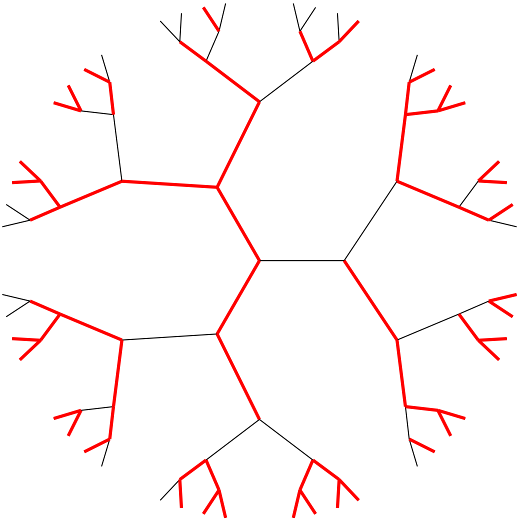


Remark Instead of a random 3-regular tree, we could have started with the complete graph.
In this case, it is more natural to take $(\tau_e)_{e\in E}$ uniformly $[0, n]$-valued.

Frozen percolation on the complete graph has been studied by Balázs Ráth (2009). Merle and Normand (2015) studied a configuration model with freezing.

The local limit of the complete graph equipped with i.i.d. times $(\tau_e)_{e\in E}$ is called the PWIT (Aldous & Steele, 2004).

Frozen percolation on the complete graph models the growth of polymers. Giant polymers are part of the gel and cannot grow further.
Related to the discrete Smoluchowski coagulation equation with a multiplicative kernel.

Exhibits *self-organized criticality*.

Frozen percolation on the oriented binary tree

Let $\mathbb{T}$ denote the space of all finite words $\mathbf{i} = i_1 \cdots i_n$ ($n \geq 0$) made up from the alphabet $\{1, 2\}$.

Elements $\mathbf{i} \in \mathbb{T}$ label oriented edges in an infinite binary tree.

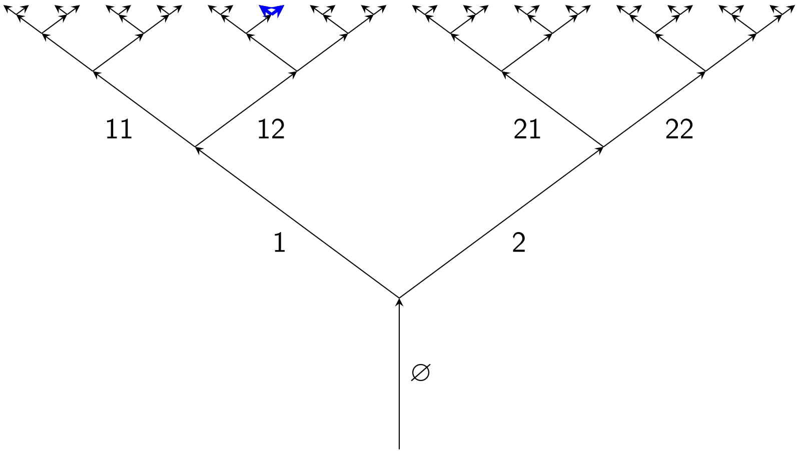
Let $(\tau_\mathbf{i})_{\mathbf{i} \in \mathbb{T}}$ be i.i.d. uniformly distributed $[0, 1]$-valued.
Aldous (2000) has constructed a process $(\vec{F}_t)_{t \in [0,1]}$ of frozen vertices $\vec{F}_t \subset \mathbb{T}$ such that:

- As soon as an infinite oriented path emerges, all its oriented edges freeze.
- At time $\tau_\mathbf{i}$, the oriented edge $\mathbf{i}$ opens if and only if neither of its descendants $\mathbf{i}1, \mathbf{i}2$ is frozen.

Unoriented frozen percolation on the 3-regular tree can be constructed from the oriented process.

Equivalent question Given, $(\tau_\mathbf{i})_{\mathbf{i} \in \mathbb{T}}$, is $(\vec{F}_t)_{t \in [0,1]}$ a.s. unique?

Frozen percolation on the oriented binary tree



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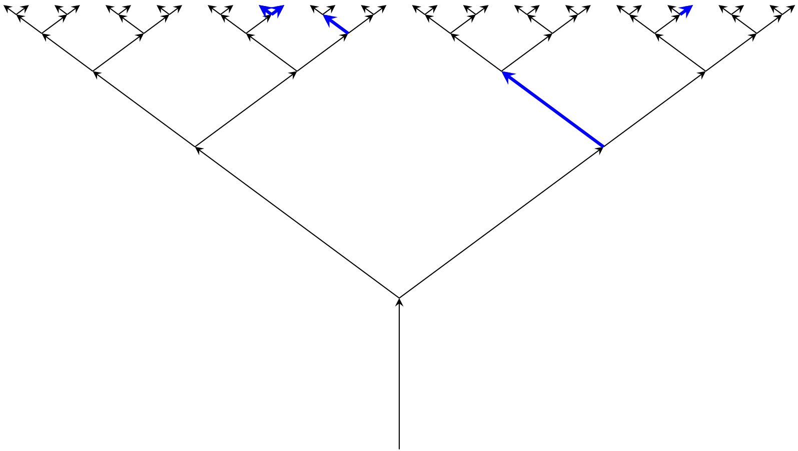

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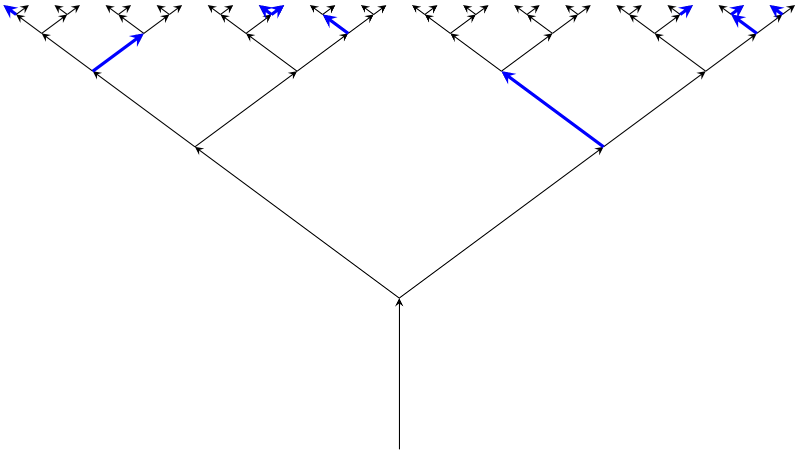

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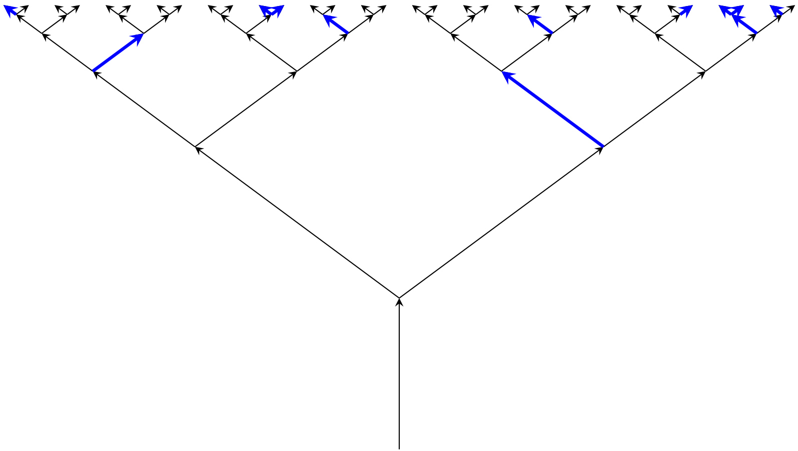

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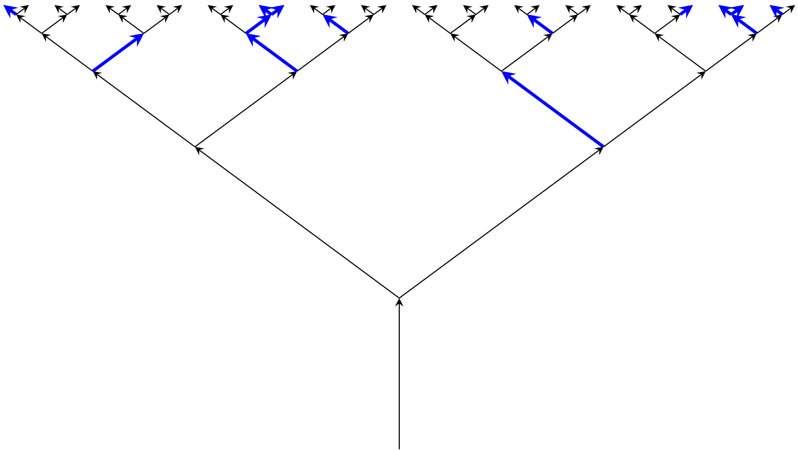

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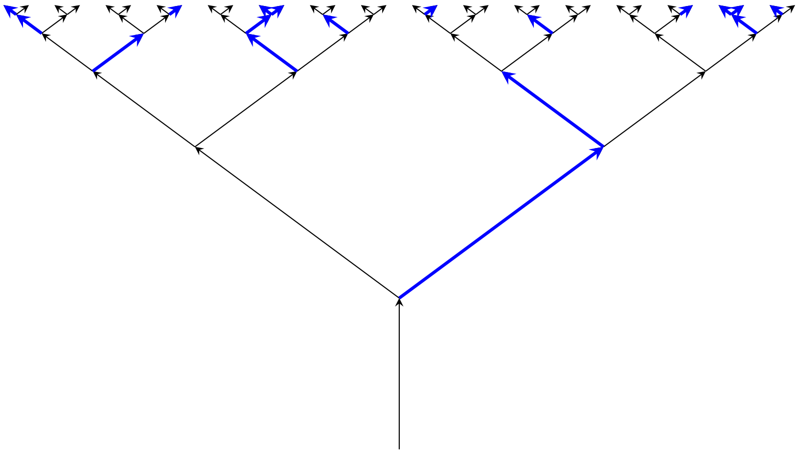

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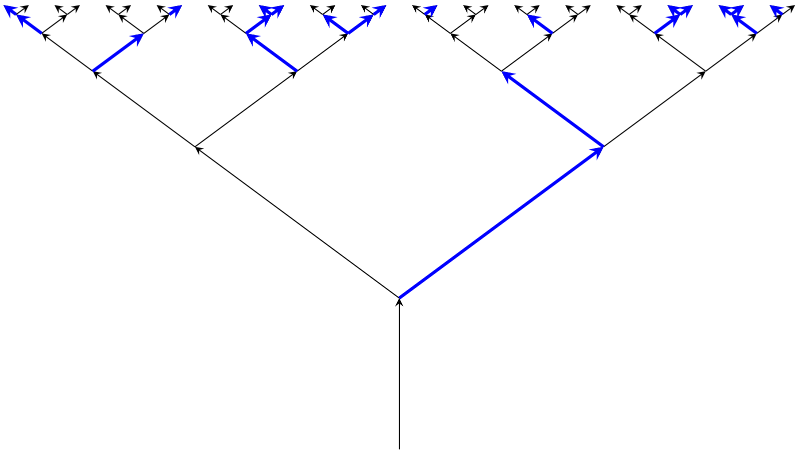

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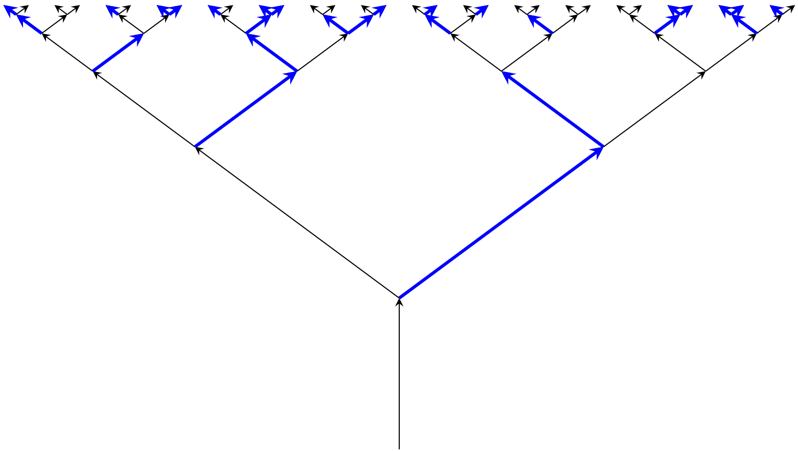

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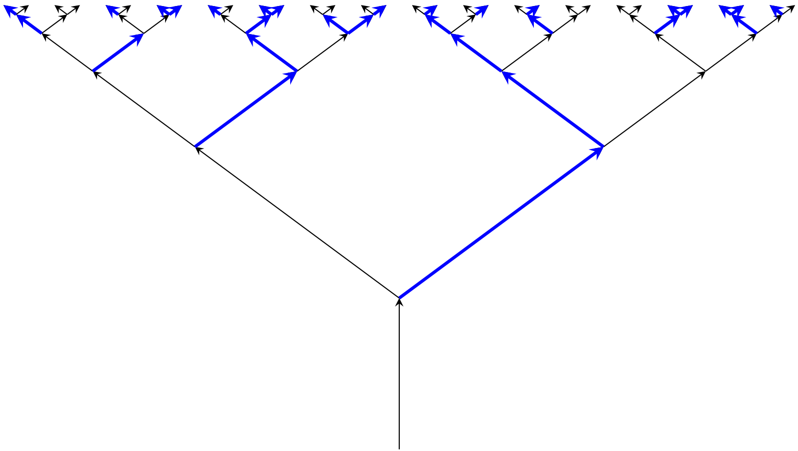

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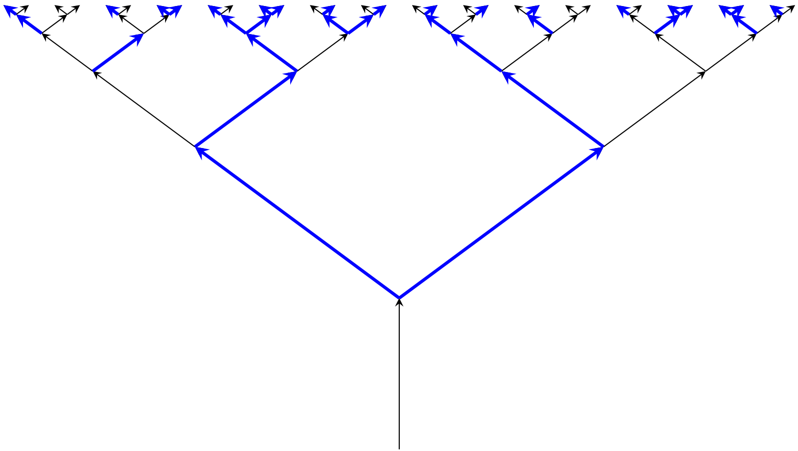
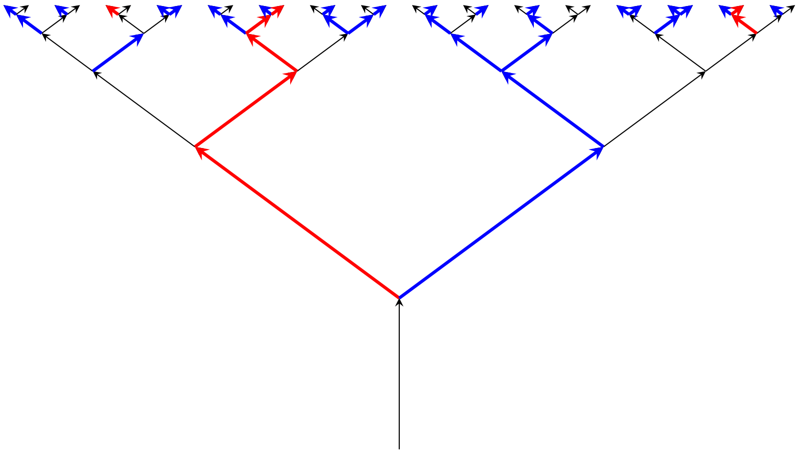


Frozen percolation on the oriented binary tree



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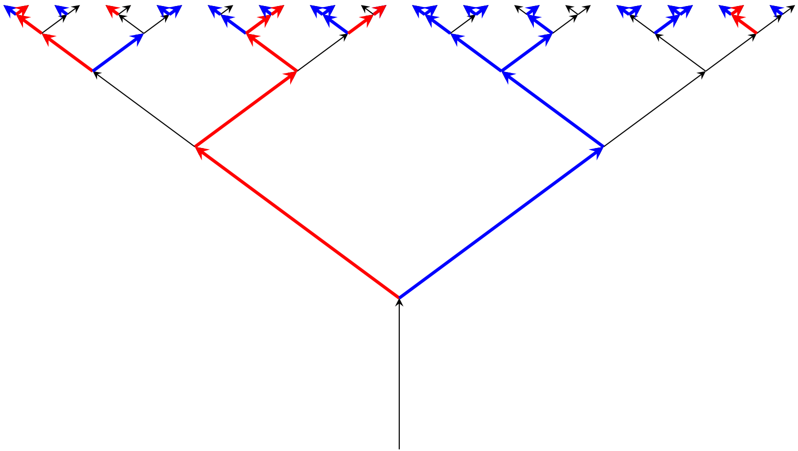

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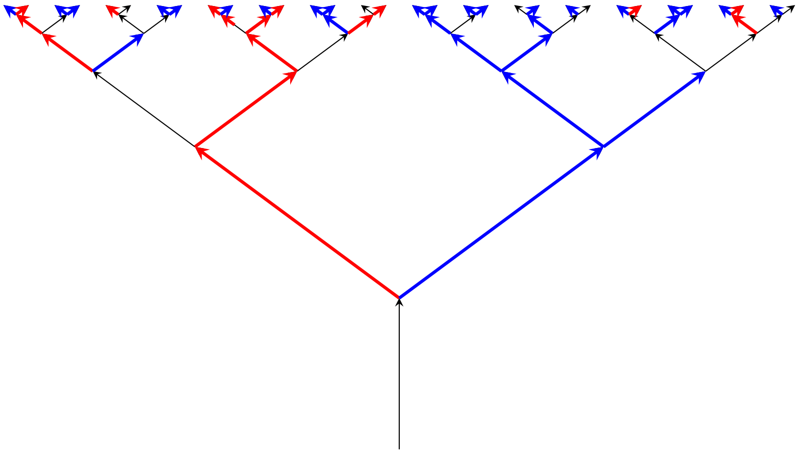
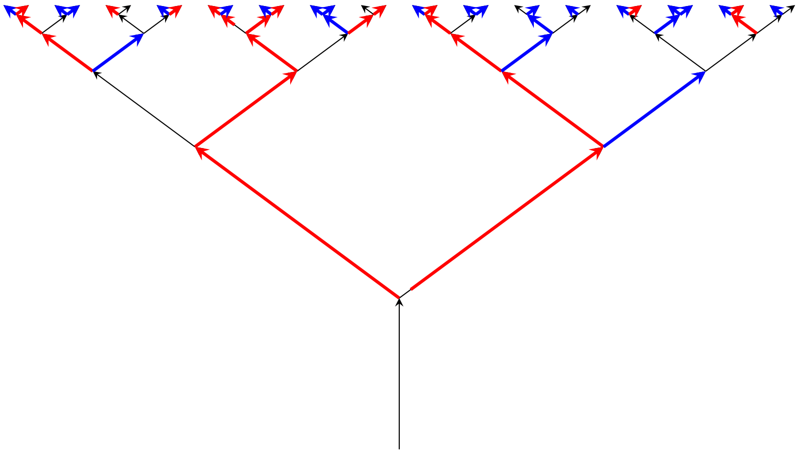


Frozen percolation on the oriented binary tree



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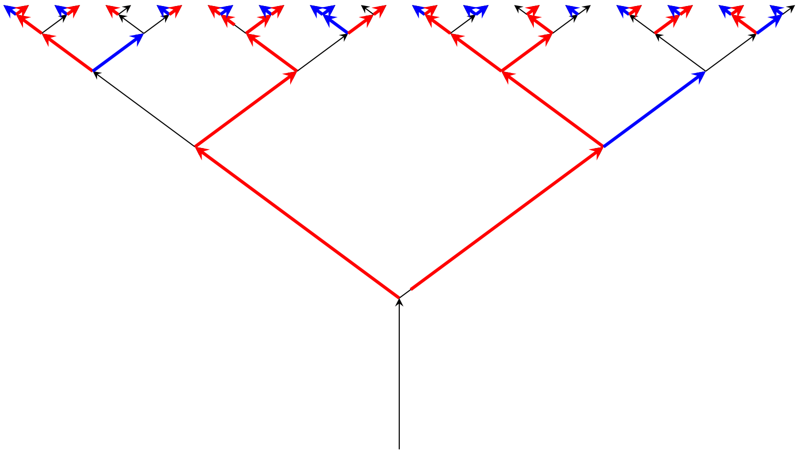
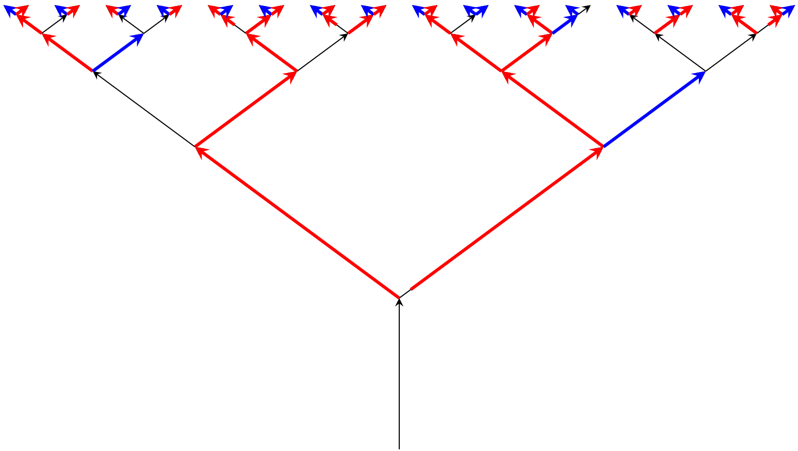
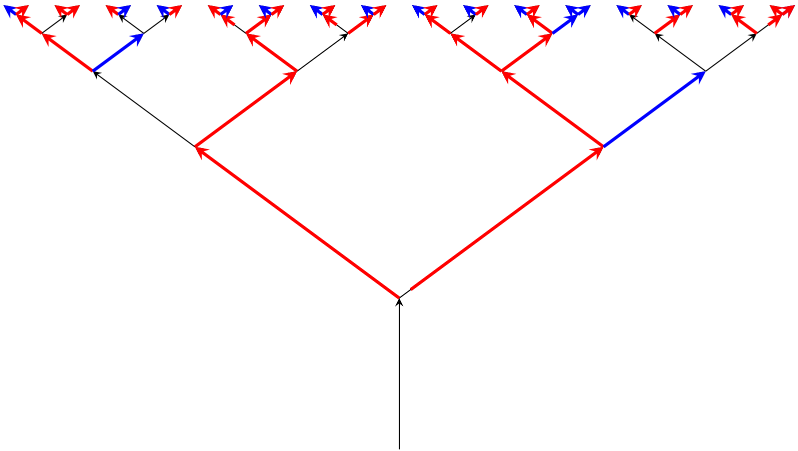


Frozen percolation on the oriented binary tree



Frozen percolation on the oriented binary tree



Frozen percolation on the oriented binary tree

Let

$$X_{\mathbf{i}} := \inf\big\{t \in [0,1] : \mathbf{i} \text{ is part of an infinite open path}\big\},$$

with $X_{\mathbf{i}} := \infty$ if this never happens. The $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ solve the inductive relation

$$X_{\mathbf{i}} = \Phi[\tau_{\mathbf{i}}](X_{\mathbf{i}1} \wedge X_{\mathbf{i}2}) \qquad (\mathbf{i} \in \mathbb{T}),$$

where Φ is the function

$$\Phi[t](x) := \begin{cases} x & \text{if } x > t, \\ \infty & \text{if } x \leq t. \end{cases}$$

A Recursive Tree Process

Let ν be the probability law on $I := [0,1] \cup \{\infty\}$ defined by

$$\nu(\mathrm{d}x) := \frac{\mathrm{d}x}{2x^2} 1_{[\frac{1}{2},1]}(x) \qquad \nu(\{\infty\}) := \tfrac{1}{2}. \tag{1}$$

Aldous (2000) has shown that ν solves the *Recursive Distributional Equation* (RDE)

$$X \stackrel{\mathrm{d}}{=} \Phi[\tau](X_1 \wedge X_2),$$

where X has law ν, X_1, X_2 are i.i.d. copies of X, and τ is independent uniform $[0,1]$-valued.

By Kolmogorov's extension theorem, there exists a *Recursive Tree Process* (RTP) $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, unique in law, such that

(i) For each finite rooted subtree $\mathbb{U} \subset \mathbb{T}$, the r.v.'s $(X_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{U}}$ are i.i.d. with common law ν and independent of $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$.

(ii) $X_{\mathbf{i}} = \Phi[\tau_{\mathbf{i}}](X_{\mathbf{i}1} \wedge X_{\mathbf{i}2}) \qquad (\mathbf{i} \in \mathbb{T})$.

Endogeny

Def The RTP is *endogenous* if $X_\varnothing$ is measurable w.r.t. the σ-field generated by $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$.

Def *bivariate map*

$$T^{(2)}(\mu^{(2)}) := \text{ the law of } \big(\Phi[\tau](X_1 \wedge X_2), \Phi[\tau](X'_1 \wedge X'_2)\big),$$

where $(X_1, X'_1), (X_2, X'_2)$ are i.i.d. with law $\mu^{(2)}$ and τ is independent uniform $[0, 1]$-valued.

Let $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to ν.
Let $(X'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be a copy of $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, conditionally independent given $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. Then

$$\underline{\nu}^{(2)} := \mathbb{P}\big[(X_\varnothing, X'_\varnothing) \in \cdot\,\big],$$
$$\overline{\nu}^{(2)} := \mathbb{P}\big[(X_\varnothing, X_\varnothing) \in \cdot\,\big],$$

solve the *bivariate RDE* $T^{(2)}(\nu^{(2)}) = \nu^{(2)}$.

Endogeny

Def $\mathcal{P}(I^2)_\nu =$ space of probability laws on I^2 whose one-dimensional marginals are given by ν.

Theorem (Aldous & Bandyopadhyay 2005)
The following statements are equivalent:

(i) The RTP $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is endogenous.

(ii) $\underline{\nu}^{(2)} = \overline{\nu}^{(2)}$.

(iii) The bivariate map $T^{(2)}$ has a unique fixed point in $\mathcal{P}(I^2)_\nu$.

(iv) $(T^{(2)})^n(\mu^{(2)}) \underset{n\to\infty}{\Longrightarrow} \overline{\nu}^{(2)}$ for all $\mu^{(2)} \in \mathcal{P}(I^2)_\nu$.

Moreover, $(T^{(2)})^n(\nu\otimes\nu) \underset{n\to\infty}{\Longrightarrow} \underline{\nu}^{(2)}$.

Reformulation of the problem To prove that frozen percolation is *not* a.s. unique, it suffices to find a nontrivial solution $\nu^{(2)} \neq \overline{\nu}^{(2)}$ to the bivariate RDE.

Endogeny

History of the problem

Aldous (2000) conjectured a.s. uniqueness (i.e., endogeny).

Bandyopahyay (2004), arXiv:math/0407175 announced a false proof.

Bandyopahyay (2005) numerical simulations
$(T^{(2)})^n(\nu \otimes \nu) \underset{n\to\infty}{\Longrightarrow} \underline{\nu}^{(2)} \neq \overline{\nu}^{(2)}$.

Antar Bandyopahyay, Tamás Terpai, and especially Balázs Ráth pursued the problem for many years...

Theorem (2019) Endogeny does *not hold*.

Proof The problem can be translated into frozen percolation on the MBBT, which is easier to handle.

The Marked Binary Branching Tree

Def The *Marked Binary Branching Tree* (MBBT) is a pair $(\mathcal{T}, \Pi)$ with:

- $\mathcal{T}$ is the family tree of a rate one continuous-time binary branching process.
- Π is a rate one Poisson process on $\mathcal{T} \times [0,1]$.

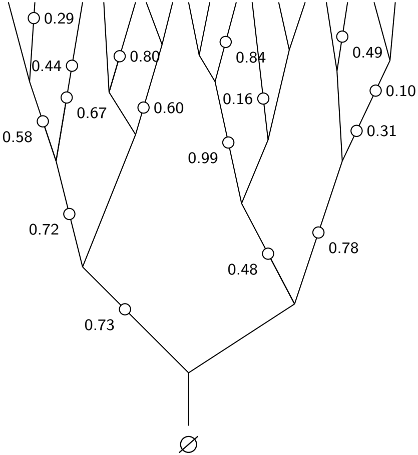
Def $\Pi_t := \{(z,\tau) \in \Pi : \tau > t\}$.

Equivalently, $\Pi = \{(z,\tau_z) : z \in \Pi_0\}$, where:

- Π_0 is a rate one Poisson process on $\mathcal{T}$,
- $(\tau_z)_{z\in\Pi_0}$ are i.i.d. uniform $[0,1]$-valued.

Interpretation Initially, points in Π_0 are closed. At time τ_z, the point z opens. Π_t set of closed points at time t.

The Marked Binary Branching Tree



The Marked Binary Branching Tree

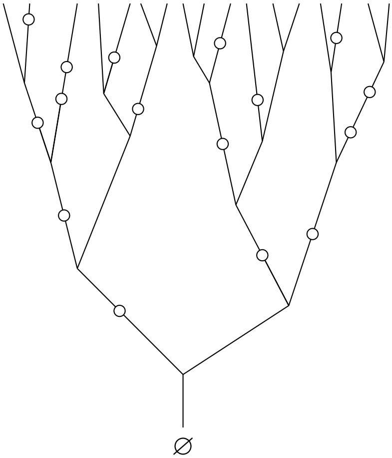
The MBBT is the *universal scaling limit* of near-critical percolation on trees.

Related to this, the MBBT itself enjoys a form of *scale invariance*: Write $z \stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow} \infty$ if at time t there is an open upward path starting at z.
Then

$$\mathcal{T}' := \{z \in \mathcal{T} : \varnothing \stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow} z \stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow} \infty\}$$

is the family tree of a rate t binary branching process.
Moreover, $\Pi' := \big\{(z, \tau_z) \in \Pi : z \in \mathcal{T}\big\}$ is a rate one Poisson process on $\mathcal{T}' \times [0, t]$.

The Marked Binary Branching Tree



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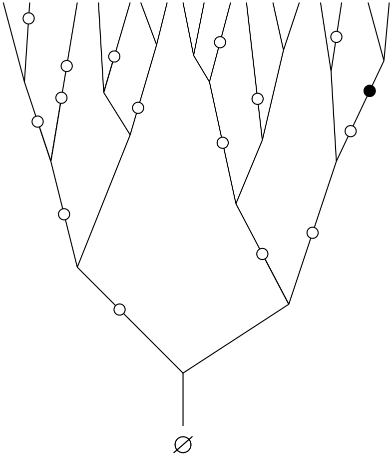

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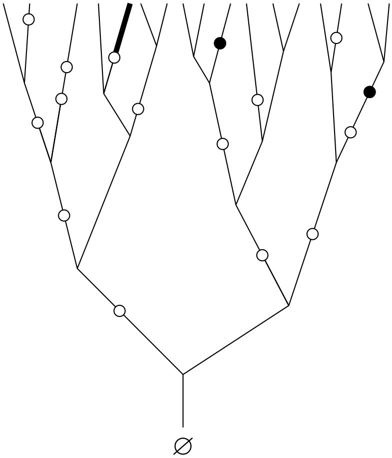

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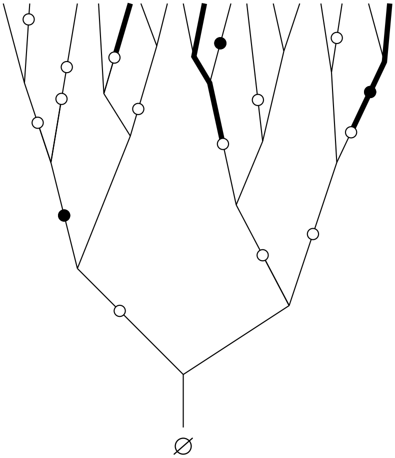

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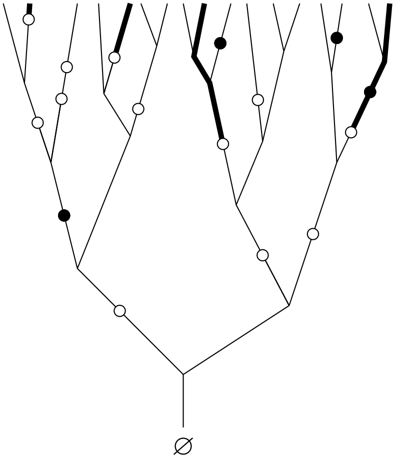

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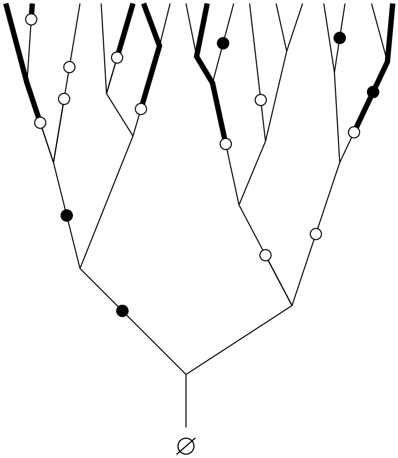

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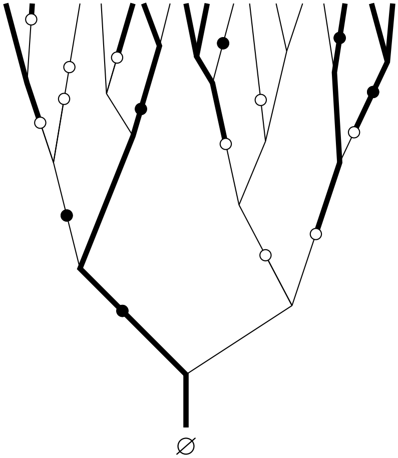

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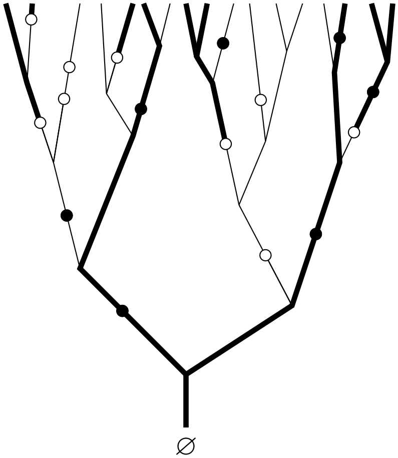

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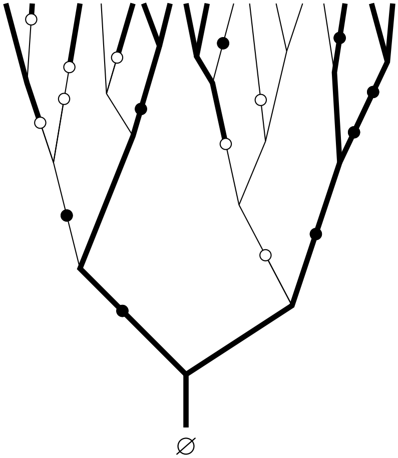

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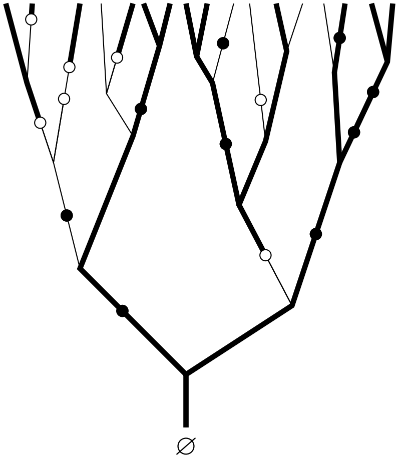

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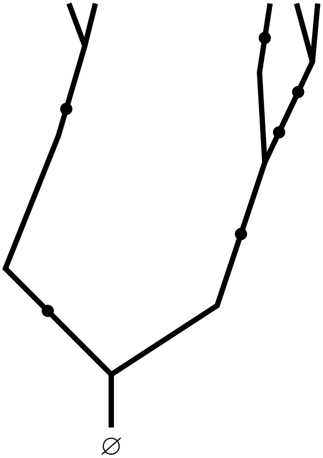

It is possible to construct frozen percolation on the MBBT such that:

At time $t = \tau_z$, the point z opens unless $z \xrightarrow{\mathcal{T}\backslash\Pi_t} \infty$.

Let $Y_\varnothing := \inf\big\{t \in [0,1] : \varnothing \xrightarrow{\mathcal{T}\backslash\Pi_t} \infty\big\}$ and $:= \infty$ if this never happens.
Then

$$\rho\big([0,t]\big) := \mathbb{P}[Y_\varnothing \le t] = \tfrac{1}{2}t \qquad \big(t \in [0,1]\big).$$

Lemma The corresponding $\underline{\rho}^{(2)}$ has the scaling property

$$\mathbb{P}\big[(Y_\varnothing, Y'_\varnothing) \in [0,tr] \times [0,ts]\big] = t\mathbb{P}\big[(Y_\varnothing, Y'_\varnothing) \in [0,r] \times [0,s]\big]$$

$(0 \le r, s, t \le 1)$.

Frozen percolation on the MBBT

Theorem For frozen percolation on the MBBT, the bivariate map has precisely two scale-invariant fixed points.
A scale invariant law $\rho^{(2)}$ on I^2 solves the bivariate RDE if and only if the function

$$F(u) := \rho^{(2)}\big(\{(y_1, y_2) \in I^2 : y_1 > u,\ y_2 \leq 1\}\big) \qquad (0 \leq u \leq 1)$$

solves the differential equation

$$\text{(i) } \frac{\partial}{\partial u} F(u) = \frac{cu}{F(u)} - \tfrac{1}{2} \qquad \big(u \in [0,1]\big),$$
$$\text{(ii) } F(0) = \tfrac{1}{2}, \qquad \text{(iii) } F(1)^2 + \tfrac{1}{2}F(1) = 2c$$

for some $c \geq 0$. There are two values $0 = \overline{c} < \underline{c} < \frac{1}{4}$ for which this equation has a solution, corresponding to $\overline{\rho}^{(2)}$ and $\underline{\rho}^{(2)}$.