Peierls bounds from Toom contours

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Toom's model

Assume that $\varphi : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}$ is monotone and depends on finitely many coordinates. Example: the North East Center majority rule on $\mathbb{Z}^2$:

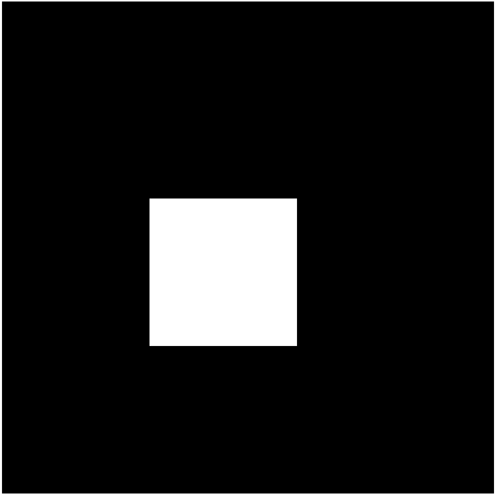
$$\varphi^{\rm NEC}(x) := \begin{cases} 1 & \text{if } x(0,0)+x(0,1)+x(1,0) \geq 2, \\ 0 & \text{if } x(0,0)+x(0,1)+x(1,0) \leq 1. \end{cases}$$

We are interested in the cellular automaton $(X_n)_{n\geq 0}$ that evolves

$$X_{n+1}(i) = \begin{cases} \varphi\big((X_n(i+j))_{j\in\mathbb{Z}^d}\big) & \text{with probability } 1-p, \\ 0 & \text{with probability } p, \end{cases}$$

independently for all $n \geq 0$ and $i \in \mathbb{Z}^d$.

Toom's model



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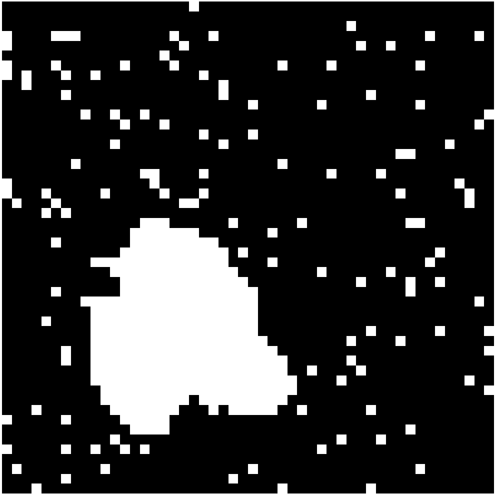

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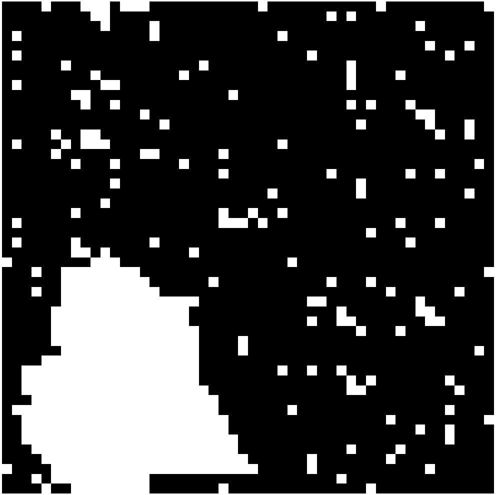

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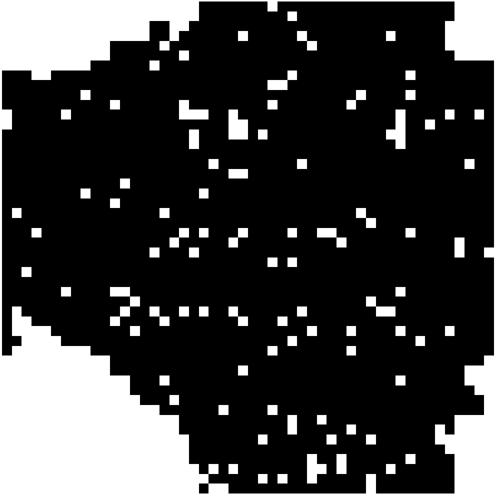

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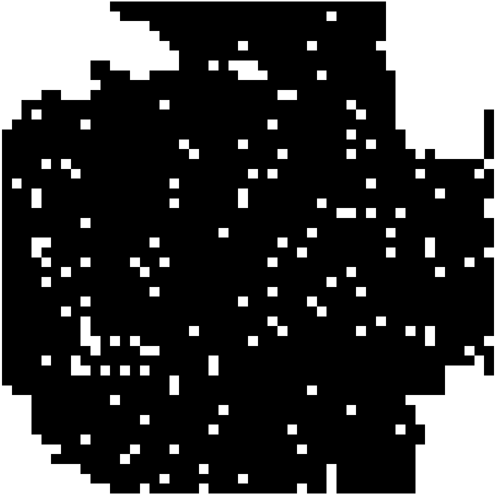

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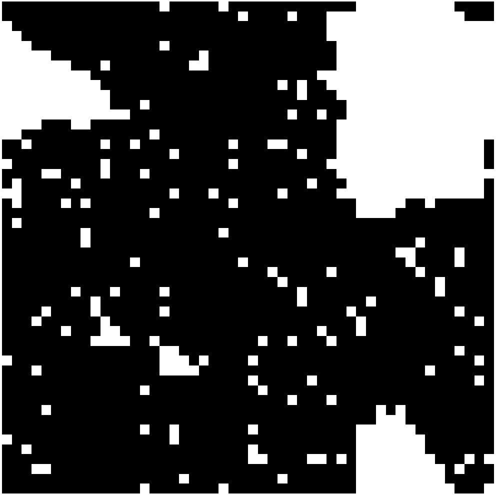

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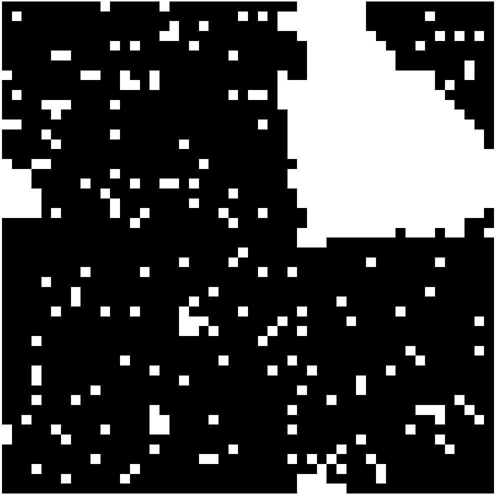

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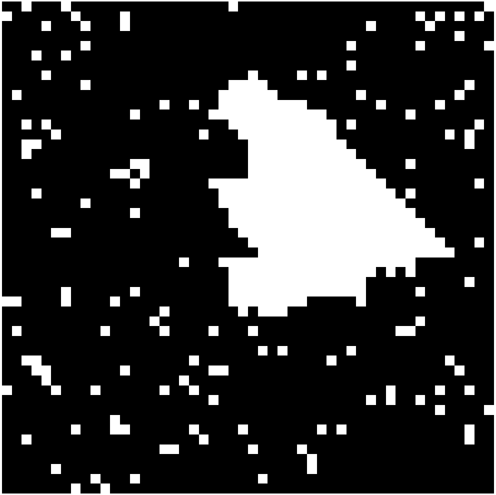

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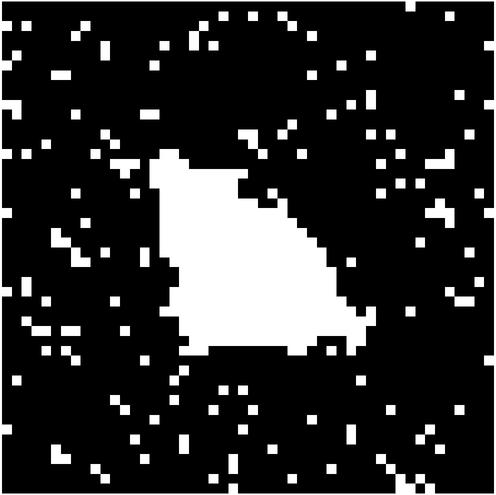

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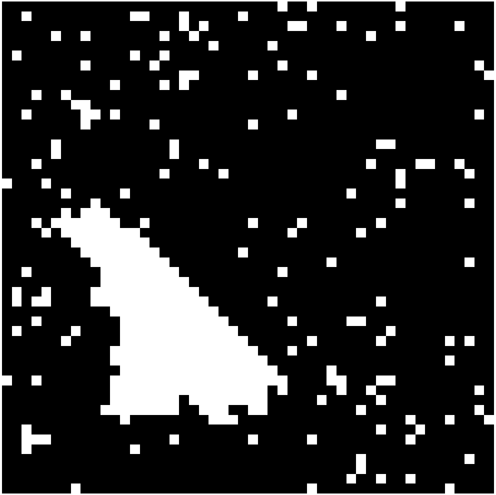

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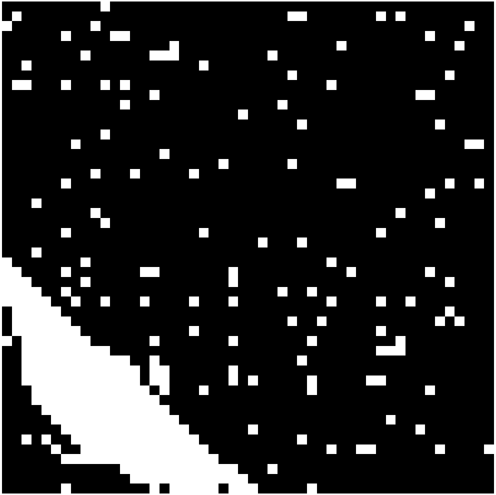

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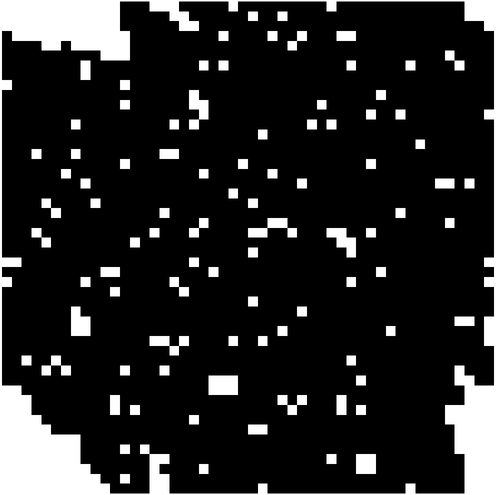

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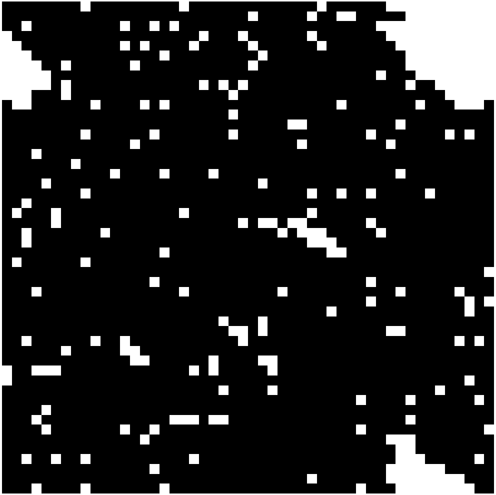

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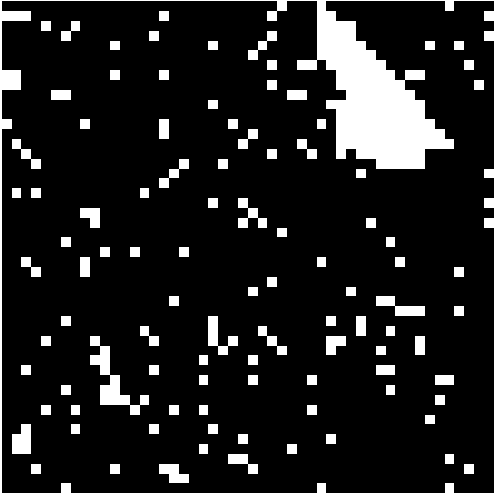

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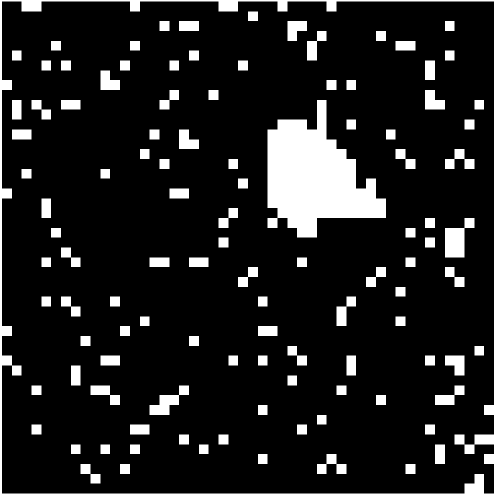

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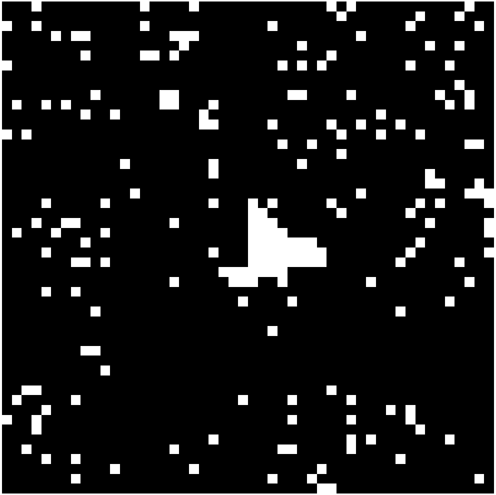

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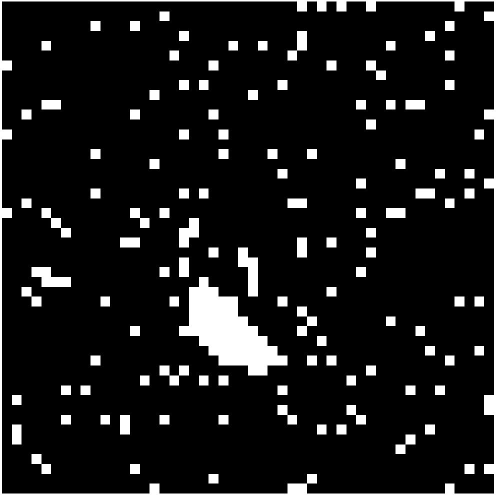

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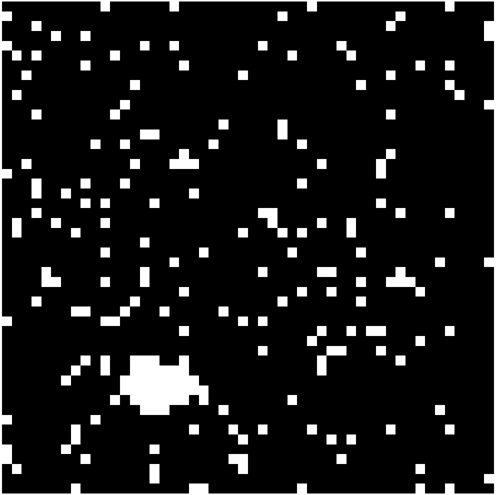

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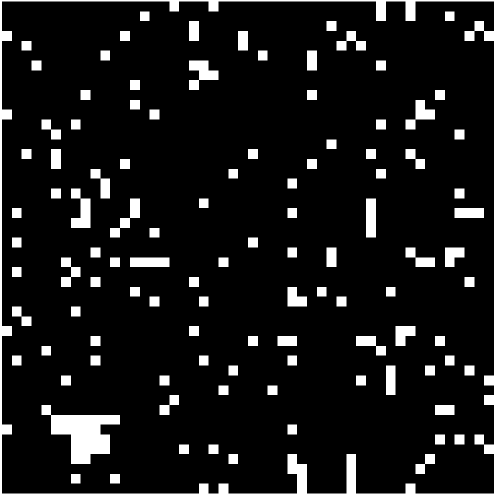

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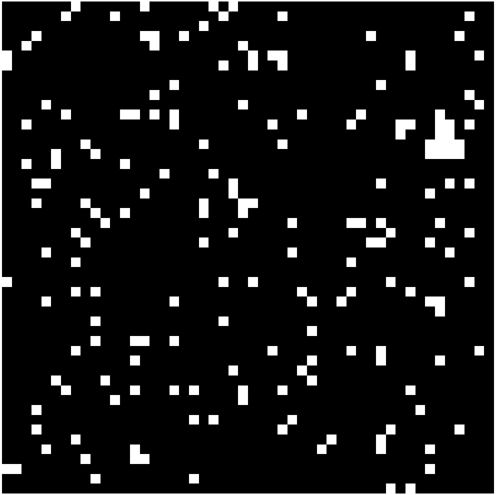

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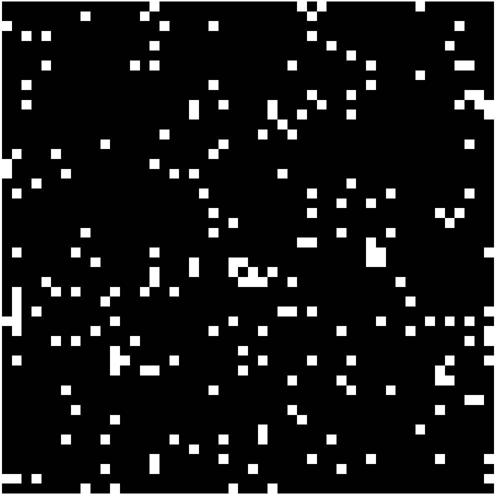


Toom's model

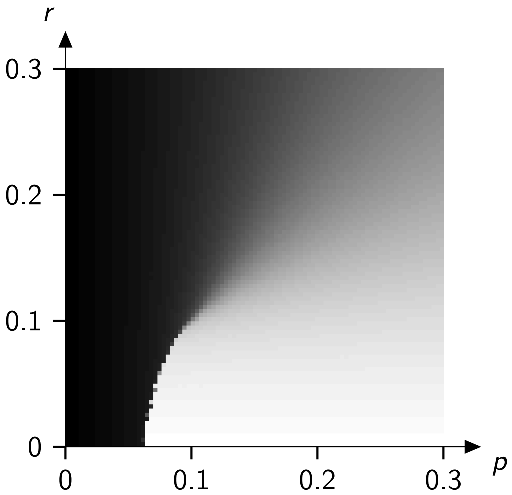
We can generalise a bit and let

$$X_{n+1}(i) = \begin{cases} \varphi\big((X_n(i+j))_{j\in\mathbb{Z}^d}\big) & \text{with probability } 1-p-r, \\ 0 & \text{with probability } p, \\ 1 & \text{with probability } r. \end{cases}$$

Let $\rho(p,r)$ denote the density of the upper invariant law.

$$\textbf{Toom (1980)} \ \lim_{p\to 0} \rho(p,0) = 1.$$

Toom's model



Density of the upper invariant law for Toom's model.

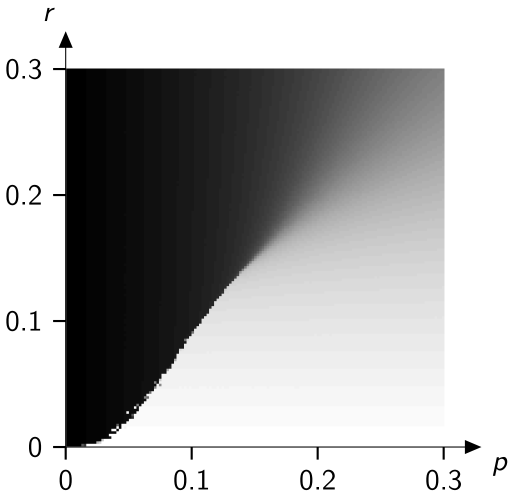
Nearest neighbour voting

The *Nearest Neighbor voting map* is defined as

$$\varphi^{\mathrm{NN}}(x) := \begin{cases} 1 & \text{if} \quad x(0,0)+x(0,1)+x(1,0) \\ & \qquad +x(0,-1)+x(-1,0) \geq 3, \\ 0 & \text{if} \quad x(0,0)+x(0,1)+x(1,0) \\ & \qquad +x(0,-1)+x(-1,0) \leq 2. \end{cases}$$

Toom (1980) $\rho(p,0) = 0$ for all $p > 0$.

Nearest neighbour voting



Density of the upper invariant law for nearest neighbour voting.

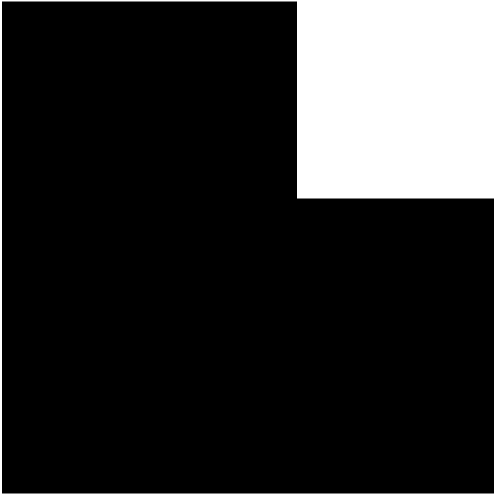
Continuous time

Consider the interacting particle system $(X_t)_{t\geq 0}$ with

$$X_t(i) \mapsto \begin{cases} \varphi^{\mathrm{NEC}}\big((X_t(i+j))_{j\in\mathbb{Z}^d}\big) & \text{with rate } 1, \\ 0 & \text{with rate } p. \end{cases}$$

Gray (1999) $\displaystyle\lim_{p\to 0} \rho(p) = 1.$

Toom's model in continuous time



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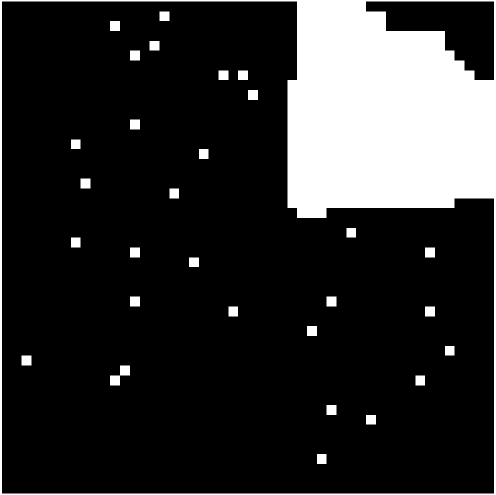

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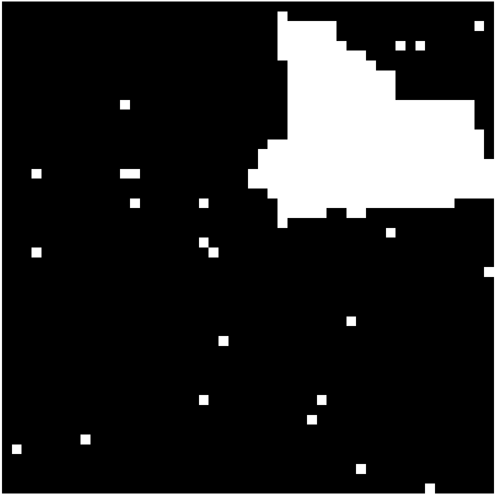

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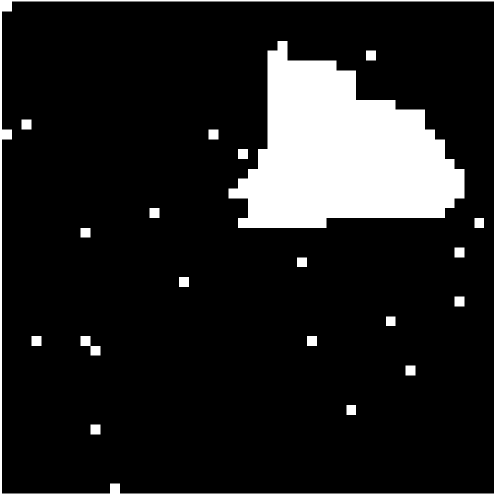

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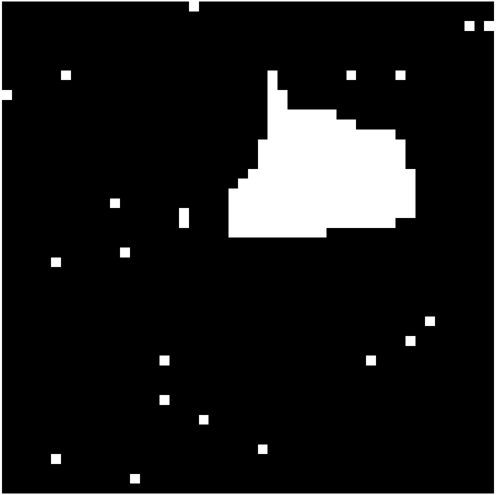

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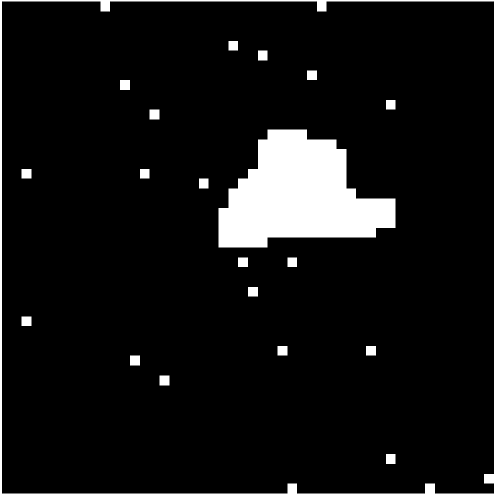

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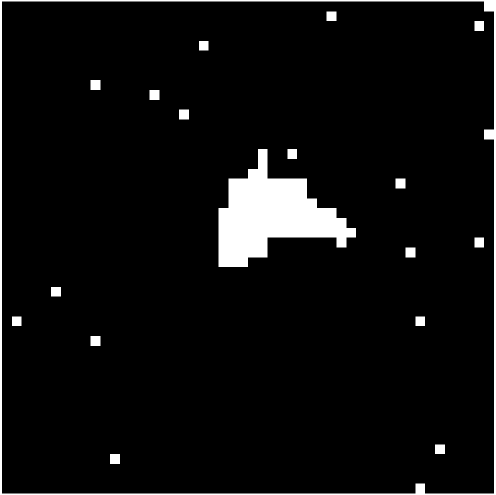

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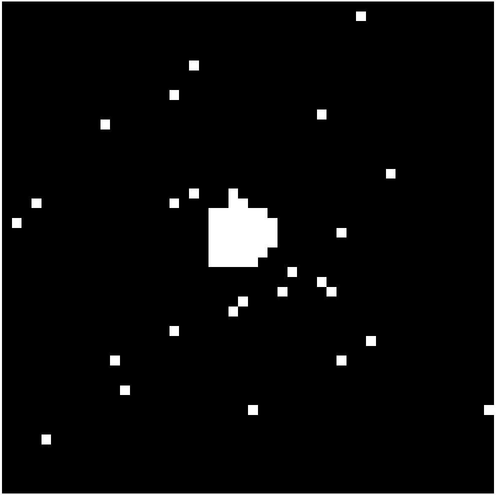

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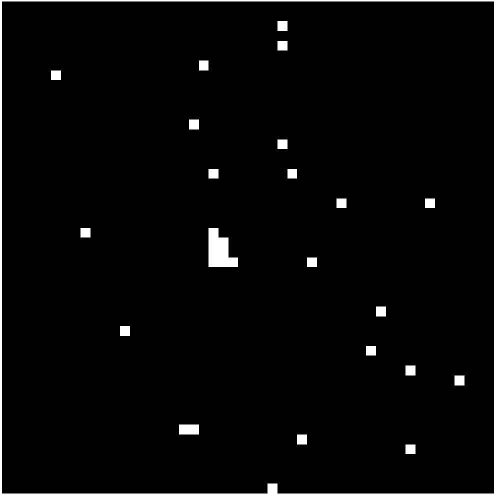

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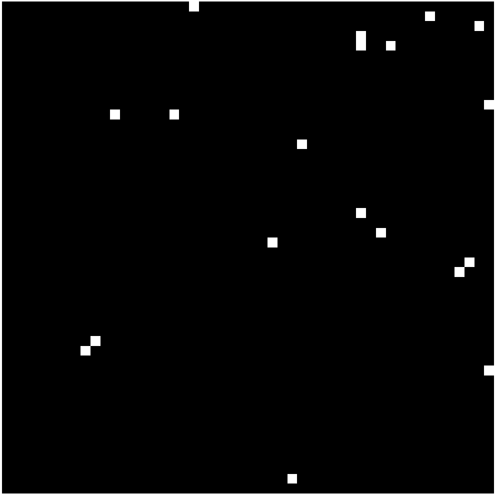


Def φ is an *eroder* if for the unperturbed cellular automaton, any finite collection of zeros disappears in finite time.

Toom's stability theorem (1980) If φ is an eroder, then $\rho(p) \to 1$ as $p \to 0$. If φ is not an eroder, then $\rho(p) = 0$ for all $p > 0$.

The eroder property

Each monotone map $\varphi : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}$ can uniquely be written as

$$\varphi(x) = \bigvee_{A \in \mathcal{A}(\varphi)} \bigwedge_{i \in A} x(i),$$

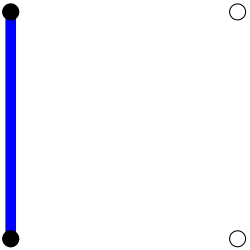
$A \in \mathcal{A}(\varphi)$ is a *minimal collection of ones* needed for $\varphi(x) = 1$.

Theorem (Toom 1980, Ponselet 2013) φ is an eroder if and only if

$$\bigcap_{A \in \mathcal{A}(\varphi)} \mathrm{Conv}(A) = \emptyset,$$

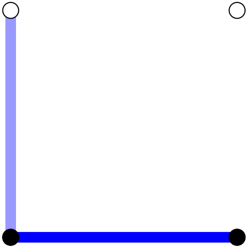
where $\mathrm{Conv}(A)$ is the convex hull of A.
By *Helly's theorem* w.l.o.g. $|\mathcal{A}(\varphi)| \leq d + 1$.

The eroder property



Toom's model φ^{NEC}

The eroder property



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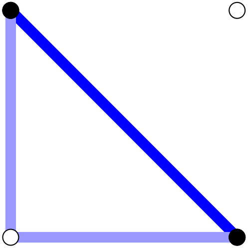

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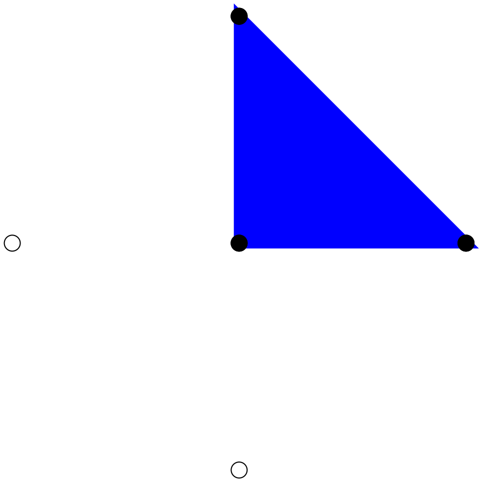
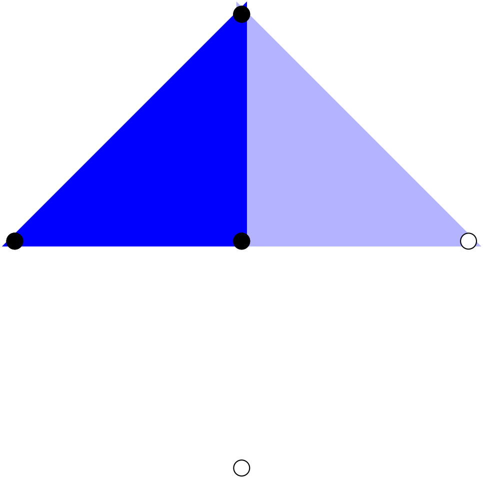


Nearest neighbour voting φ^{NN}.

The eroder property



Nearest neighbour voting φ^{NN}.

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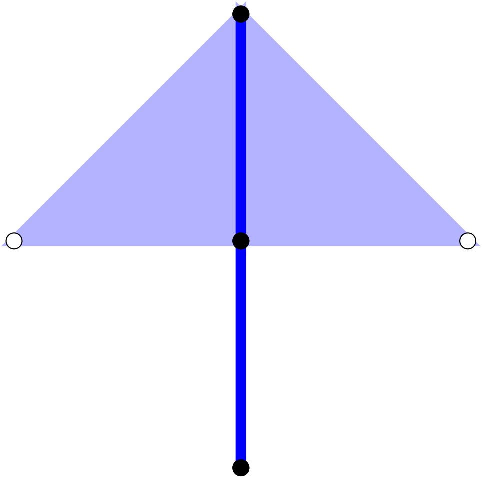

Nearest neighbour voting φ^{NN}.

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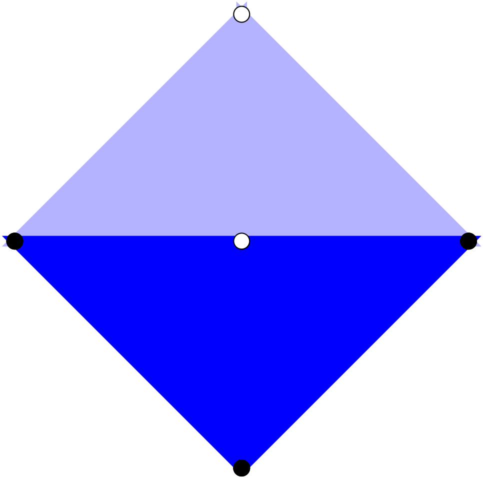


Nearest neighbour voting φ^{NN}.

The eroder property

Def A *polar function* is a linear function

$$\mathbb{R}^d \ni z \mapsto \big(L_1(z), \ldots, L_\sigma(z)\big) \in \mathbb{R}^\sigma$$

such that $\displaystyle\sum_{s=1}^{\sigma} L_s(z) = 0 \ (z \in \mathbb{R}^d)$.

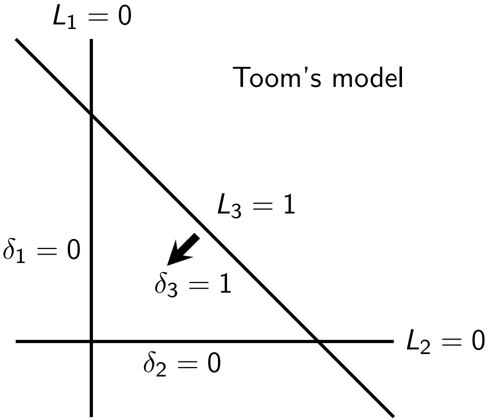
For $x \in \{0,1\}^{\mathbb{Z}^d}$, let $\displaystyle\ell_s(x) := \sup_{i \in \mathbb{Z}^d :\, x(i)=0} L_s(i)$.

Then for the unperturbed cellular automaton:

$$\ell_s(X_n) \leq \ell_s(X_0) - \delta_s n \quad \text{with} \quad \delta_s := \sup_{A \in \mathcal{A}(\varphi)} \inf_{i \in A} L_s(i).$$

The constants δ_s $(1 \leq s \leq \sigma)$ are *edge speeds*.

The eroder property



$$L_1(z) := -z_1, \quad L_2(z) := -z_2, \quad L_3(z) := z_1 + z_2.$$

The eroder property

Lemma (Toom 1980, Ponselet 2013) φ is an eroder if and only if there exists a polar function L such that

$$\delta := \sum_{s=1}^{\sigma} \delta_s > 0 \quad \text{with} \quad \delta_s := \sup_{A \in \mathcal{A}(\varphi)} \inf_{i \in A} L_s(i).$$

Proof of sufficiency Define the *extent* of x by

$$\mathrm{ext}(x) := \sum_{s=1}^{\sigma} \ell_s(x) \quad \text{with} \quad \ell_s(x) := \sup_{i \in \mathbb{Z}^d :\, x(i)=0} L_s(i).$$

Then $\mathrm{ext}(x) \geq 0$ if there is at least one zero since $\displaystyle\sum_{s=1}^{\sigma} L_s(z) = 0$.

Moreover $\mathrm{ext}(X_n) \leq \mathrm{ext}(X_0) - \delta n$. ∎

The maximal trajectory

Let $\Phi = \big(\Phi_{(i,t)}\big)_{(i,t)\in\mathbb{Z}^d\times\mathbb{Z}}$ be an i.i.d. collection of maps with

$$\mathbb{P}\big[\Phi_{(i,t)} = \varphi^0\big] = p \quad \text{and} \quad \mathbb{P}\big[\Phi_{(i,t)} = \varphi\big] = 1-p,$$

where $\varphi^0(x) := 0$ denotes the *trivial zero map*. A *trajectory* of Φ is a function $(i,t) \mapsto x_t(i)$ such that

$$x_t(i) = \Phi_{(i,t)}\big((x_{t-1}(i+j))_{j\in\mathbb{Z}^d}\big) \qquad \forall (i,t).$$

Lemma There a.s. exists a *maximal* trajectory $\overline{X}$.

Aim For small p, derive lower bound on $\rho(p) := \mathbb{P}[\overline{X}_0(0) = 1]$.

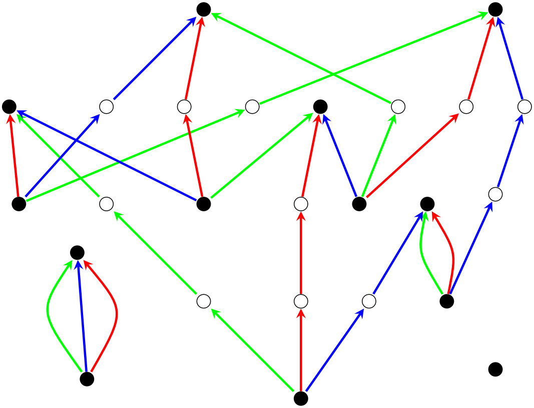
Toom contours

Def A *Toom graph* is a directed graph with edges of σ different *charges* and three types of vertices:

- At a *source*, σ directed edges emerge, one of each charge.
- At a *sink*, σ directed edges converge, one of each charge.
- At an *internal vertex*, there is one incoming edge and one outgoing edge, and they are of the same charge.

In addition, there can be *isolated vertices* which we can think of as a source and sink at the same time.

Toom contours



A Toom graph with three charges.

Peierls argument

Main idea A *Toom contour* is a connected Toom graph embedded in the plain, with one special source called the *root*.

If $\overline{X}_0(0) = 0$, then there exists a Toom contour T rooted at $(0,0)$ such that the sinks of T correspond to *defective* space-time points, where the trivial map φ^0 is applied. Then:

$$\mathbb{P}\big[\overline{X}_0(0) = 0\big] \leq \sum_T \mathbb{P}\big[\,T \text{ is present in } \Phi\big] \leq \sum_T p^{n_{\rm sink}(T)}.$$

This tends to zero as $p \to 0$ provided

$$N_n^{\rm sink} := \#\{T : n_{\rm sink}(T) = n\}$$

grows at most exponentially in n.

Peierls argument

It is not hard to show that there exists a $R < \infty$ such that

$$N_n^{\rm edge} \leq R^n \quad \text{with} \quad N_n^{\rm edge} := \#\{T : n_{\rm edge}(T) = n\}.$$

Need to show that $n_{\rm sink}(T) \geq c n_{\rm edge}(T)$ for some $c > 0$.

Idea: edges with charge s move in the direction where L_s increases, *except* for edges coming out of sources. As a result:

$$n_{\rm sink}(T) = n_{\rm source}(T) \geq c n_{\rm edge}(T)$$

for some $c > 0$.

Toom contours

Def An *embedding* of a Toom graph with vertex set V is a map

$$V \ni v \mapsto \big(\psi(v), h(v)\big) \in \mathbb{Z}^d \times \mathbb{Z}$$

- The *height* (=negative time) h increases by 1 along each directed edge.
- Sinks do not overlap with any other vertices.
- Internal vertices of the same charge do not overlap.

A *Toom contour* is an embedded connected Toom graph with one special source, the *root*, whose height is minimal among all vertices.

Toom contours

Let φ be an eroder. For each $1 \leq s \leq \sigma$, choose $A_s(\varphi) \in \mathcal{A}(\varphi)$ such that

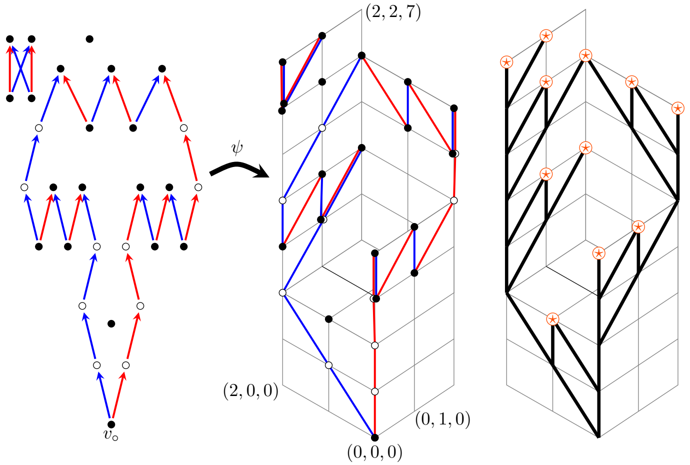
$$\delta_s := \sup_{A \in \mathcal{A}(\varphi)} \inf_{i \in A} L_s(i) = \inf_{i \in A_s(\varphi)} L_s(i).$$

Def A Toom contour is *present* in $\Phi = \big(\Phi_{(i,t)}\big)_{(i,t) \in \mathbb{Z}^d \times \mathbb{Z}}$ if:

- Sinks correspond to vertices where the trivial map φ^0 is applied.
- If (v, w) is a directed edge of charge s coming out of an internal vertex or the root, then $\psi(w) - \psi(v) \in A_s(\varphi)$.
- For directed edges emerging at other sources $\psi(w) - \psi(v) \in \bigcup_{s=1}^{\sigma} A_s(\varphi)$.

Theorem If $\overline{X}_0(0) = 0$, then there is a Toom contour rooted at $(0, 0)$ present in Φ.

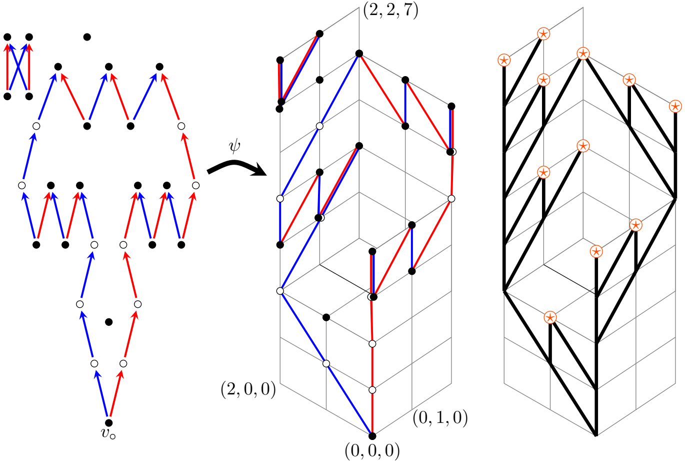
Toom contours



A Toom contour for the *cooperative branching map*

$$\varphi^{\mathrm{coop}}(x) := x(0,0) \vee \big(x(0,1) \wedge x(1,0)\big).$$

Toom contours



$$L_2(z_1, z_2) = z_1 + z_2, \quad \text{and} \quad L_1(z_1, z_2) = -z_1 - z_2.$$

The Peierls argument

Lemma There exists a $c > 0$ such that $n_{\rm sink} \geq c n_{\rm edge} + 1$.

Proof
$$\sum_{s=1}^{\sigma} \sum_{(v,w)\in E_s} \big(L_s(\psi(w)) - L_s(\psi(v))\big)$$
$$= \sum_{v\in V} \sum_{s=1}^{\sigma} \Big\{ \sum_{u:\,(u,v)\in E_s} L_s(\psi(v)) - \sum_{w:\,(v,w)\in E_s} L_s(\psi(v)) \Big\} = 0.$$

Let E°_s denote the edges of charge s out of a source different from the root and E^*_s the other edges. Then

$$0 = \sum_{s=1}^{\sigma} \sum_{(v,w)\in E^*_s} \underbrace{\big(L_s(\psi(w)) - L_s(\psi(v))\big)}_{\geq \delta_s}$$
$$+ \sum_{s=1}^{\sigma} \sum_{(v,w)\in E^\circ_s} \underbrace{\big(L_s(\psi(w)) - L_s(\psi(v))\big)}_{\geq -K}. \qquad \blacksquare$$

The Peierls argument

Lemma The number of Toom contours rooted at $(0,0)$ with N edges is bounded by R^n for some $R < \infty$.

Let $\mathcal{T}_0$ denote the set of all Toom contours rooted at $(0,0)$.
Let $n_{\rm sink}(T)$ denote the number of sinks of T.
Let $N_n^{\rm edge}$ denote the number of $T \in \mathcal{T}_0$ with n edges. Then

$$\mathbb{P}\big[\overline{X}_0(0) = 0\big] \leq \sum_{T \in \mathcal{T}_0} \mathbb{P}\big[T \text{ is present in } \Phi\big] \leq \sum_{T \in \mathcal{T}_0} p^{n_{\rm sink}(T)}$$

$$\leq p \sum_{T \in \mathcal{T}_0} p^{c n_{\rm edge}(T)} = p \sum_{n=0}^{\infty} N_n^{\rm edge} p^{cn} \leq p \sum_{n=0}^{\infty} R^n p^{cn}.$$

Continuous time

We can think of the continuous-time model as the $\varepsilon \to 0$ limit of a discrete-time model that applies three maps:

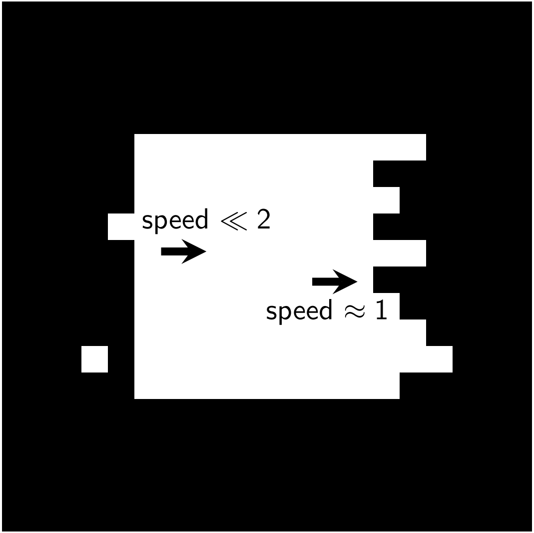
$$\begin{array}{ll} \varphi^{\rm NEC} & \text{with probability } \varepsilon, \\ \varphi^{0} & \text{with probability } \varepsilon p, \\ \varphi^{\rm id} & \text{with the remaining probability,} \end{array}$$

where $\varphi^{\rm id}(x) := x(0)$ is the *identity map*.

Gray (1999) has shown that combining the identity map with an eroder can spoil stability. Let:

$$\varphi(x) := \begin{cases} 0 & \text{if } x(-2,0) = x(-1,0) = 0, \\ 1 & \text{if } x(-3,k) = x(-2,k) = 1 \ \forall |k| \leq n, \\ x(0,0) & \text{in all other cases.} \end{cases}$$

Continuous time



Previous work

Toom (1980) Simple necessary and sufficient conditions for a monotone cellular automaton to be stable. Peierls argument.

Durrett & Gray (1985) Announce a number of deep results for cooperative branching. Referee asks for revision that never materialises.

Berman & Simon (1988), Gács & Reif (1988), Gács (1995,2021) Alternative proofs in a more restricted setting.

Bramson & Gray (1991) Alternative proof of Toom's result using a multiscale block construction.

Chen (1992,1994) Stability w.r.t. initial state & other perturbations. Proofs partly depend on [Durrett & Gray (1985)].

Gray (1999) Sufficient conditions for a monotone interacting particle system to be stable. Combines Toom's Peierls argument with the multiscale approach.

Previous work

Preskill (2007) Note on minimal explanations.

Maere & Ponselet (2011) Exponential decay of correlations.

Ponselet (2013) PhD thesis.

Our contributions

- Introduction of sources and sinks.
- Toom contours for random cellular automata that combine several maps.
- Toom contours in continuous time.
- Explicit bounds on critical values.
- A method to get sharper bounds when there are only two charges.

Explicit bounds

Cooperative branching discrete time $p_{\mathrm{c}} \geq 1/64$. (Numerics suggest $p_{\mathrm{c}} \approx 0.105$.)

Cooperative branching continuous time $\lambda_{\mathrm{c}} \leq 162$. (Durrett & Gray (1985) announced $\lambda_{\mathrm{c}} \leq 110$. Numerics suggest $\lambda_{\mathrm{c}} \approx 12.4$.)

Toom's model $p_{\mathrm{c}} \geq 3^{-21}$. (Numerics suggest $p_{\mathrm{c}} \approx 0.053$.)