An intertwining-based renormalization argument for hierarchical contact processes

Jan M. Swart

October 13, 2015

Intertwining of semigroups

[Fill '92] Let X and Y be Markov processes with state spaces S and T, semigroups $(P_t)_{t\geq 0}$ and $(P'_t)_{t\geq 0}$, and generators G and G'. Let K be a probability kernel from S to T and assume that

$$GK = KG'$$

Then one has the *intertwining relation*

$$P_t K = K P'_t \quad (t \geq 0)$$

and the processes X and Y can be coupled such that

$$\mathbb{P}[Y_t = y \,|\, (X_s)_{0\leq s\leq t}] = K(X_t, y) \quad \text{a.s.} \qquad (t \geq 0).$$

We call Y an *averaged Markov process on X*.

Example: Wright-Fisher diffusion

Let X be a Wright-Fisher diffusion with generator

$$Gf(x) = \tfrac{1}{2}x(1-x)\frac{\partial^2}{\partial x^2}f(x).$$

Let Y be a process with state space $\{0,1\}$ that jumps $0 \mapsto 1$ with rate one, i.e.,

$$G'f(y) := f(1) - f(y) \qquad (y = 0, 1).$$

Let $K : [0,1] \to \{0,1\}$ be the probability kernel

$$K(x,y) := \begin{cases} 4x(1-x) & \text{if } y = 0, \\ 1 - 4x(1-x) & \text{if } y = 1. \end{cases}$$

Then

$$GK = KG'.$$

Example: Wright-Fisher diffusion

The coupled process (X, Y) has the following description:

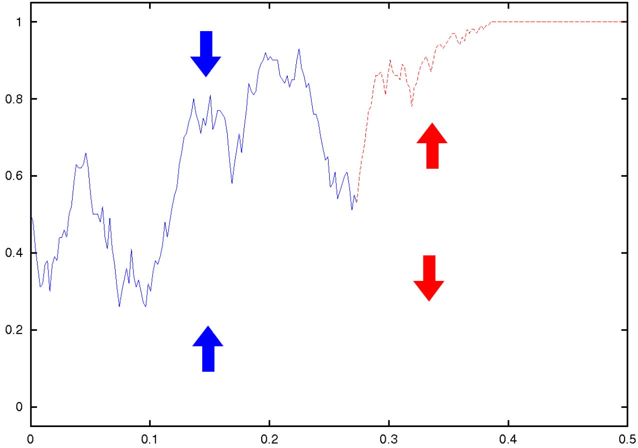
- Y evolves according to the generator G' regardless of the state of X, i.e., Y is *autonomous*.
- While Y is in the state y, the process X_t evolves according to the generator

$$G_y f(x) := \tfrac{1}{2}x(1-x)\frac{\partial^2}{\partial x^2}f(x) + b_y(x)\frac{\partial}{\partial x}f(x) \qquad (y = 0, 1),$$

where

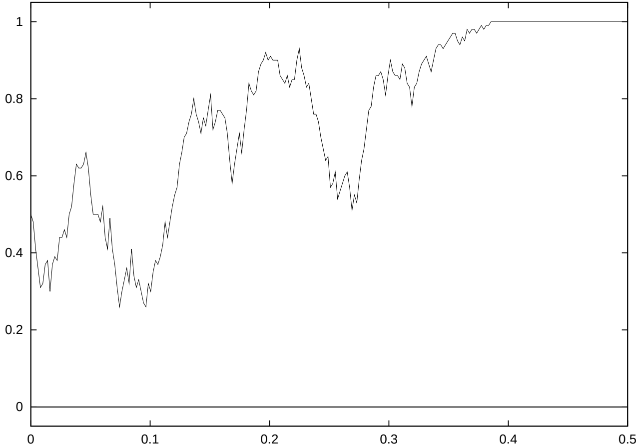
$$b_0(x) = 2(\tfrac{1}{2} - x), \quad b_1(x) = \frac{8x(1-x)(x-\frac{1}{2})}{1-4x(1-x)}.$$

Wright-Fisher diffusion with drift



As long as $Y_t = 0$, X cannot reach the boundary. When $Y_t = 1$, X cannot cross the middle.

Wright-Fisher diffusion



If we forget about the colors, then X is just a Wright-Fisher diffusion without drift!

Explanation

The process (X, Y) is Markov, but X is not autonomous, i.e., its dynamics depend on the state of Y. So how is it possible that X, on its own, is Markov?
In fact, one has

$$\mathbb{P}[Y_t = 0 \,|\, (X_s)_{0 \le s \le t}] = 4X_t(1 - X_t) \quad \text{a.s.}$$

In particular, this probability depends only on the endpoint of the path $(X_s)_{0 \le s \le t}$, and the expected drift is

$$\mathbb{E}[b_{Y_t}(X_t) \,|\, (X_s)_{0 \le s \le t}] = \\ 4X_t(1 - X_t)b_0(X_t) + \big(1 - 4X_t(1 - X_t)\big)\, b_1(X_t) = 0.$$

Intertwining is a useful tool to study *metastability*.

A generalization

[Athreya & S. '10] Let X be a Markov processes with state space S and generator G, let K be a probability kernel from S to T and let $(G'_x)_{x\in S}$ be a collection of generators of T-valued Markov processes. Assume that

$$GK = \hat{K}\,\overline{G}$$

where $\hat{K} : \mathbb{R}^{S\times T} \to \mathbb{R}^S$ and $\overline{G} : \mathbb{R}^T \to \mathbb{R}^{S\times T}$ are defined by

$$\hat{K}f(x) := \sum_{y\in S} K(x,y)f(x,y) \quad \text{and} \quad \overline{G}f(x,y) := G'_x f(y).$$

Then X can be coupled to a process Y such that (X,Y) is Markov, Y evolves according to the generator G'_x while X is in the state x, and

$$\mathbb{P}[Y_t = y \,|\, (X_s)_{0\le s\le t}] = K(X_t,y) \quad \text{a.s.} \qquad (t\ge 0).$$

A generalization

In this more general setting, the process Y is no longer autonomous, but in "good" situations its transition rates are "almost" constant as a function of the state of X.

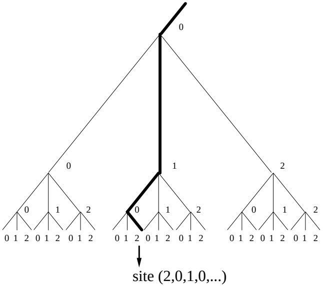
We call Y an *added-on process on* X.

The hierarchical group

By definition, the *hierarchical group with freedom N* is the set

$$\Omega_N := \big\{ i = (i_0, i_1, \ldots) : \quad i_k \in \{0, \ldots, N-1\}, \\ i_k \neq 0 \text{ for finitely many } k \big\},$$

equipped with componentwise addition modulo N. Think of sites $i \in \Omega_N$ as the leaves of an infinite tree. Then $i_0, i_1, i_2, \ldots$ are the labels of the branches on the unique path from i to the root of the tree.

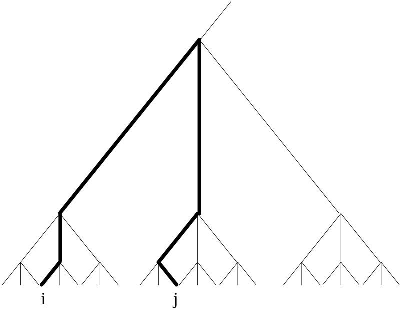


The hierarchical distance

Set

$$|i| := \inf\{k \geq 0 : i_m = 0 \ \forall m \geq k\} \qquad (i \in \Omega_N).$$

Then $|i - j|$ is the *hierarchical distance* between two elements $i, j \in \Omega_N$. In the tree picture, $|i - j|$ measures how high we must go up the tree to find the last common ancestor of i and j.



two sites at hierarchical distance 3

Hierarchical contact processes

Fix a *recovery rate* $\delta \geq 0$ and *infection rates* $\alpha_k \geq 0$ such that $\sum_{k=1}^{\infty} \alpha_k < \infty$.

Consider a contact process on Ω_N where:

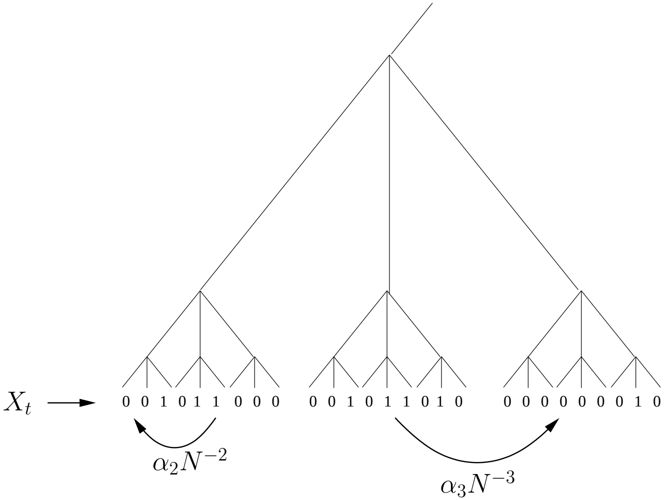
- An infected site i infects a healthy site j at hierarchical distance $k := |i - j|$ with rate $\alpha_k N^{-k}$
- Infected sites become healthy with rate $\delta \geq 0$.

Write:

$$X_t(i) = \begin{cases} 0 \\ 1 \end{cases} \text{if the site } i \text{ is} \begin{cases} \text{healthy} \\ \text{infected} \end{cases} \text{at time } t.$$

Then $(X_t)_{t \geq 0}$ with $X_t = (X_t(i))_{i \in \Omega_N}$ is a Markov process.

Hierarchical contact processes



Infection rates on the hierarchical group.

The critical recovery rate

We say that a contact process $(X_t)_{t\geq 0}$ on Ω_N with given recovery and infection rates *survives* if there is a positive probability that the process started with only one infected site never recovers completely, i.e., there are infected sites at any $t \geq 0$. For given infection rates, we let

$$\delta_{\rm c} := \sup\big\{\delta \geq 0 : \text{the contact process with infection rates } (\alpha_k)_{k\geq 1} \text{ and recovery rate } \delta \text{ survives}\big\}$$

denote the *critical recovery rate*. A simple monotone coupling argument shows that X survives for $\delta < \delta_{\rm c}$ and dies out for $\delta > \delta_{\rm c}$. It is not hard to show that $\delta_{\rm c} < \infty$. The question whether $\delta_{\rm c} > 0$ is more subtle.

(Non)triviality of the critical recovery rate

[Athreya & S. '10] Assume that $\alpha_k = e^{-\theta^k}$ $(k \geq 1)$. Then:

(a) If $N < \theta$, then $\delta_{\rm c} = 0$.

(b) If $1 < \theta < N$, then $\delta_{\rm c} > 0$.

More generally, we show that $\delta_{\rm c} = 0$ if

$$\liminf_{k\to\infty} N^{-k} \log(\beta_k) = -\infty, \quad \text{where} \quad \beta_k := \sum_{n=k}^{\infty} \alpha_n \quad (k \geq 1),$$

while $\delta_{\rm c} > 0$ if

$$\sum_{k=m}^{\infty} (N')^{-k} \log(\alpha_k) > -\infty,$$

for some $m \geq 1$ and $N' < N$.

Proof of extinction

Without loss of generality $\sum_{k=1}^{\infty} \alpha_k \leq 1$.
Let $X^{(n)}$ be the process restricted to

$$\Omega_N^n := \big\{ i = (i_0, \ldots, i_{n-1}) : i_k \in \{0, \ldots, N-1\} \big\}.$$

Comparison of $X^{(n)}$ with a process $\tilde{X}^{(n)}$ where sites jump independently from each other from 0 to 1 with rate one and from 1 to 0 with rate δ yields the estimate

$$T := \mathbb{E}^{\delta_0}\big[\inf\{ t \geq 0 : X_t^{(n)} = \underline{0} \} \big] \leq N^{-n}(1 + \delta^{-1})^{N^n}.$$

For $N < \theta$, this implies that sufficiently large blocks recover faster than they can infect other blocks of the same size, hence the result follows by comparison with subcritical branching.

Proof of survival

We use added-on Markov processes to inductively derive bounds on the finite-time survival probability of finite systems. Let

$$\Omega_2^n := \big\{ i = (i_0, \ldots, i_{n-1}) : i_k \in \{0,1\} \big\}$$

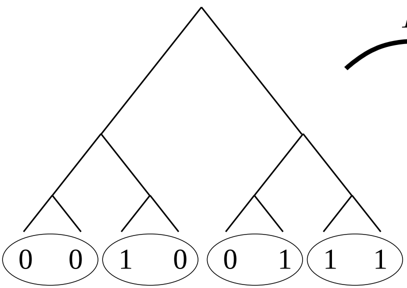
and let $S_n := \{0,1\}^{\Omega_2^n}$. We define a kernel from S_n to S_{n-1} by independently replacing blocks consisting of two spins by a single spin according to the stochastic rules:

$$00 \longrightarrow 0, \qquad 11 \longrightarrow 1,$$

$$\text{and} \quad 01 \text{ or } 10 \longrightarrow \begin{cases} 0 & \text{with probability } \xi, \\ 1 & \text{with probability } 1-\xi, \end{cases}$$

where $\xi \in (0, \frac{1}{2}]$ is a constant, to be determined later.

Renormalization kernel



The probability of this transition is $1 \cdot (1-\xi) \cdot \xi \cdot 1$.

An added-on process

Let X be a contact process on Ω_2^n with infection rates $\alpha_1, \ldots, \alpha_n$ and recover rate δ. Then X can be coupled to a process Y such that

$$\mathbb{P}[Y_t = y \,|\, (X_s)_{0 \leq s \leq t}] = K(X_t, y) \quad \text{a.s.} \qquad (t \geq 0),$$

where K is the kernel defined before, and

$$\xi := \gamma - \sqrt{\gamma^2 - \tfrac{1}{2}} \quad \text{with} \quad \gamma := \frac{1}{4}\Big(3 + \frac{\alpha_1}{2\delta}\Big).$$

The process Y is not autonomous, but “almost” so. Indeed, we can couple Y to a contact process X' on Ω_2^{n-1} with recovery rate $\delta' := 2\xi\delta$ and infection rates $\alpha'_1, \ldots, \alpha'_{n-1}$ given by $\alpha'_k := \frac{1}{2}\alpha_{k+1}$, in such a way that $X'_t \leq Y_t$ for all $t \geq 0$.

Renormalization

We may view the map $(\delta, \alpha_1, \ldots, \alpha_n) \mapsto (\delta', \alpha'_1, \ldots, \alpha'_{n-1})$ as an (approximate) renormalization transformation. By iterating this map n times, we get a sequence of recovery rates $\delta, \delta', \delta'', \ldots$, the last of which gives a upper bound on the spectral gap of the finite contact process X on Ω_2^n. Under suitable assumptions on the α_k's, we can show that this spectral gap tends to zero as $n \to \infty$, and in fact, we can derive explicit lower bounds on the probability that finite systems survive till some fixed time t.

Open problems

The renormalization procedure is only approximate, since we use the stochastic bound $X' \leq Y$ to estimate the non-autonomous process Y from below by a contact process X'.

- Can we improve our kernel K so that Y' is even closer to being autonomous?
- Can we even find an exact renormalization map, where Y' is autonomous (though possibly no longer a contact process)?
- Can we set up a similar argument for contact processes on $\mathbb{Z}$?
- Do *random* renormalization mappings have advantages over *deterministic* rules?