The Stigler-Luckock model with market makers

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Order book

Nowadays, demand and supply is often realized by electronic trading systems storing the information in databases.
Traders with access to these databases quote their offers and their summary forms a **(limit) order book**.
We define **order book** as a list of all limit orders (hence the occasional attribute **limit**), including information about their type, price and size in time.

Stigler-Luckock model: definition

Let $I = (I_-, I_+)$ be a nonempty open interval of possible prices. Every finite counting measure can be written as a finite sum of delta measures.

At any time we represent state of the order book by a pair $(\mathcal{X}^-, \mathcal{X}^+)$ of counting measures on I, where we interpret the delta measures $\mathcal{X}^-$ (resp. $\mathcal{X}^+$) is composed of as buy (resp. sell) limit orders of unit size at a given price. We also assume that:

(i) there are no such $x, y \in I$ that $x \leq y$ and $\mathcal{X}^+(x) > 0$, $\mathcal{X}^-(y) > 0$,

(ii) $\mathcal{X}^-([x, I_+)) < \infty$ and $\mathcal{X}^-((I_-, x]) < \infty$ for all $x \in I$.

Stigler-Luckock model: definition

Condition (ii) ensures that

$$\color{red}{M^-} := max(\{I_-\} \cup \{x \in I : \mathcal{X}^-(\{x\}) > 0\}),$$
$$\color{blue}{M^+} := min(\{I_+\} \cup \{x \in I : \mathcal{X}^+(\{x\}) > 0\}),$$

are well defined, here $M^\mp$ represent the current bid and ask prices.

Stigler-Luckock model: definition

The dynamics of the model are desribed by $\lambda_\pm : \bar{I} \to \mathbb{R}$, that are called **demand** (λ_-) and **supply** **functions** (λ_+). We assume that:

(A1) λ_- is nonincreasing and λ_+ is nondecreasing,

(A2) $\lambda_\pm$ are continuous functions,

(A3) function $\lambda_- - \lambda_+$ is strictly increasing,

(A4) $\lambda_\pm > 0$ on I.

Denote $d\lambda_\pm([x,y]) := \lambda_\pm(y) - \lambda_\pm(x)$, $(x, y \in I, x \le y)$.
So $d\lambda_-$ (resp. $d\lambda_+$) is a negative (resp. positive) measure.

Stigler-Luckock model: orders

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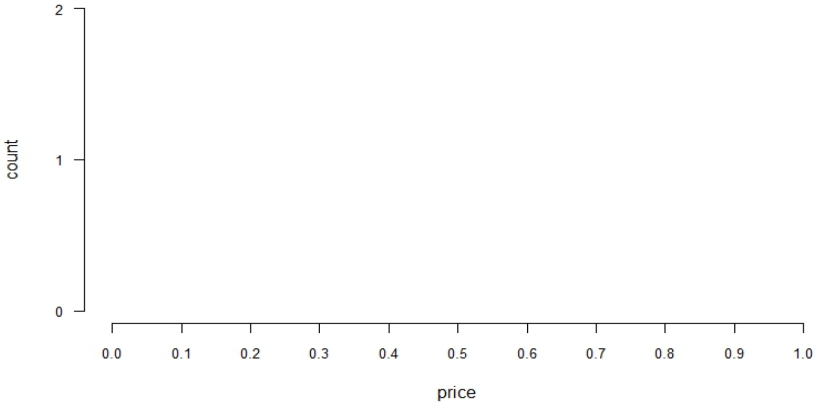
- **Market makers** arrive with Poisson intensity ρ and place one buy and one sell limit order at current bid and ask prices

Stigler-Luckock model

We assume independence of all Poisson processes governing different mechanisms (buyers/sellers and market makers).
After Stigler and Luckock we call the model the **Stigler-Luckock model** with demand and supply functions $\lambda_\pm$ and rate of market makers ρ.

Order book evolution



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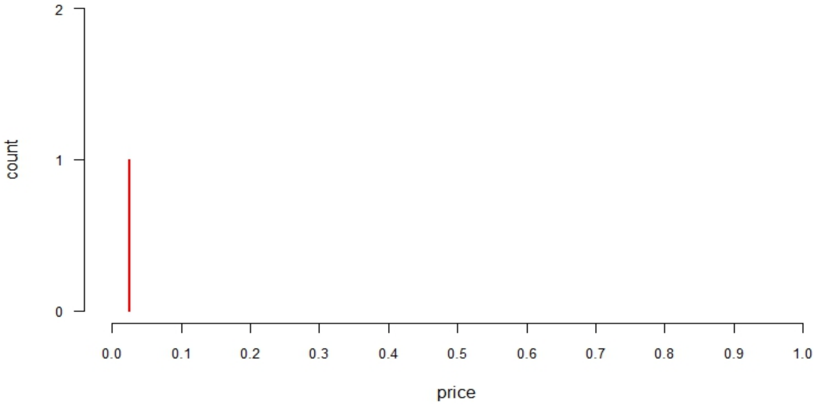

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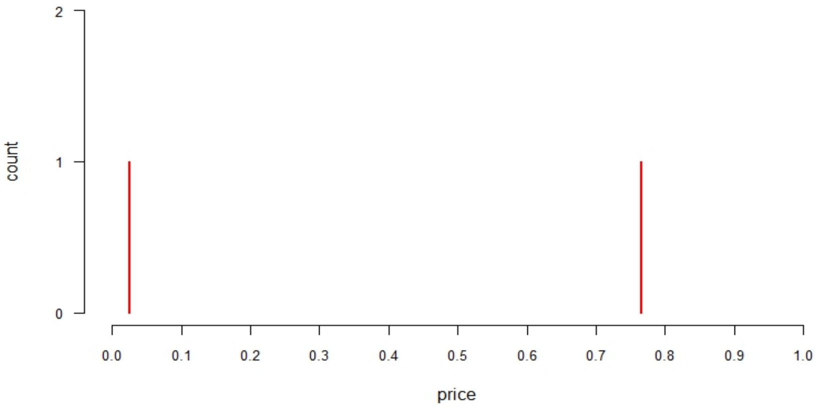

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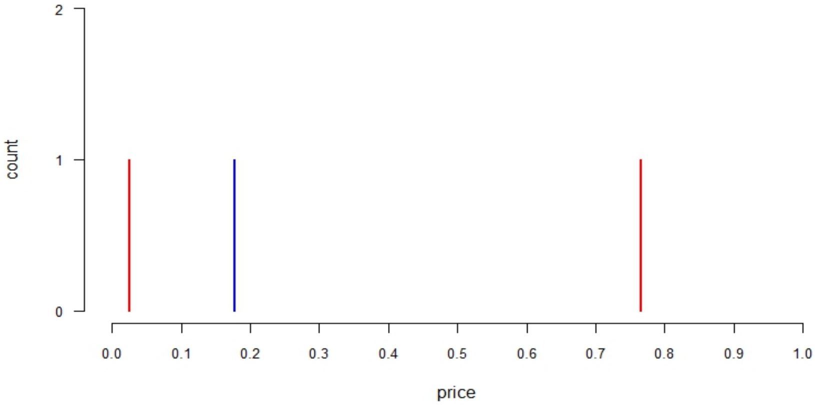

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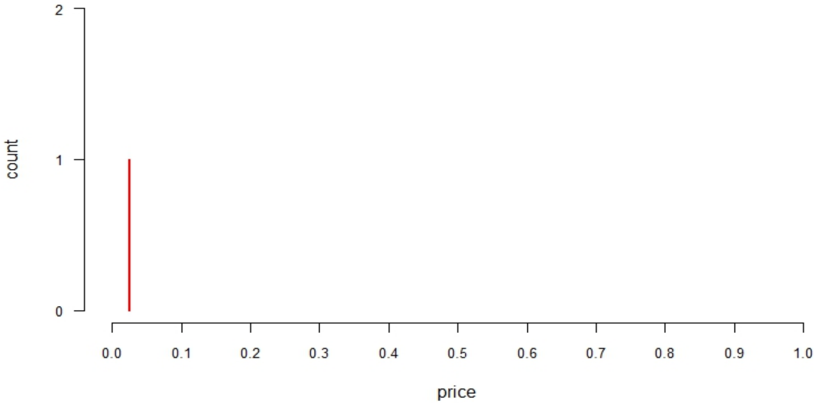

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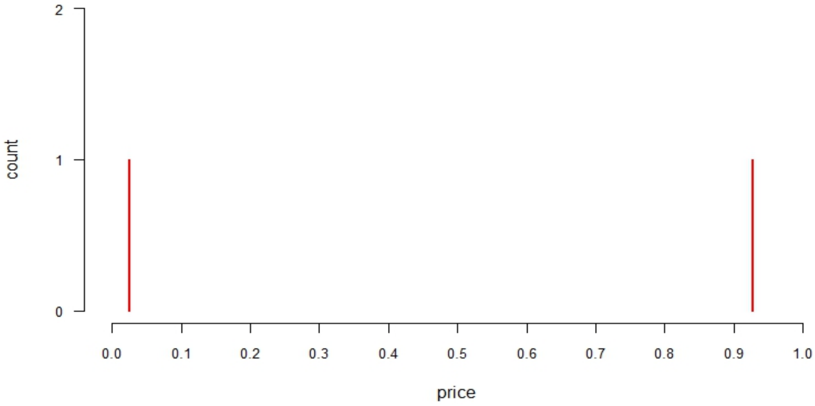

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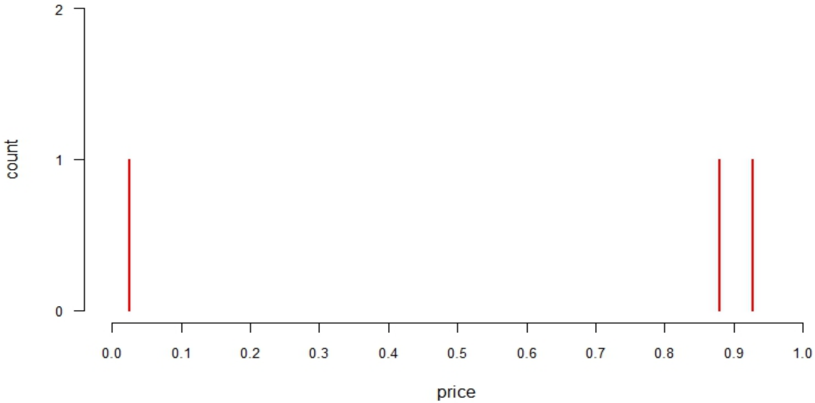

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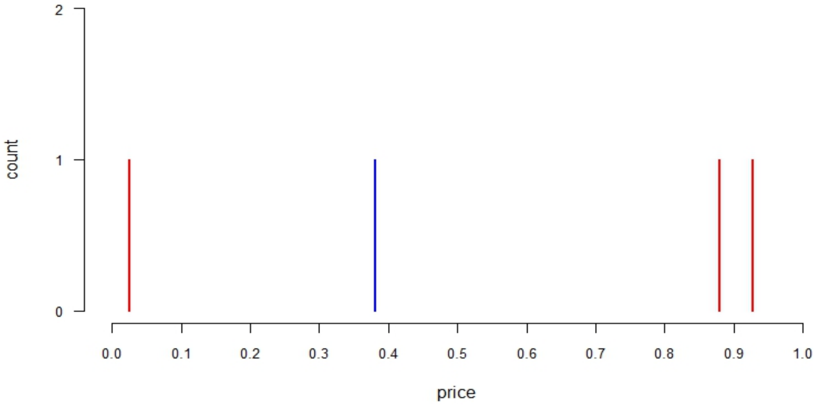

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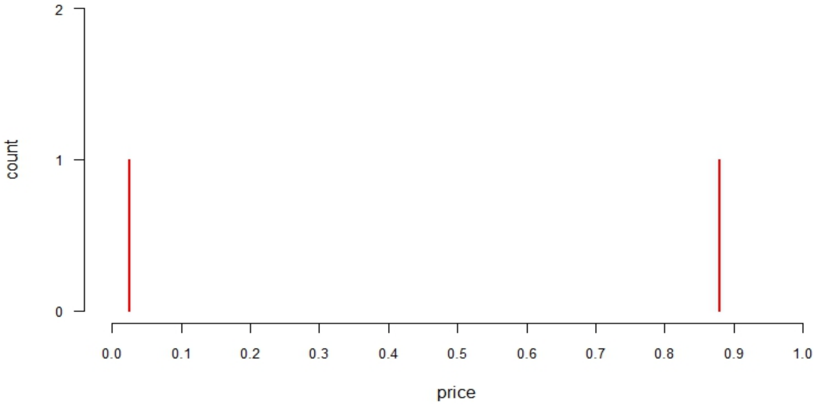

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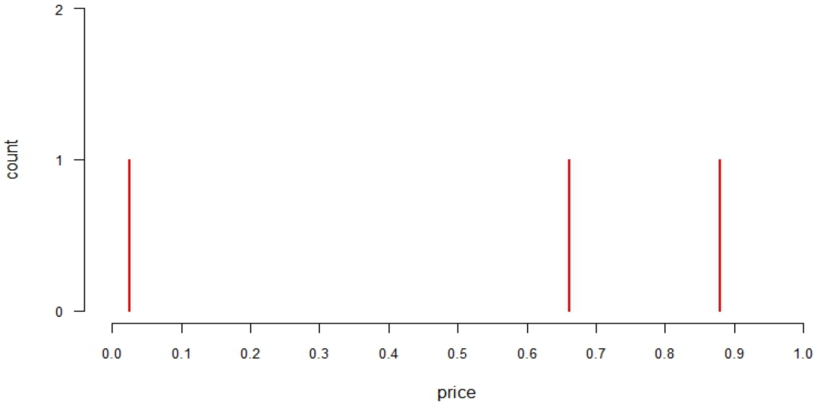

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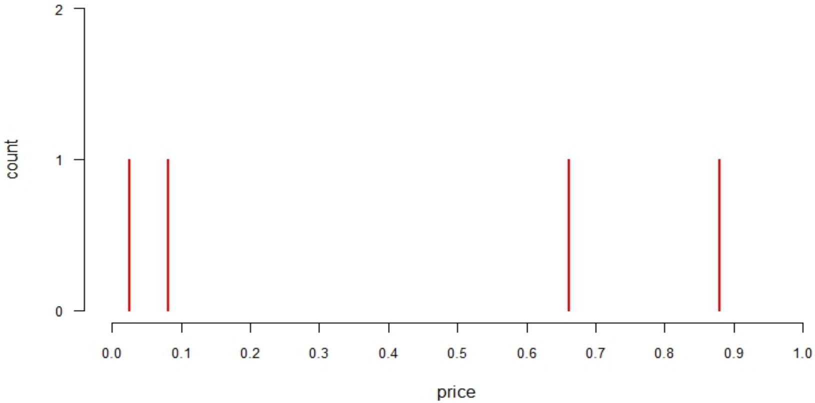

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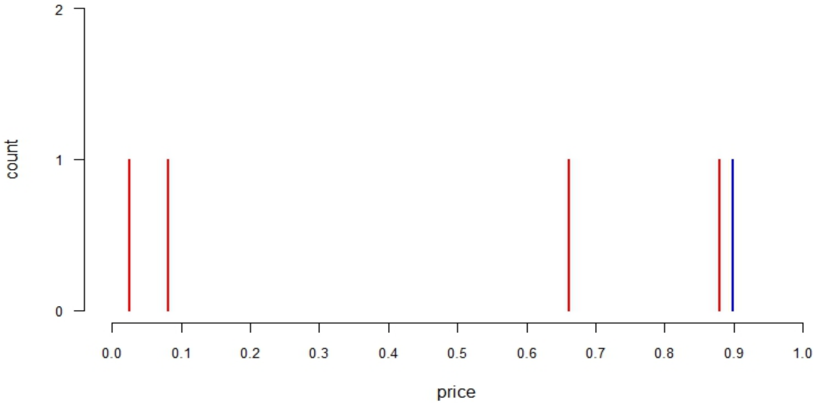

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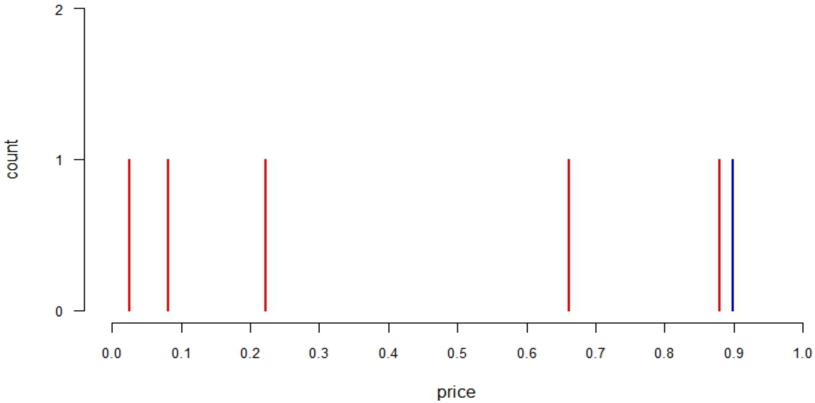

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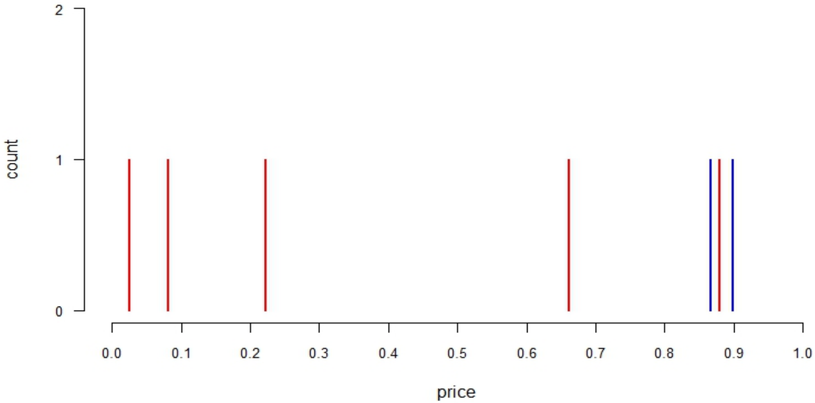

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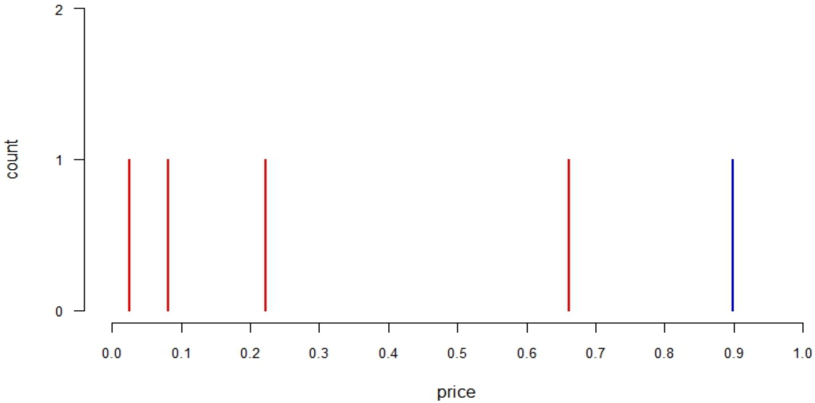

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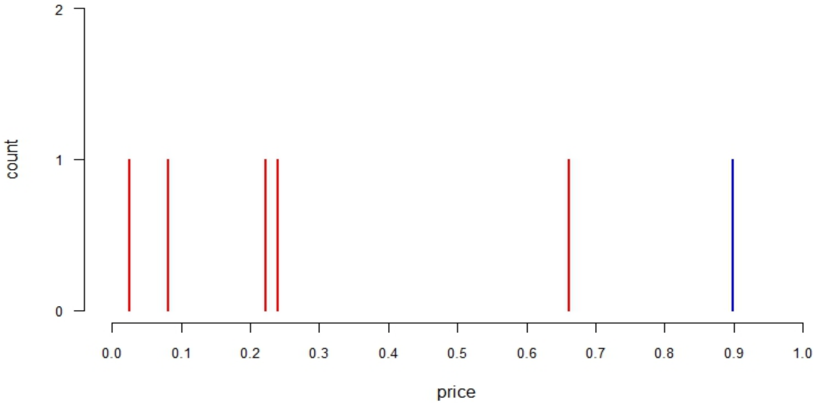

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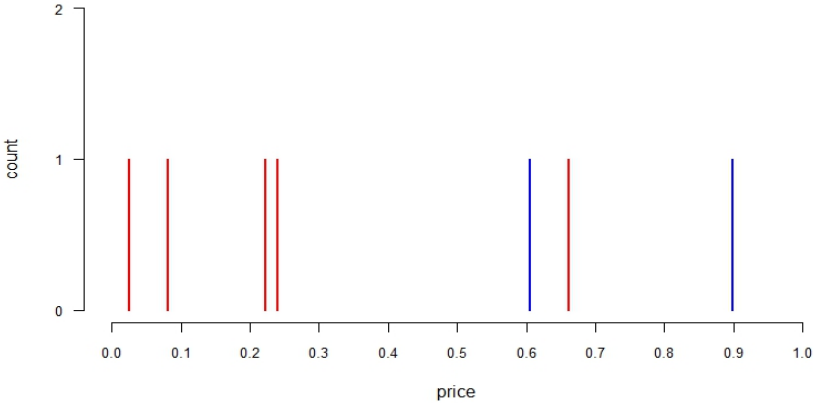

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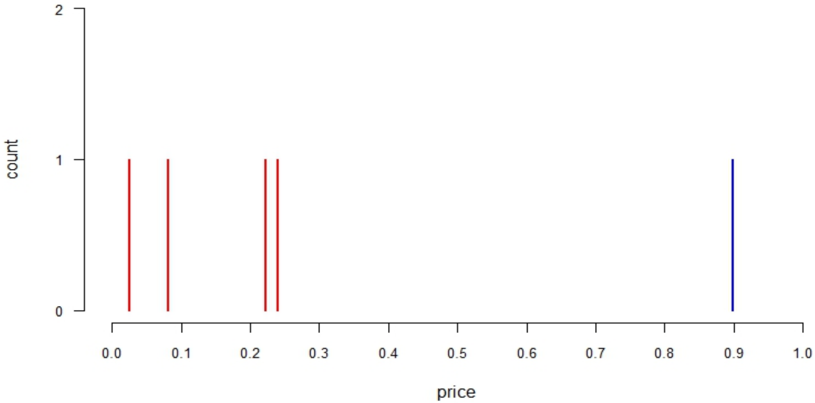

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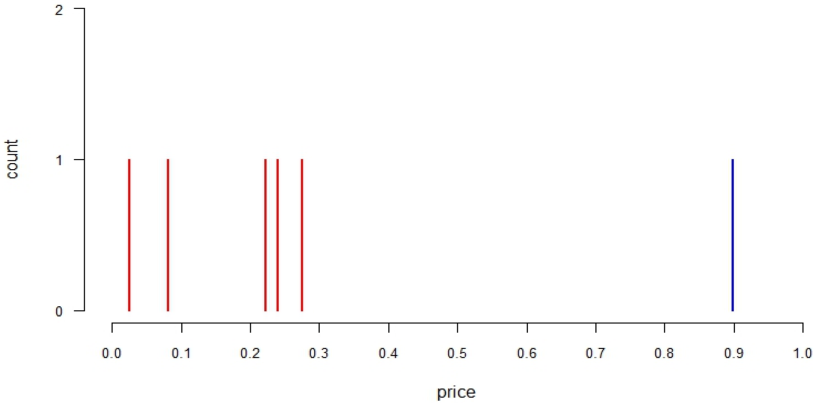

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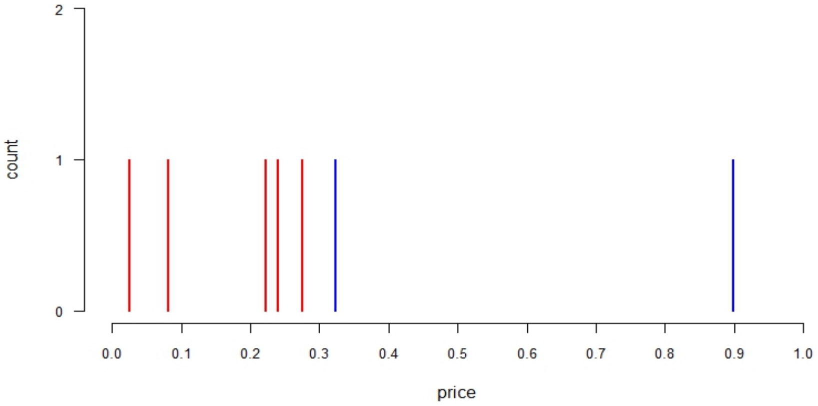

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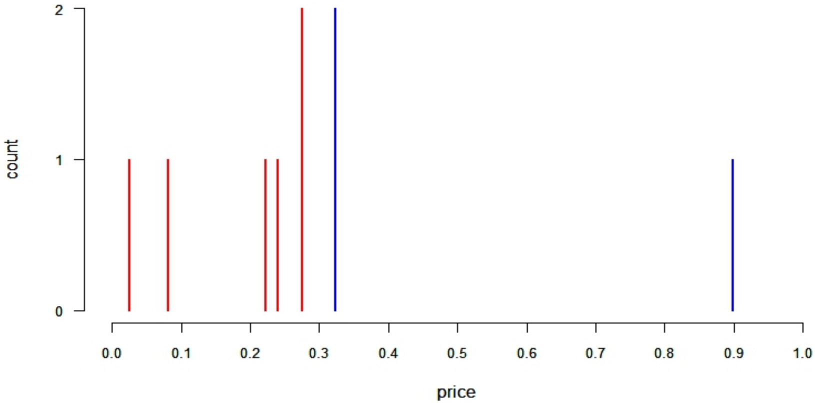

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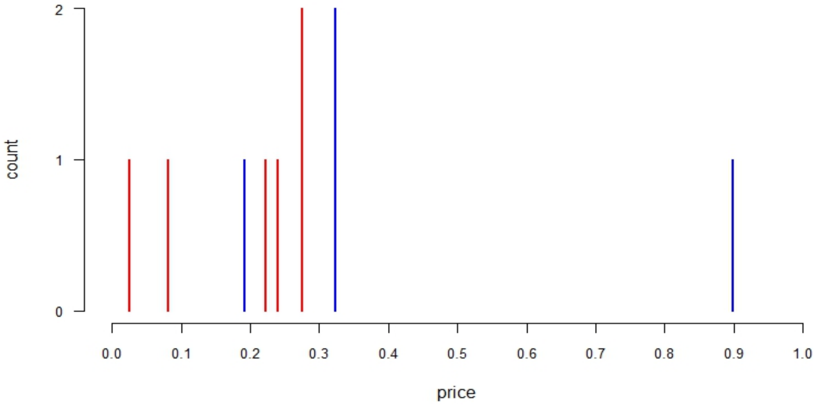

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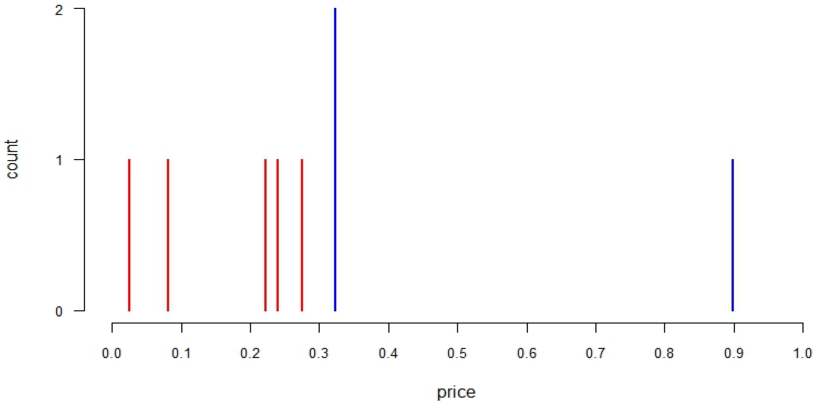

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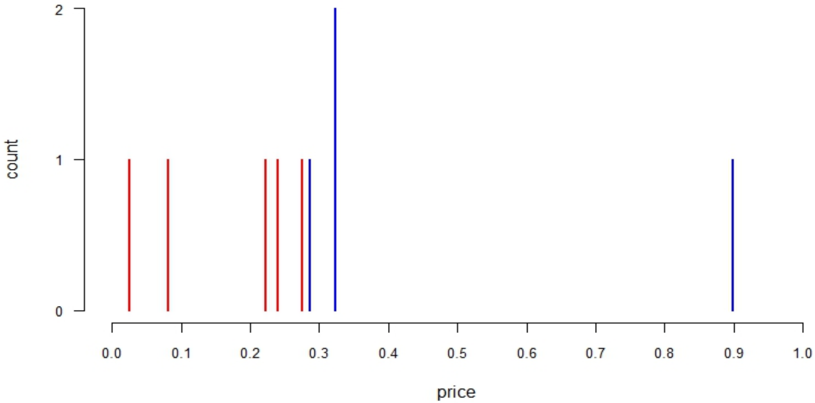

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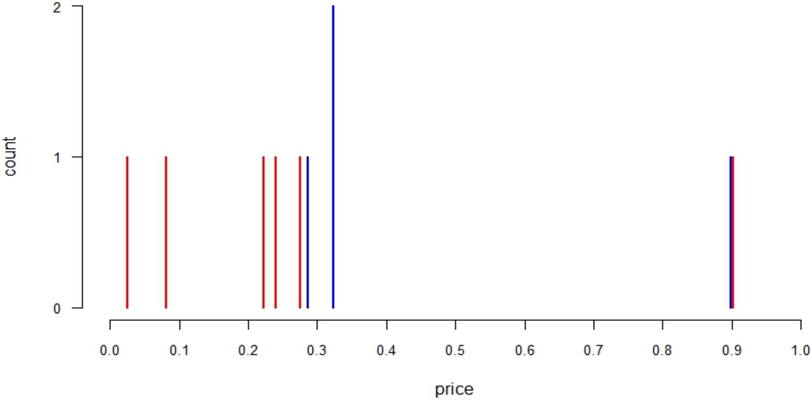

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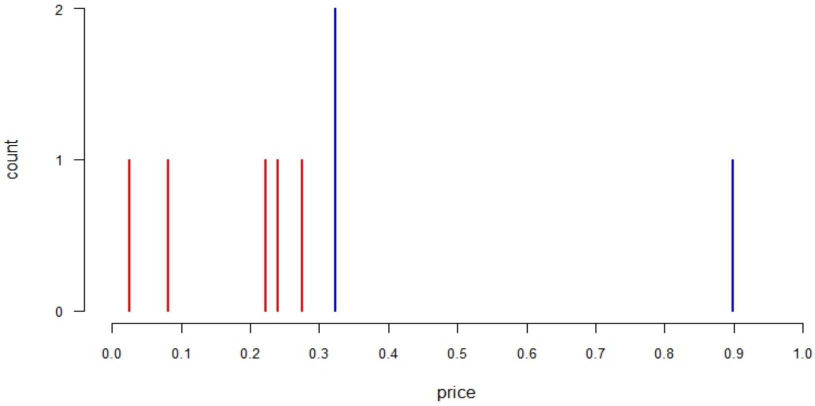
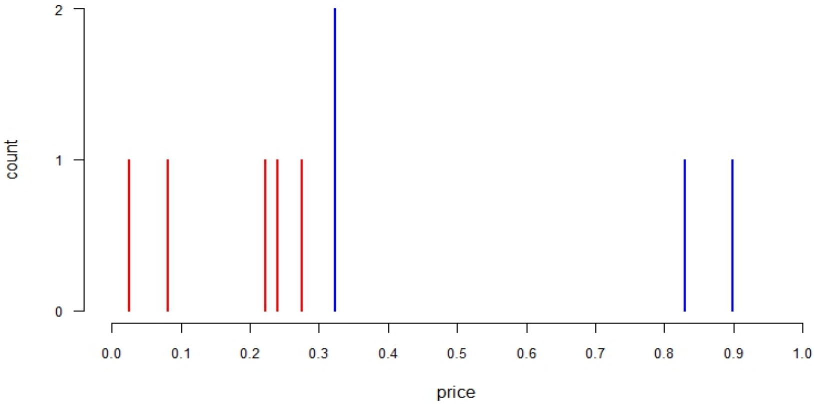


Order book evolution: end



Model history

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- **2011, J.Plačková** - no market orders or market makers, prices from discrete set of 100 values
- **2012, E.Yudovina** - no market makers, general class of $\lambda_{\pm}$ (less general than Luckock)

Economic theory

Classical economic theory (going back to Walras, 1874), the **price** of an asset is determined by its demand and supply.

Let λ_- (resp. λ_+) be the functions of demand (resp. supply).

Assume, that λ_- (resp. λ_+) is continuous and strictly increasing (resp. decreasing) and equal to zero at endpoints of interval I.

Then in a an equilibrium market, the asset is traded at the **equilibrium price** x_W and the volume of trade is V_W, defined as $V_W := \lambda_-(x_W) = \lambda_+(x_W)$.

Economic theory

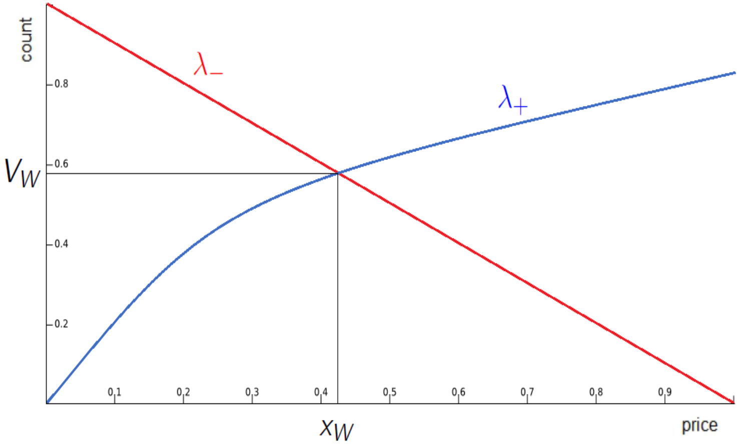
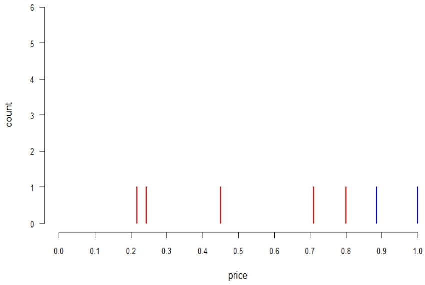


Figure: Equilibrimum price and volume of trade according to Walras.

Simulations: cumulative functions

For plotting the state of order book we use **cumulative functions**.

- Cumulative demand function: $\lambda_{-}^{c}(x) := \mathcal{X}^{-}([x, I_{+}])$
- Cumulative supply function: $\lambda_{+}^{c}(x) := \mathcal{X}^{+}([I_{-}, x])$



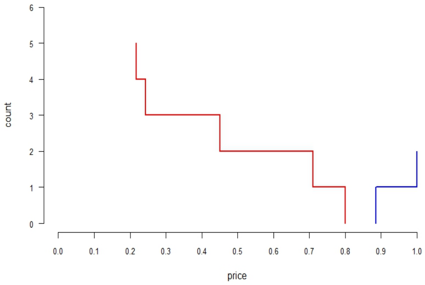


Figure: The order book after the arrival of 10 traders.

Simulations: Stigler-Luckock model without market makers

Consider a Stigler-Luckock model with $\bar{I} = [0, 1]$, demand and supply functions $\lambda_-(x) = 1 - x$ and $\lambda_+(x) = x$, $\rho = 0$.

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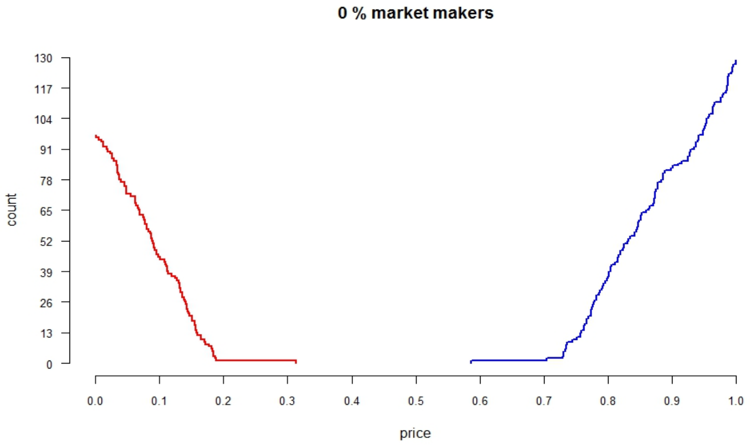

Figure: The order book after the arrival of 1,000 traders.

Simulations: Stigler-Luckock model without market makers

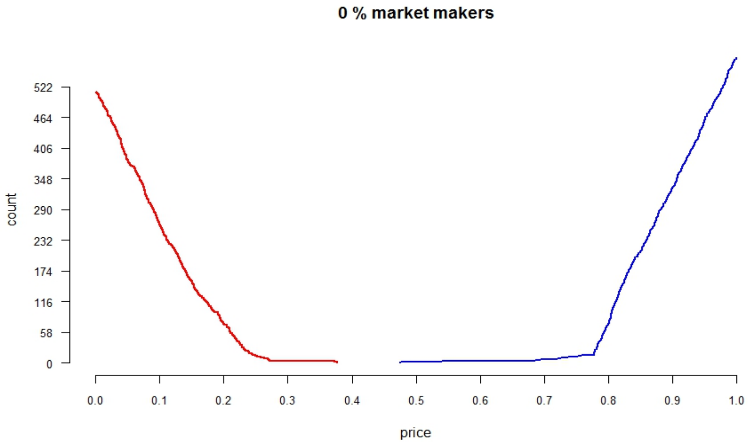

Figure: The order book after the arrival of 5,000 traders.

Simulations: Stigler-Luckock model without market makers

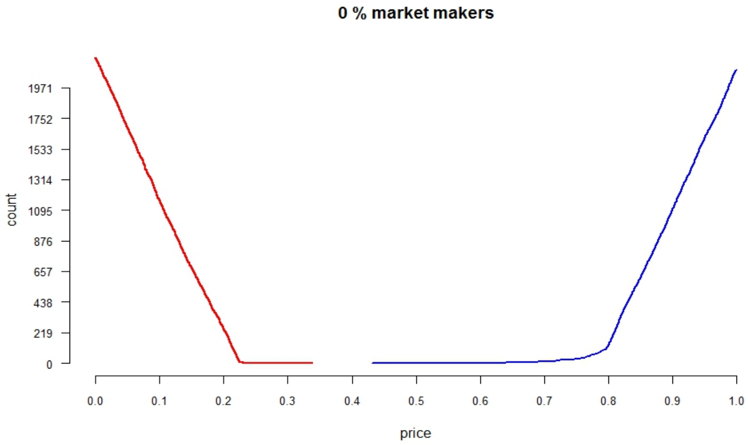

Figure: The order book after the arrival of 20,000 traders.

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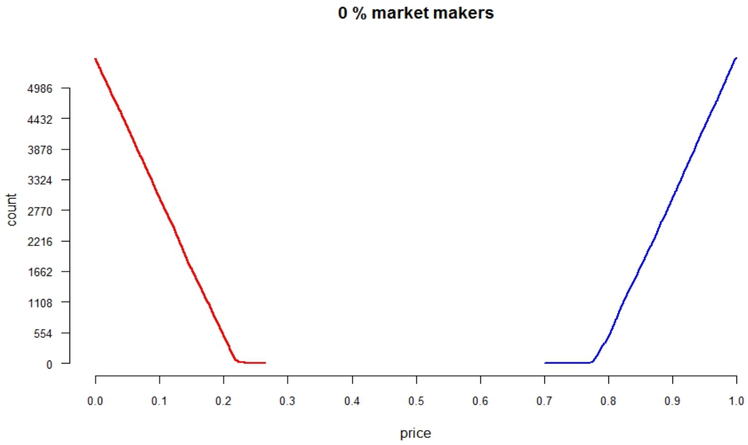

Figure: The order book after the arrival of 50,000 traders.

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- Competitive window of positive size $\approx (0.218, 0.782)$
- Luckock predicted $x_+ = 1/z$, z being the unique solution to $e^{-z} - z + 1 = 0$ (which tallies with our simulations)

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Consider a Stigler-Luckock model with $\overline{I} = [0, 1]$, demand and supply functions $\lambda_-(x) = 1 - x$ and $\lambda_+(x) = x$, $\rho > 0$.

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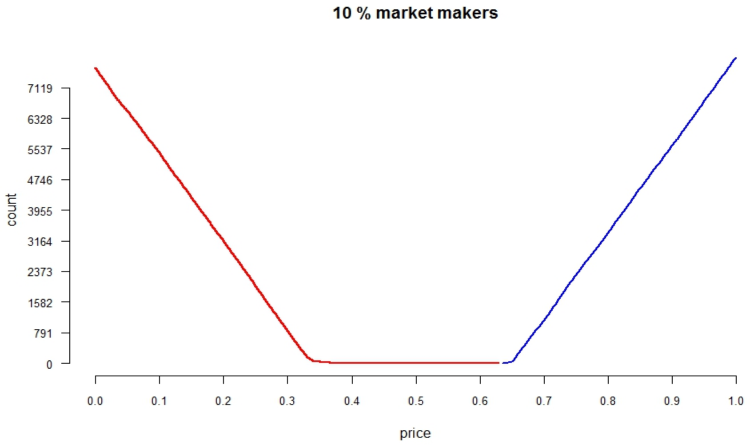

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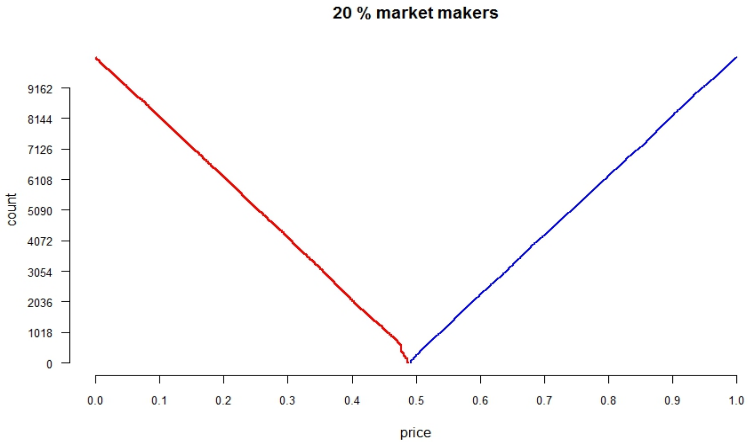

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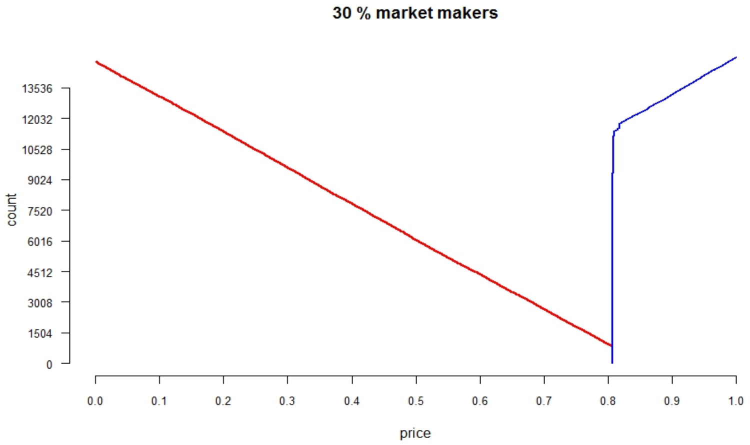

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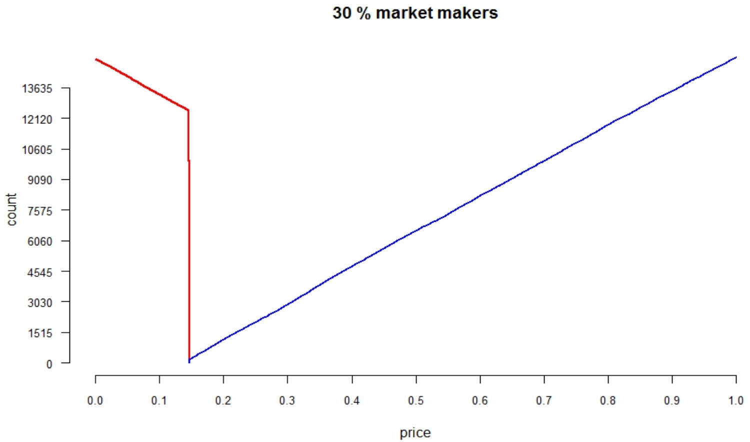

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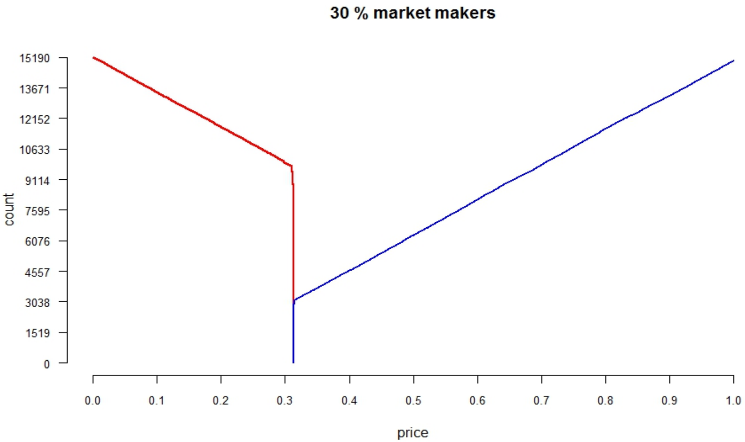

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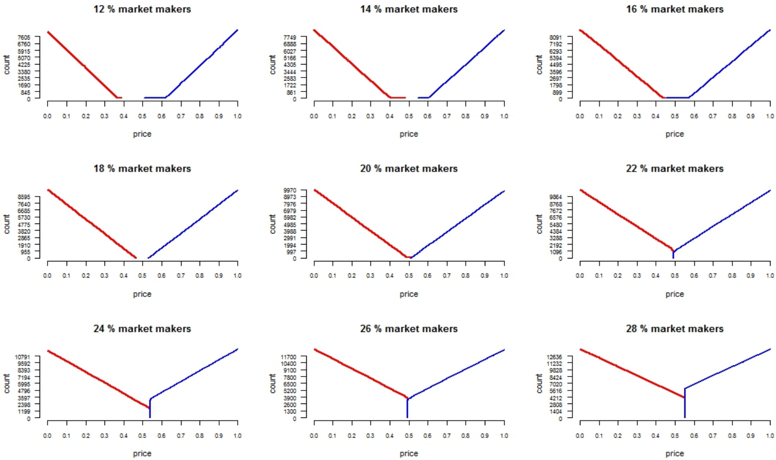

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- What happens if there is “too many” market makers?

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- As long as the bid and ask prices stay inside J, the restricted model behaves the same as the original model.

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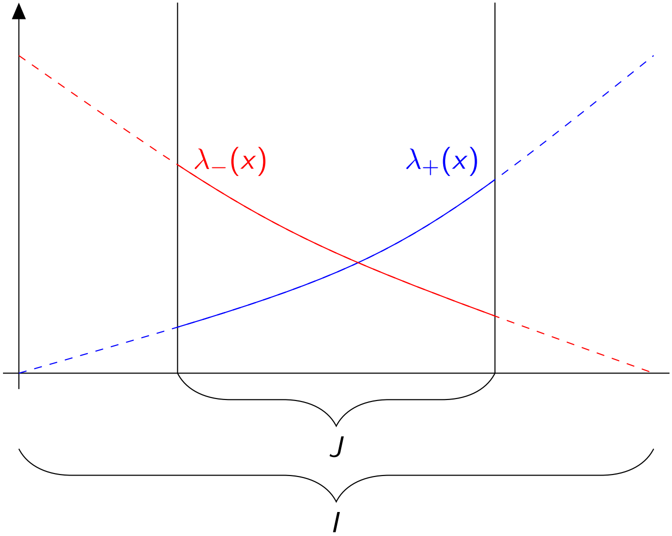
- As long as the bid and ask prices stay inside J, the restricted model behaves the same as the original model.
- For the restricted model, it is no longer true that $\lambda_\pm$ are zero at the endpoints of the interval.

Restricted models



[Luckock's differential equation] Let $M^{\pm}$ denote the price of the best buy/sell offer. Assume that the process is in equilibrium. Then

$$f_{-}(x) := \mathbb{P}[M^{-} < x] \quad \text{and} \quad f_{+}(x) := \mathbb{P}[M^{+} > x]$$

solve the differential equation

$$\begin{aligned} \text{(i)} \quad & f_{-}\mathrm{d}\lambda_{+} + (\lambda_{-} - \rho)\mathrm{d}f_{+} = 0, \\ \text{(ii)} \quad & f_{+}\mathrm{d}\lambda_{-} + (\lambda_{+} - \rho)\mathrm{d}f_{-} = 0 \\ \text{(iii)} \quad & f_{+}(J_{-}) = 1 = f_{-}(J_{+}). \end{aligned}$$

Proof: Since buy orders are added to $A \subset J$ at the same rate as they are removed,

$$\int_A \mathbb{P}[M^{-} < x]\,\mathrm{d}\lambda_{+}(\mathrm{d}x) + \rho \int_A \mathbb{P}[M^{+} \in \mathrm{d}x] = \int_A \lambda_{-}(x)\,\mathbb{P}[M^{+} \in \mathrm{d}x].$$

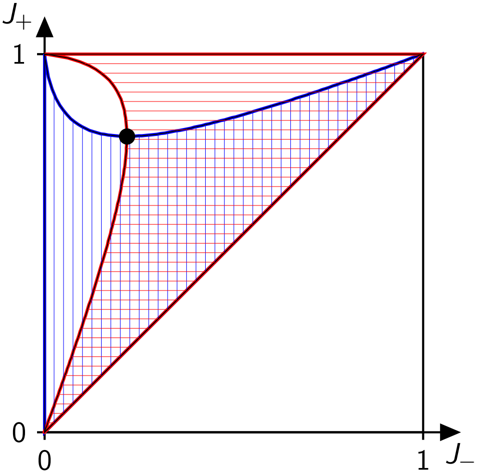
Luckock's equation

Assume $\lambda_-(J_+), \lambda_+(J_-) > 0$. Then Luckock's equation has a unique solution (f_-, f_+). In general, however, $f_\pm$ need not take values in $[0, 1]$. In such a case, by Luckock's result, no invariant law is possible.

Theorem [S. '16] Assume $\lambda_-(J_+), \lambda_+(J_-) > 0$. Then the Stigler-Luckock model on J is positive recurrent if and only if the unique solution to Luckock's equation satisfies $f_-(J_-) > 0$ and $f_+(J_+) > 0$.

Note F. Kelly and E. Yudovina have a similar, but somewhat less complete result.

Restricted models



Restrictions of the uniform model: subintervals (J_-, J_+) for which $f_-(J_-) > 0$ resp. $f_+(J_+) > 0$.

The competitive window

The dot in the previous picture indicates the *competitive window*, i.e., the unique subinterval J such that the solution of Luckock's equation on J satisfies $f_-(I_-) = 0 = f_+(I_+)$.

Look more specifically at subintervals J such that $\lambda_-(J_-) = \lambda_+(J_+)$. Define

$$J_-(V) = \sup\big\{x \in \overline{I} : \lambda_-(x) \geq V\big\},$$
$$J_+(V) = \inf\big\{x \in \overline{I} : \lambda_+(x) \geq V\big\},$$

Then $f_-(J_-)$ and $f_+(J_+)$ are either both > 0, both $= 0$, or both < 0 depending on whether $\Phi(V) < V_{\mathrm{W}}^{-2}$, $= V_{\mathrm{W}}^{-2}$, or $> V_{\mathrm{W}}^{-2}$, where

$$\Phi(V) := -\int_{V_{\mathrm{W}}}^{V} \Big\{\frac{1}{\lambda_+(J_-(U))} + \frac{1}{\lambda_-(J_+(U))}\Big\} \mathrm{d}\Big(\frac{1}{U}\Big).$$

The competitive window

The previous slide holds for models without market makers. In general, Luckock's differential equation takes the form:

$$\begin{aligned} \text{(i)}\quad & f_-\mathrm{d}\lambda_+ + (\lambda_- - \rho)\mathrm{d}f_+ = 0, \\ \text{(ii)}\quad & f_+\mathrm{d}\lambda_- + (\lambda_+ - \rho)\mathrm{d}f_- = 0 \\ \text{(iii)}\quad & f_+(J_-) = 1 = f_-(J_+). \end{aligned}$$

We can still apply the previous results provided we replace λ_- and λ_+ by

$$\tilde{\lambda}_-(x) := \lambda_-(x) - \rho \quad \text{and} \quad \tilde{\lambda}_-(x) := \lambda_-(x) - \rho.$$

Conclusion The competitive window has positive size iff $\rho < V_{\mathrm{W}}$.

Freezing

Theorem Assume that $\rho \geq V_{\mathrm{W}}$ and that λ_- and λ_+ are strictly decreasing resp. increasing. Then there exists a random variable M_∞ such that

$$\lim_{t\to\infty} M_t^- = \lim_{t\to\infty} M_t^+ = M_\infty \quad \text{a.s.}$$

The support of the law of M_∞ is given by $\{x \in \overline{I} : \lambda_-(x) \vee \lambda_+(x) \leq \rho\}$.

Proof idea Since in the long run sell limit orders are added near M_∞ at rate ρ and removed at rate $\lambda_-(M_\infty)$, no freezing is possible at a value such that $\lambda_-(M_\infty) > \rho$.
On the other hand, if $\lambda_-(x) < \rho$, then there is a positive probability that $M_t^+ \leq x$ for all $t \geq 0$. Using this and the assumption that λ_- and λ_+ are strictly decreasing resp. increasing, one can show that freezing must occur somewhere. ∎

Results of Kelly and Yudovina

Frank Kelly and Elena Yudovina. *A Markov model of a limit order book: thresholds, recurrence, and trading strategies.* Preprint (2015 and 2016) ArXiv 1504.00579.

- **Claim** For any finite initial state, $\liminf_{t\to\infty} M_t^- = J_-$ and $\limsup_{t\to\infty} M_t^+ = J_+$ a.s., where J is the competitive window.

A nice argument based on Kolmogorov's 0-1 law shows that the liminf and limsup are given by deterministic constants. The proof that they coincide with the boundary points of the competitive window is rather more involved and needs additional technical assumtions. (In particular, $c\mathrm{d}x \leq \mu_\pm \leq C\mathrm{d}x$ for some $0 < c < C < \infty$.)

Proofs use "fluid limits".

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- Study the shape of the equilibrium process near the boundary: conjecture for scaling limit made in [Formentin & S.]

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- making the supply and demand functions depend on time

Future work

A little warning: less is often more.

Before you write down the most general possible model, keep in mind that for understanding the basic principles that are at work, it is often much more useful to have a minimal working example than a model with lots of parameters.

Today we have learned that even if buyers and sellers are in one place, there is still room for market makers who make money by transporting goods not in space but in time, buying when the price is low and selling when the price is high.

And in fact, you need these people to attain Walras' equilibrium price.

Linear functionals

Let $\mathcal{X}_t^{\pm}$ denote the sets of buy and sell limit orders in the order book at time t and consider a weighted sum over the limit orders of the form

$$F(\mathcal{X}_t) := \int_J w_-(x)\mathcal{X}^-(\mathrm{d}x) + \int_J w_+(x)\mathcal{X}^+(\mathrm{d}x),$$

where $w_\pm : \overline{I} \to \mathbb{R}$ are "weight" functions.

Lemma One has

$$\tfrac{\partial}{\partial t}\mathbb{E}[F(\mathcal{X}_t)] = q_-(M_t^-) + q_+(M_t^+),$$

where $q_- : [J_-, J_+) \to \mathbb{R}$ and $q_+ : (J_+, J_-] \to \mathbb{R}$ are given by

$$\begin{aligned} q_-(x) &:= \int_x^{J_+} w_+ \mathrm{d}\lambda_+ - w_-(x)\lambda_+(x) \qquad \big(x \in [J_-, J_+)\big), \\ q_+(x) &:= -\int_{J_-}^{x} w_- \mathrm{d}\lambda_- - w_+(x)\lambda_-(x) \qquad \big(x \in (J_-, J_+]\big). \end{aligned}$$

Linear functionals

Proof of positive recurrence Assume $\lambda_-(J_+), \lambda_+(J_-) > 0$. For each $z \in \overline{J}$, there exist a unique pair of weight functions (w_-, w_+) such that

$$\frac{\partial}{\partial t}\mathbb{E}[F(\mathcal{X}_t)] = 1_{\{M_t^- \leq z\}} - f_-(z),$$

where (f_-, f_+) is the unique solution to Luckock's equation. Likewise, there exist a unique pair of weight functions (w_-, w_+) such that

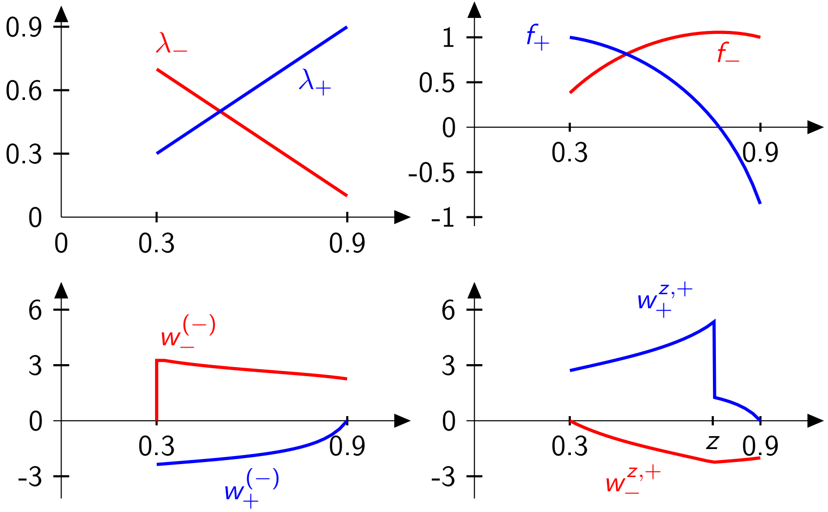
$$\frac{\partial}{\partial t}\mathbb{E}[F(\mathcal{X}_t)] = 1_{\{M_t^+ \geq z\}} - f_+(z).$$

In particular, there exist linear functionals $F^{(\pm)}$ such that

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{E}[F^{(-)}(\mathcal{X}_t)] &= 1_{\{M_t^- \leq J_-\}} - f_-(J_-),\\ \frac{\partial}{\partial t}\mathbb{E}[F^{(+)}(\mathcal{X}_t)] &= 1_{\{M_t^+ \geq J_+\}} - f_+(J_+).\end{aligned}$$

If $f_-(J_-), f_+(J_+) > 0$, then it is possible to construct a Lyapunov function from $F^{(-)}$ and $F^{(+)}$, proving positive recurrence. ∎

Linear functionals



Application of Kolmogorov's 0-1 law

Theorem [Kelly & Yudovina] $J_- := \liminf_{t\to\infty} M_t^-$ is a deterministic number.

Lemma Let $\mathcal{X}$ and $\mathcal{Y}$ be Stigler-Luckock models started in initial states such that $\mathcal{Y}_0 = \mathcal{X}_0 + \delta_x$ for some $x \in I$. If the same traders arrive in each model, then $\mathcal{Y}_t = \mathcal{X}_t + \delta_{\xi_t}$ for some $\xi_t \in I$ $(t \geq 0)$.

It follows that for any $x \in I$,

$$\mathcal{Y}_t^-((I_-, x]) \leq \mathcal{X}_t^-((I_-, x]) \leq \mathcal{Y}_t^-((I_-, x]) + 1 \qquad (t \geq 0),$$

so at any time (almost) the same buy orders below x have been removed from $\mathcal{X}$ as have been removed from $\mathcal{Y}$.

Proof of theorem Let $U_k \in I$ denote the price of the k-th trader and $\sigma_k \in \{-, +\}$ its type (buyer/seller). In view of the lemma,

$$\mathcal{E}(x) := \big\{\text{finitely many buy orders are removed from } (I_-, x]\big\}$$

is measurable w.r.t. the tail algebra of $(U_k, \sigma_k)_{k\geq 1}$. ∎