Interface tightness

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One-dimensional voter models

$\{0,1\}^{\mathbb{Z}}$ = the space of all functions $x : \mathbb{Z} \to \{0,1\}$. Interpretation:

$$x = \cdots 0000110101000110101110011111 \cdots$$

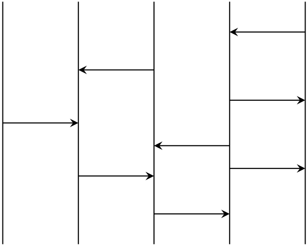
models the distribution of two genetic types of a plant, living in a one-dimensional environment (coastline, river).

$(X_t)_{t\geq 0}$ with $X_t = \big(X_t(i)\big)_{i\in\mathbb{Z}}$ continuous-time Markov process with state space $\{0,1\}^{\mathbb{Z}}$.
Dynamics: each plant lives an exponential time with mean 1, and upon death is immediately replaced by a clone of a near-by plant, at a distance chosen according to a probability distribution p.

In other words, if the present state is x, then $x(i)$ jumps:

$$\begin{array}{lll} 0 \mapsto 1 & \quad\text{with rate}\quad & \displaystyle\sum_{j\in\mathbb{Z}} p(j-i) 1_{\{x(j)=1\}}, \\ 1 \mapsto 0 & \quad\text{with rate}\quad & \displaystyle\sum_{j\in\mathbb{Z}} p(j-i) 1_{\{x(j)=0\}}. \end{array}$$

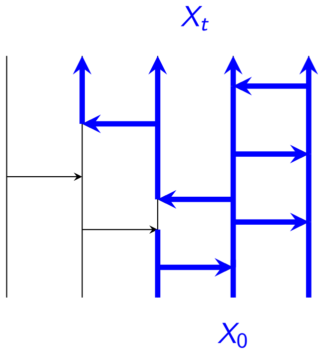
A graphical representation



We concentrate for the moment on the *nearest neighbor case*
$p(-1) = p(1) = \frac{1}{2}$.

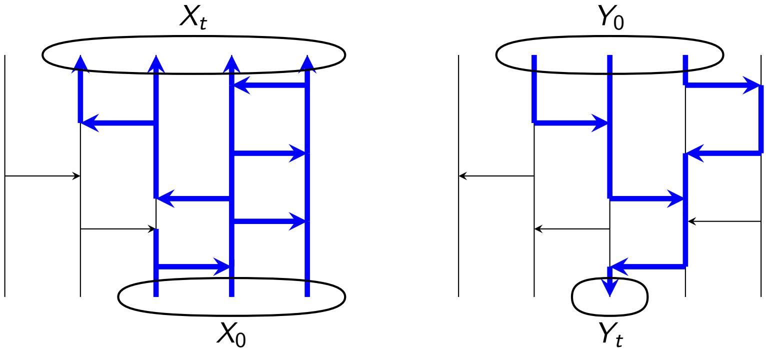
For each $i, j \in \mathbb{Z}$, at times of a Poisson process with intensity $p(j - i)$, we draw a *resampling arrow* from j to i.

A graphical representation



When there is an arrow from j to i,
the site i copies the type of site j.

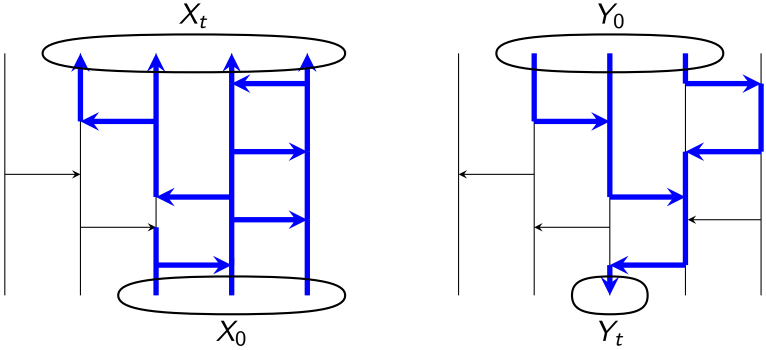
Duality



A voter model X is dual to a system of coalescing random walks Y:

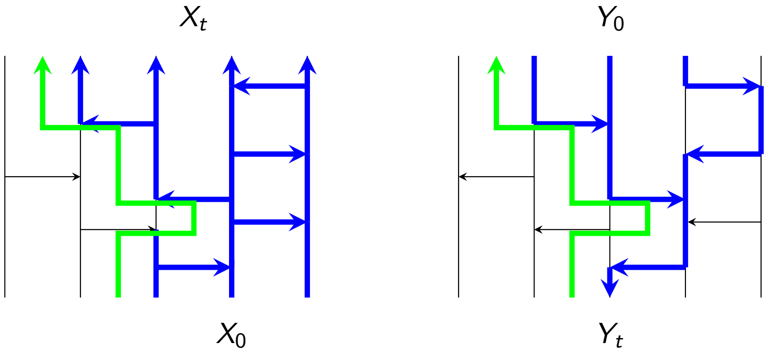
$$\mathbb{P}\big[X_t \wedge Y_0 \neq 0\big] = \mathbb{P}\big[X_0 \wedge Y_t \neq 0\big] \quad (t \geq 0).$$

Duality



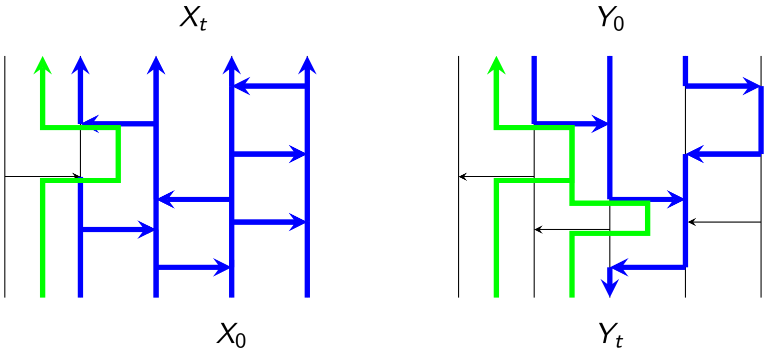
Interpretation: Set $\mathcal{Y}_t := \{i : Y_t(i) = 1\}$.
Then $\mathcal{Y}_t$ are the ancestors of $\mathcal{Y}_0$.
$X_t(i) = 1$ for some $i \in \mathcal{Y}_0$ iff $X_0(i) = 1$ for some $i \in \mathcal{Y}_t$.

Duality



Interfaces of the voter model correspond to *dual* coalescing random walks running upwards in time.

Duality

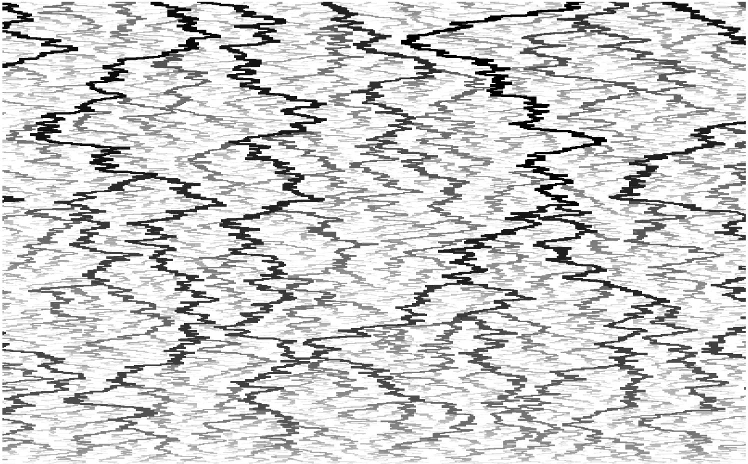


Interfaces of the voter model correspond to *dual* coalescing random walks running upwards in time.

The Brownian web

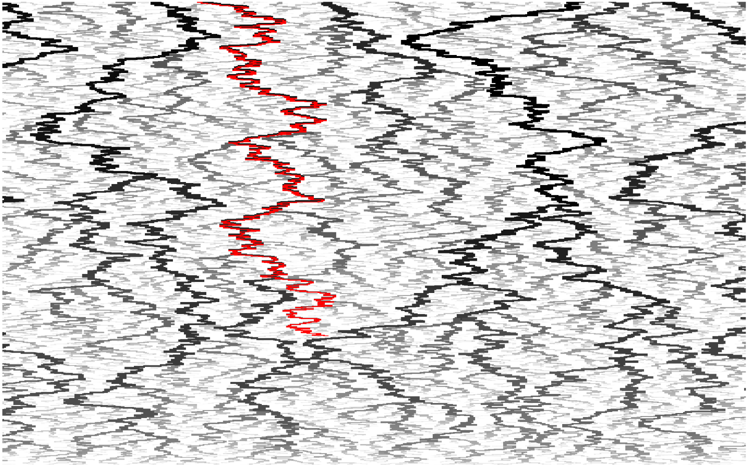
The system of coalescing random walks has a *diffusive scaling limit*, when we rescale space by ε, time by ε^2, and send $\varepsilon \downarrow 0$.

The Brownian web



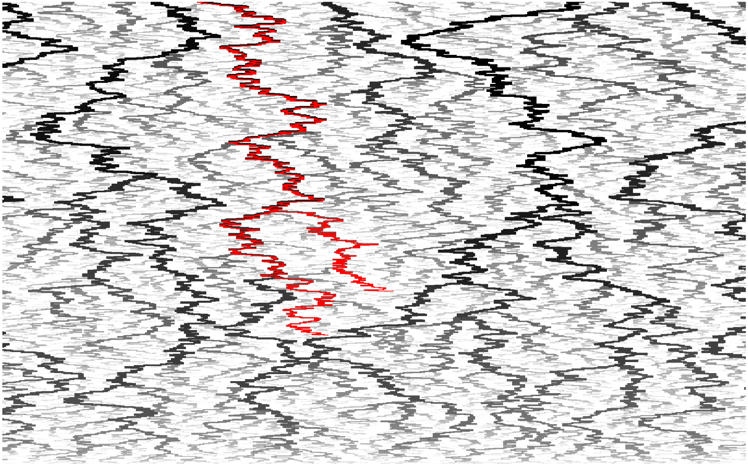
The same is true for the dual coalescing random walks running upwards.

The Brownian web



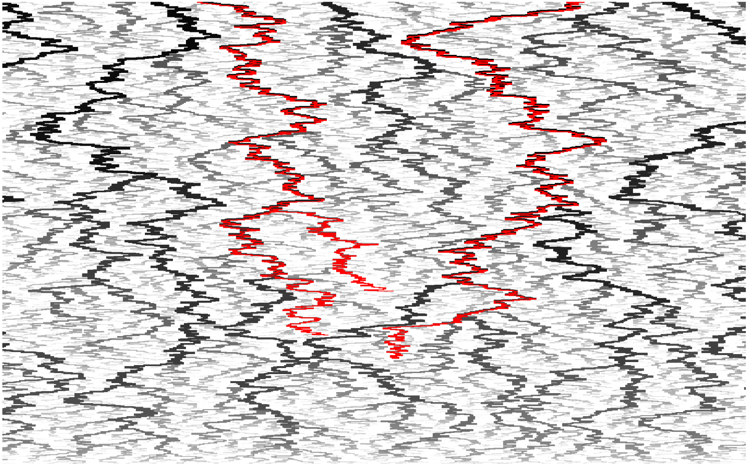
At each space-time point $(x, t) \in \mathbb{R}^2$, there starts a Brownian path.

The Brownian web



Paths started at different points coalesce.

The Brownian web



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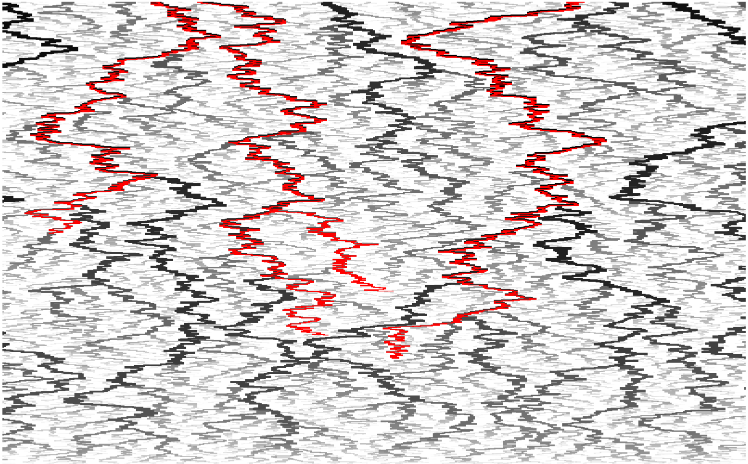

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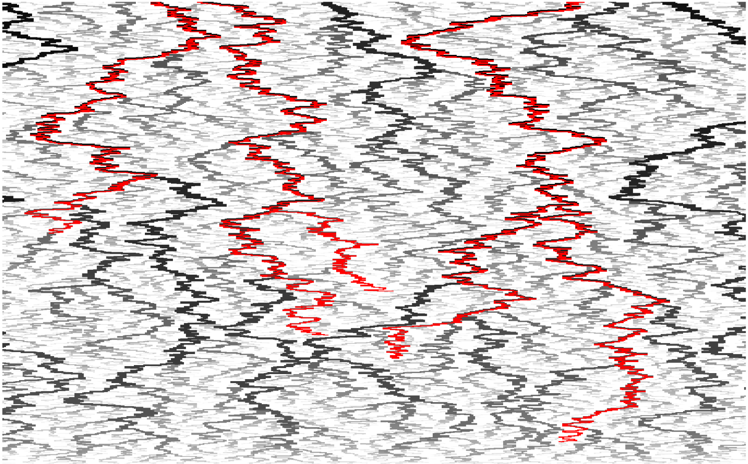

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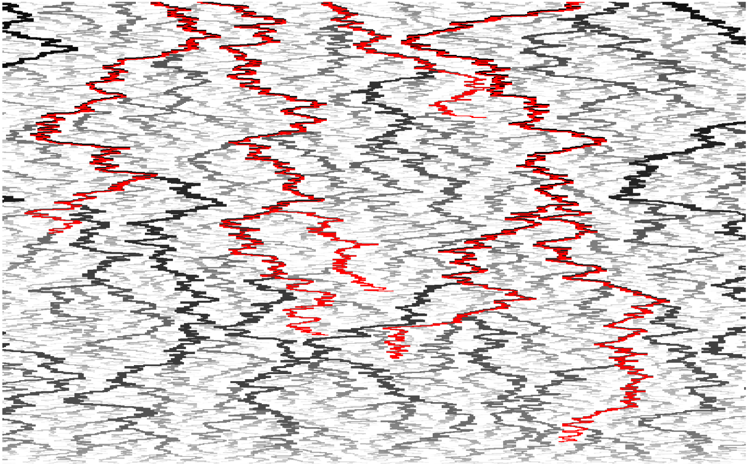

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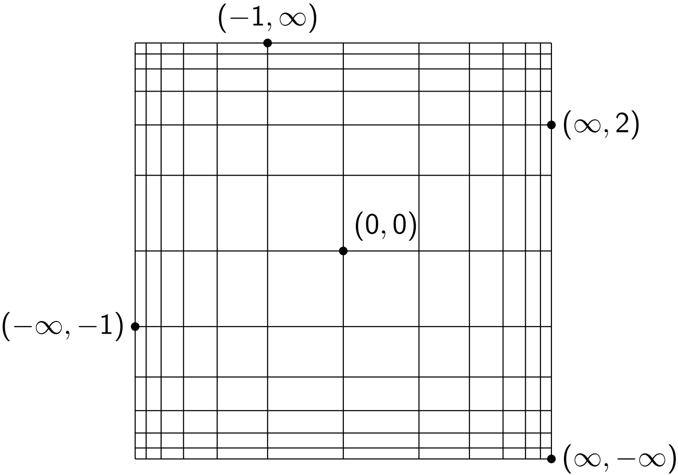
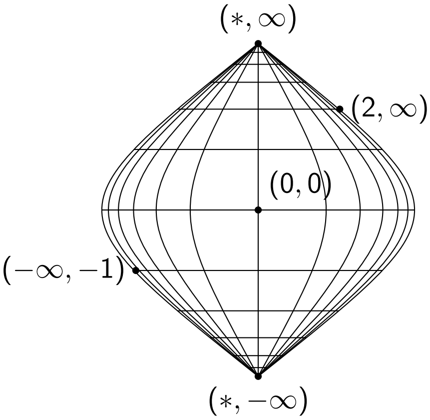
Topological matters



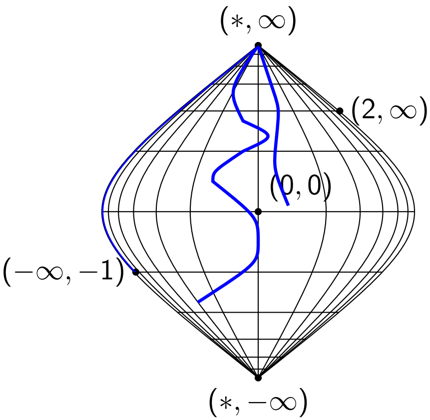
We first compactify $\mathbb{R}^2$ to $[-\infty, \infty]^2 \ldots$

Topological matters



…and then contract $[-\infty, \infty] \times \{-\infty\}$ and $[-\infty, \infty] \times \{\infty\}$ to single points.

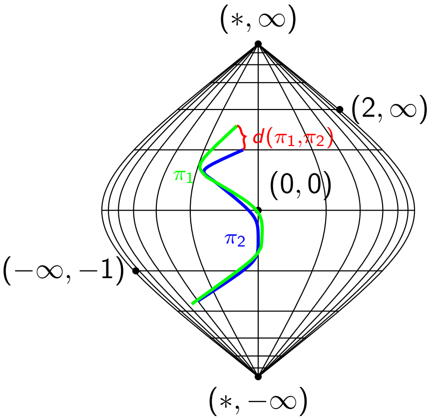
Topological matters



We identify a path $\pi:[\sigma_\pi,\infty)\to\mathbb{R}$ with (the closure of) its graph

$$\overline{\big\{(\pi(t),t):t\in[\sigma_\pi,\infty)\big\}}.$$

Topological matters



We equip the space Π of all paths with the Hausdorff metric

$$d(\pi_1, \pi_2) = \sup_{z_1 \in \pi_1} \inf_{z_2 \in \pi_2} d(z_1, z_2) \vee \sup_{z_2 \in \pi_2} \inf_{z_1 \in \pi_1} d(z_1, z_2).$$

Topological matters

We equip the space $\mathcal{K}(\Pi)$ of all compact subsets of the space of paths Π with the Hausdorff metric

$$d(\mathcal{U}_1,\mathcal{U}_2) = \sup_{\pi_1\in\mathcal{U}_1}\ \inf_{\pi_2\in\mathcal{U}_2} d(\pi_1,\pi_2) \vee \sup_{\pi_2\in\mathcal{U}_2}\ \inf_{\pi_1\in\mathcal{U}_1} d(\pi_1,\pi_2).$$

We define a diffusive scaling map S_ε by

$$S_\varepsilon(x,t) := (\varepsilon x, \varepsilon^2 t).$$

The Brownian web

Let

$$\mathcal{U} := \{\pi_{(x,s)} : x \in \mathbb{Z},\ s \in \mathbb{R}\}$$

denote the collection of coalescing random walk paths started from any point in $\mathbb{Z} \times \mathbb{R}$. By adding trivial paths that are $\equiv \pm\infty$, we can view $\mathcal{U}$ as a compact subset of Π.

[Fontes, Isopi, Newman & Ravishankar '04]

$$\mathbb{P}\big[S_\varepsilon(\mathcal{U}) \in \cdot\,\big] \underset{\varepsilon\downarrow 0}{\Longrightarrow} \mathbb{P}\big[\mathcal{W} \in \cdot\,\big]$$

where $\mathcal{W}$ is the *Brownian web*.

The Brownian web

- For each deterministic $z \in \mathbb{R}^2$, almost surely there is a unique open path $\pi_z \in \mathcal{W}$.

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The Brownian web

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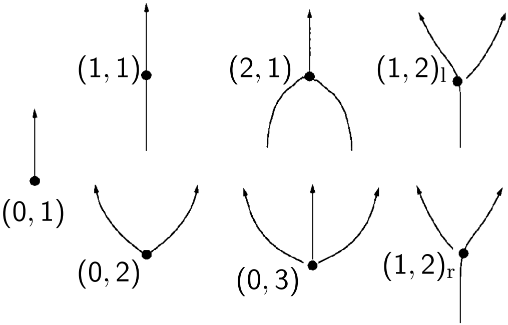
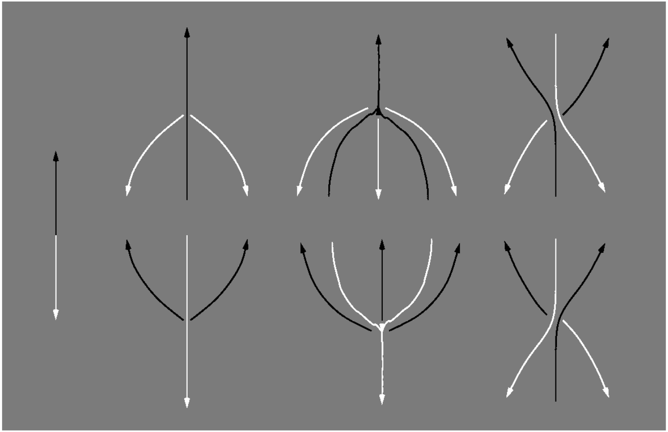
- For any deterministic finite set of points $z_1, \ldots, z_k \in \mathbb{R}^2$, the collection $(\pi_{z_1}, \ldots, \pi_{z_k})$ is distributed as coalescing Brownian motions
- For any deterministic countable dense subset $\mathcal{D} \subset \mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure of $\{\pi_z : z \in \mathcal{D}\}$.

Special points



Special points are classified according to the number of incoming and outgoing open paths. There exists 7 types of special points.

The dual Brownian web



Structure of dual open paths at special points.

The non-nearest neighbor case

[Newman, Ravishankar & Sun '05] Assume that $\sum_i |i|^{3+\delta} p(i) < \infty$ for some $\delta > 0$. Then

$$\mathbb{P}\big[S_\varepsilon(\mathcal{U}) \in \cdot\,\big] \underset{\varepsilon\downarrow 0}{\Longrightarrow} \mathbb{P}\big[\mathcal{W} \in \cdot\,\big]$$

where $\mathcal{W}$ is the Brownian web with variance $\sigma^2 := \sum_i |i|^2 p(i)$.

Proof is more difficult, because there is no (obvious) dual system of coalescing random walks.

Interfaces

$$S^{01}_{\rm int} := \Big\{x \in \{0,1\}^{\mathbb{Z}} : \exists i < j \text{ s.t. } \begin{array}{l} x(i') = 0 \ \forall i' \leq i, \\ x(j') = 1 \ \forall j' \geq j \Big\}. \end{array}$$

Interpretation: $x \in S^{01}_{\rm int}$ describes the *interface* between two infinite populations of 0's and 1's:

$$\cdots 0000000000000\underbrace{1011000110100}_{\text{interface}}11111111111111\cdots$$

Lemma
If $\displaystyle\sum_k p(k)|k| < \infty$, then $X_0 \in S^{01}_{\rm int}$ implies $X_t \in S^{01}_{\rm int}$ $\forall t \geq 0$ a.s.

Question Starting from the *Heaviside configuration*

$$x_0 := \cdots 000000000000000000000011111111111111111111\cdots$$

does the size of the interface keep growing, or does it reach some finite equilibrium size?

A voter model on $\{1,\ldots,500\}$ with periodic boundary conditions, and p the uniform distribution on $\{-2,-1,1,2\}$.
Total time elapsed 600.

Interface tightness

Def $x \sim y$ if $\exists j$ s.t. $x(i) = y(i + j)$ $(i \in \mathbb{Z})$.

Def $\overline{x} := \{y : y \sim x\}$ and $\overline{S}^{01}_{\rm int} := \{\overline{x} : x \in S^{01}_{\rm int}\}$.

Observation The *voter model modulo translations* $(\overline{X}_t)_{t \geq 0}$ is a Markov process.

Def A voter model exhibits *interface tightness on* $S^{01}_{\rm int}$ if $\overline{x}_0$ is a positive recurrent state for the Markov process $(\overline{X}_t)_{t \geq 0}$.

Theorem If $\displaystyle\sum_k p(k)|k|^2 < \infty$, then interface tightness holds on $S^{01}_{\rm int}$ and $S^{10}_{\rm int}$.

Proved when $\sum_k p(k)|k|^3 < \infty$ by Cox and Durrett (1995) and in general by Belhaouari, Mountford and Valle (2007), who moreover showed that the second moment condition is optimal.

A useful function

Cox and Durrett (1995) look at the function

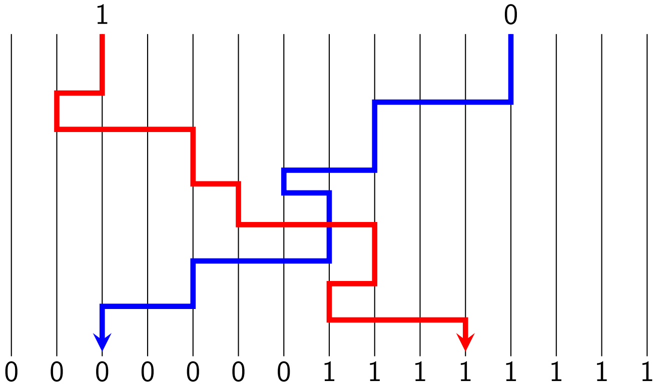
$$h(x) := \sum_{i<j} 1_{\{x(i) > x(j)\}} \qquad \big(x \in S^{01}_{\mathrm{int}}\big),$$

which counts the *number of inversions.* For the process started in the Heaviside state x_0, they used duality to prove

$$\sup_{t\geq 0} \mathbb{P}\big[h(X_t) \geq N\big] \underset{N\to\infty}{\longrightarrow} 0.$$

The function h also plays a key role in the proofs of Belhaouari, Mountford and Valle (2007).

Inversions



For an inversion, two dual coalescing random walks must cross and end up on opposite sides of the origin.

Some functions of the interface

We denote the *left* and *right boundaries* of $x \in S^{01}_{\rm int}$ by

$$L(x) := \inf\{i : x(i) = 1\} - \tfrac{1}{2} \quad \text{and} \quad R(x) := \sup\{i : x(i) = 0\} + \tfrac{1}{2},$$

and let $\ell(x) := R(x) - L(x)$ denote the *width* of the interface.

$$\cdots 0000000000000 \, \underbrace{10\overset{\overset{M(x)}{\downarrow}}{}110}_{\ell(x)=5} \, 11111111111111 \cdots$$

We also define the *midpoint* $M(x) \in \mathbb{Z} + \frac{1}{2}$ of the interface by

$$\sum_{i<M(x)} 1_{\{x(i)=1\}} = \sum_{i>M(x)} 1_{\{x(i)=0\}}.$$

The width of the interface

If interface tightness holds, then $\overline{X}_t$, started in $\overline{x}_0$, converges in law as $t \to \infty$ to some $\overline{X}_\infty$. Cox and Durrett (Theorem 6) prove that

$$\mathbb{E}\big[\ell(\overline{X}_\infty)\big] = \infty.$$

Belhaouari, Mountford, Sun and Valle (2006, Theorem 1.4) have shown that

$$\mathbb{E}\big[\ell(\overline{X}_\infty) \geq L\big] \asymp L^{-1}.$$

A pseudo-Lyapunov function

The process modulo translations $\overline{X}_t$ is a countinuous-time Markov chain with countable state space $\overline{S}^{01}_{\rm int}$.

By Foster's theorem, positive recurrence is equivalent to the existence of a Lyapunov function $V : \overline{S}^{01}_{\rm int} \to [0, \infty)$ such that

$$\begin{aligned} GV(x) &< \infty && \text{for all } x \in \overline{S}^{01}_{\rm int}, \\ GV(x) &\leq -1 && \text{for all but finitely many } x \in \overline{S}^{01}_{\rm int}, \end{aligned}$$

where G is the *generator* of $\overline{X}_t$.

For the voter model modulo translations, no such Lyapunov function has been found explicitly.

A pseudo-Lyapunov function

Sturm & S. (2008) have shown that the *number of inversions* $h(x)$ is "almost" a Lyapunov function.

More precisely,

$$Gh(x) = \frac{1}{2}\sum_{k\in\mathbb{Z}} p(k)|k|^2 - \frac{1}{2}\sum_{k\in\mathbb{Z}} p(k) I_k(x),$$

where

$$I_k(x) := \sum_{i\in\mathbb{Z}} 1_{\{x(i)\neq x(i+k)\}}$$

denotes the number of *k-boundaries.*

Since $\{\overline{x} : x \in S^{01}_{\rm int},\ Gh(x) \not\leq -1\}$ is in general not finite (except when p is almost nearest neighbor), this is not a Lyapunov function.

A pseudo-Lyapunov function

Nevertheless, it is almost as good as a Lyapunov function. One can show that if interface tightness does not hold, then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t \; \mathbb{P}\big[I_k(X_t) < N\big] = 0 \qquad (N, k \geq 1),$$

i.e., *most of the time, there are lots of k-boundaries.*

As a result, most of the time $Gh(X_t) \leq -1$, while the rest of the time $Gh(X_t) \leq \frac{1}{2} \sum_{k\in\mathbb{Z}} p(k)|k|^2 < \infty$.

This means that if interface tightness does not hold, then over long time intervals, $h(X_t)$ decreases more than it increases. Since $h \geq 0$, we arrive at a contradiction. ∎

Scaling limit of the interface

Let $L_t := L(X_t)$ and $R_t := R(X_t)$ denote the left and right boundaries of the interface and let $M_t := M(X_t)$ denote the midpoint.

Lemma If $\sum_i i^2 p(i) < \infty$, then

$$\mathbb{P}\big[\varepsilon M_{\varepsilon^{-2}t})_{t\geq 0} \in \cdot\,\big] \underset{\varepsilon\downarrow 0}{\Longrightarrow} \mathbb{P}\big[(B_t)_{t\geq 0} \in \cdot\,\big],$$

where $(B_t)_{t\geq 0}$ is Brownian motion.
If $\sum_i |i|^{3+\delta} p(i) < \infty$ for some $\delta > 0$, then moreover

$$\mathbb{P}\big[(\varepsilon L_{\varepsilon^{-2}t}, \varepsilon R_{\varepsilon^{-2}t})_{t\geq 0} \in \cdot\,\big] \underset{\varepsilon\downarrow 0}{\Longrightarrow} \mathbb{P}\big[(B_t, B_t)_{t\geq 0} \in \cdot\,\big].$$

Remark This can be used to prove that the rescaled collections of coalescing random walk paths $S_\varepsilon(\mathcal{U})$ ($\varepsilon > 0$) are tight in the Brownian web topology.

Scaling limit of the interface

Assume $\sum_k |k|^3 p(k) < \infty$.
Then the expected number of resampling arrows that start $\geq \varepsilon$ left of B_t and end $\geq \varepsilon$ right of B_t during a time interval of length one is

$$\varepsilon^{-2} \sum_{i \geq \varepsilon^{-1}} \sum_{j \geq \varepsilon^{-1}} p(i+j) = \varepsilon^{-2} \sum_{k \geq \varepsilon^{-1}} p(k)(k - \varepsilon^{-1}) \underset{\varepsilon \downarrow 0}{\longrightarrow} 0,$$

where we have used dominated convergence and

$$\varepsilon^{-2}(k - \varepsilon^{-1}) \leq k^3 \qquad (k \geq \varepsilon^{-1}).$$

Conversely, if $\sum_k |k|^3 p(k) = \infty$, we cannot expect

$$\mathbb{P}\big[(\varepsilon L_{\varepsilon^{-2}t}, \varepsilon R_{\varepsilon^{-2}t})_{t \geq 0} \in \cdot\,\big] \underset{\varepsilon \downarrow 0}{\Longrightarrow} \mathbb{P}\big[(B_t, B_t)_{t \geq 0} \in \cdot\,\big].$$

The regime where tightness fails

Recall that we identify a path $\pi : [\sigma_\pi, \infty) \to \mathbb{R}$ with the closure of its graph

$$\pi = \overline{\big\{(\pi(t), t) : t \in [\sigma_\pi, \infty)\big\}}.$$

If $\sum_k |k|^2 p(k) < \infty$ but $\sum_k |k|^3 p(k) = \infty$, we can expect

$$\mathbb{P}\big[S_\varepsilon(\mathcal{U}) \in \cdot\,\big] \underset{\varepsilon \downarrow 0}{\Longrightarrow} \mathbb{P}\big[\mathcal{W}^* \in \cdot\,\big],$$

where

$$\mathcal{W}^* := \big\{\pi \cup \{(x, \sigma_\pi)\} : \pi \in \mathcal{W},\ x \in \mathbb{R}\big\}$$

consists of all paths in the Brownian web $\mathcal{W}$ that moreover can make a jump of arbitrary size at their starting time σ_π.

Proof?

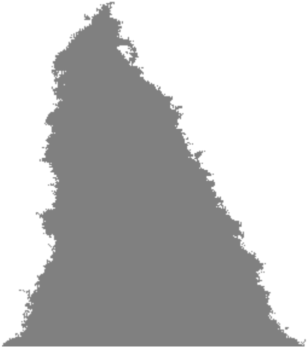
Biased voter models

In the *biased voter model* with *bias* $\varepsilon \in [0,1]$, $x(i)$ jumps:

$$
\begin{array}{lll}
0 \mapsto 1 & \text{with rate} & \displaystyle\sum_{j\in\mathbb{Z}} p(j-i) 1_{\{x(j)=1\}}, \\
1 \mapsto 0 & \text{with rate} & \displaystyle(1-\varepsilon)\sum_{j\in\mathbb{Z}} p(j-i) 1_{\{x(j)=0\}}.
\end{array}
$$

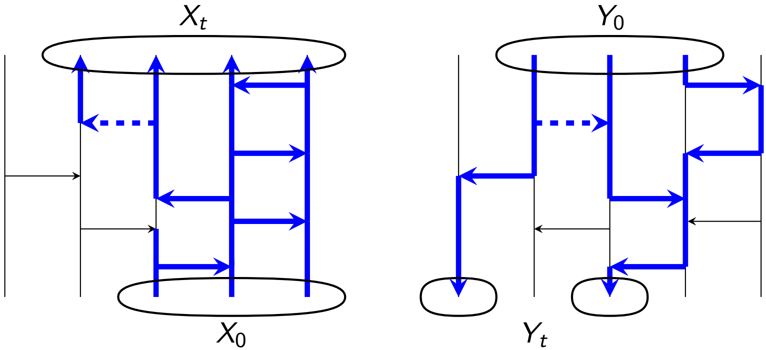
Theorem [Sun, S. & Yu '18] If $\sum_{k<0} p(k)|k| < \infty$ and $\sum_{k>0} p(k)|k|^2 < \infty$, then interface tightness holds on $S^{01}_{\rm int}$.

Numerics



A biased voter model with bias $\varepsilon = 0.3$.

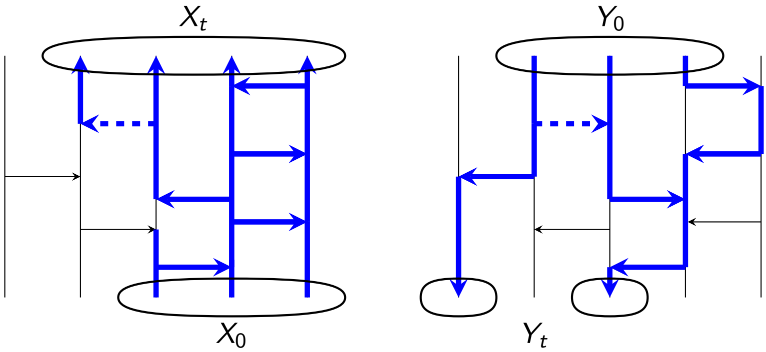
Duality for biased voter models



A biased voter model X has a branching-coalescing dual Y:

$$\mathbb{P}\big[X_t \wedge Y_0 \neq 0\big] = \mathbb{P}\big[X_0 \wedge Y_t \neq 0\big] \quad (t \geq 0).$$

Duality for biased voter models



Interpretation: Set $\mathcal{Y}_t := \{i : Y_t(i) = 1\}$.

Then $\mathcal{Y}_t$ are the *potential ancestors* of $\mathcal{Y}_0$.

$X_t(i) = 1$ for some $i \in \mathcal{Y}_0$ iff $X_0(i) = 1$ for some $i \in \mathcal{Y}_t$.

Sun, S. & Yu (2018) prove interface tightness for biased voter models using the pseudo-Lyapunov function technique of Sturm & S. Set:

$$i_0(x) := \inf\{i \in \mathbb{Z} : x(i) = 1\},$$
$$i_{n+1} := \inf\{i > i_n : x(i) = 1\}.$$

A suitable pseudo-Lyapunov function turns out to be the *weighted number of inversions*

$$h_\varepsilon(x) := \sum_{n=0}^{\infty} (1-\varepsilon)^n \sum_{j > i_n} 1_{\{x(j) = 0\}}.$$

Scaling limit

Theorem Assuming $\sum_i |i|^2 p(i) < \infty$, the equilibrium law of the width of the interface satisfies

$$\mathbb{P}\big[\ell(X^\varepsilon_\infty) \in \cdot\,\big] \underset{\varepsilon\downarrow 0}{\Longrightarrow} \mathbb{P}\big[\ell(X^0_\infty) \in \cdot\,\big]$$

Moreover, the midpoint of the interface scales to a drifted Brownian motion

$$\big(\varepsilon M(X^\varepsilon_{\varepsilon^{-2}t})\big) \underset{\varepsilon\downarrow 0}{\Longrightarrow} \big(B_t\big)_{t\geq 0}.$$

Scaling limit

Open problem Assuming $\sum_i |i|^{3+\delta} p(i) < \infty$ for some $\delta > 0$, prove that

$$\big(\varepsilon L(X^{\varepsilon}_{\varepsilon^{-2}t}), \varepsilon R(X^{\varepsilon}_{\varepsilon^{-2}t})\big) \underset{\varepsilon\downarrow 0}{\Longrightarrow} \big(B_t, B_t\big)_{t\geq 0},$$

where B_t is a drifted Brownian motion.

Open problem Our methods do not work if the resampling and selection arrows are governed by different kernels:

$$\begin{aligned} 0 \mapsto 1 \qquad &\text{with rate} \qquad (1-\varepsilon)\sum_{j\in\mathbb{Z}} p(j-i)1_{\{x(j)=1\}} \\ &\qquad\qquad +\varepsilon \sum_{j\in\mathbb{Z}} q(j-i)1_{\{x(j)=1\}}, \\ 1 \mapsto 0 \qquad &\text{with rate} \qquad (1-\varepsilon)\sum_{j\in\mathbb{Z}} p(j-i)1_{\{x(j)=0\}}. \end{aligned}$$