Stochastic Order Book dynamics

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- The model

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- Discussion

Some classical ecomomic theory

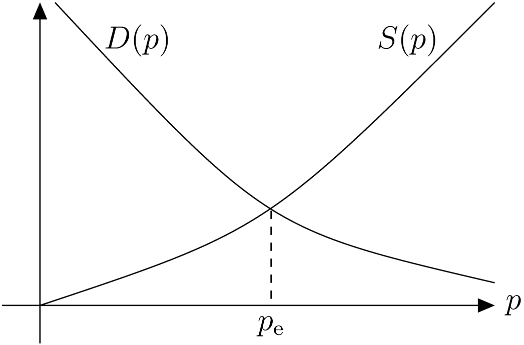
In classical economic theory, the *price* of a commodity is determined by *demand* and *supply*.

Let $D(p)$ (resp. $S(p)$) be the total *demand* (resp. *supply*) for a commodity at price level p, i.e., the total amount that could be sold (resp. bought), per unit of time, for a price of at most (resp. at least) p per unit.

Assumption $D(p)$ is strictly decreasing in p, $S(p)$ is strictly increasing in p, and there is a unique $0 < p_{\rm e} < \infty$ such that $D(p_{\rm e}) = S(p_{\rm e})$.

Postulate In an equilibrium market, the commodity is traded at the *equilibrium prize* $p_{\rm e}$.

Some classical ecomomic theory



Questions

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- How does the system evolve from one equilibrium state to another, if demand or supply change?

Stock & Commodity Exchanges & the Order Book

Stocks, as well as other derivates such as options, are usually traded at *stock exchanges*. In addition, (futures on) commodities are commonly traded at *commodity exchanges*.

On a stock exchange or commodity exchange, buyers and sellers commonly interact by means of an *order book*.

Limit and Market orders

The order book for a given asset contains a list of offers to buy or sell a given amount for a given price. Traders arriving at the market have two options.

- Place a **market order**, i.e., either *buy* (*buy market order - BMO*) or *sell* (*sell market order - SMO*) n units of the asset at the best price available in the order book.

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Market orders are matched to existing limit orders according to a mechanism that depends on the trading system.

Bid, ask, spread, midprice

- The **bid price** at time t, denoted $b(t)$, is equal to the highest price among all buy limit orders in the limit order book.
- The **ask price** at time t, denoted $a(t)$, is equal to the lowest price among all sell limit orders in the limit order book.
- The bid-ask **spread** at time t, denoted $s(t)$, is the difference between the ask and bid price: $s(t) = a(t) - b(t)$.
- The **mid price** at time t, denoted $m(t)$, is the arithemtic mean of the ask and bid price: $m(t) = (a(t) + b(t))/2$.

In equilibrium, the spread should be small and all prices should be roughly the same.

Schematic presentation of a limit order driven market

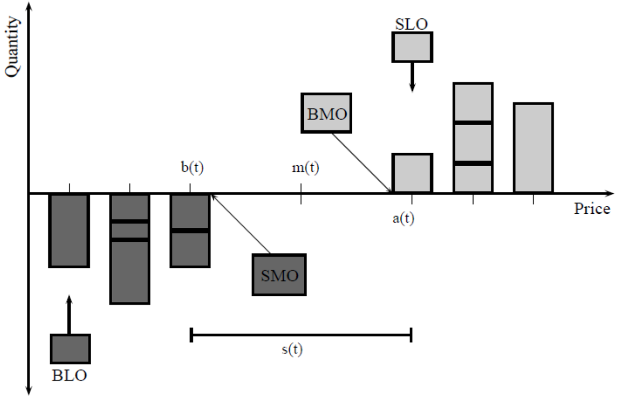

Figure 1: schematic presentation of a limit order driven market

Stochastic modeling of the price

It is observed that the prices of stocks, and to a lesser degree also commodities, evolve in a random way.

In the simplest models, the price P_t evolves as a function of (discrete) time t according to

$$\log P_t = \log P_{t-1} + r - \tfrac{1}{2}\sigma^2 + \sigma\varepsilon_t,$$

where $(\varepsilon_t)_{t\in\mathbb{Z}}$ are i.i.d. standard normal variables, r is the *interest rate* and σ the *volatility*.

GARCH models

A more realistic model for price evolution is the NGARCH model developed by Engle & Ng (1991). Here, the volatility is itself a random function of time and the price evolves according to

$$\log P_t = \log P_{t-1} + r - \tfrac{1}{2}\sigma_t^2 + \sigma_t \varepsilon_t,$$

where

$$\sigma_t^2 = \omega + \alpha\big((\sigma_{t-1}\varepsilon_{t-1})^2 - \theta\sigma_{t-1}^2\big) + \beta\sigma_{t-1}^2.$$

Here $\omega > 0$, $\alpha, \beta \geq 0$ and $\theta \in \mathbb{R}$ are parameters. If $\theta > 0$, then falling prices tend to increase volatility more than rising prices.

Even better models contain more parameters quantifying the influence of $\varepsilon_{t-2}, \varepsilon_{t-3}, \ldots$ and $\sigma_{t-2}, \sigma_{t-3}, \ldots$ on σ_t.

Aim

We wish to develop a model that:

- Is based on first principles.
- Must explain not only *how* but also *why* and prices evolve.
- Is as simple as possible.
- Need not (yet) be realistic.
- Models not only the price but also the order book.
- Explains why there is a well-defined price at all, i.e., why the spread is small.

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- If the order book contains a suitable offer, then the trader places a *market order*, i.e., sells to the highest bidder or buys from the cheapest seller.
- If the order book contains no suitable offer, then the trader places a *limit order* at his/her minimal sell or maximal buy price.

The model

A continuum version of the model: Same as before, except that minimal sell prices and maximal buy prices are uniformly distributed in $[0, 1]$.

Unrealistic elements of our model:

- One item per trader.
- Start with an empty order book.
- Uniform distribution.
- Independence, and more. . .

Expectation

What should we expect?

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In our model, the demand function is $D(p) = 100 - p$, the supply function $S(p) = p$, and the equilibrium price is $p_{\mathrm{e}} = 50$.

In spite of the greatly simplifying assumptions, we expect in great lines the right behavior, i.e., convergence to the equilibrium price. . .

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Or not??

Outcomes

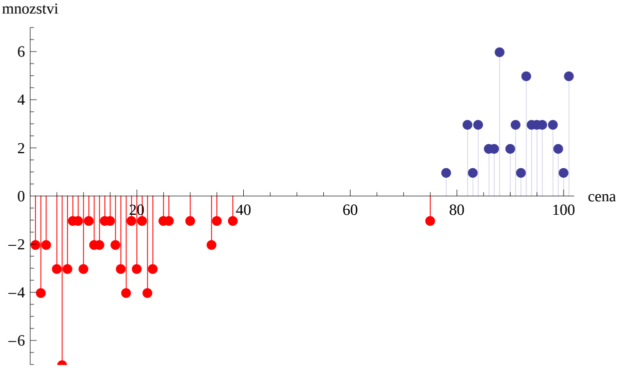

Figure: The state of the order book after 500 demands/offers.

Outcomes

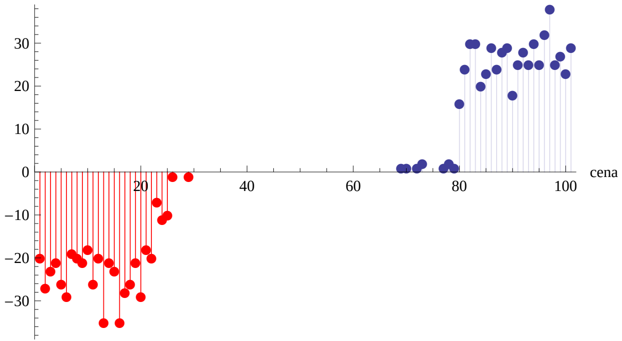

Figure: The state of the order book after 5000 demands/offers.

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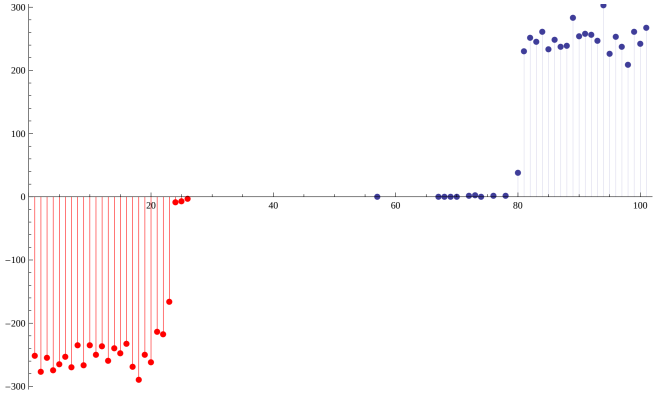

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Description of the behavior

What's happening?

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Buy limit orders (BLO's) at prices below (approximately) the 20% level are never matched by sell market orders (SMO's).

SLO's at prices above the 80% level are never matched by BMO's.

All other limit orders eventually find a seller or buyer.

Equilibrium is never attained.

Arbitrage

Interpretation 1: arbitrage

Imagine we want to buy one item for a price of at most p.
Imagine all other traders behave as in our model and a lot of time has passed since the market opened.
If $p < 20$, the trade will never happen.
If $p > 20$, *we would be stupid to buy at any price above 20, since we can get an offer at the price 20 (or perhaps 21) if we wait long enough*.

Thus, we *wait for a better price*.

In game theory: the strategy of our traders is
not a Nash equilibrium.
In economic terms: our model is *not arbitrage free*.

Impatient traders

Interpretation 2: impatient traders

Real traders must find a balance between *getting a good price* and *not waiting too long*.

Our model describes a market with *extremely impatient traders*.

Questions about the present model

- Why 20% / 80%?
- Are these exact numbers?
- Can we prove this?

Questions for a future model

- How to model patient traders?
- Can we get a model that converges to equilibrium?
- Can we get a model that is arbitrage free?

A second model

Imagine a market that is driven by *speculators* that want to *both* sell and buy, with the aim of making profit.

When new information becomes available that influences the price of a stock, most traders cancel their limit orders and the order book is for a short time (almost) empty.

Each trader assesses the new information in a different way, leading to slightly different ideas about what should be the new 'right' price of the asset.

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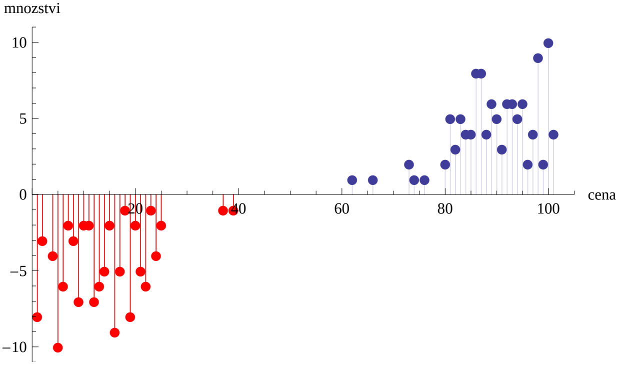
Speculators

Mathematically, this model is equivalent to a modification of the previous model, where traders always arrive in pairs of which one wants to buy and the other wants to sell, just below and above the same price.

A special feature of this new model is that the numbers of BLO's and SLO's in the order book are always equal, and strictly nondecreasing in time.

Speculators



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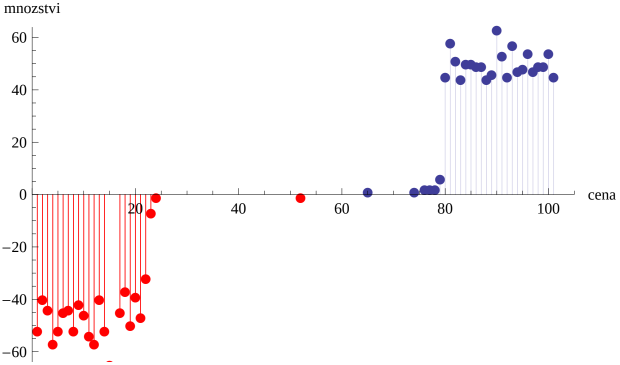

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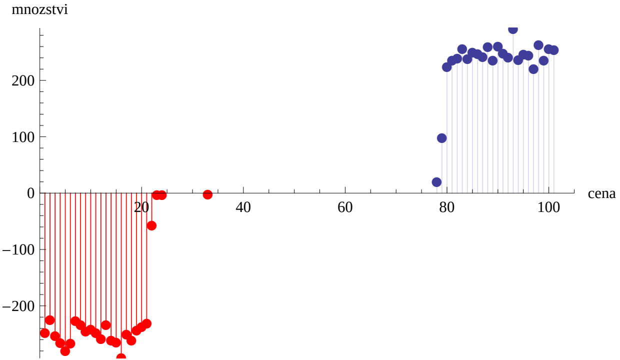

Figure: The state of the order book after 25000 demand and offers.

What's happening?

- Again 20% / 80%.
- Other statistics vary: the model has less noise.
- Explanations?