Antiferromagnetic Potts models and random colorings of planar graphs.

Jan M. Swart (Prague)

joint with A.D. Sokal (New York) and R. Kotecký (Prague)

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Gibbs measures

Let $G = (V, E)$ be a finite graph and let S be a finite set. For each *spin configuration* $\sigma : V \to S$, define a *Hamiltonian*

$$H(\sigma) := \sum_{x \in V} J_x(\sigma(x)) + \sum_{\{x,y\} \in E} J_{\{x,y\}}(\sigma(x), \sigma(y)).$$

Physically, this corresponds to the *energy* of a configuration σ. The functions J_x is represent an *external field* acting on the spin at position x while the functions $J_{\{x,y\}}$ represent an *interaction* between the spins at positions x and y.
For each *inverse temperature* $\beta \geq 0$, define a *Gibbs measure*

$$\mu_\beta(\sigma) := \frac{1}{Z_\beta} e^{-\beta H(\sigma)},$$

where the *partition sum* $Z_\beta := \sum_\sigma e^{-\beta H(\sigma)}$ is just a normalization constant.

Boundary conditions

Fix $\Lambda \subset V$ and a configuration τ. Then the conditional law

$$\mu_\beta\big(\sigma \,\big|\, \sigma = \tau \text{ on } V\backslash\Lambda\big)$$

is a Gibbs measure corresponding to the Hamiltonian

$$\begin{aligned} H(\sigma) := \sum_{x\in V} J_x(\sigma(x)) + \sum_{\substack{\{x,y\}\in E \\ x,y\in\Lambda}} J_{\{x,y\}}(\sigma(x),\sigma(y)) \\ + \sum_{\substack{\{x,y\}\in E \\ x\in\Lambda,\ y\in V\backslash\Lambda}} J_{\{x,y\}}(\sigma(x),\tau(y)). \end{aligned}$$

This can be used to define *infinite volume Gibbs measures* through the *Dobrushin-Lanford-Ruelle (DLR) conditions*.

Uniqueness of the infinite volume Gibbs measure is equivalent to the effect of the boundary conditions going to zero as $\Lambda \uparrow V$.

Motivation

Usually, G is (quasi-) transitive and H is invariant under some (quasi-) transitive subgroup of $\mathrm{Aut}(G)$. Consider large volumes $\Lambda_n \uparrow V$ and let μ_{Λ_n} be the uniform distribution on all spin configurations in Λ_n. Then we expect that the conditional laws

$$\mu_{\Lambda_n}\big(\sigma \,\big|\, H(\sigma) \approx \rho|\Lambda_n|\big)$$

locally satisfy the DLR conditions in the limit $n \to \infty$, for some suitable β depending on the *energy density* ρ. This (and related) 'facts' are known as the *equivalence of ensembles* and are related to *Large Deviations Theory*.

Potts models

We now make the choice $S = \{1, \dots, q\}$ and

$$H_\pm(\sigma) := \mp \sum_{\{x,y\} \in E} 1_{\{\sigma(x)=\sigma(y)\}}.$$

In this case, the finite-volume Gibbs measures

$$\mu^\pm_\beta(\sigma) := \frac{1}{Z_\beta} e^{-\beta H_\pm(\sigma)},$$

describe a *ferromagnetic* $(+)$ or *antiferromagnetic* $(-)$ *q-state Potts model*. In the (anti-) ferromagnetic model, neighboring spins (dis-) like to be of the same type.

Zero temperature

In the zero-temperature limit $\beta \to \infty$, the finite-volume Gibbs measure μ_β^+ converges to the uniform distribution on the ground states, which are:

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- **Antiferromagnetic:** The (proper) *q-colorings* $\{\sigma : \sigma(x) \neq \sigma(y) \ \forall \{x, y\} \in E\}$.

Ferromagnetic model

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- Phase transition of second order for small q and first order for large q.
- For $\mathbb{Z}^2$: second order for $q < 4$ and first order for $q > 4$ (proved for $q = 2$ and $q > 25$).

Antiferromagnetic model

'Facts' believed to be true:

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- For $q < q_{\mathrm{c}}$, there is a β_{c} such that the model is disordered for $\beta < \beta_{\mathrm{c}}$ and has long-range order for $\beta > \beta_{\mathrm{c}}$.
- For $\mathbb{Z}^2$, it is believed that $q_{\mathrm{c}} = 3$ and the 3-state model is critical at zero temperature.

Height mapping

Let $h : \mathbb{Z}^d \to \mathbb{Z}$ satisfy

$$|h(x) - h(y)| = 1 \quad \text{if} \quad |x - y| = 1.$$

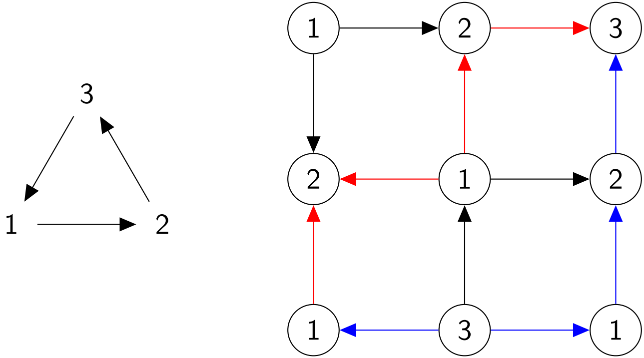
Then

$$\sigma(x) := h(x) \mod(3)$$

is a 3-coloring.

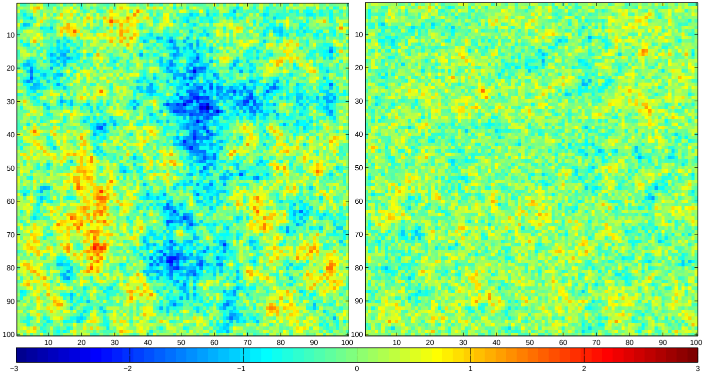
Fact: If we fix $h(x_0)$ and $\sigma(x_0)$ in one point x_0, then the mapping $h \mapsto \sigma$ is a *bijection*, i.e., we can recover h from σ.

Height mapping



The red path can be deformed into the blue path so that the height difference between the endpoints stays the same.

Height mapping



Simulations by Ron Peled of a random height mapping on a 100×100 square and the middle layer of a $100 \times 100 \times 100$ cube. Simulated using Propp-Wilson's coupling from the past.

High dimension versus dimension two

Ron Peled (preprint 2010) has proved that for sufficiently high d, a typical height-configuration is flat.
This implies (some form of) long-range order for the zero-temperature, 3-state antiferromagnetic Potts model on $\mathbb{Z}^d$.

On the other hand, on $\mathbb{Z}^2$, the fluctuations of the height model are believed to be of order log(system size). This is similar to wat is known for dimer models (R. Kenyon).

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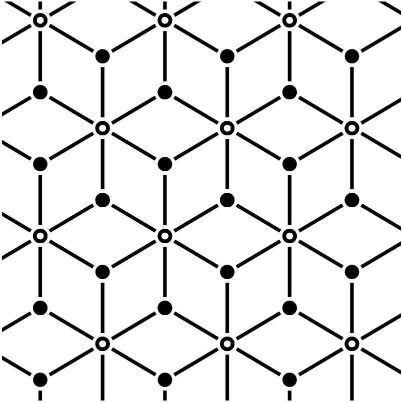
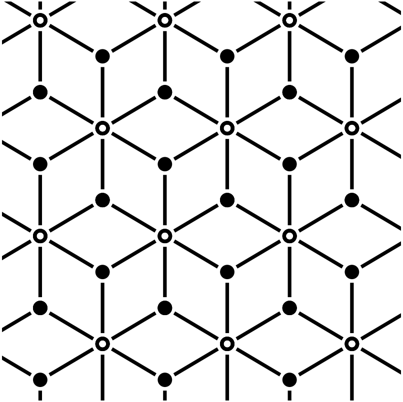
NO.

The diced lattice



Theorem (R. Kotecký, J. Salas & A.D. Sokal, 2008): The 3-state antiferromagnetic Potts model on the diced lattice has long-range order for β sufficiently large.

The diced lattice



The diced lattice:

- Is bipartite.
- Is a quadrangulation.
- Admits a height representation.

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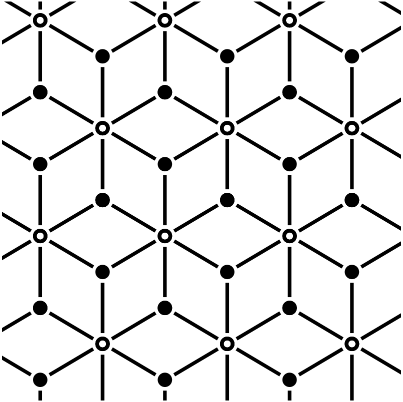

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So why is it different from $\mathbb{Z}^2$*?*

Asymmetry

Explanation 1: different densities of sublattices

For any bipartite graph, we can construct special 3-colorings by using one color for one sublattice and reserving the other two colors for the other sublattice.

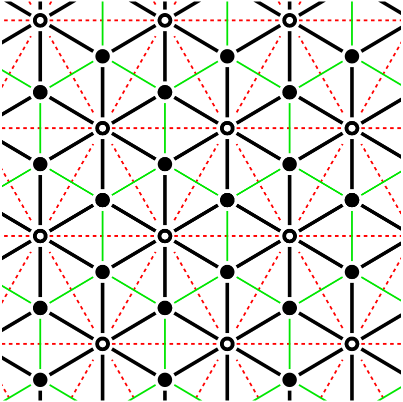
This happens *locally* on $\mathbb{Z}^2$, but on larger scales, we see infinitely many switchings between regions where one or the other sublattice is monotonely colored.

For the diced lattice, the spatial density of points of one sublattice is *twice as high* as for the other sublattice. Therefore, we can make many more configurations if we reserve two colors for this sublattice.

Effectively, this is like applying an external field that favors one sublattice.

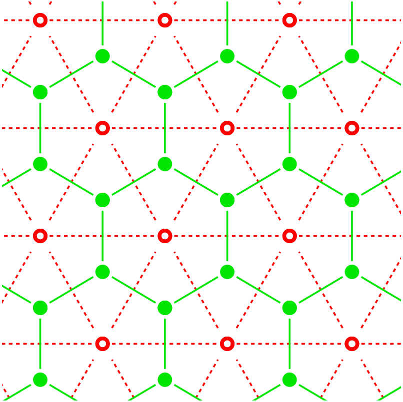
Contour model

Explanation 2: contour model



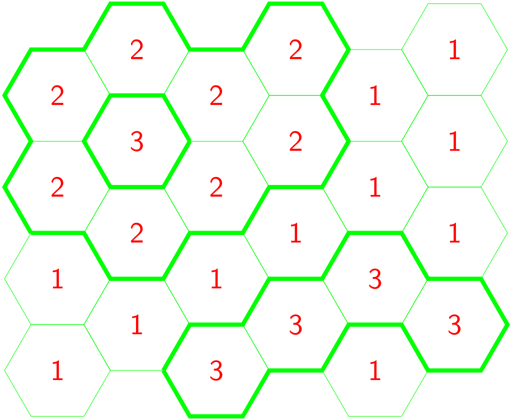
We may view the sublattices as graphs on their own, connecting vertices along the diagonals of quadrilaterals.

Contour model



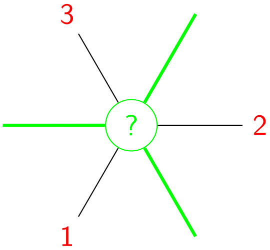
The two sublattices are dual in the sense of planar graph duality.

Contour model



We separate vertices of with different spins in the red sublattice by contours in the green sublattice.

Contour model



At zero temperature, contours are collections of simple cycles, since vertices in the green sublattice cannot be surrounded by three different types in the red sublattice.

Peierls argument

For vertices on a contour, only one type is available, while for vertices that are not on a contour, 2 types are available. As a result, for each configuration in which a given cycle is present, we can find $2^{|\gamma|}$ configurations where this contour has been removed, with $|\gamma| =$ the length of γ. Thus, the probability of a given cycle γ being present is less or equal than $2^{-|\gamma|}$ and the expected number of cycles surrounding a given vertex can be estimated by

$$\sum_{L=6}^{\infty} N(L) 2^{-L},$$

where $N(L)$ denotes the number of cycles of length L surrounding a given vertex.

Peierls argument

Let $\tilde{N}(L)$ denotes the number of self-avoiding paths of length L in the honeycomb lattice. Duminil-Copin and Smirnov (2010) have proved that

$$\lim_{L\to\infty} \tilde{N}(L)^{1/L} = \sqrt{2+\sqrt{2}},$$

i.e., the connective constant of the honeycomb lattice is $\sqrt{2+\sqrt{2}}$. Since $N(L) \leq \tilde{N}(L)$, it follows that

$$N(L) \leq \mathrm{constant} \times \big(\sqrt{2+\sqrt{2}}\big)^L.$$

Note that $\sqrt{2+\sqrt{2}} < 2$ and hence the number of cycles of length L surrounding a given vertex is bounded by

$$\mathrm{constant} \times \sum_{L=6}^{\infty} 2^{-L}\big(\sqrt{2+\sqrt{2}}\big)^L < \infty.$$

Peierls argument

Using moreover explicit counting of cycles up to length 140 due to Jensen (2006), Kotecký, Salas & Sokal (2008) were able to prove that for any vertex x in the red sublattice

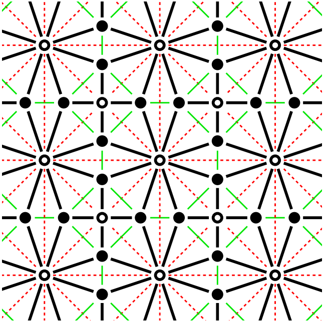
$$\mathbb{P}[x \text{ is surrounded by a cycle}] < \frac{2}{3}.$$

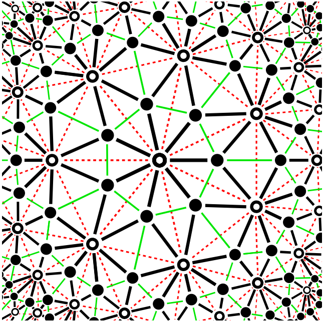
Using 1-boundary conditions on the red sublattice and letting the box size to infinity, it follows that there exists a zero-temperature infinite-volume Gibbs measure μ_∞ such that

$$\mu_\infty\big(\sigma(x) = 1\big) > \frac{1}{3}.$$

In particular, this 'positive magnetization' proves Gibbs state multiplicity and long range order.

More general lattices





We can prove positive magnetization for more general lattices, as long as the red sublattice is a triangulation.

More general lattices

Theorem Let $G = (V, E)$ be a quadrangulation of the plane, and let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be its sublattices, connected through bonds along the diagonals of quadrilaterals. Assume that G_0 is a locally finite, 3-connected, quasi-transitive triangulation with one end. Then there exist $\beta_0, C < \infty$ and $\varepsilon > 0$ such that for each inverse temperature $\beta \in [\beta_0, \infty]$ and each $k \in \{1, 2, 3\}$, there exists an infinite-volume Gibbs measure $\mu_{k,\beta}$ for the 3-state Potts antiferromagnet on G satisfying:

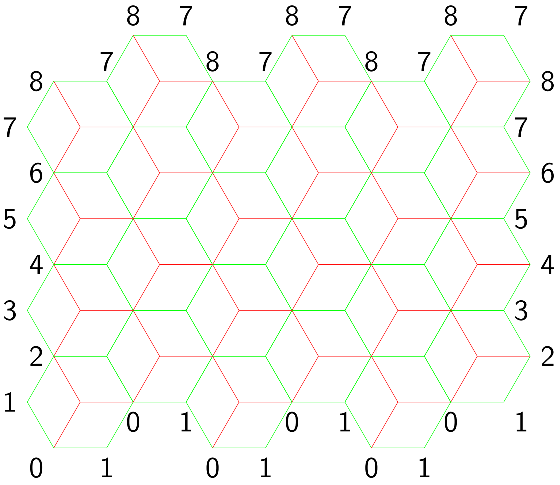
(a) For all $v_0 \in V_0$, we have $\mu_{k,\beta}(\sigma_{v_0} = k) \geq \frac{1}{3} + \varepsilon$.

(b) For all $v_1 \in V_1$, we have $\mu_{k,\beta}(\sigma_{v_1} = k) \leq \frac{1}{3} - \varepsilon$.

(c) For all $\{u, v\} \in E$, we have $\mu_{k,\beta}(\sigma_u = \sigma_v) \leq Ce^{-\beta}$.

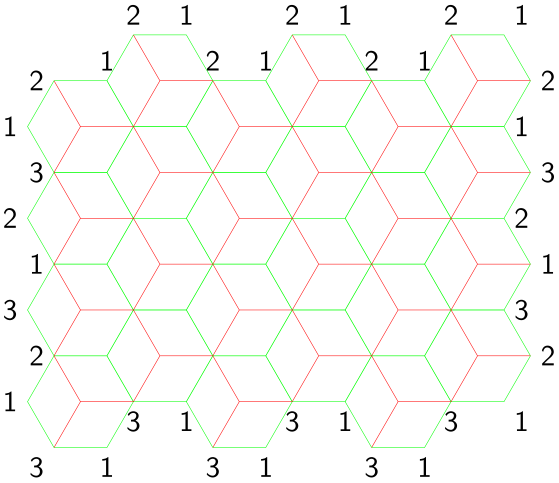
In particular, for each inverse temperature $\beta \in [\beta_0, \infty]$, the 3-state Potts antiferromagnet on G has at least three distinct extremal infinite-volume Gibbs measures.

Low temperature stability



Stiff boundary conditions for the height model...

Low temperature stability



. . . translate into b.c. for the Potts model that correspond to a unique 3-coloring of the interior.

Low temperature stability

This leads to trivial infinite-volume Gibbs measures.

But these Gibbs measures are *not* the limit of any positive temperature Gibbs measures as the temperature is sent to zero.

The reason is that at any $\beta < \infty$, we pay an energetic price of order L (surface effect) to change to more advantageous boundary conditions that lead to an entropic advantage of order L^2 (bulk effect).

Open problem: Construct such an example on a hyperbolic lattice. Stable against low-temperature perturbations?

Low temperature stability

Our Gibbs measures satisfy, for all $\beta \in [\beta_0, \infty]$:

(a) For all $v_0 \in V_0$, we have $\mu_{k,\beta}(\sigma_{v_0} = k) \geq \frac{1}{3} + \varepsilon$.

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(c) For all $\{u, v\} \in E$, we have $\mu_{k,\beta}(\sigma_u = \sigma_v) \leq Ce^{-\beta}$.

In particular, each zero-temperature Gibbs measure that is a limit of positive-temperature Gibbs measures with these properties, will also satisfy this. In particular, by (c), such a Gibbs measure is concentrated on 3-colorings.